ISSN 2304-0122 Ufa Mathematical Journal. Vol. 17. No 2 (2025). P. 1-26.

doi:10.13108/2025-17-2-1

ONE–PARAMETRIC FAMILIES OF CONFORMAL MAPPINGS OF UNBOUNDED DOUBLY CONNECTED POLYGONAL DOMAINS

A.Yu. DUTIN, S.R. NASYROV

Abstract. We propose an approximate method for finding a conformal mapping of a concentric annulus onto an arbitrary unbounded doubly connected polygonal domain. This method is based on ideas related to the parametric Löwner — Komatu method. We consider smooth one–parametric families of conformal mappings $\mathcal{F}(z,t)$ of concentric annuli onto doubly connected polygonal domains $\mathcal{D}(t)$, which are obtained from a fixed unbounded doubly connected polygonal domain $\mathcal{D}$ by making a finite number of rectilinear or, in general, polygonal slits of variable length; at the same time, we do not suppose the monotonicity of family of domains $\mathcal{D}(t)$. The integral representation for the conformal mappings $\mathcal{F}(z,t)$ includes unknown (accessory) parameters. We find a partial differential equation for these families of conformal mappings and derive from it a system of differential equations describing the dynamics of the accessory parameters as the parameter t varies and the dynamics of the conformal modulus of a given doubly connected domain as a function of the parameter t. The right hand sides of resulting system of ordinary differential equations include functions being the velocities of the end points of slits. This allows us to control completely the dynamics of slits, in particular, to achieve their consistent change in the case where more than one slit is made in the domain $\mathcal{D}$. Examples illustrating the efficiency of proposed method are provided. We mention that we have already considered the parametric method proposed in this paper but for the case of bounded doubly connected polygonal domains.

Keywords: unbounded doubly connected domains, polygonal domains, conformal moduli, conformal mappings, Schwarz — Christoffel formula, accessory parameters, one–parametric families of functions, parametric method, elliptic functions, elliptic integrals, Löwner — Komatu equation.

Mathematics Subject Classification: 30C30.

1. Introduction

Conformal mappings of polygonal domains play an important role in the geometric theory of functions and its applications. By the Riemann mapping theorem, in the simply connected case there exists a conformal mapping of a given polygonal domain onto the unit disk or half–plane. The inverse mapping is given by the Schwarz — Christoffel formula, see, for instance, [15]. In the doubly connected case the polygonal domain can be conformally mapped onto an annulus, see, for instance, [17, Ch. 5, Sect. 1], while the inverse mapping is determined by a generalization of Schwarz — Christoffel integrals, which involves elliptic functions and related theta functions, see, for instance, [2], [13]. The integral representations of such conformal

---

A.Yu. Dyutin, S.R. Nasyrov, One parameter families of conformal mappings of unbounded doubly connected polygonal domains.

The research is supported by the Russian Science Foundation, grant no. 23-11-00066, https://rscf.ru/project/23-11-00066/.

Submitted September 10, 2024.

mappings were obtained by Akhiezer [2, Ch. VIII, Sect. 48], Goluzin [3] and Komatu [19], [20]; see also [18, Sect. 17.5]. In the multiply connected case the polygonal domain can be conformally mapped onto a circular domain, that is, the domain, the boundary of which is the union of disjoint circles, the survey can be found in [16].

The undoubted advantage of the Schwarz — Christoffel formula and its generalizations to the multiply connected case is their simplicity. A significant disadvantage is that such formulas contain unknown parameters. First of all, these are the preimages of the vertices of polygonal lines enveloping the domain. In the case of unbounded domains, another unknown parameter is added, which is the preimage of the infinity. In the multiply connected case, the matter is complicated by the fact that when mapping onto a circular domain, the parameters of this domain (the so–called conformal moduli) are unknown. In the doubly connected case, there is one parameter, the conformal modulus of the domain, which is also to be determined. If a doubly connected domain $\mathcal{D}$ is conformally equivalent to the annulus $\{\tau \in \mathbb{C} : q < |\tau| < 1\}$, then its conformal modulus can be defined as

$$\mathrm{Mod}(\mathcal{D}) = \frac{1}{2\pi} \ln q^{-1}.$$

There are various methods for determining unknown (accessory) parameters in Schwarz — Christoffel integrals and their generalizations. One of the most interesting is the parametric Löwner method, which has other important applications in the theory of univalent functions, see, for example, the review [11], as well as the monograph [1]. In particular, in 1984, Louis de Branges [12] proved the famous Bieberbach conjecture on estimating the moduli of coefficients of univalent functions in the circle by the parametric method [10].

The parametric Löwner method is based on the consideration of one–parametric families of univalent functions depending on a real parameter, which conformally map the canonical domain onto a family of domains obtained from a given domain by making an extending slit, and the study of the differential equation for the functions of this family, the so–called Löwner equation and its modifications.

A review of approximate methods for finding accessory parameters based on the parametric Loewner method can be found in [16]. We mention the paper [23], in which the parametric method was proposed to employ for finding accessory parameters for simply connected polygonal domains in the case of several slits. In [16], the parametric method was used to find accessory parameters and the conformal modulus in the case of bounded doubly connected polygonal domains; the work employs the approach previously applied in [7], [8], [14], [22].

This paper is devoted to finding accessory parameters in the generalized Schwarz — Christoffel integral by the parametric method in the case of unbounded doubly connected domains, when several slits of variable length are made simultaneously in the original domain $\mathcal{D}$. We denote by $\mathcal{D}(t)$ the corresponding one–parametric family of cut polygonal domains, where t is a real parameter.

Let us describe the structure of work and the main results. In Section 2 we provide necessary facts from the theory of elliptic functions and elliptic integrals. In Section 3 we describe an integral representation of conformal mapping of a concentric annulus onto an unbounded doubly connected polygonal domain, see Theorem 3.1. This integral representation can be also obtained from similar Akhiezer and Goluzin formulas, see, for instance, [2, Ch. VIII, Sect. 48] and [3]. Then in Section 4, by means of this integral representation, we consider smooth one–parametric families $\mathcal{F}(z,t)$ of conformal mappings of concentric annuli onto unbounded doubly connected polygonal domains $\mathcal{D}(t)$. We emphasise that we make several slits at once, and we do not require the family of domains $\mathcal{D}(t)$ to be monotonic in the sense of inclusion. This allows us both to increase and decrease the lengths of some slits. We find a partial differential equation for a family of such conformal mappings, which is some modification of the Löwner — Komatu equation for the case of several slits, see Theorem 4.1. From this equation, we derive a system of ordinary differential equations, which determines the dynamics of the accessory parameters of these conformal mappings, as well as the dynamics of conformal module of a doubly connected polygon, see Theorem 4.2. Solving the Cauchy problem for this system, we find all unknown parameters in the integral representation of family $\mathcal{F}(z,t)$. We stress that we can simply control the position, velocity, and direction of motion of the ends of slits since the velocities of endpoints of slits are explicitly included in the right–hand sides of the derived differential equations.

On the base of the Carathéodory kernel convergence theorem and its generalizations, see, for instance, [17], we can cut out domains of a more complex geometry from a given unbounded domain $\mathcal{D}$. We can also repeat the procedure of cutting out a domain from a given one and obtain more complex polygonal domains in several steps. In this case, the implementation of the method consists in sequential solving several Cauchy problems for systems of ordinary differential equations. In this case, the solution (accessory parameters) obtained at each of the steps, except for the last one, determines the initial conditions for the system solved at the next step.

In conclusion, in Section 5 we demonstrate the efficiency of the proposed approximate method at examples. As an initial doubly connected domain from which the required domain are cut out, we consider the exterior of two symmetric parallel slits. In this case, the formulas defining all accessory parameters for this conformal mapping were found earlier, in Section 3, see Example 3.1. We start moving the ends of the slits along polygonal trajectories and cutting more complex polygonal domains from the initial domains, see Examples 5.1, 5.2, and 5.3. We observe that the accuracy of calculations for the accessory parameters and the conformal modulus was up to 10^{-15}, and for the vertices of the polygons it was up to 10^{-6}.

2. Preliminaries

First we recall some known facts in the theory of elliptic integrals and elliptic functions, see, for instance, [2], [13], [21].

A meromorphic in $\mathbb{C}$ function is called elliptic if it has periods ω_1 and ω_2, which are linearly independent over $\mathbb{R}$. We take an arbitrary point $z_0 \in \mathbb{C}$ and construct a parallelogram with the vertices z_0, $z_0+\omega_1$, $z_0+\omega_1+\omega_2$, $z_0+\omega_2$. This parallelogram is called the parallelogram of periods. To this parallelogram we assign all points $z_0+r_1\omega_1+r_2\omega_2$, $0\leqslant r_1<1$, $0\leqslant r_2<1$. For $z_0=0$ the parallelogram of periods is called main. The number of poles of an elliptic function in its parallelogram of periods counting the orders is called the order of elliptic function. By $\Omega=\{m_1\omega_1+m_2\omega_2\}$ we denote the lattice generated by the periods ω_1 and ω_2, m_1, $m_2\in\mathbb{Z}$. In what follows by ω we denote an arbitrary element of the lattice and suppose that $\operatorname{Im}\frac{\omega_2}{\omega_1}>0$.

One of the main elliptic functions is the Weierstrass $\wp$–function

$$\wp(z)=\wp(z;\omega_1,\omega_2)=\frac{1}{z^2}+\sum_{\omega\in\Omega,\,\omega\neq 0}\left(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right).$$

This is a meromorphic in $\mathbb{C}$ function with poles of second order at each points $\omega\in\Omega$. In the semi–periods the $\wp$–function takes mutually different values

$$\wp\left(\frac{\omega_1}{2}\right)=e_1,\qquad \wp\left(\frac{\omega_2}{2}\right)=e_2,\qquad \wp\left(\frac{\omega_1+\omega_2}{2}\right)=e_3.$$

We shall also employ the Weierstrass ζ–function

$$\zeta(z)=\zeta(z;\omega_1,\omega_2)=\frac{1}{z}+\sum_{\omega\in\Omega,\,\omega\neq 0}\left(\frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{\omega^2}\right),$$

which possesses the properties $\zeta'(z) = -\wp(z)$ and $\zeta(z+\omega_k) - \zeta(z) = \eta_k$, where $\eta_k = 2\zeta(\frac{\omega_k}{2})$, $k = 1, 2$. There is the Legendre relation for the periods ω_1 and ω_2 and the constants η_1 and η_2 [2, Ch. III, Sect. 12]:

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i. \tag{2.1}$$

Finally, the Weierstrass σ–function

$$\sigma(z) = \sigma(z;\omega_1,\omega_2) = z \prod_{\omega\in\Omega,\,\omega\neq 0} \left(1 - \frac{z}{\omega}\right) \exp\left\{\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right\}$$

has the properties

$$\frac{d\ln\sigma(z)}{dz} = \zeta(z), \qquad \frac{d^2\ln\sigma(z)}{dz^2} = -\wp(z),$$
$$\sigma(z\pm\omega_k) = -\exp\left\{\pm\eta_k\left(z\pm\frac{\omega_k}{2}\right)\right\}\sigma(z), \qquad k = 1, 2.$$

We shall also use the functions

$$\sigma_1(z) = \pm\frac{\sigma\left(z\pm\frac{\omega_1}{2}\right)}{\sigma\left(\frac{\omega_1}{2}\right)} \exp\left\{\mp\frac{\eta_1}{2}z\right\},$$
$$\sigma_2(z) = \pm\frac{\sigma\left(z\pm\frac{\omega_2}{2}\right)}{\sigma\left(\frac{\omega_2}{2}\right)} \exp\left\{\mp\frac{\eta_2}{2}z\right\},$$
$$\sigma_3(z) = \pm\frac{\sigma\left(z\pm\frac{\omega_1+\omega_2}{2}\right)}{\sigma\left(\frac{\omega_1+\omega_2}{2}\right)} \exp\left\{\mp\frac{\eta_1+\eta_2}{2}z\right\}.$$

We recall that the Weierstrass invariants g_2 and g_3 are defined as

$$g_2 = 60\sum_{\omega\in\Omega,\,\omega\neq 0}\frac{1}{\omega^4}, \qquad g_3 = 140\sum_{\omega\in\Omega,\,\omega\neq 0}\frac{1}{\omega^6}.$$

In what follows we shall need explicit expressions for the partial derivatives of the function $\zeta(z) = \zeta(z;\omega_1,\omega_2)$ with respect to the periods. In [8], the following statement was proved.

Theorem 2.1. *The partial derivatives of the function $\zeta(z) = \zeta(z;\omega_1,\omega_2)$ with respect to the periods read*

$$\frac{\partial\zeta(z)}{\partial\omega_1} = \frac{1}{2\pi i}\left[\frac{1}{2}\,\omega_2\wp'(z) + (\omega_2\zeta(z) - \eta_2 z)\wp(z) + \eta_2\zeta(z) - \frac{g_2}{12}\,\omega_2 z\right],$$
$$\frac{\partial\zeta(z)}{\partial\omega_2} = -\frac{1}{2\pi i}\left[\frac{1}{2}\,\omega_1\wp'(z) + (\omega_1\zeta(z) - \eta_1 z)\wp(z) + \eta_1\zeta(z) - \frac{g_2}{12}\,\omega_1 z\right]. \tag{2.2}$$

Finally, we shall use the Jacobi theta functions, which can be represented in terms of the Weierstrass elliptic functions, see, for instance, [2, Ch. IV, Sect. 19],

$$\vartheta_1(\tau) = \sqrt{\frac{\omega_1}{\pi}}\,(e_1-e_2)^{\frac14}(e_1-e_3)^{\frac14}(e_3-e_2)^{\frac14}\exp\left\{-\frac{\eta_1 z^2}{2\omega_1}\right\}\sigma(z), \tag{2.3}$$
$$\vartheta_2(\tau) = \sqrt{\frac{\omega_1}{\pi}}\,(e_3-e_2)^{\frac14}\exp\left\{-\frac{\eta_1 z^2}{2\omega_1}\right\}\sigma_1(z),$$
$$\vartheta_3(\tau) = \sqrt{\frac{\omega_1}{\pi}}\,(e_1-e_2)^{\frac14}\exp\left\{-\frac{\eta_1 z^2}{2\omega_1}\right\}\sigma_3(z),$$
$$\vartheta_4(\tau) = \sqrt{\frac{\omega_1}{\pi}}\,(e_1-e_3)^{\frac14}\exp\left\{-\frac{\eta_1 z^2}{2\omega_1}\right\}\sigma_2(z),$$

where $\tau = \frac{z}{\omega_1}$.

The incomplete elliptic integral of the first kind reads

$$F(z,k)=\int\limits_0^z \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}, \tag{2.4}$$

where the parameter $k\in(0,1)$ is called the modulus of this integral. Letting $z=1$ in (2.4), we obtain

$$K(k)=\int\limits_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}, \tag{2.5}$$

which is called the complete elliptic integral of the first kind. We shall use the notation $K'(k)=K(k')$, where the parameter $k'=\sqrt{1-k^2}$ is called the complementary modulus. There is the Legendre relation for the quantities K, K', E, E' in the Jacobi theory, see [2, Ch. V, Sect. 31],

$$EK'+E'K-KK'=\frac{\pi}{2}. \tag{2.6}$$

The function $z=\operatorname{sn}(w,k)$, which is inverse to the function defined by the integral (2.4), is called the Jacobi elliptic sine, its variable is expressed by the elliptic integral of the first kind (2.4). It is the elliptic functions of the second kind with main periods $4K$ and $2iK'$.

3. Integral representation

In this section we derive an integral representation for the function, which conformally maps a concentric annulus onto an unbounded doubly connected polygonal domain. We follow the notation adopted for bounded domains [16]. We consider a conformal mapping $w=\mathcal{G}(\tau)$ of the annulus $\mathcal{A}=\{\tau: q<|\tau|<1\}$ onto a doubly connected polygonal domain $\mathcal{D}$ that contains the infinity and therefore the exterior of two disjoint polygons. Then the function $\mathcal{G}$ has a unique pole in $\mathcal{A}$; we denote it by τ_0. We denote the boundary of one of the polygons by Γ_1, and the boundary of the other is denoted by Γ_2. Let Γ_1 have vertices at w_{1,i_1}, $1\leqslant i_1\leqslant n_1$, and let Γ_2 have vertices at w_{2,i_2}, $1\leqslant i_2\leqslant n_2$. Next, let $\alpha_{1,i_1}\pi$ and $\alpha_{2,i_2}\pi$ denote the interior angles of the domain $\mathcal{D}$ at the corresponding vertices w_{1,i_1} and w_{2,i_2}. The values of the angles are related by the identities

$$\sum_{i_1=1}^{n_1}\alpha_{1,i_1}=n_1+2,\qquad \sum_{i_2=1}^{n_2}\alpha_{2,i_2}=n_2+2. \tag{3.1}$$

In terms of the exponential map $\tau(z)=e^{iz}$, we can consider the map $\mathcal{F}(z)=\mathcal{G}(e^{iz})$ from the horizontal strip $\{0<\operatorname{Im} z<\ln q^{-1}\}$ onto $\mathcal{D}$. Obviously, in this strip the function $\mathcal{F}$ is a 2π–periodic function. Then the function $\mathcal{F}$ makes a conformal mapping of the rectangle $\Pi=[0,2\pi]\times(0,\ln q^{-1})$ with identified vertical sides onto $\mathcal{D}$, see Figure 1. We shall seek an integral representation for this mapping using the Weierstrass elliptic functions with the periods $\omega_1=2\pi$ and $\omega_2=2i\ln q^{-1}$.

Suppose that the mapping $\mathcal{F}$ sends a point z_0, lying in the rectangle $[0,2\pi]\times(0,\ln q^{-1})$, to the infinity. By the translation, which maps the strip $\{0<\operatorname{Im} z<\ln q^{-1}\}$ onto itself, we can achieve that the point $z=z_0$ lies on the imaginary axis, that is, $z_0=iy_0$, $0<y_0<\ln q^{-1}$. It is obvious that the function $\mathcal{F}$ is analytic in the strip $\{0<\operatorname{Im} z<\ln q^{-1}\}$ everywhere except for the point z_0, at which this function has a simple pole.

On the plane z we consider the preimages of the vertices

$$z_{1,i_1}\in[0,2\pi]\quad\text{and}\quad z_{2,i_2}=x_{2,i_2}+i\ln q^{-1}\in[i\ln q^{-1},2\pi+i\ln q^{-1}]$$

of domain $\mathcal{D}$. We denote by $\mathcal{Z}$ the set of points, which are obtained from the set of preimages of vertices by all possible translations by the vectors in the lattice Ω generated by the vectors ω_1

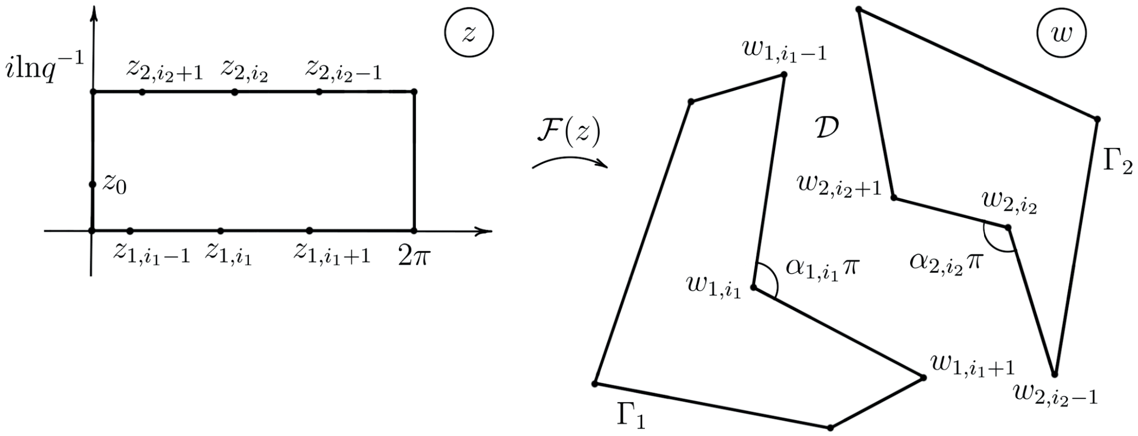

FIGURE 1. Conformal mapping of the rectangle $\Pi = [0, 2\pi] \times (0, \ln q^{-1})$ with identified vertical sides onto $\mathcal{D}$.

and ω_2. By means of the Riemann — Schwarz reflection principle we can continue $\mathcal{F}$ on $\mathbb{C} \setminus \mathcal{Z}$ as a multivalent analytic function through each interval $(z_{k,j} + \omega, z_{k,j+1} + \omega)$ parallel to the real axis. Moreover, in the first continuation, for example through the interval (z_{2,i_2}, z_{2,i_2+1}), the continued function has a simple pole at the point $\bar{z}_0 + \omega_2$, which is to be sent to the infinity by the mapping $\mathcal{F}(z)$.

It is easy to verify that the function $\dfrac{\mathcal{F}''(z)}{\mathcal{F}'(z)}$ is meromorphic and doubly periodic in $\mathbb{C}$ with the periods ω_1 and ω_2. Let us consider this function in the rectangle $\Pi_\omega = [0, \omega_1) \times [0, \frac{\omega_2}{i})$, which is its main parallelogram of periods. Then the conformal modulus of the domain $\mathcal{D}$ is equal to $\mathrm{Mod}(\mathcal{D}) = \dfrac{\omega_2}{2\omega_1 i}$. The function $\dfrac{\mathcal{F}''(z)}{\mathcal{F}'(z)}$ in the rectangle Π_ω has simple poles at the points z_{1,i_1} and z_{2,i_2} with the residues $\alpha_{1,i_1} - 1$ and $\alpha_{2,i_2} - 1$, respectively, and at the points z_0 and $\bar{z}_0 + \omega_2$ it has first–order poles with residues equal to -2. Therefore, we can reconstruct this function by its singularities using the Weierstrass elliptic function $\zeta(z) = \zeta(z; \omega_1, \omega_2)$, which has a single simple pole at a point comparable to zero modulo the periods in each parallelogram of periods. We have

$$\begin{aligned}\frac{\mathcal{F}''(z)}{\mathcal{F}'(z)} =& c + \sum_{i_1=1}^{n_1} (\alpha_{1,i_1} - 1)\, \zeta(z - z_{1,i_1}) + \sum_{i_2=1}^{n_2} (\alpha_{2,i_2} - 1) \zeta(z - z_{2,i_2}) \\ &- 2\left(\zeta(z - z_0) + \zeta(z - \bar{z}_0)\right),\end{aligned} \tag{3.2}$$

where c is some complex constant.

Reproducing the arguing from [16] for the function (3.2), we get

$$\begin{aligned}\mathcal{F}(z) =& C_1 \int_0^z \exp\{c\,\xi\} \prod_{i_1=1}^{n_1} \sigma^{\alpha_{1,i_1}-1}(\xi - z_{1,i_1}) \prod_{i_2=1}^{n_2} \sigma^{\alpha_{2,i_2}-1}(\xi - z_{2,i_2}) \\ &\cdot \left(\sigma(\xi - z_0)\sigma(\xi - \bar{z}_0)\right)^{-2} d\xi + C_2,\end{aligned} \tag{3.3}$$

where $C_1 \neq 0$ and C_2 are some complex numbers.

Let us determine the constant c in (3.3). Using the relations (2.1), (3.1) and the fact that the function $\mathcal{F}'(z)$ is ω_1–periodic, we obtain

$$c = \frac{\eta_1}{\omega_1} \left[\sum_{i_1=1}^{n_1} (\alpha_{1,i_1} - 1)\, z_{1,i_1} + \sum_{i_2=1}^{n_2} (\alpha_{2,i_2} - 1)\, x_{2,i_2} \right] + \eta_2. \tag{3.4}$$

Thus, we have proved the following theorem.

Theorem 3.1. *The function $\mathcal{F}$, which maps the annulus $\mathcal{A} = \{\tau : q < |\tau| < 1\}$ onto a doubly connected unbounded polygonal domain $\mathcal{D}$, is of the form (3.3), where $z = -i\ln\tau$. In (3.3), the points z_{1,i_1} and z_{2,i_2} are the preimages of the vertices of $\mathcal{D}$, the constant c is given by (3.4), and $C_1 \neq 0$ and C_2 are some complex constants.*

The identity (2.3) implies an integral representation for the required conformal mapping, which involves the Jacobi theta–functions.

Corollary 3.1. *The conformal mapping $\mathcal{F}$ described in Theorem 3.1 reads*

$$\begin{aligned}\mathcal{F}(z) =& C_1' \int\limits_0^z \exp\{-i\xi\} \prod_{i_1=1}^{n_1} \vartheta_1^{\alpha_{1,i_1}-1}\left(\frac{\xi - z_{1,i_1}}{\omega_1}\right) \prod_{i_2=1}^{n_2} \vartheta_1^{\alpha_{2,i_2}-1}\left(\frac{\xi - z_{2,i_2}}{\omega_1}\right) \\ &\cdot \vartheta_1^{-2}\left(\frac{\xi - z_0}{\omega_1}\right) \vartheta_1^{-2}\left(\frac{\xi - \bar{z}_0}{\omega_1}\right) d\xi + C_2.\end{aligned} \tag{3.5}$$

We observe that since the theta–functions are represented by fast converging series, the formula (3.5) is more convenient for the numerical realization of the conformal mapping than (3.3).

Now we consider examples of conformal mappings of unbounded doubly connected polygonal domains and find explicit formulas for the accessory parameters. These examples will be used below in Section 5.

Example 3.1. As a doubly connected unbounded polygonal domain $\mathcal{D}$, we take, for example, the exterior of two segments on the real axis. Let these segments be $[-\frac{1}{k}, -1]$ and $[1, \frac{1}{k}]$, where $k \in (0, 1)$. Let us describe the function $\mathcal{F}(z)$, which conformally maps the rectangle $[-\pi, \pi] \times (0, \ln q^{-1})$ with identified vertical sides onto $\mathcal{D}$, see Figure 2.

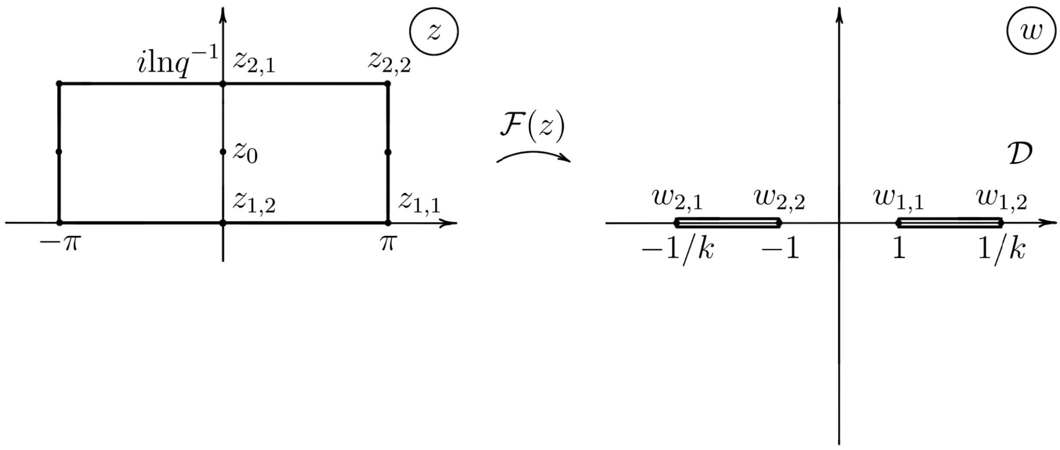

FIGURE 2. onformal mapping of rectangle $[-\pi, \pi] \times (0, \ln q^{-1})$ with identified vertical sides onto $\mathcal{D}$.

Since the domain $\mathcal{D}$ is symmetric with respect to the real axis, we can suppose that

$$z_{1,1} = \pi, \qquad z_{1,2} = 0, \qquad z_{2,1} = i\ln q^{-1}, \qquad z_{2,2} = \pi + i\ln q^{-1},$$

and moreover, $z_0 = \frac{i}{2}\ln q^{-1}$. At the same time,

$$\mathcal{F}(z_{k,j}) = w_{k,j}, \quad k, j = 1, 2,$$

where

$$w_{1,1} = 1, \qquad w_{1,2} = \frac{1}{k}, \qquad w_{2,1} = -\frac{1}{k}, \qquad w_{2,2} = -1.$$

By Theorem 3.1, the function $\mathcal{F}$, which maps the annulus $\mathcal{A} = \{\tau:\ q < |\tau| < 1\}$ onto $\mathcal{D}$, is given by the formula

$$\mathcal{F}(z) = C_1 \int_0^z \exp\{c\,\xi\} \prod_{i_1=1}^{2} \sigma(\xi - z_{1,i_1}) \prod_{i_2=1}^{2} \sigma(\xi - z_{2,i_2}) \left(\sigma(\xi - z_0)\sigma(\xi - \bar{z}_0)\right)^{-2} d\xi + C_2, \quad (3.6)$$

where $z = -i\ln\tau$, $c = \eta_1 + \eta_2$ and $\sigma(z) = \sigma(z; 2\pi, 2i\ln q^{-1})$. We observe that the above identity involves just the single unknown parameter q; we are going to find it.

In [2, Ch. VIII, Sect. 49], there was considered the conformal mapping of the annulus

$$\mathcal{A} = \{\tau:\ q(\ell) < |\tau| < 1\}$$

onto the domain $\mathcal{D}'$ being the plane cut along the segments $[-1, \alpha]$ and $[\beta, 1]$ on the real axis. The function making this conformal mapping reads

$$w(\tau; \ell) = \frac{1 - \alpha^2}{2\ \mathrm{sn}\left(\dfrac{K'(\ell)}{\pi}\ln\tau; \ell\right) + \alpha - 1} + \alpha,$$

where

$$\ell^2 = \frac{2(\beta - \alpha)}{(1 - \alpha)(1 + \beta)}, \qquad q(\ell) = \exp\left\{-\pi\,\frac{K(\ell)}{K'(\ell)}\right\}.$$

Therefore,

$$\mathrm{Mod}(\mathcal{D}') = \frac{K(\ell)}{2K'(\ell)}.$$

We observe that the function

$$z(w) = \frac{\pi}{K'(k)}\Big[F(w, k) - (K(k) + iK'(k))\Big]$$

makes a conformal mapping of the domain $\mathcal{D}$ onto the rectangle $(-\frac{2\pi K(k)}{K'(k)}, 0) \times (-\pi, \pi)$. Then

$$q(k) = \exp\left\{-2\pi\,\frac{K(k)}{K'(k)}\right\}$$

and

$$\mathrm{Mod}(\mathcal{D}) = \frac{K(k)}{K'(k)}.$$

Using the Landen transformation, we conclude that the parameters k and ℓ are related by the identity

$$k = \frac{1 - \ell'}{1 + \ell'}.$$

In Table 1 we provide the values of the parameter q and conformal modulus for some $k \in (0, 1)$.

Table 1. Some values of the parameter q and conformal modulus for the exterior of two segments $[-\frac{1}{k}, -1]$ and $[1, \frac{1}{k}]$.

k	2/3	1/2	1/3
q	0.002551352513688673	0.007360907921497923	0.01797238700896722
ω_2	11.942263325795764	9.823143989900522	8.037837508021143
$\mathrm{Mod}(\mathcal{D})$	0.9503351200027268	0.7817009613480557	0.6396307855855035

Example 3.2. As a doubly connected unbounded polygonal domain $\mathcal{D}$ we take the exterior of two parallel slits of length $2a$ separated by the distance $2b$, see Figure 3. Let us describe the function $\mathcal{F}(z)$, which maps conformally the rectangle $[-\pi,\pi]\times(0,\ln q^{-1})$ with identified vertical sides onto $\mathcal{D}$.

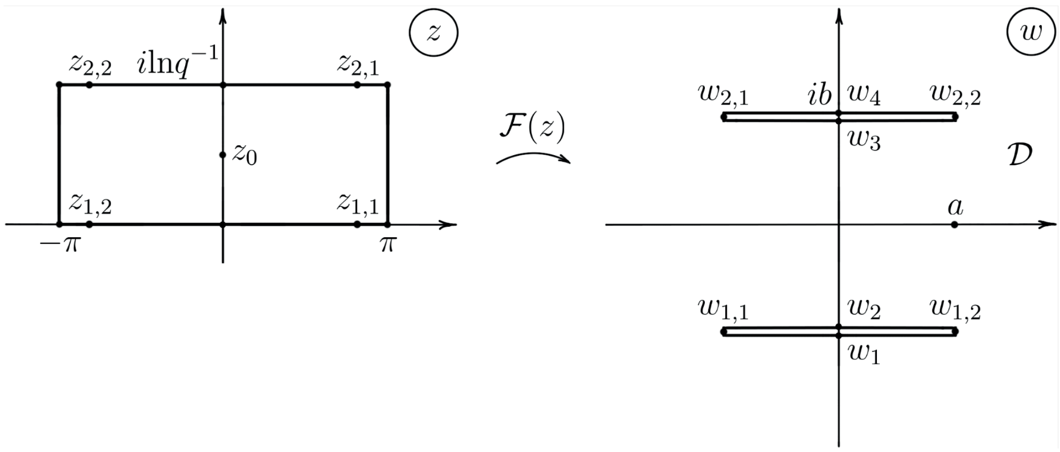

FIGURE 3. Conformal mapping of rectangle $[-\pi,\pi]\times(0,\ln q^{-1})$ with identified vertical sides onto $\mathcal{D}$.

Since the domain $\mathcal{D}$ is symmetric about the real axis, we can suppose that

$$z_{1,1}=\beta, \qquad z_{1,2}=-\beta, \qquad z_{2,1}=\beta+i\ln q^{-1}, \qquad z_{2,2}=-\beta+i\ln q^{-1}, \qquad 0<\beta<\pi,$$

and moreover, $z_0=\frac{i}{2}\ln q^{-1}$. At the same time,

$$\mathcal{F}(z_{k,j})=w_{k,j}, \qquad k,j=1,2,$$

where

$$w_{1,1}=-a+ib, \qquad w_{1,2}=a+ib, \qquad w_{2,1}=-a-ib, \qquad w_{2,2}=a-ib.$$

Then by Theorem 3.1 the function $\mathcal{F}$, which maps the annulus $\mathcal{A}=\{\tau : q<|\tau|<1\}$ onto $\mathcal{D}$, is given by the formula

$$\mathcal{F}(z)=C_1\int\limits_0^z \exp\{c\,\xi\}\prod_{i_1=1}^{2}\sigma(\xi-z_{1,i_1})\prod_{i_2=1}^{2}\sigma(\xi-z_{2,i_2})\,\left(\sigma(\xi-z_0)\,\sigma(\xi-\bar z_0)\right)^{-2}d\xi+C_2, \tag{3.7}$$

where $z=-i\ln\tau$, $c=\eta_2$ and $\sigma(z)=\sigma(z;2\pi,2i\ln q^{-1})$. Let us find the unknown parameters involved in (3.7).

Let the $k=k(q)\in(0,1)$ be given by the identity

$$k^{\frac{1}{2}}=\frac{\vartheta_2\left(0,\exp\left\{\frac{2\pi^2}{\ln q}\right\}\right)}{\vartheta_3\left(0,\exp\left\{\frac{2\pi^2}{\ln q}\right\}\right)}$$

and $c_1=\frac{2K(k)}{\ln q}$. Then the function

$$u(z)=\operatorname{sn}\left(c_1\left(-iz+i\pi-\frac{\ln q^{-1}}{2}\right),k\right)$$

conformally maps the rectangle $(0,\pi)\times(0,\ln q^{-1})$ onto the upper half–plane $\mathbb{H}^+_u$. At the same time, the vertices

$$z_1=0, \qquad z_2=\pi, \qquad z_3=\pi+i\ln q^{-1}, \qquad z_4=i\ln q^{-1}$$

of this rectangle are mapped respectively into the points

$$u_1 = -\frac{1}{k}, \qquad u_2 = -1, \qquad u_3 = 1, \qquad u_4 = \frac{1}{k}.$$

The points $z_{1,1}$ and $z_{2,1}$ are respectively mapped into the points $u_{1,1} = -\lambda$ and $u_{2,1} = \lambda$, where $\lambda \in (1, \frac{1}{k})$.

Now we consider the mapping $\mathbb{H}^+_u$ onto a part of the domain $\mathcal{D}$ located in the left half–plane, we denote this domain by $\mathcal{D}_1$. The mapping function $H(u) = H(u; \lambda, k)$ reads

$$H(u) = C \int\limits_0^u \frac{k(\xi^2 - \lambda^2)\, d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}, \tag{3.8}$$

where $C = C(k) \neq 0$ and $H(u_j) = w_j$, $1 \leqslant j \leqslant 4$. Using the elliptic integrals, we can obtain the formula, see [9],

$$H(u) = \frac{2ib}{\pi}\Big[K'(k)E(u,k) - (K'(k) - E'(k))\, F(u,k)\Big].$$

We thus have the formula for the unknown parameter $\beta = \beta(q)$ in the integral representation (3.7)

$$\beta = \frac{\ln q}{2K(k)}\, F(t, k'),$$

where

$$t = \sqrt{\frac{1 - k^2\lambda^2}{1-k^2}}, \qquad t \in (0,1).$$

The unknown parameter q can be determined, for instance, by choosing the parameter k, see Example 3.1.

We observe that

$$a = \frac{1}{k}\Big[E(t,k') - k^2\lambda^2 F(t,k')\Big], \qquad b = \frac{\pi}{2kK'(k)}.$$

In Table 2 we provide the values of parameters and conformal modulus for some $k \in (0,1)$.

TABLE 2. Some values of accessory parameters and conformal modulus of the exterior of two segments $[-1-ib, 1-ib]$ and $[-1+ib, 1+ib]$.

k	$2/3$	$1/2$	$1/3$
q	0.00255135251368	0.00736090792149	0.01797238700896
ω_2	11.9422633257957	9.82314398990052	8.03783750802114
β	1.47011356981425	1.40082495304699	1.30853828460806
Mod($\mathcal{D}$)	0.95033512000272	0.78170096134805	0.63963078558550
b	4.92429742192023	2.87161351081953	1.80004021386307

This example, like the previous one, has important applications in mechanics, for example, in the theory of compound wings [5].

For more detail on polygons with sides on straight lines, which are parallel or pass through the origin, we refer to [6, Part B, Sect. 8], as well as [14].

In the doubly connected case for rather simple symmetric domains (as presented in Examples 3.1 and 3.2) the accessory parameters can be found by using the theory of elliptic integrals and elliptic functions. For doubly connected polygonal domains with more complex geometry the problem becomes much more complicated and approximate methods are employed.

4. Families of conformal mappings

We now consider a one–parametric family of doubly connected domains $\mathcal{D}(t)$, which are obtained from a fixed unbounded doubly connected domain $\mathcal{D}$, enveloped by two closed polygonal lines, by making several disjoint rectilinear slits inside the domain and outgoing from its boundary, see Figure 4. We suppose that the end points of the slits are smooth functions of the real parameter $t \in [0, T]$. We adopt the same notation for objects and characteristics associated with the domain $\mathcal{D}$ as in Section 3, indicating in brackets, if necessary, the dependence on the parameter t.

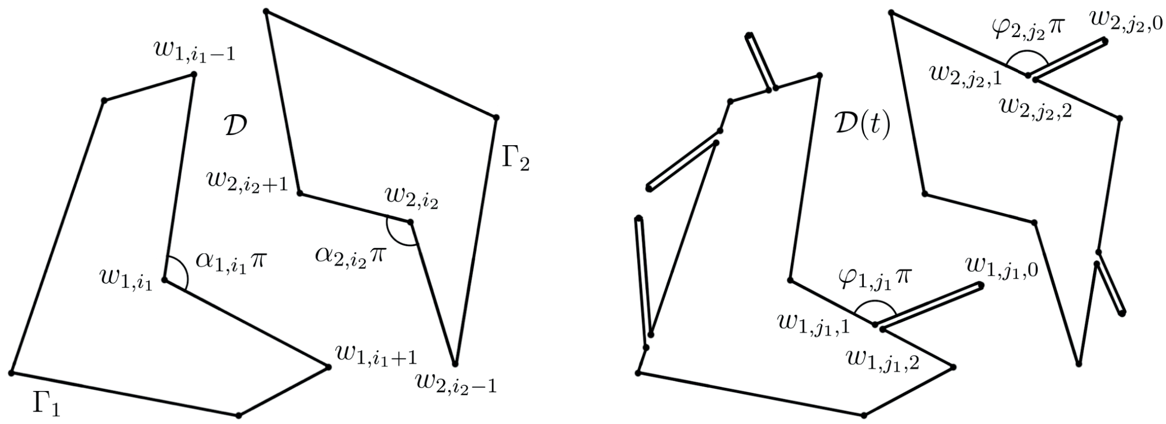

Figure 4. The doubly connected polygonal domain $\mathcal{D}$ and a domain $\mathcal{D}(t)$ obtained from $\mathcal{D}$ by making finitely many slits.

We make m_1 slits leaving pairwise distinct points of the boundary component Γ_1 at the angles $\varphi_{1,j_1}\pi$, $1 \leqslant j_1 \leqslant m_1$, and m_2 slits leaving Γ_2 at the angles $\varphi_{2,j_2}\pi$, $1 \leqslant j_2 \leqslant m_2$, see Figure 4. For simplicity we suppose that all slits leave non–corner points of the boundary; the general case can be considered similarly, see Remark 4.2 below. Let $z_{1,j_1,0}(t) \in [0, \omega_1]$ be the preimages of the ends of slits leaving the points of component Γ_1, and

$$z_{2,j_2,0}(t) = x_{2,j_2,0}(t) + \frac{\omega_2(t)}{2} \in \left[\frac{\omega_2(t)}{2}, \omega_1 + \frac{\omega_2(t)}{2}\right]$$

be the preimages of the ends of slits leaving the points of component Γ_2. By $z_{1,j_1,1}(t)$ and $z_{1,j_1,2}(t)$ we denote the preimages of the new corner points of component Γ_1 with the interior angles $\varphi_{1,j_1}\pi$ and $(1-\varphi_{1,j_1})\pi$, respectively, and by $z_{2,j_2,1}(t)$ and $z_{2,j_2,2}(t)$ we denote the preimages of the new corner points of component Γ_2 with the interior angles $\varphi_{2,j_2}\pi$ and $(1-\varphi_{2,j_2})\pi$, respectively. We are going to describe smooth one–parametric families of functions $\mathcal{F}(z,t)$ that conformally map the concentric annulus $\mathcal{A}(t) = \{\tau : \ q(t) < |\tau| < 1\}$ onto $\mathcal{D}(t)$, where $q(t) = \exp\{-2\pi m(t)\}$, $m(t)$ is the conformal modulus of the domain $\mathcal{D}(t)$.

We stress that unlike the Löwner method, we consider the case of several slits, and do not assume that the family of domains $\mathcal{D}(t)$ is monotone in the parameter t.

We shall find a partial differential equation for the family $\mathcal{F}(z,t)$. From this equation, we shall derive a system of ordinary differential equations that determines the dynamics of the accessory parameters for this family of conformal mappings. We shall also determine the dynamics of the conformal modulus $m(t)$ as the parameter t varies.

In order to derive the system of differential equations, we use the approach developed earlier for doubly connected domains being the exterior of two rectilinear slits of variable length [14], as well as for one–parameter families of conformal mappings of bounded doubly connected polygonal domains [16].

So, let us consider a smooth one-parameter family of conformal mappings

$$\begin{aligned}\mathcal{F}(z,t) =&C_1(t)\int\limits_0^z \exp\{c(t)\,\xi\}\prod_{i_1=1}^{n_1}\sigma^{\alpha_{1,i_1}-1}(\xi-z_{1,i_1}(t))\prod_{i_2=1}^{n_2}\sigma^{\alpha_{2,i_2}-1}(\xi-z_{2,i_2}(t))\\ &\cdot\prod_{j_1=1}^{m_1}s_{1,j_1}(\xi,t)\prod_{j_2=1}^{m_2}s_{2,j_2}(\xi,t)\,\left(\sigma(\xi-z_0(t))\,\sigma(\xi-\bar z_0(t))\right)^{-2}\,d\xi+C_2,\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
s_{1,j_1}(z,t) &= \sigma(z-z_{1,j_1,0}(t))\,\sigma^{\varphi_{1,j_1}-1}(z-z_{1,j_1,1}(t))\,\sigma^{-\varphi_{1,j_1}}(z-z_{1,j_1,2}(t)),\\
s_{2,j_2}(z,t) &= \sigma(z-z_{2,j_2,0}(t))\,\sigma^{\varphi_{2,j_2}-1}(z-z_{2,j_2,1}(t))\,\sigma^{-\varphi_{2,j_2}}(z-z_{2,j_2,2}(t)),\\
c(t) &= \frac{\eta_1(t)}{\omega_1}\Bigg[\sum_{i_1=1}^{n_1}(\alpha_{1,i_1}-1)\,z_{1,i_1}(t)+\sum_{i_2=1}^{n_2}(\alpha_{2,i_2}-1)\,x_{2,i_2}(t)\\
&\quad+\sum_{j_1=1}^{m_1}(z_{1,j_1,0}(t)+(\varphi_{1,j_1}-1)\,z_{1,j_1,1}(t)-\varphi_{1,j_1}\,z_{1,j_1,2}(t))\\
&\quad+\sum_{j_2=1}^{m_2}(x_{2,j_2,0}(t)+(\varphi_{2,j_2}-1)\,x_{2,j_2,1}(t)-\varphi_{2,j_2}\,x_{2,j_2,2}(t))\Bigg]+\eta_2(t).
\end{aligned}$$

Here

$$\sigma(z)=\sigma(z;\omega_1,\omega_2(t)),\qquad \eta_1(t)=2\zeta\left(\frac{\omega_1}{2};\omega_1,\omega_2(t)\right),$$
$$\eta_2(t)=2\zeta\left(\frac{\omega_2(t)}{2};\omega_1,\omega_2(t)\right),\qquad z_0(t)=iy_0(t).$$

The conformal modulus of domain $\mathcal{D}(t)$ reads

$$\mathrm{Mod}(\mathcal{D}(t))=\frac{\omega_2(t)}{2\omega_1 i}.$$

For a fixed t, the function $\mathcal{F}(z,t)$ is periodic with the period $\omega_1=2\pi$. By the symmetry principle, we can continue $\mathcal{F}(z,t)$ through the intervals, the end points of which are the preimages of vertices of boundary components. Obviously, the continuation satisfies the identities

$$\mathcal{F}(z+\omega_1,t)=\mathcal{F}(z,t),\qquad \mathcal{F}(z+\omega_2(t),t)=e^{i\alpha}\mathcal{F}(z,t)+\beta.$$

Differentiating in t and z, we obtain

$$\begin{aligned}
&\dot{\mathcal{F}}(z+\omega_1,t)=\dot{\mathcal{F}}(z,t), &&\dot\omega_2(t)\,\mathcal{F}'(z+\omega_2(t),t)+\dot{\mathcal{F}}(z+\omega_2(t),t)=e^{i\alpha}\dot{\mathcal{F}}(z,t),\\
&\mathcal{F}'(z+\omega_1,t)=\mathcal{F}'(z,t), &&\mathcal{F}'(z+\omega_2(t),t)=e^{i\alpha}\mathcal{F}'(z,t).
\end{aligned}$$

Hereinafter the dot stands for the differentiation in the parameter t, while the prime denotes the differentiation in z. We thus have

$$\begin{aligned}
&\frac{\dot{\mathcal{F}}\,(z+\omega_1,t)}{\mathcal{F}'(z+\omega_1,t)}=\frac{\dot{\mathcal{F}}(z,t)}{\mathcal{F}'(z,t)},\\
&\frac{\dot{\mathcal{F}}\,(z+\omega_2(t),t)}{\mathcal{F}'(z+\omega_2(t),t)}+\dot\omega_2(t)=\frac{\dot{\mathcal{F}}(z,t)}{\mathcal{F}'(z,t)}.
\end{aligned}$$

Therefore, the function

$$\mathcal{H}(z,t):=\frac{\dot{\mathcal{F}}(z,t)}{\mathcal{F}'(z,t)}$$

satisfies the identities

$$\mathcal{H}\left(z+\omega_1,t\right)-\mathcal{H}(z,t)=0,\qquad \mathcal{H}\left(z+\omega_2(t),t\right)-\mathcal{H}(z,t)=-\dot{\omega}_2(t). \tag{4.2}$$

We write the Puiseaux expansion for $\mathcal{F}(z,t)$ in the vicinity of points $z_{1,k_1}(t)$

$$\mathcal{F}(z,t)=A_0+A_1(t)\left(z-z_{1,k_1}(t)\right)^{\alpha_{1,k_1}}+\ldots,\qquad 1\leqslant k_1\leqslant n_1.$$

Here the quantity A_0 is independent of t. Then

$$\begin{aligned}\mathcal{F}'(z,t)&=\alpha_{1,k_1}\,A_1(t)\left(z-z_{1,k_1}(t)\right)^{\alpha_{1,k_1}-1}+\ldots,\\ \dot{\mathcal{F}}(z,t)&=-\alpha_{1,k_1}\,A_1(t)\left(z-z_{1,k_1}(t)\right)^{\alpha_{1,k_1}-1}\dot{z}_{1,k_1}(t)+\ldots\end{aligned}$$

and hence,

$$\dot{z}_{1,k_1}(t)=-\mathcal{H}(z_{1,k_1}(t),t),\qquad 1\leqslant k_1\leqslant n_1. \tag{4.3}$$

In the same way we obtain

$$\dot{z}_{2,k_2}(t)=-\mathcal{H}(z_{2,k_2}(t),t),\qquad 1\leqslant k_2\leqslant n_2, \tag{4.4}$$

$$\dot{z}_{1,\ell_1,j}(t)=-\mathcal{H}(z_{1,\ell_1,j}(t),t),\qquad 1\leqslant \ell_1\leqslant m_1,\qquad j=1,2, \tag{4.5}$$

$$\dot{z}_{2,\ell_2,j}(t)=-\mathcal{H}(z_{2,\ell_2,j}(t),t),\qquad 1\leqslant \ell_2\leqslant m_2,\qquad j=1,2. \tag{4.6}$$

Now we write the Taylor expansion for $\mathcal{F}(z,t)$ in the vicinity of the points $z_{1,\ell_1,0}(t)$

$$\mathcal{F}(z,t)=\mathcal{E}_{1,\ell_1}(t)+\frac{1}{2}\mathcal{F}''\left(z_{1,\ell_1,0}(t),t\right)\left(z-z_{1,\ell_1,0}(t)\right)^2+\ldots,\qquad 1\leqslant \ell_1\leqslant m_1, \tag{4.7}$$

where $\mathcal{E}_{1,\ell_1}(t)=\mathcal{F}(z_{1,\ell_1,0}(t),t)$ is the end of ℓ_1th slit leaving Γ_1. The expansion implies

$$\begin{aligned}\mathcal{F}'(z,t)&=\mathcal{F}''\left(z_{1,\ell_1,0}(t),t\right)\left(z-z_{1,\ell_1,0}(t)\right)+\ldots,\\ \dot{\mathcal{F}}(z,t)&=\dot{\mathcal{E}}_{1,\ell_1}(t)-\mathcal{F}''\left(z_{1,\ell_1,0}(t),t\right)\left(z-z_{1,\ell_1,0}(t)\right)\dot{z}_{1,\ell_1,0}(t)+\ldots,\end{aligned}$$

and, therefore,

$$\mathcal{H}(z,t)=\frac{\mathcal{L}_{1,\ell_1}(t)}{z-z_{1,\ell_1,0}(t)}+\mathrm{O}(1),\qquad z\to z_{1,\ell_1,0}(t),$$

where

$$\mathcal{L}_{1,\ell_1}(t):=\frac{\dot{\mathcal{E}}_{1,\ell_1}(t)}{\mathcal{F}''\left(z_{1,\ell_1,0}(t),t\right)},\qquad 1\leqslant \ell_1\leqslant m_1.$$

Similarly,

$$\mathcal{L}_{2,\ell_2}(t):=\frac{\dot{\mathcal{E}}_{2,\ell_2}(t)}{\mathcal{F}''\left(z_{2,\ell_2,0}(t),t\right)},\qquad 1\leqslant \ell_2\leqslant m_2.$$

Now we are going to find $\mathcal{F}''\left(z_{1,\ell_1,0}(t),t\right)$. We have

$$\begin{aligned}\mathcal{F}''(z,t)=&C_1(t)\exp\{c(t)\,z\}\prod_{i_1=1}^{n_1}\sigma^{\alpha_{1,i_1}-1}(z-z_{1,i_1}(t))\prod_{i_2=1}^{n_2}\sigma^{\alpha_{2,i_2}-1}(z-z_{2,i_2}(t))\\ &\cdot\prod_{j_1=1}^{m_1}s_{1,j_1}(z,t)\prod_{j_2=1}^{m_2}s_{2,j_2}(z,t)\,\left(\sigma(z-z_0(t))\,\sigma(z-\bar{z}_0(t))\right)^{-2}\\ &\cdot\left[\mathcal{Q}(z,t)+\sum_{j_1=1}^{m_1}\mathcal{Q}_{1,j_1}(z,t)+\sum_{j_2=1}^{m_2}\mathcal{Q}_{2,j_2}(z,t)+\mathcal{Q}_0(z,t)\right],\end{aligned}$$

where

$$\mathcal{Q}(z,t)=c(t)+\sum_{i_1=1}^{n_1}(\alpha_{1,i_1}-1)\,\zeta(z-z_{1,i_1}(t))+\sum_{i_2=1}^{n_2}(\alpha_{2,i_2}-1)\,\zeta(z-z_{2,i_2}(t)),$$

$$\mathcal{Q}_{k,j_k}(z,t)=\zeta(z-z_{k,j_k,0}(t))+(\varphi_{k,j_k}-1)\,\zeta(z-z_{k,j_k,1}(t))-\varphi_{k,j_k}\,\zeta(z-z_{k,j_k,2}(t)),\quad 1\leqslant k\leqslant 2,$$

$$\mathcal{Q}_0(z,t) = -2\zeta(z - z_0(t)) - 2\zeta(z - \overline{z_0(t)}).$$

Therefore, as $z \to z_{1,\ell_1,0}(t)$, we obtain

$$\begin{aligned}\mathcal{F}''\left(z_{1,\ell_1,0}(t),t\right) = C_1(t) \exp\{c(t)\, z_{1,\ell_1,0}(t)\} \\ \cdot \prod_{i_1=1}^{n_1} \sigma^{\alpha_{1,i_1}-1}(z_{1,\ell_1,0}(t) - z_{1,i_1}(t)) \prod_{i_2=1}^{n_2} \sigma^{\alpha_{2,i_2}-1}(z_{1,\ell_1,0}(t) - z_{2,i_2}(t)) \\ \cdot \prod_{j_1=1}^{m_1} \sigma^{\varphi_{1,j_1}-1}(z_{1,\ell_1,0}(t) - z_{1,j_1,1}(t))\, \sigma^{-\varphi_{1,j_1}}(z_{1,\ell_1,0}(t) - z_{1,j_1,2}(t)) \\ \cdot \prod_{j_2=1}^{m_2} s_{2,j_2}(z_{1,\ell_1,0}(t),t) \left(\sigma(z_{1,\ell_1,0}(t) - z_0(t))\, \sigma(z_{1,\ell_1,0}(t) - \overline{z_0(t)})\right)^{-2}.\end{aligned} \tag{4.8}$$

Similarly,

$$\begin{aligned}\mathcal{F}''\left(z_{2,\ell_2,0}(t),t\right) = C_1(t) \exp\{c(t)\, z_{2,\ell_2,0}(t)\} \\ \cdot \prod_{i_1=1}^{n_1} \sigma^{\alpha_{1,i_1}-1}(z_{2,\ell_2,0}(t) - z_{1,i_1}(t)) \prod_{i_2=1}^{n_2} \sigma^{\alpha_{2,i_2}-1}(z_{2,\ell_2,0}(t) - z_{2,i_2}(t)) \\ \cdot \prod_{j_2=1}^{m_2} \sigma^{\varphi_{2,j_2}-1}(z_{2,\ell_2,0}(t) - z_{2,j_2,1}(t))\, \sigma^{-\varphi_{2,j_2}}(z_{2,\ell_2,0}(t) - z_{2,j_2,2}(t)) \\ \cdot \prod_{j_1=1}^{m_1} s_{1,j_1}(z_{2,\ell_2,0}(t),t) \left(\sigma(z_{2,\ell_2,0}(t) - z_0(t))\, \sigma(z_{2,\ell_2,0}(t) - \overline{z_0(t)})\right)^{-2}.\end{aligned}$$

At the point $z_0(t)$, the function $\dot{\mathcal{F}}(z,t)$ has a pole of order at most 2 and $\mathcal{F}'(z,t)$ has a pole of order 2. Then the function $\mathcal{H}(z,t)$ has a removable singularity at this point. By $r_{-1}(t)$ we denote the residue of function $\mathcal{F}(z,t)$ at the point $z_0(t)$. We thus have

$$\begin{aligned}\mathcal{F}(z,t) &= \frac{r_{-1}(t)}{z - z_0(t)} + r_0(t) + \mathrm{O}(1), \\ \dot{\mathcal{F}}(z,t) &= \frac{r_{-1}(t)\, \dot{z}_0(t)}{(z - z_0(t))^2} + \frac{\dot{r}_{-1}(t)}{z - z_0(t)} + \mathrm{O}(1), \\ \mathcal{F}'(z,t) &= -\frac{r_{-1}(t)}{(z - z_0(t))^2} + \mathrm{O}(1).\end{aligned} \tag{4.9}$$

This shows that in the vicinity of point $z_0(t)$ the function $\mathcal{H}(z,t)$ has the expansion

$$\mathcal{H}(z,t) = -\dot{z}_0(t) + \mathrm{O}(1), \qquad z \to z_0(t). \tag{4.10}$$

Similarly, the function $\mathcal{H}(z,t)$ has a removable singularity at the point $\bar{z}_0(t)$.

The function

$$\mathcal{S}(z,t) := \mathcal{H}(z,t) - \sum_{j_1=1}^{m_1} \mathcal{L}_{1,j_1}(t)\, \zeta(z - z_{1,j_1,0}(t)) - \sum_{j_2=1}^{m_2} \mathcal{L}_{2,j_2}(t)\, \zeta(z - z_{2,j_2,0}(t))$$

has only removable singularities at the preimages of the ends of the slits and at the points $z_0(t)$, $\bar{z}_0(t)$, as well as at the points equivalent to them modulo the lattice. Hence, this function can be holomorphically continued to the entire plane $\mathbb{C}$. By (4.2) we have

$$\begin{aligned}
&\mathcal{S}(z+\omega_1,t)-\mathcal{S}(z,t)\\
&\qquad=-\sum_{j_1=1}^{m_1}\mathcal{L}_{1,j_1}(t)\,\zeta(z+\omega_1-z_{1,j_1,0}(t))-\sum_{j_2=1}^{m_2}\mathcal{L}_{2,j_2}(t)\,\zeta(z+\omega_1-z_{2,j_2,0}(t))\\
&\qquad\quad+\sum_{j_1=1}^{m_1}\mathcal{L}_{1,j_1}(t)\,\zeta(z-z_{1,j_1,0}(t))+\sum_{j_2=1}^{m_2}\mathcal{L}_{2,j_2}(t)\,\zeta(z-z_{2,j_2,0}(t))\\
&\qquad=-\,\eta_1(t)\left(\sum_{j_1=1}^{m_1}\mathcal{L}_{1,j_1}(t)+\sum_{j_2=1}^{m_2}\mathcal{L}_{2,j_2}(t)\right),
\end{aligned}\tag{4.11}$$

$$\mathcal{S}(z+\omega_2(t),t)-\mathcal{S}(z,t)=-\dot{\omega}_2(t)-\eta_2(t)\left(\sum_{j_1=1}^{m_1}\mathcal{L}_{1,j_1}(t)+\sum_{j_2=1}^{m_2}\mathcal{L}_{2,j_2}(t)\right).\tag{4.12}$$

By (4.11) and (4.12) the function $\mathcal{S}$ grows at infinity at most as a linear function and this is why $\mathcal{S}(z,t)=\alpha(t)z+\beta(t)$. Thus,

$$\mathcal{H}(z,t)=\sum_{j_1=1}^{m_1}\mathcal{L}_{1,j_1}(t)\,\zeta(z-z_{1,j_1,0}(t))+\sum_{j_2=1}^{m_2}\mathcal{L}_{2,j_2}(t)\,\zeta(z-z_{2,j_2,0}(t))+\alpha(t)\,z+\beta(t).\tag{4.13}$$

It follows from (4.11) with $z=0$ that

$$\alpha(t)=-\frac{\eta_1(t)}{\omega_1}\left(\sum_{j_1=1}^{m_1}\mathcal{L}_{1,j_1}(t)+\sum_{j_2=1}^{m_2}\mathcal{L}_{2,j_2}(t)\right).\tag{4.14}$$

By (4.10) we find

$$\begin{aligned}
\beta(t)=&-\sum_{j_1=1}^{m_1}\mathcal{L}_{1,j_1}(t)\,\zeta(z_0(t)-z_{1,j_1,0}(t))\\
&-\sum_{j_2=1}^{m_2}\mathcal{L}_{2,j_2}(t)\,\zeta(z_0(t)-z_{2,j_2,0}(t))\;-\dot{z}_0(t)-\alpha(t)\,z_0(t).
\end{aligned}\tag{4.15}$$

Finally, by (4.13), (4.14) and (4.15) we get

$$\mathcal{H}(z,t)=\sum_{j_1=1}^{m_1}\mathcal{L}_{1,j_1}(t)\,\mathcal{K}_{1,j_1}(z,t)+\sum_{j_2=1}^{m_2}\mathcal{L}_{2,j_2}(t)\mathcal{K}_{2,j_2}(z,t)-\dot{z}_0(t),\tag{4.16}$$

where

$$\mathcal{K}_{1,j_1}(z,t)=\zeta(z-z_{1,j_1,0}(t))-\frac{\eta_1(t)}{\omega_1}\,(z-z_0(t))+\zeta(z_{1,j_1,0}(t)-z_0(t)),$$

$$\mathcal{K}_{2,j_2}(z,t)=\zeta(z-z_{2,j_2,0}(t))-\frac{\eta_1(t)}{\omega_1}\,(z-z_0(t))+\zeta(z_{2,j_2,0}(t)-z_0(t)).$$

It follows from (4.12) with $z=0$ that

$$\begin{aligned}
\dot{\omega}_2(t)&=-\alpha(t)\,\omega_2(t)-\eta_2(t)\left(\sum_{j_1=1}^{m_1}\mathcal{L}_{1,j_1}(t)+\sum_{j_2=1}^{m_2}\mathcal{L}_{2,j_2}(t)\right)\\
&=\frac{\eta_1(t)\,\omega_2(t)-\eta_2(t)\,\omega_1}{\omega_1}\left(\sum_{j_1=1}^{m_1}\mathcal{L}_{1,j_1}(t)+\sum_{j_2=1}^{m_2}\mathcal{L}_{2,j_2}(t)\right),
\end{aligned}$$

and by the Legendre relation (2.1) we obtain

$$\dot{\omega}_2(t) = i\left(\sum_{j_1=1}^{m_1} \mathcal{L}_{1,j_1}(t) + \sum_{j_2=1}^{m_2} \mathcal{L}_{2,j_2}(t)\right). \tag{4.17}$$

We note that the functions $\mathcal{K}_{1,j_1}$ and $\mathcal{K}_{2,j_2}$ resemble the Villat kernel appearing in the integral representation that gives the solution to the Schwarz problem of recovering an analytic function in an annulus from the boundary values of its real part, see, for example, [2, Ch. XI].

Thus, we have proved the following result.

Theorem 4.1. *The above defined smooth family $\mathcal{F}(z,t)$ of conformal mappings satisfies the partial differential equation*

$$\frac{\dot{\mathcal{F}}(z,t)}{\mathcal{F}'(z,t)} = \mathcal{H}(z,t), \tag{4.18}$$

where the function $\mathcal{H}(z,t)$ is given by (4.16). In this case, the period ω_1 of the Weierstrass ζ–function in (4.16) is equal to 2π, and the period $\omega_2 = \omega_2(t)$ satisfies the differential equation (4.17).

Now we are going to find differential equations for $z_{1,j_1,0}(t)$. In order to do this, we write $\dot{\mathcal{F}}'(z_{1,j_1,0}(t),t)$ in two different ways. It follows from (4.7) that

$$\mathcal{F}'(z,t) = \mathcal{F}''(z_{1,j_1,0}(t),t)\,(z - z_{1,j_1,0}(t)) + \dots.$$

Then

$$\dot{z}_{1,j_1,0}(t) = -\frac{\dot{\mathcal{F}}'(z_{1,j_1,0}(t),t)}{\mathcal{F}''(z_{1,j_1,0}(t),t)}. \tag{4.19}$$

On the other hand, by Theorem 4.1, as $z \to z_{1,\ell_1,0}(t)$ we have

$$\dot{\mathcal{F}}'(z_{1,\ell_1,0}(t),t) = \mathcal{F}''(z_{1,\ell_1,0}(t),t)\,\mathcal{H}(z_{1,\ell_1,0}(t),t) + \mathcal{F}'(z_{1,\ell_1,0}(t),t)\,\mathcal{H}'(z_{1,\ell_1,0}(t),t). \tag{4.20}$$

Comparing (4.19) and (4.20), in view of (4.8) we obtain

$$\begin{aligned}\dot{z}_{1,\ell_1,0}(t) =& \dot{z}_0(t) - \sum_{j_1=1,\, j_1\neq\ell_1}^{m_1} \mathcal{L}_{1,j_1}(t)\,\mathcal{K}_{1,j_1}(z_{1,\ell_1,0}(t),t) - \sum_{j_2=1}^{m_2} \mathcal{L}_{2,j_2}(t)\mathcal{K}_{2,j_2}(z_{1,\ell_1,0}(t),t)\\ &- \mathcal{L}_{1,\ell_1}(t)\Bigg[\sum_{j_1=1, j_1\neq\ell_1}^{m_1} \mathcal{Q}_{1,j_1}(z_{1,\ell_1,0}(t),t) + \sum_{j_2=1}^{m_2} \mathcal{Q}_{2,j_2}(z_{1,\ell_1,0}(t),t)\\ &+ \mathcal{Q}(z_{1,\ell_1,0}(t),t) + \mathcal{Q}_0(z_{1,\ell_1,0}(t),t)\\ &+ (\varphi_{1,\ell_1} - 1)\zeta(z_{1,\ell_1,0}(t) - z_{1,\ell_1,1}(t)) - \varphi_{1,\ell_1}\zeta(z_{1,\ell_1,0}(t) - z_{1,\ell_1,2}(t))\\ &-\frac{\eta_1(t)}{\omega_1}(z_{1,\ell_1,0}(t) - z_0(t)) + \zeta(z_{1,\ell_1,0}(t) - z_0(t))\Bigg].\end{aligned} \tag{4.21}$$

Similarly,

$$\begin{aligned}\dot{z}_{2,\ell_2,0}(t) =& \dot{z}_0(t) - \sum_{j_1=1}^{m_1} \mathcal{L}_{1,j_1}(t)\mathcal{K}_{1,j_1}(z_{2,\ell_2,0}(t),t) - \sum_{j_2=1, j_2\neq\ell_2}^{m_2} \mathcal{L}_{2,j_2}(t)\mathcal{K}_{2,j_2}(z_{2,\ell_2,0}(t),t)\\ &- \mathcal{L}_{2,\ell_2}(t)\Bigg[\sum_{j_1=1}^{m_1} \mathcal{Q}_{1,j_1}(z_{2,\ell_2,0}(t),t) + \sum_{j_2=1, j_2\neq\ell_2}^{m_2} \mathcal{Q}_{2,j_2}(z_{2,\ell_2,0}(t),t)\\ &+ \mathcal{Q}(z_{2,\ell_2,0}(t),t) + \mathcal{Q}_0(z_{2,\ell_2,0}(t),t)\\ &+ (\varphi_{2,\ell_2} - 1)\zeta(z_{2,\ell_2,0}(t) - z_{2,\ell_2,1}(t)) - \varphi_{2,\ell_2}\zeta(z_{2,\ell_2,0}(t) - z_{2,\ell_2,2}(t))\\ &-\frac{\eta_1(t)}{\omega_1}(z_{2,\ell_2,0}(t) - z_0(t)) + \zeta(z_{2,\ell_2,0}(t) - z_0(t))\Bigg].\end{aligned} \tag{4.22}$$

Now we are going to find a differential equation for $C_1(t)$. It follows from (4.9), (4.18) and (4.10) that

$$\begin{aligned} r_{-1}(t) =& -C_1(t)\exp\{c(t)\,z_0(t)\} \\ &\cdot \prod_{i_1=1}^{n_1} \sigma^{\alpha_{1,i_1}-1}(z_0(t)-z_{1,i_1}(t)) \prod_{i_2=1}^{n_2} \sigma^{\alpha_{2,i_2}-1}(z_0(t)-z_{2,i_2}(t)) \\ &\cdot \prod_{j_1=1}^{m_1} s_{1,j_1}(z_0(t),t) \prod_{j_2=1}^{m_2} s_{2,j_2}(z_0(t),t)\sigma^{-2}(2z_0(t)). \end{aligned} \tag{4.23}$$

Since

$$\mathcal{Q}(z_0(t),t) + \sum_{j_1=1}^{m_1} \mathcal{Q}_{1,j_1}(z_0(t),t) + \sum_{j_2=1}^{m_2} \mathcal{Q}_{2,j_2}(z_0(t),t) - 2\zeta(2z_0(t)) = 0, \tag{4.24}$$

we have

$$\begin{aligned} \dot r_{-1}(t) =& C_1(t)\exp\{c(t)\,z_0(t)\} \\ &\cdot \prod_{i_1=1}^{n_1} \sigma^{\alpha_{1,i_1}-1}(z_0(t)-z_{1,i_1}(t)) \prod_{i_2=1}^{n_2} \sigma^{\alpha_{2,i_2}-1}(z_0(t)-z_{2,i_2}(t)) \\ &\cdot \prod_{j_1=1}^{m_1} s_{1,j_1}(z_0(t),t) \prod_{j_2=1}^{m_2} s_{2,j_2}(z_0(t),t)\sigma^{-2}(2z_0(t)) \\ &\cdot \left[\sum_{j_1=1}^{m_1} \mathcal{L}_{1,j_1}(t)\left(\wp(z_0(t)-z_{1,j_1,0}(t)) + \frac{\eta_1(t)}{\omega_1}\right)\right. \\ &\left.+ \sum_{j_2=1}^{m_2} \mathcal{L}_{2,j_2}(t)\left(\wp(z_0(t)-z_{2,j_2,0}(t)) + \frac{\eta_1(t)}{\omega_1}\right)\right]. \end{aligned}$$

Hence,

$$\begin{aligned} \dot p(t) =& \sum_{j_1=1}^{m_1} \mathcal{L}_{1,j_1}(t)\left(\wp(z_0(t)-z_{1,j_1,0}(t)) + \frac{\eta_1(t)}{\omega_1}\right) \\ &+ \sum_{j_2=1}^{m_2} \mathcal{L}_{2,j_2}(t)\left(\wp(z_0(t)-z_{2,j_2,0}(t)) + \frac{\eta_1(t)}{\omega_1}\right), \end{aligned} \tag{4.25}$$

where $p(t) = \ln r_{-1}(t)$.

Differentiating (4.24), we obtain

$$\begin{aligned} \dot z_0(t) =& i\,\mathrm{Im}\, \frac{\mathcal{X}(t) + \sum_{j_1=1}^{m_1} \mathcal{X}_{1,j_1}(t) + \sum_{j_2=1}^{m_2} \mathcal{X}_{2,j_2}(t)}{\mathcal{P}(t) + \sum_{j_1=1}^{m_1} \mathcal{P}_{1,j_1}(t) + \sum_{j_2=1}^{m_2} \mathcal{P}_{2,j_2}(t)} \\ &+ \dot\omega_2(t)\,\mathrm{Re}\, \frac{\mathcal{C}(t) + \mathcal{R}(t) + \sum_{j_1=1}^{m_1} \mathcal{R}_{1,j_1}(t) + \sum_{j_2=1}^{m_2} \mathcal{R}_{2,j_2}(t) + \frac{1}{2}\mathcal{Y}(t) + \frac{1}{2}\sum_{j_2=1}^{m_2} \mathcal{Y}_{2,j_2}(t)}{\mathcal{P}(t) + \sum_{j_1=1}^{m_1} \mathcal{P}_{1,j_1}(t) + \sum_{j_2=1}^{m_2} \mathcal{P}_{2,j_2}(t)}, \end{aligned} \tag{4.26}$$

where

$$
\begin{aligned}
\mathcal{X}(t) =& \sum_{i_1=1}^{n_1}(\alpha_{1,i_1}-1)\dot z_{1,i_1}(t)\left(\wp(z_0(t)-z_{1,i_1}(t))-\frac{\eta_1(t)}{\omega_1}\right)\\
&+\sum_{i_2=1}^{n_2}(\alpha_{2,i_2}-1)\dot x_{2,i_2}(t)\left(\wp(z_0(t)-z_{2,i_2}(t))-\frac{\eta_1(t)}{\omega_1}\right),\\
\mathcal{X}_{k,j_k}(t) =& \dot x_{k,j_k,0}(t)\wp(z_0(t)-z_{k,j_k,0}(t))+(\varphi_{k,j_k}-1)\dot x_{k,j_k,1}(t)\wp(z_0(t)-z_{k,j_k,1}(t))\\
&+\varphi_{k,j_k}\dot x_{k,j_k,2}(t)\wp(z_0(t)-z_{k,j_k,2}(t)), \qquad 1\leqslant k\leqslant 2,\\
\mathcal{Y}(t) =& \sum_{i_2=1}^{n_2}(\alpha_{2,i_2}-1)\wp(z_0(t)-z_{2,i_2}(t)),\\
\mathcal{Y}_{2,j_2}(t) =& \wp(z_0(t)-z_{2,j_2,0}(t))+(\varphi_{2,j_2}-1)\wp(z_0(t)-z_{2,j_2,1}(t))+\varphi_{2,j_2}\wp(z_0(t)-z_{2,j_2,2}(t)),\\
\mathcal{C}(t) =& \frac{2}{\omega_1}\frac{\partial\zeta\left(\frac{\omega_1}{2}\right)}{\partial\omega_2}\Bigg[\sum_{i_1=1}^{n_1}(\alpha_{1,i_1}-1)z_{1,i_1}(t)+\sum_{i_2=1}^{n_2}(\alpha_{2,i_2}-1)x_{2,i_2}(t)\\
&+\sum_{j_1=1}^{m_1}\left(z_{1,j_1,0}(t)+(\varphi_{1,j_1}-1)z_{1,j_1,1}(t)-\varphi_{1,j_1}z_{1,j_1,2}(t)\right)\\
&+\sum_{j_2=1}^{m_2}\left(x_{2,j_2,0}(t)+(\varphi_{2,j_2}-1)x_{2,j_2,1}(t)-\varphi_{2,j_2}x_{2,j_2,2}(t)\right)\Bigg]\\
&+2\frac{\partial\zeta\left(\frac{\omega_2}{2}\right)}{\partial\omega_2}-\wp\left(\frac{\omega_2}{2}\right),\\
\mathcal{P}(t) =& \sum_{i_1=1}^{n_1}(\alpha_{1,i_1}-1)\wp(z_0(t)-z_{1,i_1}(t))+\sum_{i_2=1}^{n_2}(\alpha_{2,i_2}-1)\wp(z_0(t)-z_{2,i_2}(t))-4\wp(2z_0(t)),\\
\mathcal{P}_{k,j_k}(t) =& \wp(z_0(t)-z_{k,j_k,0}(t))+(\varphi_{k,j_k}-1)\wp(z_0(t)-z_{k,j_k,1}(t))\\
&+\varphi_{k,j_k}\wp(z_0(t)-z_{k,j_k,2}(t)), \qquad 1\leqslant k\leqslant 2,\\
\mathcal{R}(t) =& \sum_{i_1=1}^{n_1}(\alpha_{1,i_1}-1)\frac{\partial\zeta(z_0(t)-z_{1,i_1}(t))}{\partial\omega_2}+\sum_{i_2=1}^{n_2}(\alpha_{2,i_2}-1)\frac{\partial\zeta(z_0(t)-z_{2,i_2}(t))}{\partial\omega_2}-2\frac{\partial\zeta(z_0(t))}{\partial\omega_2},\\
\mathcal{R}_{k,j_k}(t) =& \frac{\partial\zeta(z_0(t)-z_{k,j_k,0}(t))}{\partial\omega_2}+(\varphi_{k,j_k}-1)\frac{\partial\zeta(z_0(t)-z_{k,j_k,1}(t))}{\partial\omega_2}\\
&-\varphi_{k,j_k}\frac{\partial\zeta(z_0(t)-z_{k,j_k,2}(t))}{\partial\omega_2}, \qquad 1\leqslant k\leqslant 2.
\end{aligned}
$$

We thus have the following statement.

Theorem 4.2. *The accessory parameters of the family of conformal mappings $\mathcal{F}(z,t)$ satisfy the system of ordinary differential equations (4.3)–(4.6), (4.17), (4.21), (4.22), (4.25) and (4.26), where $p(t)=\ln r_{-1}(t)$ and $r_{-1}(t)$ is determined from (4.23), and the function $\mathcal{H}(z,t)$ is defined by the formula (4.16). In (4.26) the partial derivative of the function $\zeta(z)$ with respect to the period ω_2 is given by the formula (2.2).*

Corollary 4.1. *The conformal modulus $m(t)=\operatorname{Mod}(\mathcal{D}(t))$ satisfies the differential equation*

$$\dot m(t)=\frac{1}{2\omega_1}\left(\sum_{j_1=1}^{m_1}\mathcal{L}_{1,j_1}(t)+\sum_{j_2=1}^{m_2}\mathcal{L}_{2,j_2}(t)\right).$$

Remark 4.1. *We observe that the right hand sides in (4.3)–(4.6), (4.17), (4.21), (4.22), (4.25) and (4.26) have singularities at the initial moment ($t = 0$). This is due to the fact that the lengths of slits tend to zero as $t \to 0$, and hence, for each fixed $1 \leqslant k \leqslant 2$ and $1 \leqslant j_k \leqslant m_k$ the points $z_{k,j_k,0}(t)$, $z_{k,j_k,1}(t)$ and $z_{k,j_k,2}(t)$ have the same limit. In order to avoid the degeneration while solving the Cauchy problem for the system of ordinary differential equations, in practice we replace the initial values of parameters by values of order 10^{-12}.*

Remark 4.2. *Let us consider the case when we make a new (m_1+1)th slit from a jth vertice of Γ_1 (for Γ_2 the arguing is similar). Then the family of conformal mappings (4.1) is rewritten as*

$$\begin{aligned}\mathcal{F}(z,t) =& C_1(t) \int\limits_0^z \exp\{c(t)\,\xi\} \prod_{i_1=1, i_1\neq j}^{n_1} \sigma^{\alpha_{1,i_1}-1}(\xi - z_{1,i_1}(t)) \prod_{i_2=1}^{n_2} \sigma^{\alpha_{2,i_2}-1}(\xi - z_{2,i_2}(t))\\ &\cdot \prod_{j_1=1}^{m_1+1} s_{1,j_1}(\xi,t) \prod_{j_2=1}^{m_2} s_{2,j_2}(\xi,t)\,(\sigma(\xi - z_0(t))\sigma(\xi-\bar z_0(t)))^{-2}\,d\xi + C_2,\end{aligned}$$

where

$$\begin{aligned}
&s_{1,j_1}(z,t) = \sigma(z - z_{1,j_1,0}(t))\,\sigma^{\varphi_{1,j_1}-1}(z - z_{1,j_1,1}(t))\,\sigma^{-\varphi_{1,j_1}}(z - z_{1,j_1,2}(t)), \qquad 1 \leqslant j_1 \leqslant m_1,\\
&s_{1,m_1+1}(z,t) = \sigma(z - z_{1,m_1+1,0}(t))\,\sigma^{\varphi_{1,m_1+1,1}-1}(z - z_{1,m_1+1,1}(t))\,\sigma^{\varphi_{1,m_1+1,2}-1}(z - z_{1,m_1+1,2}(t)),\\
&s_{2,j_2}(z,t) = \sigma(z - z_{2,j_2,0}(t))\,\sigma^{\varphi_{2,j_2}-1}(z - z_{2,j_2,1}(t))\,\sigma^{-\varphi_{2,j_2}}(z - z_{2,j_2,2}(t)),\\
&c(t) = \frac{\eta_1(t)}{\omega_1}\Bigg[\sum_{i_1=1,\, i_1\neq j}^{n_1} (\alpha_{1,i_1}-1)\,z_{1,i_1}(t) + \sum_{i_2=1}^{n_2}(\alpha_{2,i_2}-1)\,x_{2,i_2}(t)\\
&\qquad + \sum_{j_1=1}^{m_1} (z_{1,j_1,0}(t) + (\varphi_{1,j_1}-1)\,z_{1,j_1,1}(t) - \varphi_{1,j_1}\,z_{1,j_1,2}(t))\\
&\qquad + z_{1,m_1+1,0}(t) + (\varphi_{1,m_1+1,1}-1)z_{1,m_1+1,1}(t) + (\varphi_{1,m_1+1,2}-1)z_{1,m_1+1,2}(t)\\
&\qquad + \sum_{j_2=1}^{m_2} (x_{2,j_2,0}(t) + (\varphi_{2,j_2}-1)\,x_{2,j_2,1}(t) - \varphi_{2,j_2}\,x_{2,j_2,2}(t))\Bigg] + \eta_2(t).
\end{aligned}$$

The internal angles of domain at the base of slit satisfy the relation $\varphi_{1,m_1+1,1} + \varphi_{1,m_1+1,2} = \alpha_{1,j}$. The system of ordinary differential equations for this family can be rewritten in an appropriate form.

5. Examples

In this section we consider several examples illustrating the proposed method. All calculations and constructions were performed using the Wolfram Mathematica package.

Example 5.1. We consider the problem on approximate finding a conformal mapping of the annulus $\mathcal{A} = \{\tau : q < |\tau| < 1\}$ onto a doubly connected domain $\mathcal{D}_1$, which is the exterior of two regular triangles with vertices at the points

$$w_{1,1} = -1 - ib, \qquad w_{1,2} = 1 - ib, \qquad w_{1,3} = i(-b + \sqrt{3})$$

and

$$w_{2,1} = -1 + ib, \qquad w_{2,2} = 1 + ib, \qquad w_{2,3} = i(b - \sqrt{3}),$$

see Figure 5. According to Theorem 3.1, the conformal mapping of an annulus onto $\mathcal{D}_1$ reads

$$\mathcal{F}(z) = C_1 \int_0^z \exp\{c\,\xi\} \prod_{k,j=1}^{3} \sigma^{\frac{2}{3}}(\xi - z_{k,j})\, (\sigma(\xi - z_0)\, \sigma(\xi - \bar{z}_0))^{-2}\, d\xi + C_2, \tag{5.1}$$

where $z = -i \ln \tau$ and

$$c = \frac{2}{3}\frac{\eta_1}{\omega_1}\left[\sum_{i_1=1}^{3} z_{1,i_1} + \sum_{i_2=1}^{3} x_{2,i_2}\right] + \eta_2.$$

Since the domain $\mathcal{D}_1$ is symmetric with respect to the imaginary axis, we can assume that the preimages of vertices of triangles are symmetric with respect to the imaginary axis, that is,

$$z_{1,1} = \beta, \qquad z_{1,2} = -\beta, \qquad z_{1,3} = \frac{\omega_1}{2},$$
$$x_{2,1} = \beta, \qquad x_{2,2} = -\beta, \qquad x_{2,3} = \frac{\omega_1}{2}, \qquad 0 < \beta < \frac{\omega_1}{2},$$

moreover,

$$z_0 = \frac{i \ln q^{-1}}{2}.$$

At the same time, $\mathcal{F}(z_{k,j}) = w_{k,j}$, $1 \leqslant k, j \leqslant 3$.

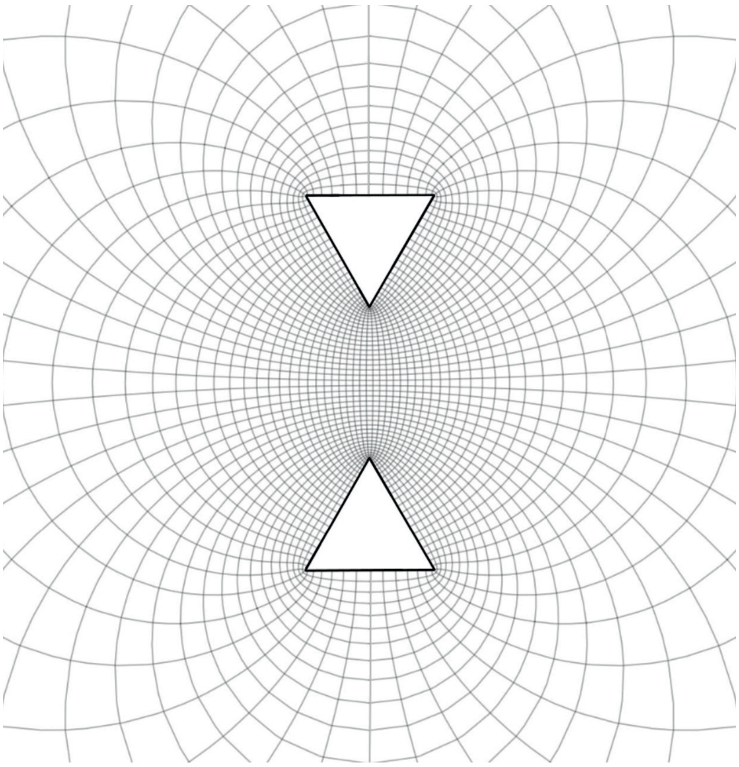

FIGURE 5. The image of the polar grid under a conformal mapping of an annulus onto the exterior of two equilateral triangles.

Our aim is to approximately find the parameters in (5.1). To do this, we consider a conformal mapping of the annulus $\mathcal{A}$ onto the domain $\mathcal{D}_0$, which is the exterior of the segments $[-1 - ib, 1 - ib]$ and $[-1 + ib, 1 + ib]$; all parameters for the conformal mapping of the annulus onto the domain $\mathcal{D}_0$ are known, see Example 3.2. We make the straight slits from the points $w_{1,1}$, $w_{1,2}$, $w_{2,1}$, $w_{2,2}$. Let the ends of these slits move with constant velocities according to the laws

$$v_{1,1}(t) = -1 - ib + (1 + i\sqrt{3})t, \qquad v_{1,2}(t) = 1 - ib + (-1 + i\sqrt{3})t,$$

$$v_{2,1}(t) = -1 + ib + (1 - i\sqrt{3})t, \qquad v_{2,2}(t) = 1 + ib + (-1 - i\sqrt{3})t, \qquad 0 \leqslant t \leqslant 1.$$

Then, for $t = 1$, the ends of the slits meet at $w_{1,3}$ and $w_{2,3}$, respectively. By $\mathcal{D}(t)$ we denote the doubly connected domain obtained from $\mathcal{D}_0$ by drawing slits along two polygonal lines: the first has vertices at $v_{1,1}(t)$, $w_{1,1}$, $w_{1,2}$, $v_{1,2}(t)$, and the second has vertices at $v_{2,1}(t)$, $w_{2,1}$, $w_{2,2}$, $v_{2,2}(t)$. We note that $\mathcal{D}(0) = \mathcal{D}_0$ and as $t \to 1$ the domains $\mathcal{D}(t)$ converge as to a kernel to the domain $\mathcal{D}_1$.

The family of conformal mappings of annuli $\mathcal{A}(t) = \{\tau : q(t) < |\tau| < 1\}$ onto $\mathcal{D}(t)$, $0 \leqslant t \leqslant 1$, read

$$\begin{aligned}\mathcal{F}(z,t) =& C_1(t) \int_0^z \exp\{c(t)\xi\}\sigma^{\frac{2}{3}}(\xi - z_{1,1}(t))\sigma^{\frac{2}{3}}(\xi - z_{1,2}(t)) \\ &\cdot \sigma(\xi - z_{1,3}(t))\sigma^{-\frac{2}{3}}(\xi - z_{1,4}(t))\sigma^{-\frac{2}{3}}(\xi - z_{1,5}(t))\sigma(\xi - z_{1,6}(t)) \\ &\cdot \sigma^{\frac{2}{3}}(\xi - z_{2,1}(t))\sigma^{\frac{2}{3}}(\xi - z_{2,2}(t))\sigma(\xi - z_{2,3}(t))\sigma^{-\frac{2}{3}}(\xi - z_{2,4}(t)) \\ &\cdot \sigma^{-\frac{2}{3}}(\xi - z_{2,5}(t))\sigma(\xi - z_{2,6}(t)) \left(\sigma(\xi - z_0(t))\sigma(\xi - \bar{z}_0(t))\right)^{-2} d\xi + C_2(t),\end{aligned} \tag{5.2}$$

where $z = -i \ln \tau$ and

$$\begin{aligned}c(t) =& \frac{\eta_1(t)}{\omega_1} \bigg[\frac{2}{3} z_{1,1}(t) + \frac{2}{3} z_{1,2}(t) + z_{1,3}(t) - \frac{2}{3} z_{1,4}(t) - \frac{2}{3} z_{1,5}(t) + z_{1,6}(t) \\ &+ \frac{2}{3} x_{2,1}(t) + \frac{2}{3} x_{2,2}(t) + x_{2,3}(t) - \frac{2}{3} x_{2,4}(t) - \frac{2}{3} x_{2,5}(t) + x_{1,6}(t)\bigg] + \eta_2(t).\end{aligned}$$

At $t = 0$ the initial values of the parameters coincide with the values for the conformal mapping of the annulus onto $\mathcal{D}_0$, they can be taken from Table 1.

Solving the Cauchy problem for the system of ordinary differential equations constructed for the family (5.2), see Theorem 4.2 and Remark 4.2, for $t = 1$ we obtain the accessory parameters $\mathcal{D}_1$; they are given in Table 3 for $b = 2.87161351081953$. We also obtained that the conformal modulus is given by the identity $m(1) = 0.52465479179157$.

TABLE 3. Some values of accessory parameters for the exterior of two equilateral triangles.

$q(1)$	$\omega_2(1)$	$z_0(1)$
0.03701236329192	6.59300655825243	1.64825163956313
$z_{1,1}(1)$	$z_{1,2}(1)$	$z_{1,3}(1)$
0.74428332332553	−0.74428332332553	−3.13964907706861
$z_{1,4}(1)$	$z_{1,5}(1)$	$z_{1,6}(1)$
−3.14159265358973	3.14159265358984	3.13964907706868
$x_{2,1}(1)$	$x_{2,2}(1)$	$x_{2,3}(1)$
0.74428332332553	−0.74428332332553	−3.13964907706784
$x_{2,4}(1)$	$x_{2,5}(1)$	$x_{2,6}(1)$
−3.14159265358959	3.14159265358998	3.13964907706810

As $t \to 1$, the accessory parameters $x_{k,3}$ and $x_{k,4}$, $1 \leqslant k \leqslant 2$, approach each other and merge into the point $-\frac{\omega_1}{2} = -\pi$, but in numerical calculations their values differ; this is caused by the non–uniform convergence of the family of conformal mappings near the specified point. Similarly, the points $x_{k,5}$ and $x_{k,6}$ merge into the point $\frac{\omega_1}{2} = \pi$ as $t \to 1$. It is interesting that since $z_{1,6} < z_{1,5} \leqslant \frac{\omega_1}{2}$, the point $z_{1,6}$ as $t \to 1$ compresses the point $z_{1,5}$ and, hence, has lower accuracy, which is indicated by numerical calculations. We note that the accuracy of calculations for accessory parameters is up to 10^{-15}, and the accuracy of calculations of triangle vertices is up to 10^{-6}. The distance between the boundaries of triangles is 1.13956922911083, and the exact value of b is $\sqrt{3} = 1.13956270325065$ ($b = 2.87161351081953$, see Example 3.2).

Example 5.2. We consider the problem on constructing the conformal mapping of an annulus onto the exterior of two rectangles

$$\Pi_1 = [-b-2, -b] \times [b, b+1]$$

and

$$\Pi_2 = [b, b+4] \times [-b-2, -b],$$

see Figure 6. We solve the problem in three steps. At each stage, we solve a Cauchy problem for the system of ordinary differential equations. The solutions (accessory parameters) obtained at each of the stages, except for the last one, determine the initial conditions for the system at the next stage. At each stage, we consider the values of the parameter t in the interval $[0, 1]$.

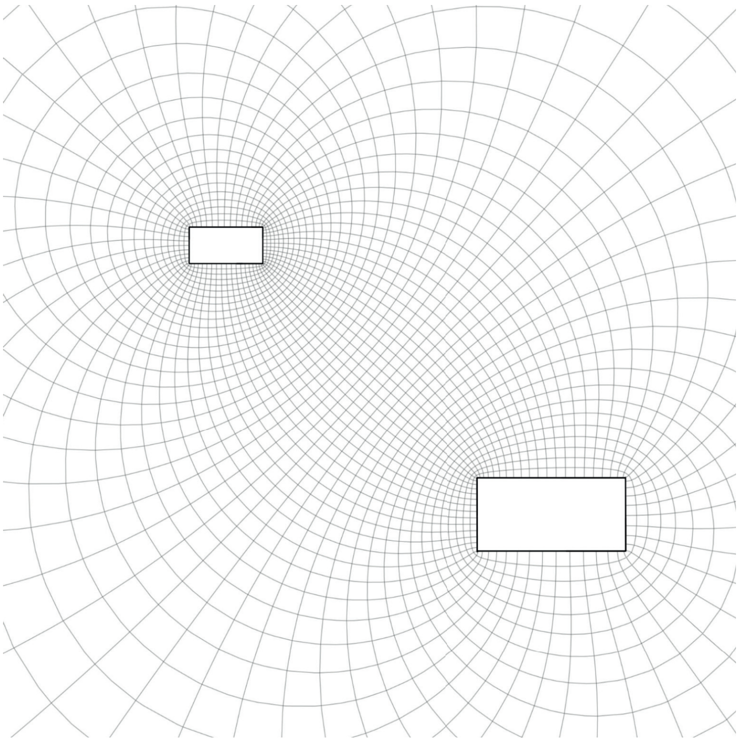

FIGURE 6. The image of the polar grid under the conformal mapping of an annulus onto the exterior of two rectangles.

Step 1. As the initial conformal mapping we take the mapping of an annulus onto the doubly connected domain, which is the exterior of the segments $[-2-b+ib, -b+ib]$ and $[-2-b-ib, -b-ib]$. In what follows we consider a family of the conformal mappings of annuli onto the exterior of two segments $[-2-b+ib, -b+ib]$ and $[-2-b+(2+2b)t-ib, -b+(4+2b)t-ib]$, see Figure 7. Here we apply the results of Section 4 and find the values of parameters for $t = 1$ assuming that the initial values (for $t = 0$) are known to us; we take them from Table 2 for $k = \frac{1}{2}$.

Step 2. We consider a family of mappings of annuli onto the exterior of two polygonal lines. The first polygonal line consists of the segments $[-2-b+i(b+t), -2-b+ib]$, $[-2-b+ib, -b+ib]$ and $[-b+i(b+t), -b+ib]$, and the second line consists of the segments $[b-i(b-2t), b-ib]$, $[b-ib, 4+b-ib]$ and $[4+b-i(b-2t), 4+b-ib]$, see Figure 8. For $t = 1$ we obtain a domain slit along the polyagonal lines, which we denote by L_1 and L_2.

Step 3. At this step we make slits from the ends of each polygonal line L_k, $k = 1, 2$, parallel to the real axis towards each other. As $t \to 1$, the contours of the obtained polygonal lines tend

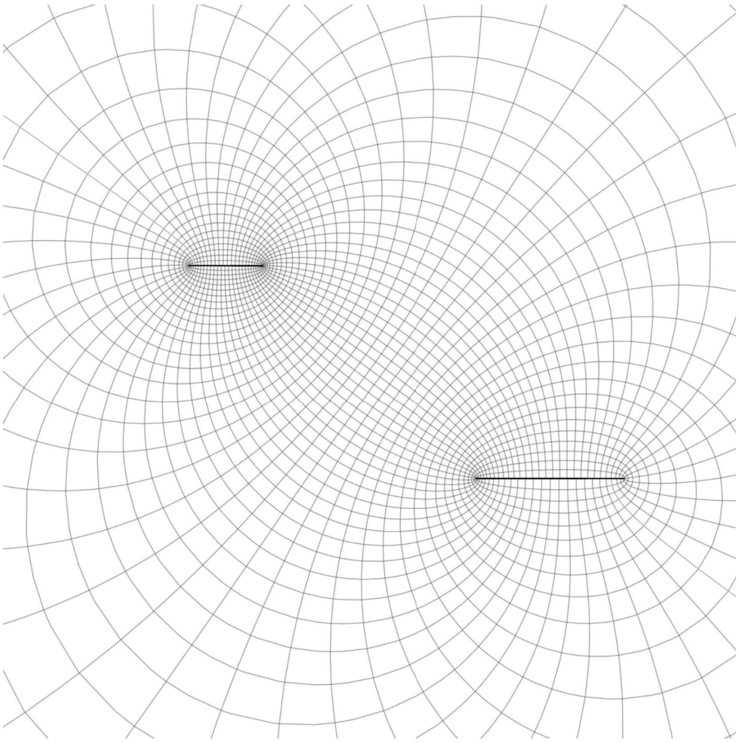

FIGURE 7. The image of polar grid under the conformal mapping of annulus onto the exterior of segments $[-2-b+ib, -b+ib]$ and $[b-ib, 4+b-ib]$.

to the boundaries of the rectangles and by the Carathéodory kernel convergence theorem the limiting function maps the annulus onto the exterior of the rectangles Π_1 and Π_2, see Figure 6. The obtained accessory parameters and conformal modulus are provided in Table 4. We note that the accuracy of the vertex calculations is up to 10^{-5}.

TABLE 4. Some values of accessory parameters and conformal modulus for the exterior of two rectangles.

$m(1)$	$q(1)$	$\omega_2(1)$	$z_0(1)$
0.69999254112105	0.01229966995971	8.79636569901417	1.86214282687967
$z_{1,1}(1)$	$z_{1,2}(1)$	$z_{1,3}(1)$	$z_{1,4}(1)$
3.04060273697885	5.24065213689680	6.23001742679993	1.49868623715722
$x_{2,1}(1)$	$x_{2,2}(1)$	$x_{2,3}(1)$	$x_{2,4}(1)$
1.18569206112100	3.23502905829918	4.63591023096649	0.06254300387664

Example 5.3. By the methods given in the previous examples, we calculate the accessory parameters for the exterior of the square $[0,1] \times [0,1]$ and the isosceles right triangle with the unit leg, see Figure 9. In Table 5 we present the values of the accessory parameters. We also obtain the conformal modulus $m = 0.37364295922581$. We note that the accuracy of vertex calculations is up to 10^{-5}.

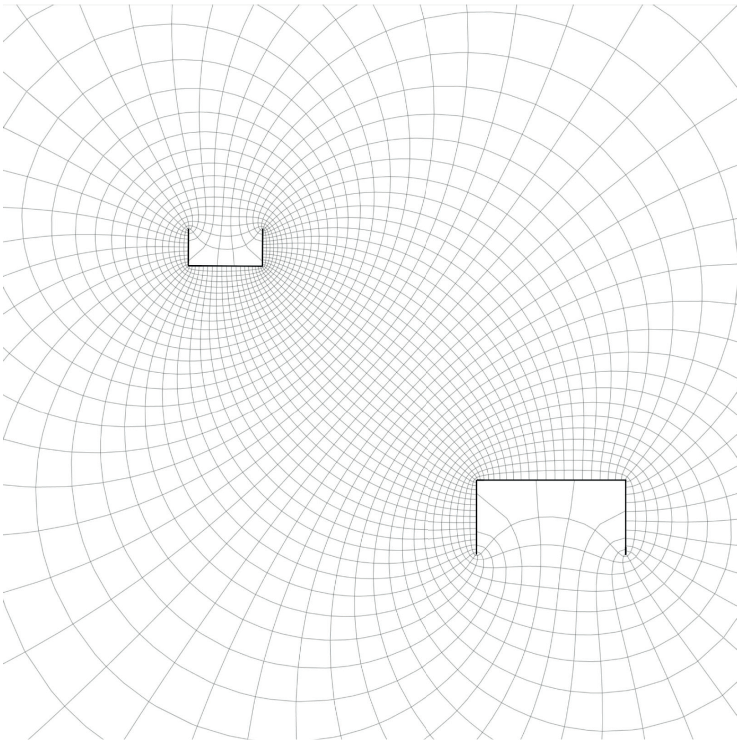

FIGURE 8. The image of the polar grid under the conformal mapping of an annulus onto the unbounded domain obtained by removing two polygonal lines composed of six segments.

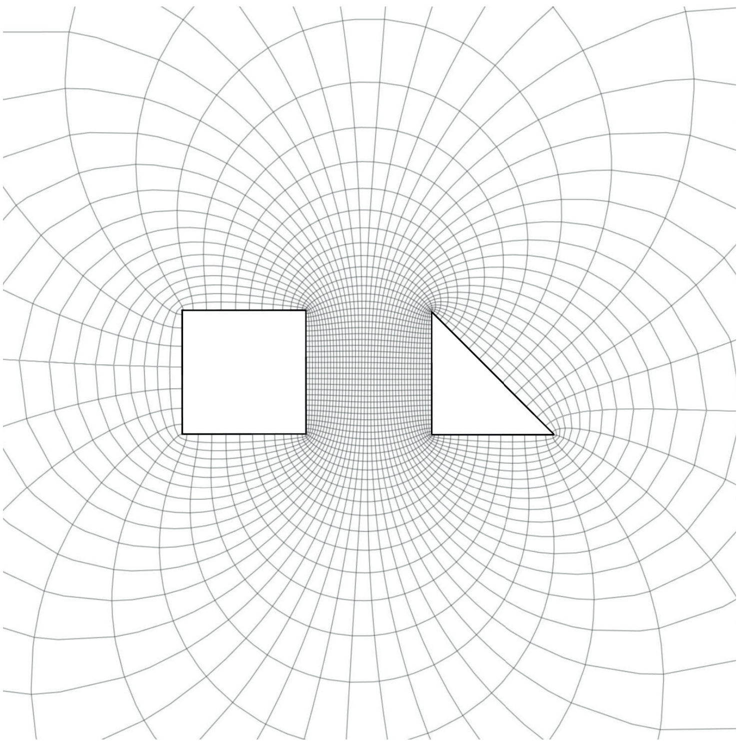

FIGURE 9. The image of the polar grid under the conformal mapping of an annulus onto the exterior of the square and isosceles right triangle.

TABLE 5. Some values of accessory parameters of the exterior of the square and isosceles right triangle.

q	ω_2
0.09559182725546	4.69533590307750
z_0	$z_{1,1}, \quad \alpha_{1,1}=\frac{3}{2}$
1.26225237317382	1.79476751461141
$z_{1,2}, \quad \alpha_{1,2}=7/4$	$z_{1,3}, \quad \alpha_{1,3}=7/4$
0.21796489082999	4.76059961486419
$z_{2,1}, \quad \alpha_{2,1}=\frac{3}{2}$	$z_{2,2}, \quad \alpha_{2,2}=\frac{3}{2}$
5.90482621712606	4.57710889452420
$z_{2,3}, \quad \alpha_{2,3}=\frac{3}{2}$	$z_{2,4}, \quad \alpha_{2,4}=\frac{3}{2}$
1.81377196659970	0.43366143113865

BIBLIOGRAPHY

1. I.A. Alexandrov. *Parametric continuations in theory of univalent functions.* Nauka, Moscow (1976). (in Russian).
2. N.I. Akhiezer. *Elements of the theory of elliptic functions.* Nauka, Moscow (1970). *English translation:* Amer. Math. Soc., Providence, RI (1990).
3. G. Goluzin, L. Kantorovich, V. Krylov, P. Melentiev, M. Muratov, N. Stenin. *Conformal mapping of simply connected and multiply connectyed domains.* ONTI, Moscow (1937). (in Russian).
4. G.M. Goluzin. *On the parametric representation of functions univalent in annulus* // Mat. Sb. **71**:2, 469–476 (1951). (in Russian).
5. V.V. Golubev. *Theory of airplane wing in plane–parallel flow.* Centr. Aero–Hydrodyn. Inst. Publ., Moscow (1927). (in Russian).
6. W. von Koppenfels, F. Stallmann. *Praxis der konformen Abbildung.* Springer–Verlag, Göttingen (1959). (in German).
7. N.N. Nakipov, S.R. Nasyrov. *A parametric method of finding accessory parameters for the generalized Schwarz — Christoffel integrals* // Uchenye Zapiski Kazanskogo Univ. Ser. Fiz.–Mat. Nauki, **158**:2, 202–220 (2016).
8. S.R. Nasyrov. *Uniformization of one–parametric families of complex tori* // Russ. Math. **61**:8, 36–45 (2017). https://doi.org/10.3103/S1066369X17080047
9. A. Betz. *Konforme Abbildung.* Springer–Verlag, Berlin (1948). https://doi.org/10.1007/978-3-642-49920-3
10. L. Bieberbach. *Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln* // S.–B. Preuss. Akad. Wiss, 940–955 (1916).
11. F. Bracci, M.D. Contreras, S. Diaz-Madrigal, A. Vasilev. *Classical and stochastic Löwner–Kufarev equations* // in "Harmonic and complex analysis and its applications", A. Vasil'ev ed.,Birkhäuser, Cham, 39–134 (2014). https://doi.org/10.1007/978-3-319-01806-5_2
12. L. de Branges. *A proof of the Bieberbach conjecture* // Acta Math. **154**:1–2, 137–151 (1985). https://doi.org/10.1007/BF02392821
13. P.F. Byrd. *Handbook of Elliptic Integrals for Engineers and Scientists.* Springer–Verlag, Berlin (1971). https://doi.org/10.1007/978-3-642-65138-0
14. D. Dautova, S. Nasyrov, M. Vuorinen. *Conformal module of the exterior of two rectilinear slits* // Comput. Methods Funct. Theory. **21**:1, 109–130 (2021). https://doi.org/10.1007/s40315-020-00315-y
15. T.A. Driscoll, L.N. Trefethen. *Schwarz — Christoffel Mapping.* Cambridge University Press, Cambridge (2002). https://doi.org/10.1017/CBO9780511546808
16. A. Dyutin, S. Nasyrov. *One parameter families of conformal mappings of bounded doubly connected polygonal domains* // Lobachevskii J. Math. **45**:1, 390–411 (2024). https://doi.org/10.1134/S1995080224010128

17. G.M. Goluzin. *Geometric theory of functions of a complex variable*. Amer. Math. Soc., Providence, RI (1969). https://doi.org/10.1090/mmono/026
18. P. Henrici. *Applied and computational complex analysis. Vol. 3*. John Wiley & Sons, New York (1986).
19. V.Y. Komatu. *Darstellungenn der in einem Kreisringe analytischen Funktionen nebst den Anwendungen auf kanforme Abbildung über Polygonalringgebiete* // Jpn. J. Math. **19**:2, 203–215 (1945). (in German). https://doi.org/10.4099/jjm1924.19.2_203
20. V.Y. Komatu. *Untersuchungen über konforme Abbilldung von zweifach zusammenhängeden Gebieten* // Proc. Phys.–Math. Soc. Japan (3) **25**, 1–42 (1943). (in German).
21. S. Lang. *Elliptic Functions*. Springer, New York (1987).
https://doi.org/10.1007/978-1-4612-4752-4
22. S.R. Nasyrov. *Uniformization of simply–connected ramified coverings of the sphere by rational functions* // Lobachevskii J. Math. **39**:2, 252–258 (2018).
https://doi.org/10.1134/S1995080218020208
23. A. Posadskii, S. Nasyrov. *One–parameter families of conformal mappings of the half–plane onto polygonal domains with several slits* // Lobachevskii J. Math. **44**:4, 1448–1463 (2023).
https://doi.org/10.1134/S1995080223040224

Andrey Yurievich Dyutin,
Kazan Federal University,
Kremlevskaya str. 18,
420008, Kazan, Russia
E-mail: `dyutin.andrei2016@yandex.ru`

Semen Rafailovich Nasyrov,
Kazan Federal University,
Kremlevskaya str. 18,
420008, Kazan, Russia
E-mail: `semen.nasyrov@yandex.ru`