

# ASYMPTOTIC REPRESENTATION OF HYPERGEOMETRIC BERNOULLI POLYNOMIALS OF ORDER 2 INSIDE DOMAINS RELATED TO THE ROOTS OF $e^w - 1 - w = 0$

L. NIGUSSA, A. NASIR

**Abstract.** Among several approaches towards the classical Bernoulli polynomials  $B_n(x)$ , one is the definition by the generating function

$$\frac{we^{xw}}{e^w - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{w^n}{n!} \quad \text{for } |w| < 2\pi.$$

As a generalization of  $B_n(x)$ , for any positive integer  $N$ , a new class of Bernoulli polynomials called Hypergeometric Bernoulli polynomials of order  $N$ ,  $B_n(N, x)$  was established. For the particular case  $N = 2$  these polynomials are given by

$$\frac{1}{2} \frac{w^2 e^{xw}}{e^w - 1 - w} = \sum_{n=0}^{\infty} B_n(2, x) \frac{w^n}{n!} \quad \text{for } |w| < 2\pi.$$

Several asymptotic formulas for the Bernoulli and Euler polynomials inside different domains related to the roots of  $\phi(w) = e^w - 1$  were found.

In this paper, we consider an integral representation for  $B_n(2, x)$  and establish a zero attractor for the re-scaled polynomials  $B_n(2, nx)$  for large values of  $n$ . We briefly discuss some analogous asymptotic formulas of  $B_n(2, x)$  inside domains related to the roots of  $\varphi(w) = e^w - 1 - w$ .

**Keywords:** Bernoulli polynomials, hypergeometric Bernoulli polynomials, integral representation, asymptotic formula, zero attractor.

**Mathematics Subject Classification:** 11M20, 11M26

## 1. INTRODUCTION

**1.1. Bernoulli Polynomials.** The classical Bernoulli polynomials  $B_n(x)$  were extensively considered by many authors and several generalizations were made, for which analogous properties were obtained. Alternative to the definition via the generating function

$$\frac{we^{xw}}{e^w - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{w^n}{n!} \quad \text{for } |w| < 2\pi, \tag{1.1}$$

these polynomials are equivalently defined by the recurrence formula

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} b_k x^{n-k}, \tag{1.2}$$

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where the  $b_k$  are the Bernoulli numbers. Equivalently, Bernoulli polynomials are also defined by an Appell sequence with zero mean, as

$$B_0(x) = 1, \quad B'_n(x) = nB_{n-1}(x), \quad \int_0^1 B_n(x) dx = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n > 0. \end{cases} \quad (1.3)$$

Each of the three definitions: the generating function (1.1), the recurrence formula (1.2) and the Appell sequence (1.3) define the same sequence  $\{B_n(x)\}$  of Bernoulli polynomials. The sequence of Bernoulli polynomials possess many interesting properties. Some of the well-known properties of  $B_n(x)$  are

- *Symmetry property*

$$B_n(1-x) = (-1)^n B_n(x), \quad n \geq 0.$$

- *Difference equation*

$$B_n(x+1) - B_n(x) = nx^{n-1}, \quad n \geq 1.$$

- *Addition formula*

$$B_n(x+y) = \sum_{k=0}^n \binom{n}{k} B_k(x) y^{n-k}.$$

- *Raabe's multiplication formula*

$$B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right),$$

where  $m$  and  $n$  are integers with  $n \geq 0$  and  $m \geq 1$ .

**1.2. Hypergeometric Bernoulli Polynomials.** On the base of the definition of  $B_n(x)$  by the generating function, several generalizations were made by different authors. Among these generalizations are the polynomials  $\{A_n(x)\}$  introduced by Howard [7]

$$\frac{1}{2} \frac{w^2 e^{xw}}{e^w - 1 - w} = \sum_{n=0}^{\infty} A_n(x) \frac{w^n}{n!} \quad \text{for } |w| < 2\pi.$$

Several authors considered similar generalizations of Bernoulli polynomials; we refer to [7], [3], [5] and [4] for some other related concepts to generalization of Bernoulli polynomials.

In this paper, we focus on the generalization made by Hassen and Nguyen [5] referred to as *hypergeometric Bernoulli polynomials of order  $N$* .

**Definition 1.1.** For any integer  $N \geq 1$ , hypergeometric Bernoulli polynomials of order  $N$ ,  $B_n(N, x)$ , are defined as

$$\frac{1}{N!} \frac{w^N e^{xw}}{e^w - T_{N-1}(w)} = \sum_{n=0}^{\infty} B_n(N, x) \frac{w^n}{n!} \quad \text{for } |w| < 2\pi, \quad (1.4)$$

where

$$T_m(w) = 1 + w + \frac{w^2}{2!} + \dots + \frac{w^m}{m!} = \sum_{k=0}^m \frac{w^k}{k!}.$$

The particular case of (1.4) with  $N = 1$  reduces to the classical Bernoulli polynomials  $B_n(x)$  given in (1.1) and for  $N = 2$  it represents the hypergeometric Bernoulli polynomials of order 2 defined as

$$\frac{1}{2} \frac{w^2 e^{xw}}{e^w - 1 - w} = \sum_{n=0}^{\infty} B_n(2, x) \frac{w^n}{n!} \quad \text{for } |w| < 2\pi. \quad (1.5)$$

As in the classical case, there are several approaches for defining hypergeometric Bernoulli polynomials. Accordingly, the hypergeometric Bernoulli polynomials  $B_n(N, x)$  are equivalently defined by a recurrence formula as

$$B_n(N, x) = \sum_{k=0}^n \binom{n}{k} B_k(N) x^{n-k}. \quad (1.6)$$

The hypergeometric Bernoulli polynomials  $B_n(N, x)$  are also defined in terms of an Appell sequence with zero moments as

$$\begin{aligned} B_0(N, x) &= 1, & B'_n(N, x) &= nB_{n-1}(N, x), \\ \int_0^1 (1-x)^{N-1} B_n(N, x) dx &= \begin{cases} \frac{1}{N} & \text{for } n = 0 \\ 0 & \text{for } n > 0. \end{cases} \end{aligned} \quad (1.7)$$

In [5], Hassen and Nguyen proved the equivalence of these different definitions of hypergeometric Bernoulli polynomials.

**Theorem 1.1** ([5]). *For each integer  $N \geq 1$ , the definitions of the hypergeometric Bernoulli polynomials  $B_n(N, x)$  via the generating function (1.4), the recurrence formula (1.6) and the Appell sequence (1.7) are equivalent.*

In [1], Asfaw and Hassen established the following properties of  $B_n(N, x)$  which are analogous to that of the classical Bernoulli polynomials  $B_n(x)$ .

- *Addition formula*

For hypergeometric Bernoulli polynomials of order  $N$  we have the addition formula

$$B_n(N, x+y) = \sum_{k=0}^n \binom{n}{k} B_k(N, x) y^{n-k}.$$

- *Difference equation*

For each  $n = 2, 3, 4, \dots$ , the polynomials  $B_n(2, x)$  satisfy the equation

$$B_n(2, x+1) - B_n(2, x) = nB_{n-1}(2, x) + \binom{n}{2} x^{n-2}.$$

- *Generalized difference equation*

For each positive integer  $m$  we have

$$B_n(2, x+m) = \sum_{k=0}^m \binom{m}{k} B_n^{(k)}(2, x) + \frac{n!}{2} \sum_{k=0}^{m-1} \sum_{j=0}^{m-1-k} \binom{m-1-k}{j} \frac{(x+k)^{n-2-j}}{(n-2-j)!},$$

where  $B_n^{(k)}(2, x) = n(n-1)\dots(n-k+1)B_{n-k}(2, x)$  is the  $k^{\text{th}}$  derivative of  $B_n(2, x)$ .

**Remark 1.1.** *The three alternative definitions; (1.4), (1.6) and (1.7) of  $B_n(N, x)$  are analogous to the three definitions of  $B_n(x)$  given in (1.1), (1.2) and (1.3), respectively. This interesting relation between  $B_n(N, x)$  and  $B_n(x)$  motivated us to study further analogous properties of these new class of polynomials, including the integral representations and asymptotic formulas of  $B_n(2, x)$ .*

## 2. PRELIMINARIES

**2.1. Some properties of roots of  $e^z - 1 - z = 0$ .** The function  $\varphi(w) = e^z - 1 - z$  appears in the generating function of  $B_n(2, a)$  in (1.5). The roots of this function are basic quantities

for the series representation of  $B_n(2, x)$ . In [6], Hassen and Nguyen discussed several concepts related to the roots of  $\varphi(z)$ .

If  $z_k = x_k + iy_k$  is a root of  $\varphi(z)$ , then its complex conjugate  $\bar{z}_k = x_k - iy_k$  is also a root, and we usually list all the roots in pairs as  $\{z_k, \bar{z}_k\}$ . In fact, all the roots  $z_k = x_k + iy_k$  and  $\bar{z}_k = x_k - iy_k$  of  $\varphi$  lie inside the right half-plane.

The following results are proved in [6] with slight modifications made in [1].

**Lemma 2.1.** *Let  $z_k = x_k + iy_k = r_k e^{i\theta_k}$  be the roots of  $\varphi(z)$  that lie in the upper half of the complex plane and such that  $0 < r_1 < r_2 < r_3 < \dots$ .*

(i) *The inequality*

$$2\pi k + \frac{\pi}{3} < y_k < 2\pi k + \frac{\pi}{2}, \quad k = 1, 2, \dots$$

*holds. Moreover, for each  $k$  there exists exactly one root  $z_k$  with the imaginary part obeying this inequality, and there are no other zeros elsewhere in the complex plane.*

(ii) *The arguments  $\theta_k$  of  $z_k$  obey the inequalities  $\theta_1 < \theta_2 < \theta_3 < \dots < \frac{\pi}{2}$ , and  $\theta_k \rightarrow \frac{\pi}{2}$  as  $k \rightarrow \infty$ .*

(iii) *As  $k$  increases, all the quantities  $x_k$ ,  $y_k$  and  $\theta_k$  increase. In addition, we have  $y_k < e^{x_k}$  and  $y_k \rightarrow e^{x_k}$  as  $k \rightarrow \infty$ .*

**2.2. Dominant roots.** There are two roots of  $\varphi(z)$  with minimal moduluses. These are  $z_1 = x_1 + iy_1 = r_1 e^{i\theta_1}$  and  $\bar{z}_1 = x_1 - iy_1 = r_1 e^{-i\theta_1}$ , and we call them dominant roots. The approximate values of  $x_1$ ,  $y_1$ ,  $r_1$  and  $\theta_1$  (as calculated by Mathematica) are  $x_1 \approx 2.0888$ ,  $y_1 \approx 7.4615$ ,  $r_1 \approx 7.7484$  and  $\theta_1 \approx 1.2978$ . That is,

$$z_1 \approx 2.0888 + i 7.4615, \quad \bar{z}_1 \approx 2.0888 - i 7.4615. \quad (2.1)$$

We also list approximate values for a few first roots  $z_k = x_k + iy_k = r_k e^{i\theta_k}$ .

k	$x_k$	$y_k$	$r_k$	$\theta_k$
1	2.0888	7.4615	7.7484	1.2978
2	2.6641	13.879	14.132	1.3812
3	3.0263	20.224	20.449	1.4223
4	3.2917	26.543	26.747	1.4474
5	3.5013	32.851	33.037	1.4646
6	3.6745	39.151	39.323	1.4772
7	3.8222	45.447	45.608	1.4869

TABLE 1. Few approximate values of  $z_k = x_k + iy_k = r_k e^{i\theta_k}$

On Figure 1 below we plot a few first roots  $z_k$ .

**2.3. Szegő curves related to the roots of  $\varphi(z)$ .** Consider the polynomials  $T_n(x)$ , the Taylor polynomials of the function  $e^x$ ,

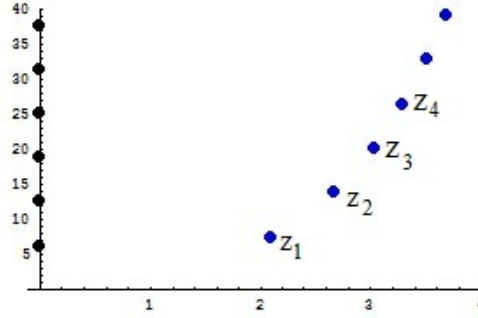
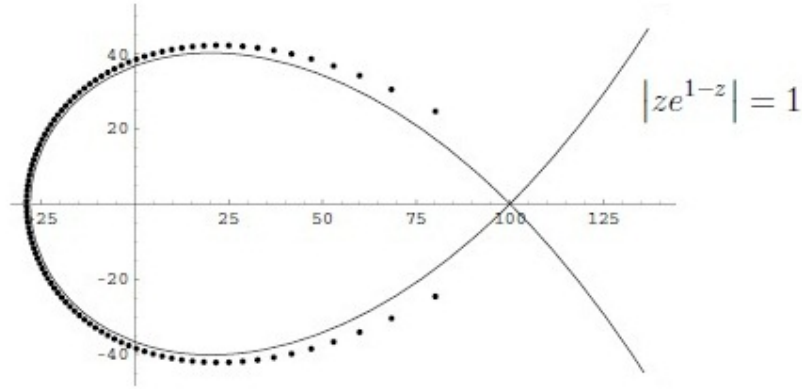
$$T_n(x) = \sum_{k=0}^n \frac{x^k}{k!}.$$

In terms of generating functions, the functions  $T_n(x)$  are given by

$$\frac{e^{xw}}{1-w} = \sum_{n=0}^{\infty} T_n(x) w^n.$$

The zeros of  $T_n(x)$  have an interesting asymptotic behavior: the zeros of  $T_n(nx)$  approach the curve  $|ze^{1-z}| = 1$  for large values of  $n$ , see Figure 2.

The curve  $|ze^{1-z}| = 1$  was first introduced by Gabor Szegő in 1924.

FIGURE 1. Plots of some roots of  $\varphi(z)$ FIGURE 2. The complex zeros of  $T_{100}(100x)$  along the curve  $|ze^{1-z}| = 1$ 

**Definition 2.1** (Standard Szegő curve). Let  $\phi(z) = ze^{1-z}$ . The curve  $\mathbb{S}$  in the complex plane defined by

$$\mathbb{S} = \{z \in \mathbb{C} : |ze^{1-z}| = 1, |z| \leq 1\} \quad (2.2)$$

is called standard Szegő curve.

We observe that the function  $\varphi(z)$  appears in (1.5) and recall its roots  $z_k$  for  $k = 1, 2, 3, \dots$ . Then we consider the function  $\phi(z_k w) = wz_k e^{1-z_k w}$  and we define different Szegő curves related to the roots of  $\varphi(z)$ .

**Definition 2.2** (Szegő curves). Let  $z_k = x_k + iy_k$ ,  $k = 1, 2, 3, \dots$  be the roots of  $\varphi(z)w$  with  $|z_k| = r_k$ . The curves  $\frac{1}{z_k}\mathbb{S}$ , called Szegő curves related to the roots of  $\varphi(z)$ , are defined as

$$\frac{1}{z_k}\mathbb{S} = \left\{z \in \mathbb{C} : |zz_k e^{1-z_k z}| = 1, |z| \leq \frac{1}{r_k}\right\}. \quad (2.3)$$

The two special Szegő curves related to the dominant roots  $z_1$  and  $\bar{z}_1$  are called dominant Szegő curves. We denote the dominant Szegő curves by  $\mathbb{S}_1 = \frac{1}{z_1}\mathbb{S}$  and  $\hat{\mathbb{S}}_1 = \frac{1}{\bar{z}_1}\mathbb{S}$ , respectively,

$$\mathbb{S}_1 = \left\{z \in \mathbb{C} : |zz_1 e^{1-z_1 z}| = 1, |z| \leq \frac{1}{r_1}\right\}, \quad (2.4)$$

$$\hat{\mathbb{S}}_1 = \left\{z \in \mathbb{C} : |z\bar{z}_1 e^{1-\bar{z}_1 z}| = 1, |z| \leq \frac{1}{r_1}\right\}. \quad (2.5)$$

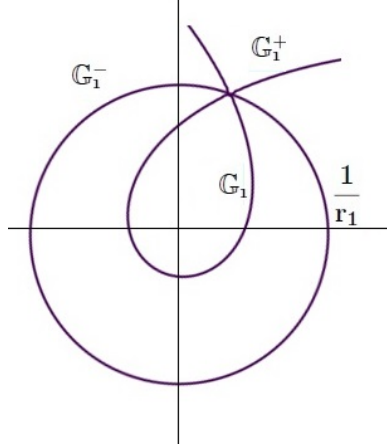


FIGURE 3. Three different open domains in the complex plane.

Let the domains  $\mathbb{G}_1 = \mathbb{G}_{z_1}$  and  $\hat{\mathbb{G}}_1 = \mathbb{G}_{\bar{z}_1}$  be the interiors of  $\mathbb{S}_1$  and  $\hat{\mathbb{S}}_1$ , respectively, and let  $\mathbb{G} = \mathbb{G}_1 \cup \hat{\mathbb{G}}_1$ . In other words,  $\mathbb{S}_1$  and  $\hat{\mathbb{S}}_1$  are the boundaries of  $\mathbb{G}_1$  and  $\hat{\mathbb{G}}_1$ . That is,  $\mathbb{S}_1 = \partial\mathbb{G}_1$  and  $\hat{\mathbb{S}}_1 = \partial\hat{\mathbb{G}}_1$ .

**Remark 2.1.** We observe that each Szegő curve given in (2.3) is obtained from the standard Szegő curve (2.2) by a dilatation by  $\frac{1}{r_k}$  and a rotation by  $\pm\theta_k$ . Indeed, the function  $\phi(z)$  is conformal in the unit disk  $B(0, 1)$ , hence both  $\phi(z_1 z)$  and  $\phi(\bar{z}_1 z)$  are also conformal in the disk  $B(0, \frac{1}{r_1})$ .

For particular values of  $z_1$  and  $\bar{z}_1$  given in (2.1), the curves  $\mathbb{S}_1$  and  $\hat{\mathbb{S}}_1$  and their interiors  $\mathbb{G}_1$  and  $\hat{\mathbb{G}}_1$  are roughly sketched as shown on Figure 4.

If we omit the restriction  $|z| \leq \frac{1}{r_1}$  in (2.4) and (2.5), the equations  $|\phi(z_1 z)| = 1$  and  $|\phi(\bar{z}_1 z)| = 1$  define unbounded curves in the complex plane. We denote these unbounded Szegő curves related to  $z_1$  and  $\bar{z}_1$  by  $\Gamma_1$  and  $\hat{\Gamma}_1$ , respectively,

$$\Gamma_1 = \{z \in \mathbb{C} : |\phi(z_1 z)| = 1\}, \quad \hat{\Gamma}_1 = \{z \in \mathbb{C} : |\phi(\bar{z}_1 z)| = 1\}.$$

Then  $\mathbb{S}_1 \subseteq \Gamma_1$  and  $\hat{\mathbb{S}}_1 \subseteq \hat{\Gamma}_1$ . Indeed,  $\mathbb{S}_1$  and  $\hat{\mathbb{S}}_1$  are the portions of  $\Gamma_1$  and  $\hat{\Gamma}_1$  that lie in the closed disk  $\bar{B}\left(0, \frac{1}{r_1}\right)$ , respectively. As an alternative to (2.4) and (2.5), the dominant Szegő curves  $\mathbb{S}_1$  and  $\hat{\mathbb{S}}_1$  can equivalently be described as

$$|\phi(z_1 z)| = 1, \operatorname{Re}(z_1 z) \leq 1, \quad |\phi(\bar{z}_1 z)| = 1, \operatorname{Re}(\bar{z}_1 z) \leq 1. \quad (2.6)$$

Here  $\mathbb{S}_1$  is the portion of  $\Gamma_1$  that lies in the closed half-plane  $\operatorname{Re}(z_1 z) \leq 1$ . In other words, both (2.4) and (2.6) define the same curve, the Szegő curve  $\mathbb{S}_1$ .

The curve  $\Gamma_1$  divides the complex plane into three different domains. We denote these domains by  $\mathbb{G}_1$ ,  $\mathbb{G}_1^+$  and  $\mathbb{G}_1^-$ , where

1.  $\mathbb{G}_1$  is the interior of the Szegő curve  $\mathbb{S}_1$ , given by

$$\mathbb{G}_1 = \left\{z \in \mathbb{C} : |\phi(z_1 z)| < 1, |z| \leq \frac{1}{r_1}\right\}. \quad (2.7)$$

2.  $\mathbb{G}_1^+$  is the unbounded domain given by

$$\mathbb{G}_1^+ = \left\{z \in \mathbb{C} : |\phi(z_1 z)| < 1, |z| > \frac{1}{r_1}\right\}. \quad (2.8)$$

3.  $\mathbb{G}_1^-$  is the unbounded domain given by

$$\mathbb{G}_1^- = \{z \in \mathbb{C} : |\phi(z_1 z)| > 1\}. \quad (2.9)$$

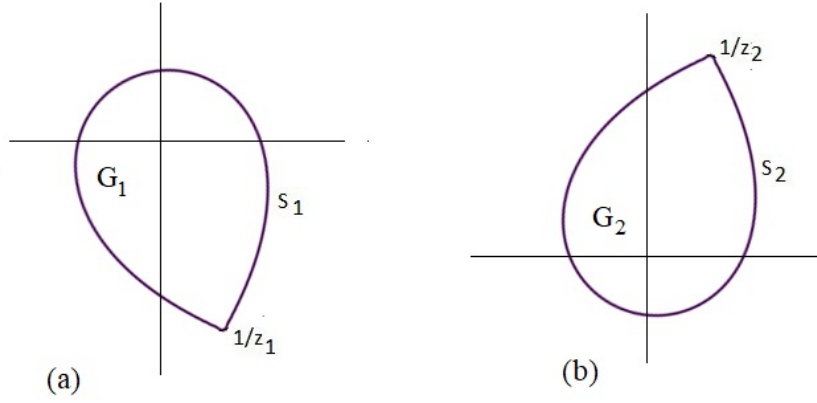


FIGURE 4. The Szegő curves  $|z_1 z e^{1-z_1 z}| = 1$  and  $|\bar{z}_1 z e^{1-\bar{z}_1 z}| = 1$  for  $|z| \leq \frac{1}{r_1}$ .

For the dominant root  $\bar{z}_1$ , we simply replace  $z_1$  by  $\bar{z}_1$  in (2.7), (2.8) and (2.9) to define the corresponding domains  $\hat{\mathbb{G}}_1$ ,  $\hat{\mathbb{G}}_1^+$  and  $\hat{\mathbb{G}}_1^-$ , respectively. The introduced domains are shown on Figure 3.

**Lemma 2.2.** *The Szegő curves  $\mathbb{S}_1$  and  $\hat{\mathbb{S}}_1$  intersect each other at exactly two points, which are located on the real axis.*

*Proof.* These two curves intersect at point  $z \in \mathbb{C}$  if and only if  $|\phi(z_1 z)| = |\phi(\bar{z}_1 z)|$ . That is,  $z = x + iy$  is an intersection point only if it satisfies

$$|e^{1-(x_1+iy_1)(x+iy)}| = |e^{1-(x_1-iy_1)(x+iy)}|.$$

This yields  $e^{y_1 y} = e^{-y_1 y}$ , which implies  $\text{Im}(z) = y = 0$ , hence  $z$  lies on the real axis.  $\square$

Let  $P$  and  $Q$  be the two points of intersection of  $\mathbb{S}_1$  and  $\hat{\mathbb{S}}_1$ . Then the line connecting  $P$  and  $Q$ ,  $L_{P,Q}$ , is the real axis. Hence,  $L_{P,Q}$  divides the complex plane into the upper half-plane  $\mathcal{H}^+$  and lower half-plane  $\mathcal{H}^-$ . We can express these half-planes as

$$\mathcal{H}^+ = \{z \in \mathbb{C} : |\phi(z_1 z)| > |\phi(\bar{z}_1 z)|\}, \quad \mathcal{H}^- = \{z \in \mathbb{C} : |\phi(z_1 z)| < |\phi(\bar{z}_1 z)|\}.$$

**Definition 2.3.** *Let  $z_1$  and  $\bar{z}_1$  be the two dominant zeros of  $\varphi(z)$ . If  $\mathbb{S}_1$  and  $\hat{\mathbb{S}}_1$  are their corresponding Szegő curves, then we define the domains  $\mathbb{D}_1 = \mathbb{D}_{z_1}$  and  $\hat{\mathbb{D}}_1 = \mathbb{D}_{\bar{z}_1}$  as*

$$\mathbb{D}_1 = \mathbb{G}_1 \cap \mathcal{H}^-, \quad \hat{\mathbb{D}}_1 = \hat{\mathbb{G}}_1 \cap \mathcal{H}^+,$$

where  $\mathbb{G}_1$  and  $\hat{\mathbb{G}}_1$  are the interiors of  $\mathbb{S}_1$  and  $\hat{\mathbb{S}}_1$ , respectively. We define the domains  $\mathbb{D}$  as the union of  $\mathbb{D}_1$  and  $\hat{\mathbb{D}}_1$ . That is,

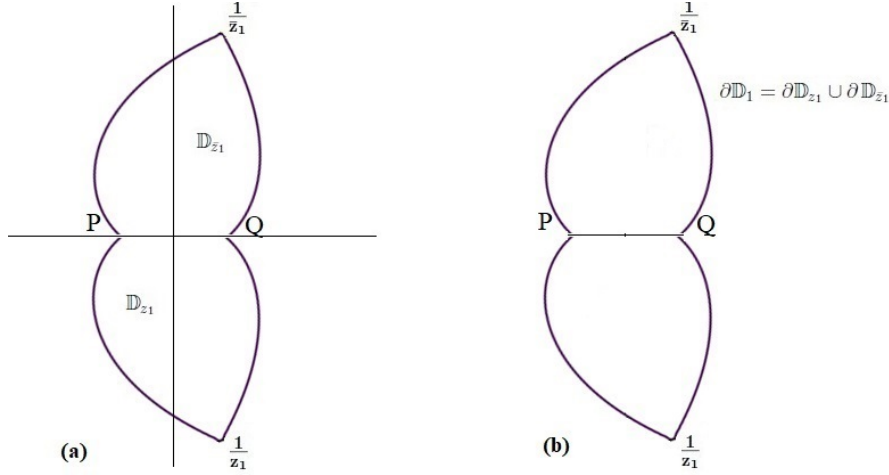
$$\mathbb{D} = \mathbb{D}_1 \cup \hat{\mathbb{D}}_1.$$

The Szegő curves  $\mathbb{S}_1$  and  $\hat{\mathbb{S}}_1$  intersect each other at two points, say  $P$  and  $Q$ , which are on the real axis. The domains  $\mathbb{D}_1$  and  $\hat{\mathbb{D}}_1$  are shown in Figure 5 below.

Both  $\mathbb{D}_1$  and  $\hat{\mathbb{D}}_1$  are disjoint open sets. That is,  $\mathbb{D}_1 \cap \hat{\mathbb{D}}_1 = \emptyset$ , but they have line segment  $PQ$  as their common boundary. Indeed, the boundary  $\partial \mathbb{D} = \partial \mathbb{D}_1 \cup \partial \hat{\mathbb{D}}_1$  includes the outer boundary points and all points of the line segment  $PQ$ . Moreover, the domains  $\mathbb{D}_1$  and  $\hat{\mathbb{D}}_1$  can be expressed as

$$\mathbb{D}_1 = \{z \in \mathbb{G}_1 : |\phi(z_1 z)| < |\phi(\bar{z}_1 z)|\}, \quad \hat{\mathbb{D}}_1 = \{z \in \hat{\mathbb{G}}_1 : |\phi(\bar{z}_1 z)| < |\phi(z_1 z)|\}.$$

Note that  $\mathbb{D} = \mathbb{D}_1 \cup \hat{\mathbb{D}}_1 \subseteq \mathbb{G} = \mathbb{G}_1 \cup \hat{\mathbb{G}}_1$ . The domains  $\mathbb{D}$  and  $\mathbb{G}$  are different but they have identical closures. That is,  $\overline{\mathbb{D}} = \overline{\mathbb{G}}$ .

FIGURE 5. Szegő domains  $\mathbb{D}_{z_1}$  and  $\mathbb{D}_{\bar{z}_1}$  and their boundaries

We now present the fact that  $z_1$  and  $\bar{z}_1$  dominate all the other roots of  $\varphi(w) = e^w - 1 - w$ . In fact, this is one of our results which we think is new in our study.

**Theorem 2.1.** *If  $z_k = x_k + iy_k = r_k e^{i\theta_k}$  is an arbitrary zero of  $\varphi(z)$  with  $y_k > 0$  and  $|z_k| = r_k \geq r_2$ , then  $\frac{1}{z_k}$  lies inside the domain  $\mathbb{D}_1$ . More generally, for each root  $z_k$  of  $\varphi(z)$  with  $|z_k| > r_1$  we have  $\frac{1}{z_k} \in \mathbb{D} = \mathbb{D}_1 \cup \hat{\mathbb{D}}_1$ .*

*Proof.* Let  $z_k = x_k + iy_k = r_k e^{i\theta_k}$  be as given in the hypothesis. Then the relation

$$\left| \frac{1}{z_k} \right| = \frac{1}{r_k} < \frac{1}{r_1}$$

shows that  $\frac{1}{z_k} \in B(0, \frac{1}{r_1})$  for all  $k \geq 2$ . This yields  $\frac{1}{z_k} \notin \bar{\mathbb{G}}_1^+$ . For the function  $\phi(z_1 z) = z z_1 e^{1-z_1 z}$  we evaluate  $|\phi(z_1 z)|$  at  $z = \frac{1}{z_k}$ , observe that  $r_1 < r_2 < r_k$  for all  $k > 2$ , and we get

$$\left| \phi \left( z_1 \frac{1}{z_k} \right) \right| = \frac{r_1}{r_k} e^{1 - \frac{(x_1 x_k + y_1 y_k)}{r_k^2}} \leq \frac{r_1}{r_2} e^{1 - \frac{(x_1 x_2 + y_1 y_2)}{r_k^2}} \quad \text{for all } k \geq 2.$$

But we can use the values of  $x_1, y_1, r_1, x_2, y_2$  and  $r_2$  and we easily see that

$$\frac{r_1}{r_2} e^{1 - \frac{(x_1 x_2 + y_1 y_2)}{r_k^2}} < 1.$$

Thus,  $\left| \phi \left( z_1 \frac{1}{z_k} \right) \right| < 1$  and  $\left| \frac{1}{z_k} \right| < \frac{1}{r_1}$  so that  $\frac{1}{z_k} \in \mathbb{G}_1$ . Moreover, since  $y_k > 0$ , we have  $\frac{1}{z_k} \in \mathcal{H}^-$ . Hence,  $\frac{1}{z_k} \in \mathbb{D}_1$ . Similarly, if we assume  $y_k < 0$ , then we evaluate  $|\phi(\bar{z}_1 z)|$  at  $z = \frac{1}{z_k}$  and get  $\frac{1}{z_k} \in \mathbb{D}_2$ . Therefore,  $\frac{1}{z_k} \in \mathbb{D} = \mathbb{D}_1 \cup \hat{\mathbb{D}}_1$  for each  $k \geq 2$ .  $\square$

### 3. MAIN RESULTS

**3.1. Integral representation for  $B_n(2, x)$ .** We denote

$$g_2(w) := 2 \frac{e^w - 1 - w}{w^2},$$

and we rewrite the definition (1.5) of hypergeometric Bernoulli polynomials of order 2 as

$$\frac{e^{zw}}{g_2(w)} = \sum_{k=0}^{\infty} B_k(2, z) \frac{w^k}{k!}.$$



Let  $\gamma$  be the key-hole contour given on Figure 6 below with  $\delta = 1$ . Clearly, the curve  $\gamma$  passes through the point  $(1, 0)$  and  $\delta = 1 < r_1$  so that a neighborhood of  $\gamma$  contains no zeroes of the function  $\varphi(z)$ .

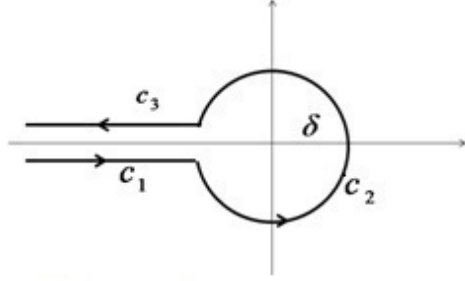


FIGURE 6. The key-hole contour about the origin with  $\delta = 1$

**Lemma 3.1** (Integral representation). *The hypergeometric Bernoulli polynomials  $B_n(2, z)$  satisfy the integral representation*

$$B_n(2, z) = \frac{n!}{4\pi i} \int_{\gamma} \frac{e^{zw}}{w^n} \frac{w}{e^w - 1 - w} dw.$$

The re-scaled polynomials  $B_n(2, nz)$  satisfy the representations

$$B_n(2, nz) = \frac{n!}{4\pi i} \int_{\gamma} \left( \frac{e^{zw}}{w} \right)^n \frac{w}{e^w - 1 - w} dw. \quad (3.1)$$

*Proof.* We divide Equation (1.5) by  $2\pi i w^{n+1}$  and integrate the result over the curve  $\gamma$

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} \frac{w^2 e^{zw}}{2(e^w - 1 - w) w^{n+1}} dw &= \frac{1}{2\pi i} \int_{\gamma} \sum_{k=0}^{\infty} \frac{B_k(2, z) w^k}{k! w^{n+1}} dw \\ &= \sum_{k=0}^{\infty} \frac{B_k(2, z)}{k!} \left( \frac{1}{2\pi i} \int_{\gamma} \frac{1}{w^{n+1-k}} dw \right) = \frac{B_n(2, z)}{n!}, \end{aligned}$$

here we have employed that

$$\int_{\gamma} \frac{1}{w^{n+1-k}} dw = 2\pi i$$

for  $k = n$ , and by the Cauchy integral theorem the integral vanishes for all  $k \neq n$ . Thus,

$$B_n(2, z) = \frac{n!}{4\pi i} \int_{\gamma} \frac{e^{zw}}{w^n} \frac{w}{e^w - 1 - w} dw.$$

Since both sides of this latter equation are entire functions of  $z$ , we can replace  $z$  by  $nz$  and this gives the desired result.  $\square$

**Remark 3.1.** In Equation (3.1), the substitution of  $w$  by  $w/z$  changes the curve of integration  $\gamma$  to  $\gamma_2$ , where  $\gamma_2$  represents the curve  $\gamma$ , in which the radius of  $C_2$  is changed to  $\delta = |z|$ . If  $\varphi(z)$  has zeros at  $w = z_k$ , then the zeros of  $\varphi(w/z)$  are  $w = z_k z$ . Since  $\frac{1}{r_1} \geq \frac{1}{r_k}$ , for each  $z \in \mathbb{C}$  such that  $\frac{1}{r_1} < |z| < \infty$ , we have  $|z_k z| > \frac{r_k}{r_1} \geq 1$ . This shows that  $\frac{w}{\varphi(w/z)}$  is analytic in a disk of radius greater than 1 and we can deform the curve  $\gamma_2$  to  $\gamma$  without changing the value of the

integral. Moreover, the integrals over the curves  $C_1$  and  $C_3$  cancel each other so that the curve of integration reduces to  $C_2$ . Therefore, replacement  $w$  by  $\frac{w}{z}$  in Equation (3.1) yields

$$B_n(2, nz) = \frac{n!z^{n-2}}{4\pi i} \int_{|w|=1} \left(\frac{e^w}{w}\right)^n \frac{w}{\varphi\left(\frac{w}{z}\right)} dw. \quad (3.2)$$

**3.2. Asymptotic representation of  $B_n(2, z)$  outside the disk  $B\left(0, \frac{1}{r_1}\right)$ .** Let

$$T_n(z) = \sum_{k=0}^n \frac{z^k}{k!}$$

be the Taylor polynomials of  $e^z$ . In [2], the polynomials  $T_n(z)$  are expressed by their integral representation as

$$T_{n-1}(nz) = \frac{1}{2\pi i} \int_{|w|=\delta} \left(\frac{e^{zw}}{w}\right)^n \frac{1}{1-w} dw \quad (0 < \delta < 1).$$

In [2], Boyer and Goh used this integral form for the asymptotic representation of  $T_n(z)$ , the generalized Szegő asymptotics.

**Theorem 3.1** (Generalized Szegő asymptotics). *Let  $\frac{1}{3} < \alpha < \frac{1}{2}$ . For the sequence of polynomials  $\{T_n(z)\}$ , we have*

$$T_{n-1}(nz) = -\frac{(ze)^n}{\sqrt{2\pi n}(1-z)} (1 + \mathcal{O}(n^{1-3\alpha})), \quad |z| > 1, \quad (3.3)$$

$$T_{n-1}(nz) = e^{nz} - \frac{(ze)^n}{\sqrt{2\pi n}(1-z)} (1 + \mathcal{O}(n^{1-3\alpha})), \quad \operatorname{Re}(z) < 1. \quad (3.4)$$

We follow similar procedures and use the integral form (3.1) in order to find the asymptotic representation for  $B_n(2, nz)$ . Of course, the expressions (3.3) and (3.4) are very important for our asymptotic representations of  $B_n(2, z)$  inside different domains of the complex plane.

Let  $z_k$  be the roots of  $\varphi(z)$ , which lie in the upper half-plane with  $r_1 < r_2 < r_3 < \dots$ , where  $r_k = |z_k|$ . Then  $\bar{z}_k$  are also roots of  $\varphi(z)$  with modulus  $|\bar{z}_k| = |z_k| = r_k$ . We consider several domains of the complex plane in which we establish asymptotic representation for  $B_n(2, z)$ .

One of such domains is the exterior of the disk  $B\left(0, \frac{1}{r_k}\right)$ , which we define by  $\mathbb{A}_k$ ,

$$\mathbb{A}_k = \left\{ z \in \mathbb{C} : \frac{1}{r_k} < |z| < \infty \right\}.$$

We focus on the case when  $k = 1$ ,  $\mathbb{A}_1 = \left\{ z \in \mathbb{C} : \frac{1}{r_1} < |z| < \infty \right\}$ , the unbounded domain which is the exterior of the disk  $B\left(0, \frac{1}{r_1}\right)$ . Also from the interior of this disk, we consider  $\overline{B}\left(0, \frac{1}{r_1}\right) \setminus \mathbb{D}$  and the domain  $\mathbb{D}$  itself, and we give asymptotic representations of  $B_n(2, nz)$  in each of these three domains.

We let

$$F_n(z) = 2\sqrt{2\pi n} \frac{B_n(2, nz)}{n!(ze)^n}, \quad \mathbf{Z}(\varphi) = \{z \in \mathbb{C} : \varphi(z) = 0\}.$$

**Theorem 3.2.** *For each  $z \in \mathbb{A}_1 = \left\{ z \in \mathbb{C} : \frac{1}{r_1} < |z| < \infty \right\}$  we have*

$$B_n(2, nz) = \frac{n!(ze)^n}{2\sqrt{2\pi n}z^2\varphi\left(\frac{1}{z}\right)} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right). \quad (3.5)$$

Moreover, the asymptotic formula (3.5) holds uniformly for all  $z$  in any compact subset  $K$  of  $\mathbb{A}_1$ .

*Proof.* We rewrite the representation (3.2) as

$$B_n(2, nz) = \frac{n! z^{n-2}}{2\pi i} \mathcal{I}(n),$$

where

$$\mathcal{I}(n) = \frac{1}{2} \int_{|w|=1} \left( \frac{e^w}{w} \right)^n \frac{w}{\varphi\left(\frac{w}{z}\right)} dw.$$

We use the saddle-point method (or Laplace method) to derive the asymptotic representation for  $\mathcal{I}(n)$ . We let  $f(w) = w - \log(w)$  and  $h(w) = \frac{w/2}{\varphi\left(\frac{w}{z}\right)}$ , then  $\mathcal{I}(n)$  becomes

$$\mathcal{I}(n) = \int_{\gamma} e^{nf(w)} h(w) dw.$$

Both functions  $f(w)$  and  $h(w)$  are analytic in a neighborhood of  $\gamma$ , say  $\mathcal{N}(\gamma)$ , hence, both have power series in a neighborhood  $\mathcal{N}(1) \subseteq \mathcal{N}(\gamma)$  of the point  $w_0 = 1$ . For the points  $w = 1 + re^{i\theta}$  in  $\mathcal{N}(1)$ , we may express the series of  $f$  in terms of the real and imaginary parts as

$$f(w) = 1 + \frac{r^2}{2} \cos(2\theta) + \frac{r^3}{6} \cos(3\theta) + \dots + i \left( \frac{r^2}{2} \sin(2\theta) + \frac{r^3}{6} \sin(3\theta) + \dots \right).$$

As  $|e^f| = e^{\operatorname{Re}(f)}$ , we see that the main contribution of  $f(w)$  to the integral  $\mathcal{I}(n)$  comes from its real part  $\operatorname{Re}(f)$ . Also we deform  $\gamma$  so that  $\sin(2\theta) = 0$  and  $\cos(2\theta) = 1$  for points  $w$  on  $\gamma$ .

At  $w = 1$  we have  $r = 0$  and the integration over the portion of  $\gamma$  inside  $\mathcal{N}(1)$  becomes an integral with respect to the variable  $r$ . We treat the integral over the portion of  $\gamma$  that lie below the abscissa axis as the integral over  $r$  from negative values to  $r = 0$  and the integral over the portion above the abscissa axis as from  $r = 0$  to positive values of  $r$ . This gives

$$\mathcal{I}(n) = h(1)e^n \int_{-a}^a e^{\frac{nr^2}{2} + \frac{nr^3}{6} + \dots} \left( 1 + \frac{h'(1)}{h(1)}r + \frac{h''(1)}{2h(1)}r^2 + \dots \right) dr,$$

for a positive real number  $a$ . If we make a substitution  $r = \frac{t}{\sqrt{n}}$ , then we get

$$\mathcal{I}(n) = \frac{h(1)e^n}{\sqrt{n}} \int_{-a\sqrt{n}}^{a\sqrt{n}} e^{\frac{t^2}{2} + \frac{t^3}{6\sqrt{n}} + \dots} \left( 1 + \frac{h'(1)}{h(1)} \frac{t}{\sqrt{n}} + \frac{h''(1)}{2h(1)} \frac{t^2}{n} + \dots \right) dt.$$

Now we use the fact that

$$e^{\frac{t^3}{6\sqrt{n}} + \dots} \leq 1 + 2 \left( \frac{t^3}{6\sqrt{n}} \right) + \dots$$

and express the product of the two series as an asymptotic term. Then for sufficiently large values of  $n$ , the above integral becomes

$$\mathcal{I}(n) = \frac{h(1)e^n}{\sqrt{n}} \int_{-\infty}^{\infty} e^{\frac{t^2}{2}} dt \left( 1 + \mathcal{O}\left(\frac{1}{n}\right) \right).$$

Then we evaluate the integral and get

$$\mathcal{I}(n) = \frac{h(1)e^n}{\sqrt{n}} \sqrt{-2\pi} \left( 1 + \mathcal{O}\left(\frac{1}{n}\right) \right).$$

Finally, noting that  $h(1) = \frac{1/2}{e^{1/z} - 1 - \frac{1}{z}} = \frac{1/2}{\varphi(\frac{1}{z})}$  in  $\mathcal{I}(n)$ , we obtain

$$B_n(2, nz) = \frac{n!(ze)^n}{2\sqrt{2\pi n}} \frac{1}{z^2 \varphi(\frac{1}{z})} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \quad \text{for all } z \in \mathbb{A}_1.$$

□

**3.3. Asymptotic representation for  $B_n(2, z)$  inside the disk  $B\left(0, \frac{1}{r_1}\right)$ .** First we define some domains  $R_k$  inside  $B\left(0, \frac{1}{r_1}\right)$ . Let  $\mu > 0$  be such that  $r_1 < r_2 < r_3 < \dots < r_m < \mu < r_{m+1}$ . Fix some  $k$  such that  $|z_k| = |\bar{z}_k| = r_k < \mu$  and let  $\mathcal{D}_{z_k} = B\left(\frac{1}{z_k}, \delta_k\right)$  where  $\delta_k > 0$  is small enough so that  $B\left(\frac{1}{w}, \delta_w\right)$  are disjoint for distinct  $w$  in  $\mathbf{Z}(\varphi)$ . Let  $\mathcal{T}_{z_k}$  be the tangent to  $B\left(0, \frac{1}{r_k}\right)$  at  $\frac{1}{z_k}$  and let  $\mathcal{H}_{z_k}$  be the half-plane determined by  $\mathcal{T}_{z_k}$  and containing the origin. If  $\epsilon_k > 0$  is such that  $\epsilon_k < \sqrt{\frac{1}{r_k^2} + \delta_k^2} - \frac{1}{r_k}$ , then the disk  $B\left(0, \frac{1}{r_k} + \epsilon_k\right)$  intersects  $\mathcal{T}_{z_k}$  only inside the disk  $\mathcal{D}_{z_k}$ .

**Definition 3.1.** Let  $\mu > 0$  be fixed and let  $z_k$  and  $\bar{z}_k$  be roots of  $\varphi(z)$  of modulus  $r_k$ ,  $r_k < \mu$ . We define the domains

$$\begin{aligned} \mathcal{D}_k &= B\left(\frac{1}{z_k}, \delta_k\right) \cup B\left(\frac{1}{\bar{z}_k}, \delta_k\right), \quad \mathcal{H}_k = \mathcal{H}_{z_k} \cap \mathcal{H}_{\bar{z}_k}, \\ R_k &= [\mathcal{H}_k \setminus \mathcal{D}_k] \cap \left[ B\left(0, \frac{1}{r_k} + \epsilon_k\right) \setminus \left( B\left(0, \frac{1}{r_{k+1}} + \epsilon_{k+1}\right) \cup \mathcal{D}_{k+1} \right) \right], \\ R_\mu &= \mathcal{H}_1 \setminus \left[ B\left(0, \frac{1}{\mu}\right) \cup \left( \bigcup_{k=1}^m \mathcal{D}_k \right) \right]. \end{aligned}$$

Consider the function  $h(w) = \frac{w}{e^w - 1 - w}$  and define  $H(w)$  by

$$H(w) = h(w) - \sum_{j=1}^m \left[ \frac{1}{w - z_j} + \frac{1}{w - \bar{z}_j} \right].$$

In (3.1) we use

$$h(w) = H(w) + \sum_{j=1}^m \left[ \frac{1}{w - z_j} + \frac{1}{w - \bar{z}_j} \right],$$

and we get

$$B_n(2, nz) = I(z, n) + \sum_{j=1}^m J_j(z, n), \quad (3.6)$$

where

$$\begin{aligned} I(z, n) &= \frac{n!}{4\pi i} \int_{|w|=1} \left( \frac{e^{zw}}{w} \right)^n H(w) dw, \\ J_j(z, n) &= \frac{n!}{4\pi i} \int_{|w|=1} \left( \frac{e^{zw}}{w} \right)^n \left[ \frac{1}{w - z_j} + \frac{1}{w - \bar{z}_j} \right] dw. \end{aligned}$$

**Lemma 3.2.** Let  $\mu > 0$  be such that  $r_1 < r_2 < r_3 < \dots < r_m < \mu < r_{m+1}$  for some  $m \in \mathbb{N}$ . The function  $H(w)$  is analytic in the disk  $|w| < \mu$ , and the integral  $I(z, n)$  in (3.6) satisfies

$$I(z, n) = \frac{n!(ze)^n}{2\sqrt{2\pi n}} \frac{1}{z} H\left(\frac{1}{z}\right) \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right)$$

uniformly on compact subsets of  $\mathbb{A}_\mu = \{z \in \mathbb{C} : \frac{1}{\mu} < |z| < \infty\}$ .

*Proof.* Clearly, the non-trivial zeros  $z_j$  of  $\varphi(z)$  (poles of  $h(w)$ ) are simple poles of  $h(w)$  with the residue  $\text{Res}(h(w); z_j) = 1$ . Hence, the function  $H(w)$  is analytic in the disk  $|w| < \mu$ .

Let  $z \in K \subseteq \mathbb{A}_\mu$ . By replacing  $w$  by  $w/z$ , we can express  $I(z, n)$  as

$$I(z, n) = \frac{n!z^{n-1}}{2\pi i} \int_{|w|=|z|} \left(\frac{e^w}{w}\right)^n H\left(\frac{w}{z}\right) dw.$$

Here again we can deform the curve of integration to  $|w| = 1$  without changing its value. Then in the same way as we did in the proof of Theorem 3.2, we apply the saddle-point method and obtain the required result.  $\square$

**Lemma 3.3.** *Let  $\frac{1}{3} < \alpha < \frac{1}{2}$ ,  $z_j \in \mathbf{Z}(\varphi)$  and  $\mathcal{H}_j = \{z : \text{Re}(z_j z) < 1\}$ . If  $z \in K \subseteq \mathcal{H}_j$ , then*

$$\frac{1}{2\pi i} \int_{|w|=1} \left(\frac{e^{zw}}{w}\right)^n \frac{1}{w - z_j} dw = -z_j^{-n} e^{nz_j z} + \frac{(ze)^n}{\sqrt{2\pi n}(1 - z_j z)} (1 + \mathcal{O}(n^{1-3\alpha})).$$

*Proof.* We replace  $w$  by  $z_j w$ , and we get

$$\begin{aligned} \frac{1}{2\pi i} \int_{|w|=1} \left(\frac{e^{zw}}{w}\right)^n \frac{1}{w - z_j} dw &= -z_j^{-n} \frac{1}{2\pi i} \int_{|w|=\frac{1}{|z_j|}} \left(\frac{e^{zz_j w}}{w}\right)^n \frac{1}{1 - w} dw \\ &= -z_j^{-n} T_{n-1}(nz_j z). \end{aligned}$$

Since  $z \in \mathcal{H}_j$ , we have  $\text{Re}(z_j z) < 1$  so that we can use (3.4) to express  $T_{n-1}(nz_j z)$  asymptotically and get the desired result.  $\square$

**Corollary 3.1.** *For  $z_j \in \mathbf{Z}(\varphi)$  such that  $|z_j| \leq r_k$ , the functions  $J_j(z, n)$  defined in (3.6) can be expressed asymptotically as*

$$J_j(z, n) = \frac{n!(ze)^n}{2\sqrt{2\pi n}} \left[ \frac{-\sqrt{2\pi n}}{\phi(z z_j)^n} + \frac{-\sqrt{2\pi n}}{\phi(z \bar{z}_j)^n} + \left( \frac{1}{1 - z_j z} + \frac{1}{1 - \bar{z}_j z} \right) (1 + \mathcal{O}(n^{1-3\alpha})) \right] \quad (3.7)$$

uniformly on compact subsets of  $R_k$ .

*Proof.* By definition,  $z \in R_k$  and  $|z_j| \leq r_k$  implies  $\text{Re}(z_j z) < 1$ . Thus,  $R_k \subseteq \mathcal{H}_j$  so that we use Lemma 3.3 and get

$$\frac{1}{2\pi i} \int_{|w|=1} \left(\frac{e^{zw}}{w}\right)^n \frac{1}{w - z_j} dw = -z_j^{-n} e^{nz_j z} + \frac{(ze)^n}{\sqrt{2\pi n}(1 - z_j z)} (1 + \mathcal{O}(n^{1-3\alpha})).$$

We get the same equation with  $z_j$  replaced by  $\bar{z}_j$ . Finally,  $J_j(z, n)$  will be the sum of the two equations multiplied by  $\frac{n!}{2}$ .  $\square$

**Lemma 3.4.** *Let  $z_j \in \mathbf{Z}(\varphi)$  and let  $z \in K$ , where  $K \subseteq \{z : \frac{1}{|z_j|} < |z| < \infty\}$ . Then*

$$\frac{1}{2\pi i} \int_{|w|=1} \left(\frac{e^{zw}}{w}\right)^n \frac{1}{w - z_j} dw = \frac{(ze)^n}{\sqrt{2\pi n}(1 - z_j z)} (1 + \mathcal{O}(n^{1-3\alpha})).$$

*Proof.* As we have the condition  $|z_j z| > 1$  from the assumptions, we use (3.3) and the procedure is similar to the proof of Lemma 3.3.  $\square$

**Corollary 3.2.** For  $z_j \in \mathbf{Z}(\varphi)$  with  $r_k < |z_j| < \mu$ , we have

$$J_j(z, n) = \frac{n!(ze)^n}{2\sqrt{2\pi n}} \left[ \frac{1}{1 - z_j z} + \frac{1}{1 - \bar{z}_j z} \right] (1 + \mathcal{O}(n^{1-3\alpha})) \quad (3.8)$$

uniformly on compact subsets of  $R_k$ .

*Proof.*  $z \in R_k$  and  $r_{k+1} \leq |z_j|$  implies  $|z_j z| > 1$  so that it follows from the Lemma 3.4.  $\square$

Let  $\mu > 0$  be fixed and let  $z_1, z_2, \dots, z_m$  be roots of  $\varphi(z)$  with modulus  $|z_j| = r_j$  such that  $r_1 \leq r_2 \leq \dots \leq r_m < \mu < r_{m+1}$ . We choose  $k$  with  $r_k < \mu$  and we give an asymptotic representation of  $B_n(2, nz)$  for all  $z$  in the domain  $R_k$ .

**Theorem 3.3.** Let  $\frac{1}{3} < \alpha < \frac{1}{2}$ ,  $\mu > 0$  and  $k$  such that  $|z_k| < \mu$  be fixed. Then

$$B_n(2, nz) = \frac{n!(ze)^n}{2\sqrt{2\pi n}} \left[ \frac{1}{z^2 \varphi\left(\frac{1}{z}\right)} - \sqrt{2\pi n} \sum_{j=1}^k \left( \frac{1}{\phi(z z_j)^n} + \frac{1}{\phi(z \bar{z}_j)^n} \right) + \mathcal{O}(n^{1-3\alpha}) \right]$$

for all  $z \in R_k$ .

*Proof.* We may express equation (3.6) as  $B_n(2, nz) = I(z, n) + J(z, n)$ , where

$$J(z, n) = \sum_{j=1}^m J_j(z, n).$$

For any  $z \in R_k$ , we have  $\text{Re}(z_j z) < 1$  if  $1 \leq j \leq k$  and  $|z_j z| > 1$  if  $k < j \leq m$ . Hence, for any  $z \in R_k$ , we can use either Corollary 3.1 or Corollary 3.2 for asymptotic representation of  $J_j(z, n)$ . That is, we get

$$J(z, n) = \sum_{j=1}^m J_j(z, n) = \sum_{j=1}^k J_j(z, n) + \sum_{j=k+1}^m J_j(z, n),$$

where  $J_j(z, n)$  is given by (3.7) if  $1 \leq j \leq k$ , and  $J_j(z, n)$  is given by (3.8) if  $k+1 \leq j \leq m$ . Thus,

$$J(z, n) = \frac{n!(ze)^n}{2\sqrt{2\pi n}} \left[ -\sqrt{2\pi n} \sum_{j=1}^k \left( \frac{1}{\phi(z z_j)^n} + \frac{1}{\phi(z \bar{z}_j)^n} \right) + \sum_{j=1}^m \left( \frac{1}{1 - z_j z} + \frac{1}{1 - \bar{z}_j z} \right) (1 + \mathcal{O}(n^{1-3\alpha})) \right].$$

Also since  $R_k \subseteq \{w : |w| < \mu\}$ , Lemma 3.2 holds for  $z \in R_k$ . Moreover, by noting that

$$\frac{1}{z} H\left(\frac{1}{z}\right) = \frac{1}{z^2 \varphi\left(\frac{1}{z}\right)} - \sum_{j=1}^m \left( \frac{1}{1 - z_j z} + \frac{1}{1 - \bar{z}_j z} \right),$$

we get

$$I(z, n) = \frac{n!(ze)^n}{2\sqrt{2\pi n}} \left[ \frac{1}{z^2 \varphi\left(\frac{1}{z}\right)} - \sum_{j=1}^m \left( \frac{1}{1 - z_j z} + \frac{1}{1 - \bar{z}_j z} \right) \right] \left( 1 + \mathcal{O}\left(\frac{1}{n}\right) \right).$$

Finally, since  $\frac{1}{n} < n^{1-3\alpha}$ , the term  $\mathcal{O}\left(\frac{1}{n}\right)$  will be absorbed in  $\mathcal{O}(n^{1-3\alpha})$  and combining the above two equations, we get

$$B_n(2, nz) = \frac{n!(ze)^n}{2\sqrt{2\pi n}} \left[ \frac{1}{z^2 \varphi\left(\frac{1}{z}\right)} - \sqrt{2\pi n} \sum_{j=1}^k \left( \frac{1}{\phi(z z_j)^n} + \frac{1}{\phi(z \bar{z}_j)^n} \right) + \mathcal{O}(n^{1-3\alpha}) \right].$$

$\square$

#### 4. OPEN PROBLEM

The open problem related to our work is to determine the asymptotic real and complex zeros of hypergeometric Bernoulli polynomials of order  $N = 3$  and establish similar asymptotic formulas for  $B_n(3, x)$ . Generalizing this concept for hypergeometric Bernoulli polynomials of arbitrary order,  $B_n(N, x)$  is also one more open problem.

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