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NONLINEAR INTEGRABLE LATTICES WITH THREE INDEPENDENT VARIABLES

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Abstract. We suggest an algorithm for deriving nonlinear integrable equations of the form

$$u_{n,x}^{j+1} = F(u_{n,x}^{j}, u_n^{j+1}, u_{n+1}^{j}, u_n^{j}, u_{n-1}^{j+1})$$

with three independent variables; the algorithm uses the known list of Toda type integrable equations. The algorithm is based on the Darboux integrable finite field reductions, construction of a complete set of characteristic integrals and dicretization via integrals.

Keywords: characteristic integrals, integrability in sense of Darboux, Lax pairs.

Mathematics Subject Classification: 35Q51, 39A14

1. Introduction

Nowadays, the problem of classification of integrable nonlinear partial differential equations and their discrete analogues in 1+1 dimensions is well–studied. Within the framework of the symmetry approach, there was obtained a complete description of integrable representatives of a number of classes of equations that are interesting from the point of view of application, see [17], [34], [26], [2]. The problem of exhaustive classification of integrable equations containing a large number of independent variables remains less studied due to its extreme complexity. The symmetry approach, which has proven to be the most effective tool for classifying equations of dimension 1+1, is not quite suitable for integrable classification of multidimensional equations. As it is noted in [27], in this problem the symmetry approach loses its efficiency due to problems with nonlocalities involved in higher symmetries.

Various alternative approaches to the problem of searching and classifying integrable multidimensional equations are known in the literature. A description of these approaches, as well as the current state of arts in this area, can be found in the papers [7], [8], [4], [28], [3], [35] and references therein.

The following important conclusion follows from the results of the papers [12], [29], [13], [21], [15], [14], [19] devoted to the study of integrable nonlinear differential–difference equations with three independent variables, where at least one of the variables is discrete (such equations are usually called lattices). Any integrable equation in this class admits a special boundary condition in one of the discrete directions, which, when imposed at two arbitrary non–coinciding points, reduces the three–dimensional lattice to a two–dimensional system of hyperbolic equations admitting complete sets of integrals in each of the characteristic directions. Recall that systems admitting complete sets of characteristic integrals are referred to as integrable in the

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sense of Darboux. In fact, the existence of a hierarchy of Darboux integrable reductions is a criterion for the integrability of a nonlinear three–dimensional lattice.

This assumption was convincingly confirmed for all three types of lattices: for the class of equations of two–dimensional Toda chain type in [12], [29], [13], for semi–discrete lattices with one continuous and two discrete variables in [21], [15] and, finally, for fully discrete Hirota type equations in [14], [19].

The first examples of a hierarchy of integrable reductions of nonlinear lattices were considered in the nineteenth century by G. Darboux. For example, he investigated finite reductions of the equation

$$(\log u_n)_{xy} = 2u_n - u_{n+1} - u_{n-1}$$

obtained by imposing the cutoff conditions $u_{N+1} = 0$, $u_0 = 0$, for which he found general solution in an explicit form.

An interest in this topic was revived at the end of the 20th century in the studies of A.V. Mikhailov, A.B. Shabat, A.N. Leznov, M.A. Olshanetsky and others in connection with the discovery of the inverse scattering transform method. Particular attention was paid to nonlinear chains corresponding to Cartan matrices of simple Lie algebras, see, for example, [30], [25], [23]. The concept of characteristic algebra, introduced in [30] by A.B. Shabat and R.I. Yamilov, has become an effective tool for studying Darboux integrable systems. It was proved in the mognography [36] that the criterion for the existence of a complete set of integrals of an arbitrary hyperbolic system is a finite dimensional of its characteristic algebras. Integrable semi–discrete models related to the Cartan matrices were studied also in [10], [31].

Below we investigate the relationship between the following two classes of nonlinear integrable lattices with three independent variables

$$u_{n,xy} = f(u_{n,x}, u_{n,y}, u_{n+1}, u_n, u_{n-1}) \tag{1.1}$$

and

$$u^{j+1}_{n,x} = F(u^j_{n,x}, u^{j+1}_n, u^j_{n+1}, u^j_n, u^{j+1}_{n-1}). \tag{1.2}$$

In [29], [13], [20] the problem of complete description of integrable cases of equation (1.1) with a linear dependence on the derivatives $u_{n,x}$, $u_{n,y}$ was solved. More precisely, it was assumed that (1.1) has the form

$$u_{n,xy} = A_n u_{n,x} u_{n,y} + B_n u_{n,x} + C_n u_{n,y} + E_n, \tag{1.3}$$

where the coefficients A_n, B_n, C_n, E_n depend on the dynamical variables u_{n+1}, u_n, u_{n-1}. To the best of our knowledge, in the literature there are no examples of integrable equations of the form (1.1), which are nonlinear in the derivatives. Up to point changes of variables, the list of integrable lattices of the class (1.3) has the form

(E1) $u_{n,xy} = e^{u_{n+1}-2u_n+u_{n-1}}$,

(E2) $u_{n,xy} = e^{u_{n+1}} - 2e^{u_n} + e^{u_{n-1}}$,

(E3) $u_{n,xy} = e^{u_{n+1}-u_n} - e^{u_n-u_{n-1}}$,

(E4) $u_{n,xy} = (u_{n+1} - 2u_n + u_{n-1})\, u_{n,x}$,

(E5) $u_{n,xy} = \left(e^{u_{n+1}-u_n} - e^{u_n-u_{n-1}}\right) u_{n,x}$,

(E6) $u_{n,xy} = \alpha_n u_{n,x} u_{n,y}, \quad \alpha_n = \dfrac{1}{u_n - u_{n-1}} - \dfrac{1}{u_{n+1} - u_n}$,

(E7) $u_{n,xy} = \alpha_n (u_{n,x} + u_n^2 - 1)(u_{n,y} + u_n^2 - 1) - 2u_n(u_{n,x} + u_{n,y} + u_n^2 - 1)$.

To classify the lattices of form (1.3), the aforementioned method of Darboux integrable reductions was used. In the special case $A_n = B_n = C_n = 0$ all integrable lattices of the form (1.3) were previously listed in the paper [30]. The problem of classifying chains of the form (1.2) by the same method of Darboux integrable reductions using characteristic Lie algebras turned out to be more difficult, and remains unsolved. Some intermediate results were obtained in [21], [15].

In this paper, we propose a method for obtaining integrable equations of the form (1.2) by discretizing integrable representatives of the more thoroughly studied class (1.1). To test the method, we derive discrete versions of several quasilinear equations presented in the list above (equations (E1)–(E3)). The discretization algorithm consists of the following steps. First, for a given equation from the list, we seek its finite field Darboux integrable reduction in the form of a system of partial differential equations, and construct complete sets of characteristic integrals in the x and y directions for it. Next, we assume that all involved dynamical variables depend on an additional discrete variable, and for each set of constructed characteristic integrals, we seek a finite system of a different type, namely, a system of differential–difference equations, for which this set is also a set of integrals. It is curious that for each of the two sets of integrals under consideration, such a system is uniquely determined, with the exception of some degenerate cases. Then, using the found system of differential–difference equations, we find the explicit form of the desired lattice of class (1.2). For scalar 1+1 dimensional Liouville type models method of discretization via integrals was suggested in [11]. Some important properties of characteristic integrals of Liouville type equations were noted in [33].

Let us briefly outline the contents of the paper. In the second section, we give the definition of a complete set of characteristic integrals of a hyperbolic system and present a completeness criterion. We discuss methods for constructing integrals. In Section 3, we explain in detail the proposed algorithm for discretizing Toda type lattices using finite field reductions and characteristic integrals. In Section 4 and Section 5, we apply the discretization algorithm to the lattices (E3) and (E1). In Section 6, we obtain a discrete version of the chain (E2) using a nonlocal Lax pair. In this case, for the equation (6.6), we constructed a Lax pair that also depends on a nonlocal variable.

2. Complete sets of characteristic integrals

We consider a hyperbolic system of the form

$$u_{i,xy} = F_i(u_1, \dots, u_N, u_{1,x}, \dots, u_{N,x}, u_{1,y}, \dots, u_{N,y}), \qquad i = 1, 2, \dots, N, \tag{2.1}$$

where F_i are analytic functions defined on a domain in the space $\mathbb{C}^{3N}$.

We recall some basic definitions (see, for example, [36]). Denote $u_{k,[s]} := \frac{\partial^s}{\partial x^s} u_k$. Then a smooth function of the form

$$I = I(u_1, \dots, u_N, u_{1,x}, \dots, u_{N,x}, u_{1,xx}, \dots, u_{N,xx}, \dots, u_{1,r}, \dots, u_{N,r}),$$

is called an x–integral of order r of system (2.1), if $D_y I = 0$, where D_y is the derivative with respect to y by virtue of (2.1). In other words, for the x–integral we have the relation

$$D_y I = \sum_i \left(u_{i,y} \frac{\partial}{\partial u_i} + F_i \frac{\partial}{\partial u_{i,x}} + D_x(F_i) \frac{\partial}{\partial u_{i,xx}} + \cdots \right) I = YI = 0.$$

A set of x–integrals $\{I_i\}_{i=1}^{i=N}$ with the orders r_i for (2.1) forms a complete set of independent x–integrals if the condition

$$\det\left(\frac{\partial I_j}{\partial u_{i,m}}\right) = \begin{vmatrix} \frac{\partial I_1}{\partial u_{1,r_1}} & \frac{\partial I_1}{\partial u_{2,r_1}} & \dots & \frac{\partial I_1}{\partial u_{N,r_1}} \\ \dots & \dots & \dots & \dots \\ \frac{\partial I_N}{\partial u_{1,r_N}} & \frac{\partial I_N}{\partial u_{2,r_N}} & \dots & \frac{\partial I_N}{\partial u_{N,r_N}} \end{vmatrix} \neq 0 \tag{2.2}$$

holds. Let us formulate an effective algebraic criterion [36] for the existence of a complete set of integrals of system (2.1).

Theorem 2.1. *System of equations (2.1) admits a complete set of independent x–integrals if and only if the characteristic Lie algebra over the ring of locally analytic functions of the variables $\bar{u}_y$, $\bar{u}$, $\bar{u}_x$, $\bar{u}_{xx}$, ..., where $\bar{u} = (u_1, u_2, \dots, u_N)$, generated by the characteristic operators*

$$X_j = \frac{\partial}{\partial u_{j,y}}, \quad j = 1, \dots, N; \qquad Y = \sum_i u_{i,y}\frac{\partial}{\partial u_i} + F_i\frac{\partial}{\partial u_{i,x}} + D_x(F_i)\frac{\partial}{\partial u_{i,xx}} + \cdots$$

is finite–dimensional.

Remark 2.1. *In Theorem 2.1, stating that «Lie algebra is finite–dimensional», we mean its dimension as a left module over the ring of locally analytic functions.*

Integrals and characteristic algebra in the direction y are defined similarly. Alternative methods for constructing characteristic integrals were discussed in [32], [6], [16]. Below, to construct complete sets of characteristic integrals, we use an algorithm based on the concept of Lax pair.

3. Algorithm for discretization of lattices in 3D using integrals

Let us consider the rule for finding a differential–difference equation for a given integral of a partial differential equation using a simple example. As a touchstone, we take the Liouville equation

$$u_{xy} = e^u. \tag{3.1}$$

As it is known, the Liouville equation has characteristic integrals of the form

$$I = u_{xx} - \frac{1}{2}u_x^2, \qquad J = u_{yy} - \frac{1}{2}u_y^2. \tag{3.2}$$

It is easy to verify the identities $D_y I = 0$, $D_x J = 0$, where the derivatives D_y and D_x are calculated by virtue of Equation (3.1). We assume that the dynamical variables u, u_x, u_{xx}, ... depend on an additional discrete variable n. By discretization of Equation (3.1) using the integral (3.2) we mean an equation of the form

$$u_{n+1,x} = H(u_{nx}, u_{n+1}, u_n) \tag{3.3}$$

such that the function

$$\bar{I}^{(n)} = u_{nxx} - \frac{1}{2}u_{nx}^2$$

is an integral in the direction n for this equation. In other words, the relation

$$D_n\bar{I}^{(n)} = \bar{I}^{(n)} \tag{3.4}$$

holds, where the shift is due to the equation (3.3). It is easy to verify (see [11]) that the function H is uniquely determined from the condition (3.4). The desired equation has the form

$$u_{n+1,x} = u_{n,x} + Ce^{\frac{1}{2}(u_{n+1}+u_n)}, \tag{3.5}$$

where C is an arbitrary constant. Note that Equation (3.5) admits an integral in another direction as well

$$J = e^{\frac{1}{2}(u_{n+1}-u_n)} + e^{\frac{1}{2}(u_{n+1}-u_{n+2})}.$$

To verify this, we check that the identity $D_x J = 0$ holds, where the derivative is taken by virtue of Equation (3.5). The example shows that the discretization by means of the integral preserves integrability in the sense of Darboux.

Based on this and many other similar examples, we propose the following discretization algorithm in the class of integrable lattices with three independent variables. Consider an integrable nonlinear lattice of the form (1.1). Take its finite field reduction

$$\begin{cases} u_{1,x,y} = f_1\left(u_{1,x}, u_{1,y}, u_2, u_1\right), \\ u_{2,x,y} = f\left(u_{2,x}, u_{2,y}, u_3, u_2, u_1\right), \\ \qquad\dots \\ u_{m-1,x,y} = f\left(u_{m-1,x}, u_{m-1,y}, u_m, u_{m-1}, u_{m-2}\right), \\ u_{m,x,y} = f_2\left(u_{m,x}, u_{m,y}, u_m, u_{m-1}\right), \end{cases} \tag{3.6}$$

where the functions f_1 and f_2 are chosen so that this system is Darboux integrable, i.e., it has complete sets of characteristic integrals in both directions. Boundary conditions for the chains (E1)–(E7), which ensure the Darboux integrable reductions, were found in [29], [13], [20]. Suppose that set of functions I_1, I_2, $\dots$, I_m constitutes a complete set of characteristic integrals of minimal orders in one of the directions (the definition of minimality can be found in [36]); for definiteness, we will assume that these are x–integrals, i.e. they have the form

$$I_k = I_k(\bar{u}, \bar{u}_x, \bar{u}_{xx}, \dots), \quad k = \overline{1, m}.$$

Here we use the notation $\bar{u} = (u_1, u_2, \dots, u_m)$. Recall that the x–integral satisfies the relation $D_y I_k = 0$, where D_y denotes the total derivative operator, and $D_y I_k$ is calculated due to (3.6).

In what follows, we assume that the dynamical variables u_1, u_2, $\dots$, u_m depend on an additional discrete variable, and we introduce the superscript $u_1^{(j)}$, $u_2^{(j)}$, $\dots$, $u_m^{(j)}$. The dynamics with respect to this variable is determined by means of the differential–difference system

$$\begin{cases} u_{1,x}^{(j+1)} = F_1\left(u_{1,x}^{(j)}, u_1^{(j+1)}, u_2^{(j)}, u_1^{(j)}\right), \\ u_{2,x,y}^{(j+1)} = F\left(u_{2,x}^{(j)}, u_2^{(j+1)}, u_3^{(j)}, u_2^{(j)}, u_1^{(j+1)}\right), \\ \qquad\dots \\ u_{m-1,x,y}^{(j+1)} = F\left(u_{m-1,x}^{(j)}, u_{m-1}^{(j+1)}, u_m^{(j)}, u_{m-1}^{(j)}, u_{m-2}^{(j+1)}\right), \\ u_{m,x,y}^{(j+1)} = F_2\left(u_{m,x}^{(j)}, u_m^{(j+1)}, u_m^{(j)}, u_{m-1}^{(j+1)}\right). \end{cases} \tag{3.7}$$

Then we look for a specific form of the functions F_1, F_2, and F by assuming that the functions

$$\tilde{I}_k = I_k(\bar{u}^{(j)}, \bar{u}_x^{(j)}, \bar{u}_{xx}^{(j)}, \dots)$$

are x–integrals of the system (3.7), i.e., they satisfy the condition $D_j I_k = I_k$, where operator D_j is the operator, which shifts the variable j. Note that under the discretization we have to adhere to another requirement, which excludes a trivial result: it is as follows.

(A). The functions $\tilde{I}_1$, $\tilde{I}_2$, $\dots$, $\tilde{I}_m$ provide a complete set of independent integrals of minimal orders for (3.7).

From the found system, we reconstruct a semi–discrete 3D chain of the form (1.2). Examples show that the solution to the discretization problem is independent of the choice of $m \geqslant 3$. Therefore, we can limit ourselves to considering the case $m = 3$.

Let us consider the linear systems associated with the lattices (E1)–(E3), the discrete versions of which will be presented below. We start with (E2), which has a Lax pair of the form (see [24])

$$\psi_{n,x} = u_{n,x}\psi_n + \psi_{n+1}, \qquad \psi_{n,y} = -e^{u_n - u_{n-1}}\psi_{n-1}. \tag{3.8}$$

The function $u_n = u_n(x,y)$ in the coefficients of this linear system solves the lattice (E2) if and only if system (3.8) is consistent. Recall that Equations (E1)–(E3) can be rewritten as

$$\text{(E1)} \quad v_{n,xy} = e^{v_{n+1}-2v_n+v_{n-1}},$$
$$\text{(E2)} \quad u_{n,xy} = e^{u_{n+1}-u_n} - e^{u_n-u_{n-1}},$$
$$\text{(E3)} \quad w_{n,xy} = e^{w_{n+1}} - 2e^{w_n} + e^{w_{n-1}}$$

are mutually related by the linear transformations

$$w_n = u_{n+1} - u_n, \qquad u_n = v_{n+1} - v_n. \tag{3.9}$$

The linear systems associated with (E1) and (E3) can be obtained from (3.8) by using the substitutions (3.9). For example, for (E1) we have

$$\psi_{n,x} = (v_{n+1,x} - v_{n,x})\psi_n + \psi_{n+1}, \qquad \psi_{n,y} = -e^{v_{n+1}-2v_n+v_{n-1}}\psi_{n-1}. \tag{3.10}$$

However, the compatibility condition of the system (3.10) is only necessary, but not sufficient for the fulfillment of Equation (E1).

For the lattice (E3) we obtain the system

$$\psi_{n,x} = u_{n,x}\psi_n + \psi_{n+1}, \qquad \psi_{n,y} = -e^{w_{n-1}}\psi_{n-1}, \tag{3.11}$$

which contains a nonlocal variable u_n determined by the equation

$$u_{n+1,x} - u_{n,x} = w_{n,x},$$

where the compatibility of the system is a necessary and sufficient condition for the fulfillment of Equation (E3).

4. Discretization of equation ($E3$)

In this section we discretize the lattice (E3) by using integrals of finite field systems of equations, which are obtained from the considered lattice by imposing special truncation conditions. By setting $u_0 = \infty$, $u_4 = -\infty$ in ($E3$) we obtain the following integrable in the sense of Darboux hyperbolic system

$$\begin{cases} u_{xy} = e^{v-u}, \\ v_{xy} = e^{w-v} - e^{v-u}, \\ w_{xy} = -e^{w-v}, \end{cases} \tag{4.1}$$

where $u = u_1$, $v = u_2$, $w = u_3$.

Let us find a complete set of characteristic y–integrals of the system (4.1). To this end we use the generating function of integrals defined by the operator [32]

$$B = (D_x - w_x)(D_x - v_x)(D_x - u_x).$$

The coefficients of polynomial

$$B = D_x^3 - I_1 D_x^2 - I_2 D_x - I_3$$

form a complete set of indepedent y–integrals of the system (4.1). Obviously, they have the form

$$I_1 = u_x + v_x + w_x,$$
$$I_2 = 2u_{xx} + v_{xx} - u_x v_x - u_x w_x - v_x w_x,$$
$$I_3 = u_{xxx} - u_x v_{xx} - v_x u_{xx} - w_x u_{xx} + u_x v_x w_x.$$

Note that the completeness of set of integrals is easily verified by calculating the determinant (2.2)

$$\begin{vmatrix} \frac{\partial I_1}{\partial u_x} & \frac{\partial I_1}{\partial v_x} & \frac{\partial I_1}{\partial w_x} \\ \frac{\partial I_2}{\partial u_{xx}} & \frac{\partial I_2}{\partial v_{xx}} & \frac{\partial I_2}{\partial w_{xx}} \\ \frac{\partial I_3}{\partial u_{xxx}} & \frac{\partial I_3}{\partial v_{xxx}} & \frac{\partial I_3}{\partial w_{xxx}} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 1 \neq 0.$$

Our next goal is to find a system of differential–difference equations of the form

$$\begin{cases} u_x^{j+1} = f_1\left(u_x^j, u^{j+1}, u^j, v^j\right), \\ v_x^{j+1} = f_2\left(v_x^j, v^{j+1}, v^j, w^j, u^{j+1}\right), \\ w_x^{j+1} = f_3\left(w_x^j, w^{j+1}, w^j, v^{j+1}\right), \end{cases} \tag{4.2}$$

for which functions I_1, I_2, I_3 are j–integrals, i.e. they satisfy the conditions

$$\left(D_j - 1\right) I_i = 0 \quad \text{for} \quad i = 1, 2, 3. \tag{4.3}$$

In what follows we assume that the variables u, v and w depend on the continuous variable x and discrete variable j, therefore we set $u = u^j(x)$, $v = v^j(x)$ and $w = w^j(x)$. Then we rewrite Equation (4.3) in an enlarged form and obtain three equations

$$\begin{aligned} & u_x^{j+1} + v_x^{j+1} + w_x^{j+1} - u_x^j - v_x^j - w_x^j = 0, \\ & 2u_{xx}^{j+1} + v_{xx}^{j+1} - u_x^{j+1}v_x^{j+1} - u_x^{j+1}w_x^{j+1} - v_x^{j+1}w_x^{j+1} \\ & \qquad\qquad - 2u_{xx}^j - v_{xx}^j + u_x^j v_x^j + u_x^j w_x^j + v_x^j w_x^j = 0, \\ & u_{xxx}^{j+1} - u_x^{j+1}v_{xx}^{j+1} - v_x^{j+1}u_{xx}^{j+1} - w_x^{j+1}u_{xx}^{j+1} + u_x^{j+1}v_x^{j+1}w_x^{j+1} \\ & \qquad\qquad - u_{xxx}^j + u_x^j v_{xx}^j + v_x^j u_{xx}^j + w_x^j u_{xx}^j - u_x^j v_x^j w_x^j = 0 \end{aligned} \tag{4.4}$$

to find the unknown system (4.2). In Equations (4.4) we replace the derivatives of variables u^{j+1}, v^{j+1} and w^{j+1} with respect to x due to the system (4.2). For example, after the replacement, the first equation of system (4.4) becomes

$$\begin{aligned} f_1\left(u_x^j, u^{j+1}, u^j, v^j\right) + f_2\left(v_x^j, v^{j+1}, v^j, w^j, u^{j+1}\right) \\ + f_3\left(w_x^j, w^{j+1}, w^j, v^{j+1}\right) - u_x^j - v_x^j - w_x^j = 0. \end{aligned} \tag{4.5}$$

We shall not write the remaining two equations of the system (4.4) explicitly since they are quite cumbersome.

Let us concentrate on the study of Equation (4.5). Differentiating Equation (4.5) with respect to variables u_x^j, v_x^j and w_x^j, we respectively obtain three equations

$$\begin{aligned} & \frac{\partial}{\partial u_x^j} f_1\left(u_x^j, u^{j+1}, u^j, v^j\right) - 1 = 0, \\ & \frac{\partial}{\partial v_x^j} f_2\left(v_x^j, v^{j+1}, v^j, w^j, u^{j+1}\right) - 1 = 0, \\ & \frac{\partial}{\partial w_x^j} f_3\left(w_x^j, w^{j+1}, w^j, v^{j+1}\right) - 1 = 0, \end{aligned}$$

from which we have

$$\begin{aligned} & f_1\left(u_x^j, u^{j+1}, u^j, v^j\right) = u_x^j + f_{11}\left(u^{j+1}, u^j, v^j\right), \\ & f_2\left(v_x^j, v^{j+1}, v^j, w^j, u^{j+1}\right) = v_x^j + f_{21}\left(v^{j+1}, v^j, w^j, u^{j+1}\right), \\ & f_3\left(w_x^j, w^{j+1}, w^j, v^{j+1}\right) = w_x^j + f_{31}\left(w^{j+1}, w^j, v^{j+1}\right), \end{aligned}$$

where f_{11}, f_{21}, f_{31} are new sought functions.

Due to the found values of functions f_1, f_2, f_3, Equation (4.5) becomes

$$f_{11}\left(u^{j+1},u^j,v^j\right)+f_{21}\left(v^{j+1},v^j,w^j,u^{j+1}\right)+f_{31}\left(w^{j+1},w^j,v^{j+1}\right)=0. \tag{4.6}$$

Note that in the above equation the variables u^j and w^{j+1} are involved only in functions $f_{11}\left(u^{j+1},u^j,v^j\right)$ and $f_{31}\left(w^{j+1},w^j,v^{j+1}\right)$, respectively. This fact implies

$$f_{11}\left(u^{j+1},u^j,v^j\right)=f_{11}\left(u^{j+1},v^j\right),$$
$$f_{31}\left(w^{j+1},w^j,v^{j+1}\right)=f_{31}\left(w^j,v^{j+1}\right).$$

Further analysis of equation (4.6) makes it clear that the function $f_{21}\left(v^{j+1},v^j,w^j,u^{j+1}\right)$ can be eliminated

$$f_{21}\left(v^{j+1},v^j,w^j,u^{j+1}\right)=-f_{11}\left(u^{j+1},v^j\right)-f_{31}\left(w^j,v^{j+1}\right).$$

We proceed to studying the second equation in (4.4), which, in view of the above calculations, casts into the form

$$\begin{aligned}
&\left(\frac{\partial}{\partial u^{j+1}}f_{11}\left(u^{j+1},v^j\right)+f_{11}\left(u^{j+1},v^j\right)\right)\left(u^j_x+f_{11}\left(u^{j+1},v^j\right)\right)\\
&+\left(\frac{\partial}{\partial v^j}f_{11}\left(u^{j+1},v^j\right)-\frac{\partial}{\partial v^{j+1}}f_{31}\left(w^j,v^{j+1}\right)-f_{11}\left(u^{j+1},v^j\right)-f_{31}\left(w^j,v^{j+1}\right)\right)v^j_x\\
&-\left(\frac{\partial}{\partial w_n}f_{31}\left(w^j,v^{j+1}\right)-f_{31}\left(w^j,v^{j+1}\right)\right)w^j_x\\
&+\left(\frac{\partial}{\partial v^{j+1}}f_{31}\left(w^j,v^{j+1}\right)+f_{31}\left(w^j,v^{j+1}\right)\right)\left(f_{11}\left(u^{j+1},v^j\right)+f_{31}\left(w^j,v^{j+1}\right)\right)=0.
\end{aligned}$$

Collecting the coefficients at the independent variables in the latter equation, we obtain the following four equations

$$\frac{\partial}{\partial u^{j+1}}f_{11}\left(u^{j+1},v^j\right)+f_{11}\left(u^{j+1},v^j\right)=0, \tag{4.7}$$

$$\frac{\partial}{\partial w^j}f_{31}\left(w^j,v^{j+1}\right)-f_{31}\left(w^j,v^{j+1}\right)=0, \tag{4.8}$$

$$\frac{\partial}{\partial v^j}f_{11}\left(u^{j+1},v^j\right)-\frac{\partial}{\partial v^{j+1}}f_{31}\left(w^j,v^{j+1}\right)-f_{11}\left(u^{j+1},v^j\right)-f_{31}\left(w^j,v^{j+1}\right)=0, \tag{4.9}$$

$$\left(\frac{\partial}{\partial v^{j+1}}f_{31}\left(w^j,v^{j+1}\right)+f_{31}\left(w^j,v^{j+1}\right)\right)\left(f_{11}\left(u^{j+1},v^j\right)+f_{31}\left(w^j,v^{j+1}\right)\right)=0.$$

By Equations (4.7) and (4.8) we find

$$f_{11}\left(u^{j+1},v^j\right)=f_{12}\left(v^j\right)e^{-u^{j+1}},\qquad f_{31}\left(w^j,v^{j+1}\right)=f_{32}\left(v^{j+1}\right)e^{w^j},$$

where $f_{12}\left(v^j\right)$ and $f_{32}\left(v^{j+1}\right)$ are some functions to be determined. We rewrite Equation (4.9) as

$$\left(\frac{\partial}{\partial v^j}f_{12}\left(v^j\right)-f_{12}\left(v^j\right)\right)e^{-u^{j+1}}-\left(\frac{\partial}{\partial v^{j+1}}f_{32}\left(v^{j+1}\right)+f_{31}\left(v^{j+1}\right)\right)e^{w^j}=0,$$

and it can be easily integrated

$$f_{12}\left(v^j\right)=C_1e^{v^j},\qquad f_{32}\left(v^{j+1}\right)=C_2e^{-v^{j+1}},$$

where C_1, C_2 are arbitrary constants.

Thus, we finally obtain the sought system of equations

$$\begin{cases}u^{j+1}_x=u^j_x+C_1e^{v^j-u^{j+1}},\\ v^{j+1}_x=v^j_x-C_1e^{v^j-u^{j+1}}-C_2e^{w^j-v^{j+1}},\\ w^{j+1}_x=w^j_x+C_2e^{w^j-v^{j+1}}.\end{cases} \tag{4.10}$$

This system is completely determined due to the first two equations in (4.4). The third equation is satisfied immediately.

We assume that the system (4.10) is obtained from a three–dimensional lattice

$$u_{n,x}^{j+1} = f(u_{n,x}^{j}, u_{n}^{j+1}, u_{n}^{j}, u_{n+1}^{j}, u_{n-1}^{j+1}), \quad -\infty < n, j < \infty,$$

by imposing truncation conditions and we then conclude that the lattice reads

$$u_{n,x}^{j+1} = u_{n,x}^{j} + Ce^{u_n^j - u_{n-1}^{j+1}} - Ce^{u_{n+1}^j - u_n^{j+1}},$$

where $C_1 = -C_2 =: C$. We can put $C = 1$, since C can be removed by the dilatation $x = C\bar{x}$.

Thus, we have obtained the discretization of the lattice (E3)

$$u_{n,x}^{j+1} = u_{n,x}^{j} + e^{u_n^j - u_{n-1}^{j+1}} - e^{u_{n+1}^j - u_n^{j+1}}.$$

This lattice is known to be integrable, see [5].

5. Discretization of equation $(E1)$

We consider the lattice

$$u_{n,xy} = e^{u_{n+1} - 2u_n + u_{n-1}}. \tag{5.1}$$

The finite–field reduction of the lattice (5.1) is

$$\begin{cases} u_{xy} = e^{v-2u}, \\ v_{xy} = e^{w-2v+u}, \\ w_{xy} = e^{v-2w}, \end{cases} \tag{5.2}$$

where $u := u_1$, $v := u_2$, $w := u_3$. It is obtained by imposing the truncation conditions $u_0 = 0$, $u_4 = 0$.

The system (5.2) was previously studied in [30], and it was shown that the system admits complete sets of integrals in both characteristic directions. Let us construct in explicit form a complete set of the y–integrals of the system (5.2). To this purpose, we use the linear system

$$\begin{cases} \varphi_{n,x} = (u_{n+1,x} - u_{n,x})\,\varphi_n + \varphi_{n+1}, \\ \varphi_{n,y} = -e^{u_{n+1} - 2u_n + u_{n-1}}\varphi_{n-1}, \end{cases}$$

associated with the lattice (5.1). Terminating the linear system in accordance with the boundary conditions $u_0 = 0$, $u_4 = 0$ of the chain (5.1), we obtain

$$\begin{cases} \varphi_{0,x} = u_x\varphi_0 + \varphi_1, \\ \varphi_{1,x} = (v_x - u_x)\,\varphi_1 + \varphi_2, \\ \varphi_{2,x} = (w_x - v_x)\,\varphi_2 + \varphi_3, \\ \varphi_{3,x} = -w_x\varphi_3, \end{cases} \qquad \begin{cases} \varphi_{0,y} = 0, \\ \varphi_{1,y} = -e^{v-2u}\varphi_0, \\ \varphi_{2,y} = -e^{w-2v+u}\varphi_1, \\ \varphi_{3,y} = -e^{-2w+v}\varphi_2, \end{cases} \tag{5.3}$$

where $u := u_1$, $v := u_2$, $w := u_3$. System of equations (5.3) is compatible if and only if its coefficients satisfy the system (5.2), i.e. (5.3) is a Lax pair for (5.2).

From (5.3) we obtain the operator

$$B = (D_x + w_x)\,(D_x - w_x + v_x)\,(D_x - v_x + u_x)\,(D_x - u_x)$$

that can be used as a generating function of the y–integrals. We expand the operator B in the polynomial form and get three y–integrals

$$\begin{aligned}
I_1 =& u_{xx}+v_{xx}+w_{xx}+u_x^2+v_x^2+w_x^2-u_xv_x-v_xw_x,\\
I_2 =& 2u_{xxx}+v_{xxx}+4u_xu_{xx}-v_xu_{xx}-2u_xv_{xx}+2v_xv_{xx}-v_xw_{xx}\\
&+u_x^2v_x-u_xv_x^2+v_x^2w_x-v_xw_x^2,\\
I_3 =& u_{xxxx}+2u_xu_{xxx}-u_xv_{xxx}+2u_xv_xu_{xx}-v_x^2u_{xx}+v_xw_xu_{xx}\\
&-w_x^2u_{xx}+u_x^2v_{xx}-u_x^2w_{xx}-2u_xv_xv_{xx}+u_xv_xw_{xx}+2u_{xx}^2\\
&-u_{xx}v_{xx}-u_{xx}w_{xx}+u_x^2v_xw_x-u_x^2w_x^2-u_xv_x^2w_x+u_xv_xw_x^2.
\end{aligned}$$

Let us check whether the integrals I_1–I_3 form a complete set of independent integrals.We calculate the determinant (2.2)

$$\begin{vmatrix}\frac{\partial I_1}{\partial u_{xx}} & \frac{\partial I_1}{\partial v_{xx}} & \frac{\partial I_1}{\partial w_{xx}}\\ \frac{\partial I_2}{\partial u_{xxx}} & \frac{\partial I_2}{\partial v_{xxx}} & \frac{\partial I_2}{\partial w_{xxx}}\\ \frac{\partial I_3}{\partial u_{xxxx}} & \frac{\partial I_3}{\partial v_{xxxx}} & \frac{\partial I_3}{\partial w_{xxxx}}\end{vmatrix} = \begin{vmatrix}1 & 1 & 1\\ 2 & 1 & 0\\ 1 & 0 & 0\end{vmatrix} = 1 \neq 0.$$

We look for a system of equations of the form

$$\begin{cases} u_x^{j+1} = f_1\left(u_x^j, u^{j+1}, u^j, v^j\right),\\ v_x^{j+1} = f_2\left(v_x^j, v^{j+1}, v^j, w^j, u^{j+1}\right),\\ w_x^{j+1} = f_3\left(w_x^j, w^{j+1}, w^j, v^{j+1}\right)\end{cases}$$

for which the functions I_1, I_2, I_3 are j–integrals, i.e. relations of the form

$$(D_j-1)\,I_i = 0, \qquad i=1,2,3, \tag{5.4}$$

hold. We present one of these equations explicitly

$$\begin{aligned}
(D_j-1)\,I_1 =& u_{xx}^{j+1}+v_{xx}^{j+1}+w_{xx}^{j+1}+(u_x^{j+1})^2+(v_x^{j+1})^2+(w_x^{j+1})^2-u_x^{j+1}v_x^{j+1}-v_x^{j+1}w_x^{j+1}\\
&-\left(u_{xx}^j+v_{xx}^j+w_{xx}^j+(u_x^j)^2+(v_x^j)^2+(w_x^j)^2-u_x^jv_x^j-v_x^jw_x^j\right)\\
=&\left((f_1)_{u_x^j}-1\right)u_{xx}^j+\left((f_1)_{v_x^j}-1\right)v_{xx}^j+\left((f_1)_{w_x^j}-1\right)w_{xx}^j\\
&+(f_3)_{w^j}w_x^j+(f_3)_{v^{j+1}}f_2+(f_2)_{v^{j+1}}f_2+(f_2)_{u^{j+1}}f_1+(f_1)_{u^{j+1}}f_1\\
&+(f_3)_{w^{j+1}}f_3+(f_1)_{u^j}u_x^j+(f_1)_{v^j}v_x^j+(f_2)_{v^j}v_x^j+(f_2)_{w^j}w_x^j+u_x^jv_x^j\\
&-f_1f_2-f_2f_3+f_1^2+f_2^2+f_3^2+v_x^jw_x^j-(u_x^j)^2-(v_x^j)^2-(w_x^j)^2=0.
\end{aligned} \tag{5.5}$$

Since the variables u_{xx}^j, v_{xx}^j and w_{xx}^j are independent, it follows from (5.5) that

$$(f_1)_{u_x^j}-1=0, \qquad (f_1)_{v_x^j}-1=0, \qquad (f_1)_{w_x^j}-1=0$$

or, what is the same,

$$\begin{aligned}
f_1\left(u_x^j, u^{j+1}, u^j, v^j\right) &= u_x^j + f_{11}\left(u^{j+1}, u^j, v^j\right),\\
f_2\left(v_x^j, v^{j+1}, v^j, w^j, u^{j+1}\right) &= v_x^j + f_{21}\left(v^{j+1}, v^j, w^j, u^{j+1}\right),\\
f_3\left(w_x^j, w^{j+1}, w^j, v^{j+1}\right) &= w_x^j + f_{31}\left(w^{j+1}, w^j, v^{j+1}\right).
\end{aligned}$$

Afterwards we analyze Equations (5.4) in a way similar to the previous case. As a result we find that the desired functions f_1, f_2, f_3 read

$$\begin{aligned} f_1\left(u_x^j, u^{j+1}, u^j, v^j\right) &= u_x^j + C_1 e^{-u^{j+1}-u^j+v^j},\\ f_2\left(v_x^j, v^{j+1}, v^j, w^j, u^{j+1}\right) &= v_x^j + C_2 e^{-v^{j+1}-v^j+w^j+u^{j+1}},\\ f_3\left(w_x^j, w^{j+1}, w^j, v^{j+1}\right) &= w_x^j + C_3 e^{-w^{j+1}-w^j+v^{j+1}}. \end{aligned}$$

Finally, we get a three–dimensional lattice

$$u_{n,x}^{j+1} = u_{n,x}^j + C_n e^{u_{n-1}^{j+1}-u_n^{j+1}-u_n^j+u_{n+1}^j}, \tag{5.6}$$

where $C_n \neq 0$ is an arbitrary function of n. Apparently, this lattice is integrable in a sense, it is associated with the linear system

$$\begin{cases} \varphi_{n,x}^j = -e^{u_{n+1}^j-2u_n^j+u_{n-1}^j}\varphi_{n-1}^j,\\ \varphi_n^{j+1} = \varphi_{n+1}^j + \frac{1}{C_n} e^{u_n^j+u_{n+1}^{j+1}-u_n^{j+1}-u_{n+1}^j}\varphi_n^j. \end{cases}$$

In this article we are interested only on autonomous lattices in the class (1.2), therefore we set $C_n = C = const$ in (5.6); the nonautonomous case requires further investigation. We put $C = 1$, since C can be removed by the change $x = C\bar{x}$, and we get

$$u_{n,x}^{j+1} = u_{n,x}^j + e^{u_{n-1}^{j+1}-u_n^{j+1}-u_n^j+u_{n+1}^j},$$

that was found earlier in [1]. It is obvious that the above lattice is related with the lattice

$$w_{n,x}^{j+1} = w_{n,x}^j + e^{w_{n+1}^j-w_n^{j+1}} - e^{w_n^j-w_{n-1}^{j+1}}$$

by means of the substitution $u_n^{j+1} - u_{n+1}^j = w_n^{j+1}$.

6. Discretization of the lattices in 3D via Lax pairs

Discretization of Equation (E3) via integrals formally leads to a lattice of the trivial form $w_{n,y}^{j+1} = w_{n,y}^j$. However, in this case one of our requirements is violated (see Condition (A)). The employed complete set of integrals of minimal orders defined by (E3) is not a set of integrals of minimal orders for the obtained lattice. This shows that difficulties arise when discretizing the lattice (E3) using characteristic integrals. Apparently, they are related to the presence of a nonlocal variable in its Lax pair, see (3.11). An alternative approach to the problem of discretizing integrable equations was proposed many years ago in [22]. This approach can be called discretization by Lax pair. We follow the ideas of [22] to discretize the lattice (E3)

$$w_{n,xy} = e^{w_{n+1}} - 2e^{w_n} + e^{w_{n-1}}. \tag{6.1}$$

Let us recall that the Lax pair for this equation is

$$\psi_{n,y} = -e^{w_{n-1}}\psi_{n-1}, \qquad \psi_{n,x} = q_{n,x}\psi_n + \psi_{n+1}, \tag{6.2}$$

where the additional (nonlocal) variable q_n is determined by the condition

$$w_n = q_{n+1} - q_n.$$

We look for a lattice of the form

$$v_{n,y}^{j+1} = F(v_{n,y}^j, v_{n+1}^j, v_{n-1}^{j+1}, v_n^{j+1}, v_n^j)$$

and at the same time determine a Lax pair for it by specifying its structure

$$\psi_{n,y}^j = -e^{w_{n-1}^j}\psi_{n-1}^j, \qquad \psi_n^{j+1} = e^{b_n^j}\psi_n^j + A_n^j\psi_{n+1}^j. \tag{6.3}$$

Here the sought function $v_n^j = v_n^j(y)$ depends on two discrete variables n, j and one continuous variable y. The unknown functional parameters b_n^j and A_n^j are supposed to be found from the compatibility condition of the system (6.3)

$$D_j \psi^j_{n,y} = D_y \psi_n^{j+1},$$

where D_j and D_y denote the operator, which shifts the variable j, and the operator of the total derivative with respect to the variable y. The compatibility condition implies the equation

$$-D_j(e^{w^j_{n-1}}\psi^j_{n-1}) = D_y(e^{b^j_n}\psi^j_n + A^j_n\psi^j_{n+1}).$$

By this equation we easily obtain three relations

$$\begin{aligned}
&\frac{\partial}{\partial y}A^j_n = 0 \quad \Rightarrow \quad A^j_n = const,\\
&b^j_n + w^j_{n-1} = b^j_{n-1} + w^{j+1}_{n-1},\\
&\frac{\partial}{\partial y}b^j_n = e^{w^j_n - b^j_n} - e^{w^{j+1}_{n-1} - b^j_n}.
\end{aligned}$$

The first relation yields that the function A^j_n is independent on the dynamical variables and we can set $A^j_n = 1$ without loss of generality. The second relation is reduced to

$$(D_n - 1)b^j_{n-1} = (D_j - 1)w^j_{n-1}.$$

The latter allows us to introduce a new function X^j_n as a solution to the equations

$$b^j_{n-1} = (D_j - 1)X^j_n, \qquad w^j_{n-1} = (D_n - 1)X^j_n.$$

As a result, the third relation becomes the semi–discrete Toda lattice from the class (1.2)

$$X^{j+1}_{n,y} - X^j_{n,y} = e^{X^j_{n+1} - X^{j+1}_n} - e^{X^j_n - X^{j+1}_{n-1}}, \tag{6.4}$$

derived in Section 3, see also [9]. We obtain the well–known Lax pair for Equation (6.4) by substituting the above representations for the parameters A^j_n, b^j_n and w^j_n into (6.3), see [9],

$$\begin{cases}
\psi^j_{n,y} = -e^{X^j_{n+1} - X^j_n}\psi^j_{n-1},\\
\psi^{j+1}_n = e^{X^{j+1}_{n+1} - X^j_{n+1}}\psi^j_n + \psi^j_{n+1}.
\end{cases} \tag{6.5}$$

Under substitution $v^j_n = X^j_n - X^{j+1}_{n-1}$, Equation (6.4) becomes

$$v^{j+1}_{n,y} - v^j_{n,y} = e^{v^j_{n+1}} + e^{v^{j+1}_{n-1}} - e^{v^{j+1}_n} - e^{v^j_n}. \tag{6.6}$$

It is easy to verify that the obtained equation is a discretization of Equation (6.1). Equation (6.6) was first derived in [5], and then it was studied in a series of papers, see, for example, [18]. However, to the best of our knowledge, the Lax pair for it has not been constructed yet. Our goal in this section is to present the Lax pair for (6.6). In the system of linear equations (6.5) we make a discrete substitution by assuming

$$\bar\varphi^j_n = \psi^{j+1}_n - \psi^j_{n+1}.$$

Then it follows from the second equation of the system that the relation

$$\psi^j_n = e^{X^j_{n+1} - X^{j+1}_{n+1}}\bar\varphi^j_n$$

holds, which allows us to exclude the function ψ^j_n from the system (6.5). As a result, (6.5) becomes

$$\begin{cases}
\bar\varphi^j_{n,y} = e^{v^j_{n+2} - v^j_{n+1}}\bar\varphi^j_n - e^{X^{j+1}_{n+1} - X^{j+1}_n}\bar\varphi^j_{n-1},\\
\bar\varphi^{j+1}_n = e^{X^{j+2}_{n+1} - X^{j+1}_{n+1}}\bar\varphi^j_n + e^{v^j_{n+2} - v^{j+1}_{n+2}}\bar\varphi^j_{n+1}.
\end{cases}$$

We simplify the system using a point change of the variables

$$\varphi^j_{n+2} = e^{-X^{j+1}_{n+1}}\bar{\varphi}^j_n.$$

As a result, we obtain the desired Lax pair for Equation (6.6)

$$\begin{cases}\varphi^j_{n,y} = (e^{v^j_n} - e^{v^j_{n-1}} - X^{j+1}_{n-1,y})\varphi^j_n - \varphi^j_{n-1},\\ \varphi^{j+1}_n = \varphi^j_n + e^{v^j_n}\varphi^j_{n+1}.\end{cases}$$

As it has been expected, it contains the nonlocality $X^{j+1}_{n-1,y}$, which, however, occurs only in one of the Lax equations, as in the original Lax pair (6.2) for the chain (6.1). The nonlocal variable is defined by the equation $v^j_n = X^j_n - X^{j+1}_{n-1}$.

Conclusions

The problem on finding an exhaustive list of integrable chains of the form (1.2) is extremely difficult and remains open. Therefore, it is relevant to expand the list of integrable representatives of this class and to study their specific features in detail. In this paper, we discuss two methods for obtaining integrable examples of (1.2) by discretizing the equations of the Toda lattice type (1.1). The first discretization method is based on the Lax pair, and the second one is based on Darboux integrable reductions and discretization with respect to characteristic integrals. Using discretization of Equations (E1)–(E3), we obtain three chains of the form (1.2)

$$\begin{aligned}
u^{j+1}_{n,x} &= u^j_{n,x} + e^{u^j_n - u^{j+1}_{n-1}} - e^{u^j_{n+1} - u^{j+1}_n},\\
u^{j+1}_{n,x} &= u^j_{n,x} + e^{u^{j+1}_{n-1} - u^{j+1}_n - u^j_n + u^j_{n+1}},\\
u^{j+1}_{n,x} &= u^j_{n,x} + e^{u^j_{n+1}} + e^{u^{j+1}_{n-1}} - e^{u^{j+1}_n} - e^{u^j_n}.
\end{aligned} \tag{6.7}$$

The corresponding Lax pairs are

$$\begin{cases}\varphi^j_{n+1} = -\varphi^{j+1}_n + e^{u^j_n - u^j_{n+1}}\varphi^j_n,\\ \varphi^j_{n,x} = -e^{u^{j-1}_n - u^j_n}\varphi^{j-1}_n;\end{cases} \tag{6.8}$$

$$\begin{cases}\varphi^j_{n,x} = -e^{u^j_{n+1} - 2u^j_n + u^j_{n-1}}\varphi^j_{n-1},\\ \varphi^{j+1}_n = \varphi^j_{n+1} + e^{u^j_n + u^{j+1}_{n+1} - u^{j+1}_n - u^j_{n+1}}\varphi^j_n;\end{cases} \tag{6.9}$$

$$\begin{cases}\varphi^j_{n,x} = (e^{u^j_n} - e^{u^j_{n-1}} - X^{j+1}_{n-1,x})\varphi^j_n - \varphi^j_{n-1},\\ \varphi^{j+1}_n = \varphi^j_n + e^{u^j_n}\varphi^j_{n+1}, \qquad u^j_n = X^j_n - X^{j+1}_{n-1}.\end{cases}$$

The Lax pairs (6.8) and (6.9) were found earlier in [9], while the Lax pair for (6.7) is new.

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