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ATTRACTORS OF MODIFIED KELVIN — VOIGT MODEL WITH MEMORY ALONG FLUID TRAJECTORIES

M.V. TURBIN, A.S. USTIUZHANINOVA

Abstract. In the work we prove the existence of trajectory and global attractors for the modified Kelvin — Voigt model with memory along fluid trajectories. The proof is based on approximate–topological approach to study problems in the hydrodynamics.

Namely, first we introduce the needed functional spaces and give an operator interpretation of the considered problem. Then we pose an approximation problem and prove its solvability on a finite segment and on the semi–axis. Under certain conditions for the coefficients of the problem we establish exponential estimates of solutions, and these estimates are independent on the approximation parameter. After that, on the base of limit passage, we show the existence of a weak solution to the original problem on the semi–axis. Then we determine the trajectory space for the considered problem, show that the definition is well–defined and prove the existence theorem for minimal trajectory and global attractors.

Keywords: trajectory attractor, global attractor, modified Kelvin — Voigt model, regular Lagrange flow, apriori estimate, existence theorem.

Mathematics Subject Classification: 35B41, 35Q35, 76A10

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a convex bounded domain with a smooth boundary. The system of equations corresponding to the modified Kelvin — Voigt model with memory along fluid trajectories reads

$$\begin{aligned}&\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \nu \Delta v - \varkappa \frac{\partial \Delta v}{\partial t} - \varkappa \sum_{i=1}^{n} v_i \frac{\partial \Delta v}{\partial x_i} \\ &\quad - \sum_{i=1}^{L} \int_0^t \beta_i e^{-\alpha_i (t-s)} \Delta v(s, z(s,t,x))\, ds + \nabla p = f, \quad (t,x) \in Q_T = [0,T] \times \Omega;\end{aligned} \tag{1.1}$$

$$\operatorname{div} v = 0, \quad (t,x) \in Q_T; \tag{1.2}$$

$$z(\tau; t, x) = x + \int_t^\tau v(s, z(s; t, x)) ds, \quad 0 \leqslant t, \tau \leqslant T, \quad x \in \Omega. \tag{1.3}$$

Here v is the velocity vector of the fluid particles, p is the fluid pressure, f is the vector of density of external forces, $\nu > 0$, $\varkappa > 0$ are the viscosity of fluid and the delay, respectively, β_i, α_i, $i = \overline{1, L}$, are some constants. In view of the physical meaning we assume that the constant α_i, $i = \overline{1, L}$, are different, real and positive. The function $z(\tau; t, x)$ is the trajectory of fluid particles corresponding to the velocity field v.

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For the system (1.1)–(1.3) we consider the initial boundary value problem with the initial and boundary conditions

$$v(0,x) = a(x), \quad x \in \Omega, \quad v|_{[0,T]\times\partial\Omega} = 0. \tag{1.4}$$

The Kelvin — Voigt model describes the motion of various polymer solutions and melts [1] and it was justified experimentally [2], [3]. This is one of the models of linear viscoelastic fluids with finitely many discretely distributed relaxation and retardation times. The general theory of such fluids, which includes the Kelvin — Voigt model, was constructed on the base of the Boltzmann superposition principle, according to which all effects on the medium are independent and additive, and the reactions of the medium to external effects are linear. For more details on the system of equations corresponding to the Kelvin — Voigt model, see [5], [6]. The modified Kelvin — Voigt model is obtained from the Kelvin — Voigt model in a way similar to the work [1]. Namely, by virtue of the principle of smallness of relative strain rates, one neglects the terms containing products of derivatives with respect to the spatial variables of the fluid velocity.

Since (1.1) involves the integral, which is taken along the trajectories of fluid motion, this model is more physical in comparison with the standard models, which are obtained from the reological relation with the partial derivatives in time. Such models describe the behavior of the fluid more accurately. But this is precisely the main problem in proving the existence of weak solutions to the corresponding initial boundary value problem. The fact is that to find the trajectories of the fluid particles, it is necessary to solve the Cauchy problem (1.3). But the smoothness of the weak solution is usually insufficient for the classical solvability of the Cauchy problem. The way to resolve this issue is to use the theory of regular Lagrangian flows created in the work [7]. In the work [8], this theory was successfully applied to the Oldroyd–type model. Also, based on this theory, the existence of weak solutions was proved for the original Kelvin — Voigt model in the work [9].

The study of the limiting behavior of solutions, namely the behavior of solutions as the time tends to infinity, is of particular interest in studying hydrodynamic problems. In such problems, the solutions can tend to a certain set in the phase space. Here, the phase space is understood as a set, the elements of which are identified with the states of the system. That is, regardless of the initial conditions of the problem, its solutions end up in this set, possibly after a sufficiently long time. Such sets are called attractors, since the solutions are attracted to them.

Since it is not always possible to establish the uniqueness of solutions in hydrodynamic problems, the classical approach based on the theory of semigroups (see, for example, [10], [11]) turns out to be not applicable. The solution to this problem was the theory of trajectory attractors created by Vishik and Chepyzhov [12], [13], and independently of them, a similar theory for the three–dimensional Navier — Stokes system was created by Sell and You [14].

In the theory of trajectory attractors, instead of a semigroup of evolutionary operators, a certain set of functions is considered, which depend on time and take values in the phase space. This set of functions is called the trajectory space, and the individual functions belonging to it are called trajectories. Each trajectory represents a certain variant of system evolution. The trajectory space allows one to bypass the requirement of uniqueness of solution. In the case under consideration, several trajectories can originate from a certain point in the phase space, or, what is the same, several solutions can exist for the same initial condition.

Later, the theory of trajectory attractors was developed in the works of Zvyagin and Vorotnikov [15], [16] and it was applied to a number of problems of mathematical hydrodynamics [17]–[22]. In particular, it was possible to omit the condition of translational invariance of trajectory space. This condition is too restrictive and often it is not satisfied in hydrodynamic problems. The matter is that the trajectory spaces in the theory under consideration are usually constructed on the base of energy estimates. It is not always possible to obtain the necessary translational invariant estimate. But it is often possible to establish an exponential estimate, which, thanks to the results by V.G. Zvyagin and D.A. Vorotnikov, turns out to be quite sufficient.

In this paper, on the base of attractor theory of non–invariant trajectory spaces, we prove the existence of minimal trajectory and global attractors for the modified Kelvin — Voigt model with the memory along the trajectories of fluid motion. Namely, for the studied model, we introduce the concept of a weak solution on a finite segment and on a semi–axis. After that, on the base on the approximate–topological approach to the study of problems of mathematical hydrodynamics, see, for example, [6], [23]–[26], we prove the existence of solutions. Then, on the base of the established exponential estimates of solutions, we introduce the space of trajectories, prove that its definition is correct, and prove the existence of minimal trajectory and global attractors of the studied problem.

2. Necessary facts from attractor theory

In the work we use the following notions and statements from the attractor theory, for more detail, see, for instance, the monograph [15], as well as the papers [16], [19].

Let E, E_0 be two Banach spaces, the space E is reflexive and the embedding $E \subset E_0$ is continuous. By $\mathbb{R}_+$ we denote the nonnegative semi–axis of real line $\mathbb{R}$.

The space $C(\mathbb{R}_+; E_0)$ consists of continuous functions defined on $\mathbb{R}_+$ and taking values in E_0. Since the half–line $\mathbb{R}_+$ is non–compact, in the linear space $C(\mathbb{R}_+; E_0)$ we can not define the standard norm of space of continuous functions. In the space $C(\mathbb{R}_+; E_0)$ we introduce the family of semi–norms

$$\|u\|_n = \|u\|_{C([0,n],E_0)}, \quad n \in \mathbb{N}.$$

The sequence $\{u_m\}$ from $C(\mathbb{R}_+; E_0)$ converges to the function u as $m \to \infty$ if $\|u_m - u\|_n \to 0$ for each $n \in \mathbb{N}$. Thus, the space $C(\mathbb{R}_+; E_0)$ is countably–normed space. The topology of local uniform convergence in the space $C(\mathbb{R}_+; E_0)$ is metrizable with respect to the metrics

$$\rho(u, v) = \|u - v\|_{C(\mathbb{R}_+;E_0)} = \sum_{n=1}^{\infty} 2^{-n} \frac{\|u - v\|_n}{1 + \|u - v\|_n}.$$

The obtained metric space is complete.

In the work we use the notation $\|u - v\|_{C(\mathbb{R}_+;E_0)}$ for the metrics in $C(\mathbb{R}_+; E_0)$, which is conventional in works on attractors of non–invariant trajectory space. This is related with using the abstract notions and statements from the works [15], [16], [19], in which this notation is used. At the same time, the functional $\| \cdot \|_{C(\mathbb{R}_+;E_0)}$ is not a norm since

$$\|\lambda v\|_{C(\mathbb{R}_+;E_0)} \neq |\lambda| \|v\|_{C(\mathbb{R}_+;E_0)}$$

as $\lambda \neq \pm 1$.

Let Π_M $(M \geqslant 0)$ be the restriction operator of functions defined on $\mathbb{R}_+$ to the segment $[0, M]$. The next lemma is true.

Lemma 2.1. *The set $P \subset C(\mathbb{R}_+; E_0)$ is relatively compact in $C(\mathbb{R}_+; E_0)$ if and only if for each $M > 0$ the set $\Pi_M P$ is relatively compact in $C([0, M], E_0)$.*

We denote by $L_\infty(\mathbb{R}_+; E)$ the space of essentially bounded functions defined on $\mathbb{R}_+$ and taking values in the space E. The space $L_\infty(\mathbb{R}_+; E)$ is Banach with the norm [27]

$$\|u\|_{L_\infty(\mathbb{R}_+;E)} = \operatorname{vrai} \max_{t \in \mathbb{R}_+} \|u(t)\|_E.$$

Definition 2.1. *Let J be a finite or infinite interval of real axis and $\overline{J}$ be its closure. Let Y be a Banach space. The function $u : \overline{J} \to Y$ is called weakly continuous if $t_n \to t$, $t_n \in \overline{J}$,* *implies* $u(t_n) \rightharpoonup u(t)$ *weakly in* Y. *The set of weakly continuous functions* $u : \overline{J} \to Y$ *we denote by* $C_w(\overline{J}, Y)$.

In the work we shall employ the following theorem, see, for instance, [28].

Theorem 2.1. *Let* E *and* E_0 *be two Banach space such that* $E \subset E_0$ *and the embedding is continuous. If the function* v *belongs to* $L_\infty(0, T; E)$ *and is continuous as a function with the values in* E_0, *then the function* v *is weakly continuous as a function with values in* E, *that is,* $v \in C_w([0, T], E)$.

Therefore, the function $v \in C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$ is weakly continuous as a function with the values in E (and this is why $v(t) \in E$ for all $t \in \mathbb{R}_+$), the function v is bounded as a function with the values E, and the identity

$$\|v\|_{C(\mathbb{R}_+;E_0)\cap L_\infty(\mathbb{R}_+;E)} = \sup_{t\in\mathbb{R}_+} \|v(t)\|_E$$

holds.

By $T(h)$ ($h \geqslant 0$) we denote the translation operators, each of which maps a function f into a function $T(h)f$ such that $T(h)f(t) = f(t + h)$. The identity $T(h_1)T(h_2) = T(h_1 + h_2)$ holds.

Let $\mathcal{H}^+ \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$ be a non–empty family of functions. The set $\mathcal{H}^+$ is called the trajectory space, the elements in $\mathcal{H}^+$ are called trajectories.

Definition 2.2. *The set* $P \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$ *is called the attracting set for the trajectory space* $\mathcal{H}^+$ *if for each set* $B \subset \mathcal{H}^+$ *bounded in* $L_\infty(\mathbb{R}_+; E)$ *the condition*

$$\sup_{u\in B} \inf_{v\in P} \|T(h)u - v\|_{C(\mathbb{R}_+;E_0)} \to 0 \quad as \quad h \to \infty$$

holds.

Definition 2.3. *The set* $P \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$ *is called the absorbing set for the space of trajectories* $\mathcal{H}^+$ *if for each set* $B \subset \mathcal{H}^+$ *bounded in* $L_\infty(\mathbb{R}_+; E)$ *there exists* $h \geqslant 0$ *such that for all* $t \geqslant h$ *the embedding* $T(t)B \subset P$ *holds.*

Each absorbing set is attracting.

Definition 2.4. *The set* $P \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$ *is called the trajectory semi–attractor of trajectory space* $\mathcal{H}^+$ *if*

(i) *The set* P *is compact in* $C(\mathbb{R}_+; E_0)$ *and bounded in* $L_\infty(\mathbb{R}_+; E)$.
(ii) *The embedding* $T(t)P \subset P$ *holds for all* $t \geqslant 0$.
(iii) *The set* P *is attracting.*

Definition 2.5. *The set* $P \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$ *is called the trajectory attractor of trajectory space* $\mathcal{H}^+$ *if*

(i) *The set* P *is compact in* $C(\mathbb{R}_+; E_0)$ *and bounded in* $L_\infty(\mathbb{R}_+; E)$.
(ii) *The identity* $T(t)P = P$ *holds for all* $t \geqslant 0$.
(iii) *The set* P *is attracting.*

Definition 2.6. *The minimal trajectory attractor of trajectory space* $\mathcal{H}^+$ *is the minimal in embedding trajectory attractor.*

Definition 2.7. *The set* $\mathcal{A} \subset E$ *is called the global attractor (in* E_0*) of trajectory space* $\mathcal{H}^+$ *if it satisfies the following conditions:*

(i) *The set* $\mathcal{A}$ *is compact in* E_0 *and bounded in* E.
(ii) *For each bounded in* $L_\infty(\mathbb{R}_+; E)$ *set* $B \subset \mathcal{H}^+$ *the attraction condition*

$$\sup_{u\in B} \inf_{y\in\mathcal{A}} \|u(t) - y\|_{E_0} \to 0 \quad as \quad t \to \infty$$

holds.

(iii) *The set $\mathcal{A}$ is minimal in embedding obeying Conditions (i) and (ii).*

If there exists a minimal trajectory attractor and global attractor, then it is unique.
In the work we also employ the following statement [15].

Lemma 2.2. *Let P be relatively compact in $C(\mathbb{R}_+; E_0)$ and bounded in $L_\infty(\mathbb{R}_+; E)$ absorbing set for the trajectory space $\mathcal{H}_+$. Then its closure $\overline{P}$ in the space $C(\mathbb{R}_+; E_0)$ is compact in $C(\mathbb{R}_+; E_0)$ and bounded in $L_\infty(\mathbb{R}_+; E)$ absorbing set for trajectory space $\mathcal{H}_+$. If, in addition, the inclusion $T(t)P \subset P$ holds for all $t \geqslant 0$, then $\overline{P}$ is a semi–attractor.*

The following theorem on existence of minimal trajectory and global attractors is true [15].

Theorem 2.2. *Let there exists a trajectory semi–attractor P of trajectory space $\mathcal{H}^+$. Then there exists a minimal trajectory attractor $\mathcal{U}$ and global attractor $\mathcal{A}$ of trajectory space $\mathcal{H}^+$, and the relation $\mathcal{A} = \mathcal{U}(t)$, $t \geqslant 0$, holds.*

3. Functional spaces

To define the notion of weak solution, we need to introduce some spaces. As usually, $C_0^\infty(\Omega)^n$ is the set of functions defined on Ω with the values in $\mathbb{R}^n$ from the class C^∞ with compact supports contained in Ω. Let

$$\mathcal{V} = \{v(x) = (v_1, \dots, v_n) \in C_0^\infty(\Omega)^n : \operatorname{div} v = 0\}.$$

The space V^0 is the completion of $\mathcal{V}$ by the norm $L_2(\Omega)^n$, V^1 is the completion of $\mathcal{V}$ by the norm $H^1(\Omega)^n$, $V^2 = H^2(\Omega)^n \cap V^1$.

By the Weyl decomposition of vector fields in $L_2(\Omega)^n$, see, for instance, [28], [29],

$$L_2(\Omega)^n = V^0 \oplus \nabla H^1(\Omega).$$

Here $\nabla H^1(\Omega) = \{\nabla p : p \in H^1(\Omega)\}$.

Let $\pi : L_2(\Omega)^n \to V^0$ be the Leray projection. In the space $\mathcal{V}$ we consider the operator $A = -\pi\Delta$. As it is known, see [30], [31], the operator A is extended to a closed operator in the space V^0, and this closed operator is self–adjoint, positive and has a completely continuous inverse operator. The domain of A coincides with V^2. By the Hilbert theorem on spectral decomposition of completely continuous operators, the eigenfunctions $\{e_j\}$ of the operator A form an orthonormal basis in V^0.

Let $0 < \zeta_1 \leqslant \zeta_2 \leqslant \zeta_3 \leqslant \ldots \leqslant \zeta_k \leqslant \ldots$ be the eigenvalues of the operator A, and E_∞ be the set of finite linear combinations formed by e_j. The space V^α, $\alpha \in \mathbb{R}$, is defined as the completion of E_∞ by the norm

$$\|v\|_{V^\alpha} = \left(\sum_{k=1}^{\infty} \zeta_k^\alpha |v_k|^2\right)^{\frac{1}{2}},$$

where $v_k = (v, e_k)$ are the Fourier coefficients of the function v over the system of eigenfunctions $\{e_k\}$, $(\cdot,\cdot)$ is the scalar product in V^0.

For $\alpha = 0, 1, 2$ the spaces V^α coincide with the above introduced spaces V^0, V^1, and V^2, respectively. It was show in [6] that the mentioned norms in the spaces V^α, $\alpha \in \mathbb{N}$, are equivalent to the norms

$$\|v\|_{V^\alpha} = \|A^{\alpha/2} v\|_{V^0}. \tag{3.1}$$

Hereafter the spaces V^α are equipped with the norms (3.1).

To define the weak solution to the original and approximation problem on the segment, we introduce the spaces

$$W_1[0,T] = \left\{u : u \in L_\infty(0,T; V^2), u' \in L_\infty(0,T; V^1)\right\};$$

$$W_2[0,T] = \left\{u : u \in C([0,T],V^5), u' \in L_\infty(0,T;V^5)\right\},$$

with the corresponding norms

$$\|u\|_{W_1[0,T]} = \|u\|_{L_\infty(0,T;V^2)} + \|u'\|_{L_\infty(0,T;V^1)};$$
$$\|u\|_{W_2[0,T]} = \|u\|_{C([0,T],V^5)} + \|u'\|_{L_\infty(0,T;V^5)}.$$

To define the weak solution on the half–line $\mathbb{R}_+$ we consider the space $W_1^{\rm loc}(\mathbb{R}_+)$, which consists of the functions v defined almost everywhere on $\mathbb{R}_+$ and taking the values in V^2 such the restriction of v to each segment $[0,T]$ belongs to $W_1[0,T]$. We also consider the space $W_2^{\rm loc}(\mathbb{R}_+)$, which consists of functions v from the class $C(\mathbb{R}_+,V^5)$ such that the restriction of v on each segment $[0,T]$ belongs to $W_2[0,T]$.

In the work we also employ the Aubin — Dubinsky — Simon theorem [32].

Theorem 3.1. *Let $X \subset E \subset Y$ be Banach spaces and the embedding $X \subset E$ be completely continuous, while the embedding $E \subset Y$ be continuous. Let $F \subset L_p(0,T;X)$, $1 \leqslant p \leqslant \infty$. We suppose that for each $f \in F$ its generalized derivative in the space $D'(0,T;Y)$ belongs to $L_r(0,T;Y)$, $1 \leqslant r \leqslant \infty$. Let*

1. *the set F be bounded in $L_p(0,T;X)$,*
2. *the set $\{f' : f \in F\}$ is bounded in $L_r(0,T;Y)$.*

Then for $p < \infty$ the set F is relatively compact in $L_p(0,T;E)$, while for $p = \infty$ and $r > 1$ the set F is relatively compact in $C([0,T],E)$.

We shall also employ the following Leray — Schauder theorem.

Theorem 3.2. *Let G be an open bounded subset of Banach space X, $0 \in G$, and let $\Xi(\tau,\cdot) : \overline{G} \to X$, $\tau \in [0,1]$, be a one–parametric family of mappings obeying the conditions*

1. *The mapping $\Xi : [0,1] \times \overline{G} \to X$ is compact in its variables.*
2. *$\Xi(\tau,x) \neq x$ for all $\tau \in [0,1]$ and $x \in \partial G$, that is, the mapping $\Xi(\tau,\cdot)$ has no fixed points at the boundary G.*
3. *$\Xi(0,\cdot) \equiv 0$.*

Then the mapping $\Xi(1,\cdot)$ has a fixed point $x_1 \in G$, that is, $x_1 = \Xi(1,x_1)$.

We provide needed statement on the solvability of problem

$$z(\tau;t,x) = x + \int_t^\tau v(s,z(s;t,x))\,ds, \quad 0 \leqslant t,\tau \leqslant T, \quad x \in \overline{\Omega}. \tag{3.2}$$

We shall assume that $v \in L_1(0,T;W_1^1(\Omega)^n)$, $\operatorname{div} v = 0$ and $v \cdot n = 0$ on $\partial\Omega$, where n is the normal vector.

Definition 3.1. *The function $z(\tau;t,x) : [0,T] \times [0,T] \times \overline{\Omega} \to \overline{\Omega}$ is a regular Lagrangian flow associated with v if the following conditions are satisfied:*

1) *For almost all x and each $t \in [0,T]$ the function $\gamma(\tau) = z(\tau;t,x)$ is absolutely continuous and satisfies Equation (3.2).*
2) *For all $\tau,t \in [0,T]$ and arbitrary Lebesgue measurable set $B \subset \overline{\Omega}$ with the Lebesgue measure $m(B)$ the identity $m(z(\tau;t,B)) = m(B)$ holds.*
3) *For all t_1, t_2, $t_3 \in [0,T]$ and almost all $x \in \overline{\Omega}$ the identity*

$$z(t_3;t_1,x) = z(t_3;t_2,z(t_2;t_1,x)) \tag{3.3}$$

holds.

Here $z(\tau;t,B)$ is the image of the set B, that is,

$$z(\tau;t,B) = \bigcup_{x\in B} z(\tau;t,x);$$

for more details on regular Lagrangian flows see, for instance, [7]. Here we consider a partial case of a bounded domain Ω and a divergence–free function v. At the same time, in the case of a smooth vector field v the regular Lagrangian flow coincides with the classical solution to the Cauchy problem (3.2).

The following theorems are true [7].

Theorem 3.3. *Let*

$$v \in L_1(0,T;W_p^1(\Omega)^n), \quad 1 \leqslant p \leqslant +\infty, \quad \operatorname{div} v = 0, \quad v|_{\partial\Omega} = 0.$$

Then there exists a unique regular Lagrangian flow z corresponding to v. Moreover,

$$\frac{\partial}{\partial\tau} z(\tau;t,x) = v(\tau, z(\tau;t,x)), \quad t,\tau \in \Omega, \quad \textit{for almost all} \quad x \in \Omega,$$
$$z(\tau;t,\overline{\Omega}) = \overline{\Omega} \quad \textit{(up to zero measure set).}$$

Theorem 3.4. *Let*

$$v, v^m \in L_1(0,T;W_1^p(\Omega)^n), \qquad m = 1,2,\dots$$

for some $p > 1$. Let

$$\text{div } v^m = 0, \qquad v^m|_{\partial\Omega} = 0, \qquad \text{div } v = 0, \quad v|_{\partial\Omega} = 0$$

and the inequalities

$$\|\nabla v\|_{L_1(0,T;L_p(\Omega)^{n^2})} + \|v\|_{L_1(0,T;L_1(\Omega)^n)} \leqslant M,$$
$$\|\nabla v^m\|_{L_1(0,T;L_p(\Omega)^{n^2})} + \|v^m\|_{L_1(0,T;L_1(\Omega)^n)} \leqslant M,$$

hold. Let v^m converges to v in $L_1(Q_T)^n$ as $m \to \infty$. Let z^m and z be regular Lagrangian flows corresponding to v^m and v. Then the sequence z^m converges to z in the Lebesgue measure on $[0,T] \times \Omega$ in the variables (τ, x) uniformly in $t \in [0,T]$.

Throughout the work the constants are defined by the symbol C with a subscript. The constants essential for the proof are written explicitly and sometimes are denoted by the symbol K with a subscript. The symbol ':' stands for a pointwise product of matrices.

4. Definition of weak solution

Let $a \in V^2, f \in V^0$.

Definition 4.1. *A weak solution to the initial boundary value problem* (1.1)–(1.4) *on the segment* $[0,T]$ *is a function* $v \in W_1[0,T]$ *obeying the identity*

$$\int_\Omega v'\varphi dx - \sum_{i,j=1}^n \int_\Omega v_i v_j \frac{\partial \varphi_j}{\partial x_i} dx + \nu \int_\Omega \nabla v : \nabla \varphi dx + \varkappa \int_\Omega \nabla v' : \nabla \varphi dx$$
$$+ \varkappa \sum_{i,j=1}^n \int_\Omega v_i \Delta v_j \frac{\partial \varphi_j}{\partial x_i} dx - \sum_{i=1}^L \int_0^t \beta_i e^{-\alpha_i(t-s)} \int_\Omega \Delta v(s, z(s;t,x))\varphi dx ds = \int_\Omega f\varphi dx \tag{4.1}$$

for each test function $\varphi \in V^1$ *for almost each* $t \in (0,T)$ *and obeying the initial condition*

$$v(0) = a. \tag{4.2}$$

Here z is the regular Lagrangian flow generated by v, which exists due to Theorem 3.3.

Definition 4.2. *A weak solution to the problem* (1.1)–(1.4) *on the semi–axis* $\mathbb{R}_+$ *is a function* $v \in W_1^{loc}(\mathbb{R}_+)$ *such that for each* $T > 0$ *the restriction of* v *to the segment* $[0, T]$ *is a weak solution to the problem* (1.1)–(1.4) *on the segment* $[0, T]$.

5. Approximation problem

By the definition of the norms in the spaces V^0, V^1 and V^2 the inequalities

$$\|u\|_{V^0}^2 \leqslant K_1 \|u\|_{V^1}^2, \quad \|u\|_{V^1}^2 \leqslant K_1 \|u\|_{V^2}^2, \quad u \in V^2, \tag{5.1}$$

hold. Here $K_1 = 1/\zeta_1$, where ζ_1 is the lowest eigenvalue of the operator A.

Let K_2 be a constant determined by the following identity

$$K_2 = \frac{\nu \varkappa}{K_1^2 + 2\varkappa K_1 + \varkappa}. \tag{5.2}$$

Let $\varepsilon > 0$ and γ be a constant, for which the inequality

$$0 < \gamma \leqslant \min \left(K_2, \alpha_1, \alpha_2, \dots \alpha_L\right) \tag{5.3}$$

holds. The exact choice of γ was described in the proof of Theorem 6.1. We consider the following approximation problem

$$\begin{aligned} &\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \nu \Delta v - \varkappa \frac{\partial \Delta v}{\partial t} - \varepsilon e^{-\gamma t} \frac{\partial \Delta^3 v}{\partial t} - \varkappa \sum_{i=1}^{n} v_i \frac{\partial \Delta v}{\partial x_i} \\ &\quad - \sum_{i=1}^{L} \int_0^t \beta_i e^{-\alpha_i (t-s)} \Delta v(s, z(s; t, x))\, ds + \nabla p = f; \quad \operatorname{div} v = 0, \quad (t, x) \in Q_T; \end{aligned} \tag{5.4}$$

$$z(\tau; t, x) = x + \int_t^\tau v(s, z(s; t, x)) ds, \quad 0 \leqslant t, \tau \leqslant T, \quad x \in \Omega; \tag{5.5}$$

$$v(0, x) = b(x), \quad x \in \Omega; \quad v|_{[0,T]\times\partial\Omega} = \Delta v|_{[0,T]\times\partial\Omega} = \Delta^2 v|_{[0,T]\times\partial\Omega} = 0. \tag{5.6}$$

Let $b \in V^5, f \in V^0$.

Definition 5.1. *A function* $v \in W_2[0, T]$ *is a called the solution to the approximation problem* (5.4)–(5.6) *if it satisfies the identity*

$$\begin{aligned} &\int_\Omega v' \varphi dx - \sum_{i,j=1}^{n} \int_\Omega v_i v_j \frac{\partial \varphi_j}{\partial x_i}\, dx + \nu \int_\Omega \nabla v : \nabla \varphi\, dx + \varkappa \int_\Omega \nabla v' : \nabla \varphi dx \\ &\quad + \varepsilon e^{-\gamma t} \int_\Omega \nabla(\Delta^2 v') : \nabla \varphi dx + \varkappa \sum_{i,j=1}^{n} \int_\Omega v_i \Delta v_j \frac{\partial \varphi_j}{\partial x_i} dx \\ &\quad - \sum_{i=1}^{L} \int_0^t \beta_i e^{-\alpha_i (t-s)} \int_\Omega \Delta v(s, z(s; t, x)) \varphi dx ds = \int_\Omega f \varphi dx \end{aligned} \tag{5.7}$$

for each function $\varphi \in V^1$ *for almost each* $t \in (0, T)$ *and it obeys the initial condition*

$$v(0) = b. \tag{5.8}$$

Here z is a solution to the problem (5.5). By the continuous embedding $V^5 \subset C^1(\overline{\Omega})^n$, which holds for $n = 2, 3$, the problem (5.5) has a unique classical solution.

Definition 5.2. *A solution to the approximation problem* (5.4)–(5.6) *on the semi–axis* $\mathbb{R}_+$ *is a function* $v \in W_2^{loc}(\mathbb{R}_+)$ *such that for each* $T > 0$ *the restriction* v *to the segment* $[0, T]$ *solves the approximation problem* (5.4)–(5.6) *on this segment.*

In order to pass to the operator formulation of the problem, we introduce the operators by the identities

$$A : V^1 \to V^{-1}, \quad \langle Au, \varphi\rangle = \int_\Omega \nabla u : \nabla\varphi dx, \quad \forall u, \varphi \in V^1;$$

$$J : V^1 \to V^{-1}, \quad \langle Ju, \varphi\rangle = \int_\Omega u\varphi dx, \quad \forall u, \varphi \in V^1;$$

$$A^3 : V^5 \to V^{-1}, \quad \langle A^3 u, \varphi\rangle = \int_\Omega \nabla(\Delta^2 u) : \nabla\varphi dx, \quad \forall u \in V^5, \quad \varphi \in V^1;$$

$$B_1 : L_4(\Omega)^n \to V^{-1}, \quad \langle B_1(u), \varphi\rangle = \sum_{i,j=1}^n \int_\Omega u_i u_j \frac{\partial\varphi_j}{\partial x_i} dx, \quad \forall u \in L_4(\Omega)^n, \quad \varphi \in V^1;$$

$$B_2 : V^2 \to V^{-1}, \quad \langle B_2(u), \varphi\rangle = \sum_{i,j=1}^n \int_\Omega u_i \Delta u_j \frac{\partial\varphi_j}{\partial x_i} dx, \quad \forall u \in V^2, \quad \varphi \in V^1;$$

$$C : L_2(0, T; V^2) \to L_2(0, T; V^{-1}), \quad \forall u \in L_2(0, T; V^2), \quad \varphi \in V^1;$$

$$\langle C(u)(t), \varphi\rangle = \sum_{i=1}^L \int_0^t \beta_i e^{-\alpha_i(t-s)} \int_\Omega \Delta u(s, z(s; t, x))\varphi dx ds.$$

In terms of the introduced operators we can give an equivalent definition of the solution to approximation problem.

Definition 5.3. *A solution to the problem* (5.4)–(5.6) *on the segment* $[0, T]$ *is the function* $v \in W_2[0, T]$ *obeying the operator equation*

$$(J + \varepsilon e^{-\gamma t} A^3 + \varkappa A)v'(t) + \nu Av(t) - B_1(v)(t) + \varkappa B_2(v)(t) - C(v)(t) = f \tag{5.9}$$

for almost each in $t \in [0, T]$ *and the initial condition* (5.8).

The following lemma on the properties of operators holds [24].

Lemma 5.1.

1. *For a function* $g \in L_2(0, T; V^1)$ *we have* $Ag \in L_2(0, T; V^{-1})$, *the operator* $A : L_2(0, T; V^1) \to L_2(0, T; V^{-1})$ *is continuous and for almost each* $t \in (0, T)$ *the estimate*
$$\|Ag(t)\|_{V^{-1}} \leqslant \|g(t)\|_{V^1} \tag{5.10}$$
holds.
2. *The operator* $(J + \varkappa A) : V^1 \to V^{-1}$ *is continuous and invertible. For each function* $g \in L_2(0, T; V^1)$ *we have* $(J+\varkappa A)g \in L_2(0, T; V^{-1})$, *the operator* $(J+\varkappa A) : L_2(0, T; V^1) \to L_2(0, T; V^{-1})$ *is continuous and for almost each* $t \in (0, T)$ *the estimate*
$$\varkappa\|g(t)\|_{V^1} \leqslant \|(J + \varkappa A)g(t)\|_{V^{-1}} \tag{5.11}$$
holds.
3. *For* $g \in L_2(0, T; V^5)$ *we have* $(J + \varepsilon e^{-\gamma t} A^3 + \varkappa A)g \in L_2(0, T; V^{-1})$, *the operator* $(J + \varepsilon e^{-\gamma t} A^3 + \varkappa A) : L_2(0, T; V^5) \to L_2(0, T; V^{-1})$ *is continuous, invertible and for almost all* $t \in (0, T)$ *the estimate*
$$\varepsilon e^{-\gamma t}\|g(t)\|_{V^5} \leqslant \|(J + \varepsilon e^{-\gamma t} A^3 + \varkappa A)g(t)\|_{V^{-1}} \tag{5.12}$$
holds.

4. *For a function* $g \in L_2(0,T;V^1)$ *we have* $B_1(g) \in L_2(0,T;V^{-1})$, *the mapping* $B_1 : L_2(0,T;V^1) \to L_2(0,T;V^{-1})$ *is continuous and for almost all* $t \in (0,T)$ *the estimate*
$$\|B_1(g)(t)\|_{V^{-1}} \leqslant C_1\|g(t)\|^2_{V^1} \tag{5.13}$$
holds. Here the constant C_1 *depends on the domain* Ω *and is independent of the function* g.
5. *For a function* $g \in L_2(0,T;V^2)$ *we have* $B_2(g) \in L_2(0,T;V^{-1})$, *the mapping* $B_2 : L_2(0,T;V^2) \to L_2(0,T;V^{-1})$ *is continuous and for almost all* $t \in (0,T)$ *the estimate*
$$\|B_2(g)(t)\|_{V^{-1}} \leqslant C_2\|g(t)\|^2_{V^2} \tag{5.14}$$
holds. Here the constant C_2 *depends on the domain* Ω *and is independent of the function* g.

Lemma 5.2. *The mapping*
$$C : L_2(0,T;V^2) \to L_2(0,T;V^{-1})$$
is continuous and for almost all $t \in (0,T)$ *the inequality*
$$\|C(g)(t)\|_{V^{-1}} \leqslant C_3\left(\int_0^t e^{-\gamma(t-s)}\|g(s)\|^2_{V^2}dt\right)^{\frac{1}{2}} \tag{5.15}$$
holds. Here the constant C_3 *depends on* K_1, γ, α_i, β_i, $i = \overline{1,L}$.

Proof. By the definition of the operator C for each function $g \in L_2(0,T;V^2)$ for almost all $t \in (0,T)$ for each $\varphi \in V^1$ by the Hölder inequality we have
$$|\langle C(g)(t),\varphi\rangle| = \left|\int_0^t \sum_{i=1}^L \beta_i e^{-\alpha_i(t-s)} \int_\Omega \Delta g(s,z(s;t,x))\varphi dxds\right|$$
$$\leqslant \sum_{i=1}^L |\beta_i| \int_0^t e^{-\alpha_i(t-s)} \left(\int_\Omega |\Delta g(s,z(s;t,x))|^2dx\right)^{\frac{1}{2}} \left(\int_\Omega |\varphi|^2dx\right)^{\frac{1}{2}} ds.$$
In first integral in the right hand side we make the change of variables $y = z(s;t,x)$ (the inverse change $x = z(t;s,y)$). Since $\operatorname{div} g = 0$, we have $\det\frac{\partial z}{\partial x} = 1$. This is why
$$\int_\Omega |\Delta g(s,z(s;t,x))|^2dx = \int_\Omega |\Delta g(s,y)|^2dy = \|\Delta g(s)\|^2_{L_2(\Omega)^n}.$$

Thus, by the Poincaré inequality (5.1) we get
$$|\langle C(g)(t),\varphi\rangle| \leqslant \sqrt{K_1}\sum_{i=1}^L |\beta_i| \int_0^t e^{-\alpha_i(t-s)}\|g(s)\|_{V^2}ds\|\varphi\|_{V^1}.$$
This yields
$$\|C(g)(t)\|_{V^{-1}} \leqslant \sqrt{K_1}\sum_{i=1}^L |\beta_i| \int_0^t e^{-\alpha_i(t-s)}\|g(s)\|_{V^2}ds. \tag{5.16}$$

For each $i = \overline{1,L}$ by the Hölder inequality and (5.3) we find
$$\int_0^t e^{-\alpha_i(t-s)}\|g(s)\|_{V^2}ds = \int_0^t e^{-(\alpha_i-\frac{\gamma}{2})(t-s)}e^{-(t-s)\frac{\gamma}{2}}\|g(s)\|_{V^2}ds$$

$$\leqslant \left(\int_0^t e^{-2(\alpha_i -' \frac{\gamma}{2})(t-s)} ds \right)^{\frac{1}{2}} \left(\int_0^t e^{-\gamma(t-s)} \|g(s)\|_{V^2}^2 ds \right)^{\frac{1}{2}}$$
$$= \left(\frac{1 - e^{-2(\alpha_i - \frac{\gamma}{2})t}}{2\alpha_i - \gamma} \right)^{\frac{1}{2}} \left(\int_0^t e^{-\gamma(t-s)} \|g(s)\|_{V^2}^2 ds \right)^{\frac{1}{2}}$$
$$\leqslant \frac{1}{\sqrt{2\alpha_i - \gamma}} \left(\int_0^t e^{-\gamma(t-s)} \|g(s)\|_{V^2}^2 ds \right)^{\frac{1}{2}}.$$

Together with (5.16) this implies the desired inequality (5.15). The proof is complete. □

6. Estimates

Theorem 6.1. *Let v be the solution to the approximation equation* (5.9) *and the coefficients of the problem* (5.4)–(5.6) *satisfy the conditions*

$$\varkappa K_2 \alpha_i > 2L|\beta_i|, \qquad i = \overline{1, L}. \tag{6.1}$$

Then for all $t \in [0, T]$ the estimate

$$\begin{aligned} &\varkappa^2 \|v(t)\|_{V^2}^2 + K_2 \lambda_1 \varkappa^2 \int_0^t e^{-\gamma(t-s)} \|v(s)\|_{V^2}^2 ds \\ &\leqslant C_5 + e^{-\gamma t} \left(C_4 \|v(0)\|_{V^2}^2 + \varepsilon \|v(0)\|_{V^3}^2 + \varepsilon\varkappa \|v(0)\|_{V^4}^2 \right) \end{aligned} \tag{6.2}$$

holds. Here λ_1, λ_2 are some constants such that

$$\lambda_1 > 0, \qquad \lambda_2 > 0, \qquad 0 < \lambda_1 + \lambda_2 < 1,$$

the constant K_2 is determined by the identity (5.2),

$$C_4 = K_1^2 + 2\varkappa K_1 + \varkappa^2, \qquad C_5 = \frac{\|f\|_{V^0}^2}{\gamma K_2 (1 - \lambda_1 - \lambda_2)}.$$

Proof. Let $v \in W_2[0, T]$ be a solution of Equation (5.9). Then for all $s \in [0, T]$ we have $v(s) \in V^5$. By the continuity of the embedding $V^5 \subset V^1$, the continuity of the operator $A : V^5 \to V^3$, which holds due to the construction of scale of spaces V^α, and the continuity of embedding $V^3 \subset V^1$ we obtain that $(v + \varkappa Av)(s) \in V^1$ for all $s \in [0, T]$. We apply (5.9) to $(v + \varkappa Av)(s)$, $s \in [0, T]$. We get

$$\left\langle (J + \varepsilon e^{-\gamma s} A^3 + \varkappa A) v' + \nu A v - B_1(v) + \varkappa B_2(v) - C(v), v + \varkappa A v \right\rangle = \langle f, v + \varkappa A v \rangle.$$

By the definition of the operators B_1 and B_2 and by the Green formula we have

$$\begin{aligned} \langle -B_1(v) + \varkappa B_2(v), v + \varkappa A v \rangle &= - \sum_{i,j=1}^n \int_\Omega v_i v_j \frac{\partial (v - \varkappa \Delta v)_j}{\partial x_i} dx + \varkappa \sum_{i,j=1}^n \int_\Omega v_i \Delta v_j \frac{\partial (v - \varkappa \Delta v)_j}{\partial x_i} dx \\ &= - \sum_{i,j=1}^n \int_\Omega v_i (v - \varkappa \Delta v)_j \frac{\partial (v - \varkappa \Delta v)_j}{\partial x_i} dx \\ &= - \frac{1}{2} \int_\Omega \sum_{i,j=1}^n v_i \frac{\partial \left((v - \varkappa \Delta v)_j (v - \varkappa \Delta v)_j \right)}{\partial x_i} dx \end{aligned}$$

$$=\frac{1}{2}\int\limits_{\Omega}|v-\varkappa\Delta v|^2\mathrm{div}v dx=0.$$

By the definition of the operator $(J+\varepsilon e^{-\gamma s}A^3+\varkappa A)$ and the Green formula we obtain

$$\begin{aligned}\langle\big(J+\varepsilon e^{-\gamma s}A^3+\varkappa A\big)\,v',v+\varkappa Av\rangle=&\frac{1}{2}\frac{d}{ds}\|v(s)\|^2_{V^0}+\varkappa\frac{d}{ds}\|v(s)\|^2_{V^1}+\frac{\varkappa^2}{2}\frac{d}{ds}\|v(s)\|^2_{V^2}\\&+e^{-\gamma s}\frac{\varepsilon}{2}\frac{d}{ds}\|v(s)\|^2_{V^3}+e^{-\gamma s}\frac{\varepsilon\varkappa}{2}\frac{d}{ds}\|v(s)\|^2_{V^4}.\end{aligned}$$

For the next term we have

$$\langle\nu Av,v+\varkappa Av\rangle=\nu\int\limits_{\Omega}\nabla v:\nabla v dx+\nu\varkappa\int\limits_{\Omega}\Delta v\Delta v dx=\nu\big(\|v(s)\|^2_{V^1}ds+\varkappa\|v(s)\|^2_{V^2}\big).$$

The latter term in the left hand side can be estimated from above by the Hölder inequality

$$\begin{aligned}|\langle C(v),v+\varkappa Av\rangle|=&\left|\int\limits_0^s\sum_{i=1}^L\beta_ie^{-\alpha_i(s-\xi)}\int\limits_{\Omega}\Delta v(\xi,z(\xi;s,x))(v+\varkappa Av)(s)dxd\xi\right|\\\leqslant&\int\limits_0^s\sum_{i=1}^L|\beta_i|e^{-\alpha_i(s-\xi)}\left(\int\limits_{\Omega}|\Delta v(\xi,z(\xi;s,x))|^2\,dx\right)^{\frac12}\left(\int\limits_{\Omega}|(v+\varkappa Av)(s)|^2\,dx\right)^{\frac12}d\xi.\end{aligned}$$

We make the change $x=z(s;\xi,y)$ in the first factor. Then similarly to the proof of Lemma 5.2 we find

$$\begin{aligned}|\langle C(v),v+\varkappa Av\rangle|\leqslant&\int\limits_0^s\sum_{i=1}^L|\beta_i|e^{-\alpha_i(s-\xi)}\left(\int\limits_{\Omega}|\Delta v(\xi,y))|^2\,dy\right)^{\frac12}\|(v+\varkappa Av)(s)\|_{V^0}d\xi\\=&\sum_{i=1}^L|\beta_i|\int\limits_0^s e^{-\alpha_i(s-\xi)}\|v(\xi)\|_{V^2}d\xi\|(v+\varkappa Av)(s)\|_{V^0}.\end{aligned}$$

Thus, we get the inequality

$$\begin{aligned}&\frac{d}{ds}\|v(s)\|^2_{V^0}+2\varkappa\frac{d}{ds}\|v(s)\|^2_{V^1}+\varkappa^2\frac{d}{ds}\|v(s)\|^2_{V^2}+e^{-\gamma s}\varepsilon\frac{d}{ds}\|v(s)\|^2_{V^3}\\&\quad+e^{-\gamma s}\varepsilon\varkappa\frac{d}{ds}\|v(s)\|^2_{V^4}+2\nu\left(\|v(s)\|^2_{V^1}+\varkappa\|v(s)\|^2_{V^2}\right)\\&\leqslant2\sum_{i=1}^L|\beta_i|\int\limits_0^s e^{-\alpha_i(s-\xi)}\|v(\xi)\|_{V^2}d\xi\|(v+\varkappa Av)(s)\|_{V^0}+2\langle f,(v+\varkappa Av)(s)\rangle.\end{aligned}$$

On the space V^2 we consider the auxiliary norm

$$\|u\|^2=\|u\|^2_{V^0}+2\varkappa\|u\|^2_{V^1}+\varkappa^2\|u\|^2_{V^2},$$

which is equivalent to the norm $\|\cdot\|_{V^2}$; indeed, the estimates (5.1) imply the estimates

$$\varkappa^2\|u\|^2_{V^2}\leqslant\|u\|^2\leqslant(K_1^2+2\varkappa K_1+\varkappa^2)\|u\|^2_{V^2}.\tag{6.3}$$

Then by (5.3) we have

$$\begin{aligned}\nu(\|u(s)\|^2_{V^1}+\varkappa\|u(s)\|^2_{V^2})&\geqslant\nu\varkappa\|u(s)\|^2_{V^2}\\&\geqslant\frac{\nu\varkappa}{(K_1^2+2\varkappa K_1+\varkappa^2)}\|u(s)\|^2=K_2\|u(s)\|^2\geqslant\gamma\|u(s)\|^2.\end{aligned}\tag{6.4}$$

This yields

$$\frac{d}{ds}\|v(s)\|^2 + e^{-\gamma s}\varepsilon\frac{d}{ds}\|v(s)\|^2_{V^3} + e^{-\gamma s}\varepsilon\varkappa\frac{d}{ds}\|v(s)\|^2_{V^4} + 2K_2\|v(s)\|^2$$
$$\leqslant \frac{2}{\varkappa}\sum_{i=1}^{L}|\beta_i|\int_0^s e^{-\alpha_i(s-\xi)}\|v(\xi)\|d\xi\|v(s)\| + 2\langle f, (v+\varkappa Av)(s)\rangle.$$

We estimate the latter term in the right hand side

$$\langle f, (v+\varkappa Av)(s)\rangle \leqslant \|f\|_{V^0}\|(v+\varkappa Av)(s)\|_{V^0} \leqslant \frac{\|f\|^2_{V^0}}{2K_2(1-\lambda_1-\lambda_2)} + \frac{K_2(1-\lambda_1-\lambda_2)}{2}\|v(s)\|^2,$$

where λ_1, λ_2 are some constants, $\lambda_1 > 0$, $\lambda_2 > 0$, $0 < \lambda_1 + \lambda_2 < 1$.

Thus,

$$\frac{d}{ds}\|v(s)\|^2 + e^{-\gamma s}\varepsilon\frac{d}{ds}\|v(s)\|^2_{V^3} + e^{-\gamma s}\varepsilon\varkappa\frac{d}{ds}\|v(s)\|^2_{V^4} + K_2\|v(s)\|^2 + K_2\lambda_1\|v(s)\|^2$$
$$+ K_2\lambda_2\|v(s)\|^2 - \frac{2}{\varkappa}\sum_{i=1}^{L}|\beta_i|\int_0^s e^{-\alpha_i(s-\xi)}\|v(\xi)\|d\xi\|v(s)\| \leqslant \frac{\|f\|^2_{V^0}}{K_2(1-\lambda_1-\lambda_2)}.$$

Estimating the left hand side by means of (6.4) and denoting

$$G(s) = K_2\lambda_2\|v(s)\|^2 - \frac{2}{\varkappa}\sum_{i=1}^{L}|\beta_i|\int_0^s e^{-\alpha_i(s-\xi)}\|v(\xi)\|d\xi\|v(s)\|;$$
$$F = \frac{\|f\|^2_{V^0}}{K_2(1-\lambda_1-\lambda_2)},$$

we get

$$\frac{d}{ds}\|v(s)\|^2 + \gamma\|v(s)\|^2 + e^{-\gamma s}\varepsilon\frac{d}{ds}\|v(s)\|^2_{V^3} + e^{-\gamma s}\varepsilon\varkappa\frac{d}{ds}\|v(s)\|^2_{V^4} + K_2\lambda_1\|v(s)\|^2 + G(s) \leqslant F.$$

In the first and second term in the left hand side of the inequality we make the change $v(s) = e^{-\frac{\gamma s}{2}}\bar{v}(s)$. We obtain

$$\frac{d}{ds}\|e^{-\frac{\gamma s}{2}}\bar{v}(s)\|^2 + \gamma e^{-\gamma s}\|\bar{v}(s)\|^2 + e^{-\gamma s}\varepsilon\frac{d}{ds}\|v(s)\|^2_{V^3}$$
$$+ e^{-\gamma s}\varepsilon\varkappa\frac{d}{ds}\|v(s)\|^2_{V^4} + K_2\lambda_1\|v(s)\|^2 + G(s) \leqslant F.$$

Therefore,

$$-\gamma e^{-\gamma s}\|\bar{v}(s)\|^2 + e^{-\gamma s}\frac{d}{ds}\|\bar{v}(s)\|^2 + \gamma e^{-\gamma s}\|\bar{v}(s)\|^2 + e^{-\gamma s}\varepsilon\frac{d}{ds}\|v(s)\|^2_{V^3}$$
$$+ e^{-\gamma s}\varepsilon\varkappa\frac{d}{dt}\|v(s)\|^2_{V^4} + K_2\lambda_1\|v(s)\|^2 + G(s) \leqslant F.$$

We multiply the latter inequality by $e^{\gamma s}$

$$\frac{d}{ds}\|\bar{v}(s)\|^2 + \varepsilon\frac{d}{ds}\|v(s)\|^2_{V^3} + \varepsilon\varkappa\frac{d}{ds}\|v(s)\|^2_{V^4} + e^{\gamma s}K_2\lambda_1\|v(s)\|^2 + e^{\gamma s}G(s) \leqslant e^{\gamma s}F.$$

We integrate this inequality in s from 0 to t, $t \in [0, T]$, and estimate the right hand side from above

$$\|\bar{v}(t)\|^2 + \varepsilon\|v(t)\|^2_{V^3} + \varepsilon\varkappa\|v(t)\|^2_{V^4} + K_2\lambda_1\int_0^t e^{\gamma\tau}\|v(\tau)\|^2d\tau + \int_0^t e^{\gamma\tau}G(\tau)d\tau$$

$$\leqslant \int_0^t e^{\gamma\tau} F d\tau + \|\bar{v}(0)\|^2 + \varepsilon\|v(0)\|_{V^3}^2 + \varepsilon\varkappa\|v(0)\|_{V^4}^2$$
$$= \frac{F}{\gamma}(e^{\gamma t} - 1) + \|\bar{v}(0)\|^2 + \varepsilon\|v(0)\|_{V^3}^2 + \varepsilon\varkappa\|v(0)\|_{V^4}^2$$
$$\leqslant e^{\gamma t}\frac{F}{\gamma} + \|\bar{v}(0)\|^2 + \varepsilon\|v(0)\|_{V^3}^2 + \varepsilon\varkappa\|v(0)\|_{V^4}^2.$$

Multiplying by $e^{-\gamma t}$, we find

$$e^{-\gamma t}\|\bar{v}(t)\|^2 + e^{-\gamma t}\varepsilon\|v(t)\|_{V^3}^2 + e^{-\gamma t}\varepsilon\varkappa\|v(t)\|_{V^4}^2 + K_2\lambda_1 \int_0^t e^{-\gamma(t-\tau)}\|v(\tau)\|^2 d\tau$$
$$+ \int_0^t e^{-\gamma(t-\tau)} G(\tau) d\tau \leqslant \frac{F}{\gamma} + e^{-\gamma t}\left(\|\bar{v}(0)\|^2 + \varepsilon\|v(0)\|_{V^3}^2 + \varepsilon\varkappa\|v(0)\|_{V^4}^2\right). \tag{6.5}$$

We are going to establish the non–negativity of the latter term in the left hand side. Recalling the above introduced notations, we have

$$\int_0^t e^{-\gamma(t-\tau)} G(\tau) d\tau = \int_0^t e^{-\gamma(t-\tau)} \left(K_2\lambda_2\|v(\tau)\|^2 - \frac{2}{\varkappa}\sum_{i=1}^L |\beta_i| \int_0^\tau e^{-\alpha_i(\tau-s)}\|v(s)\|ds\|v(\tau)\| \right) d\tau.$$

We introduce the auxiliary functions

$$h(\tau) = \|v(\tau)\|; \qquad g_i(\tau) = \int_0^\tau e^{-\alpha_i(\tau-s)}\|v(s)\|ds = \int_0^\tau e^{-\alpha_i(\tau-s)} h(s) ds, \quad i = \overline{1, L}.$$

The function h is continuous on the segment $[0, T]$, while the functions g_i, $i = \overline{1, L}$, are continuously differentiable on this segment. The straightforward calculations give

$$g_i'(\tau) = h(\tau) - \alpha_i \int_0^\tau e^{-\alpha_i(\tau-s)} h(s) ds = h(\tau) - \alpha_i g_i(\tau), \quad i = \overline{1, L}.$$

Therefore,

$$g_i'(\tau) + \alpha_i g_i(\tau) = h(\tau); \quad g_i(0) = 0, \quad i = \overline{1, L}.$$

Then we have

$$G(\tau) = K_2\lambda_2 h^2(\tau) - \frac{2}{\varkappa} h(\tau) \sum_{i=1}^L |\beta_i| g_i(\tau)$$
$$= \sum_{i=1}^L \left(\frac{K_2\lambda_2}{L}\left(g_i'(\tau) + \alpha_i g_i(\tau)\right)^2 - \frac{2|\beta_i|}{\varkappa}\left(g_i'(\tau) + \alpha_i g_i(\tau)\right) g_i(\tau) \right)$$
$$= \sum_{i=1}^L \left(\frac{K_2\lambda_2}{L}(g_i'(\tau))^2 + \left(\frac{2\alpha_i K_2\lambda_2}{L} - \frac{2|\beta_i|}{\varkappa} \right) g_i'(\tau) g_i(\tau) + \left(\frac{\alpha_i^2 K_2\lambda_2}{L} - \frac{2\alpha_i|\beta_i|}{\varkappa} \right) g_i^2(\tau) \right).$$

Integrating by parts for each $i = \overline{1, L}$ we get

$$2\int_0^t e^{-\gamma(t-\tau)} g_i'(\tau) g_i(\tau) d\tau = e^{-\gamma t}\left(e^{\gamma\tau} g_i^2(\tau)\right)\Big|_0^t - \gamma e^{-\gamma t} \int_0^t e^{\gamma\tau} g_i^2(\tau) d\tau$$

$$= g_i^2(t) - \gamma \int\limits_0^t e^{-\gamma(t-\tau)} g_i^2(\tau) d\tau.$$

This implies

$$\int\limits_0^t e^{-\gamma(t-\tau)} G(\tau) d\tau = \sum_{i=1}^{L} \Big(\frac{K_2\lambda_2}{L} \int\limits_0^t e^{-\gamma(t-\tau)} (g_i'(\tau))^2 d\tau + \left(\frac{\alpha_i K_2 \lambda_2}{L} - \frac{|\beta_i|}{\varkappa} \right) g_i^2(\tau)$$
$$+ \left(\alpha_i \left(\frac{\alpha_i K_2 \lambda_2}{L} - \frac{2|\beta_i|}{\varkappa} \right) - \gamma \left(\frac{\alpha_i K_2 \lambda_2}{L} - \frac{|\beta_i|}{\varkappa} \right) \right) \int\limits_0^t e^{-\gamma(t-\tau)} g_i^2(\tau) d\tau \Big).$$

Let us show that for each $i = \overline{1, L}$ under the conditions and for an appropriate choice of the positive number μ_i the expression

$$\frac{K_2\lambda_2}{L} \int\limits_0^t e^{-\gamma(t-\tau)} (g_i'(\tau))^2 d\tau + \left(\frac{\alpha_i K_2 \lambda_2}{L} - \frac{|\beta_i|}{\varkappa} \right) g_i^2(\tau)$$
$$+ \left(\alpha_i \left(\frac{\alpha_i K_2 \lambda_2}{L} - \frac{2|\beta_i|}{\varkappa} \right) - \mu_i \left(\frac{\alpha_i K_2 \lambda_2}{L} - \frac{|\beta_i|}{\varkappa} \right) \right) \int\limits_0^t e^{-\gamma(t-\tau)} g_i^2(\tau) d\tau \Big)$$

is non–negative. For the first term in the right hand side we have

$$\frac{K_2\lambda_2}{L} \int\limits_0^t e^{-\gamma(t-\tau)} (g_i'(\tau))^2 d\tau \geqslant 0.$$

By (6.1) we have

$$\frac{K_2\alpha_i}{L} > \frac{2|\beta_i|}{\varkappa}, \qquad i = \overline{1, L},$$

and this is why we can choose λ_2, probably rather close to 1 such that

$$\frac{\alpha_i K_2 \lambda_2}{L} - \frac{2|\beta_i|}{\varkappa} > 0 \quad \text{for all} \quad i = \overline{1, L}. \tag{6.6}$$

Hence,

$$\left(\frac{\alpha_i K_2 \lambda_2}{L} - \frac{|\beta_i|}{\varkappa} \right) g_i^2(\tau) \geqslant 0.$$

Since

$$\frac{\alpha_i K_2 \lambda_2}{L} - \frac{2|\beta_i|}{\varkappa} > 0, \qquad \frac{\alpha_i K_2 \lambda_2}{L} - \frac{|\beta_i|}{\varkappa} > 0,$$

we can choose μ_i such that $0 < \mu_i \leqslant \gamma$ and

$$\left(\alpha_i \left(\frac{\alpha_i K_2 \lambda_2}{L} - \frac{2|\beta_i|}{\varkappa} \right) - \mu_i \left(\frac{\alpha_i K_2 \lambda_2}{L} - \frac{|\beta_i|}{\varkappa} \right) \right) g_i^2(\tau) \geqslant 0.$$

This implies

$$\int\limits_0^t e^{-\gamma(t-\tau)} G(\tau) d\tau \geqslant 0.$$

Thus, using the non–negativity of the terms, by the estimate (6.5) we get

$$e^{-\gamma t} \|\bar{v}(t)\|^2 + K_2\lambda_1 \int\limits_0^t e^{-\gamma(t-s)} \|v(s)\|^2 ds \leqslant \frac{F}{\gamma} + e^{-\gamma t} \left(\|\bar{v}(0)\|^2 + \varepsilon \|v(0)\|_{V^3}^2 + \varepsilon\varkappa \|v(0)\|_{V^4}^2 \right).$$

Making the inverse change $\bar{v}(t) = e^{\gamma t/2}v(t)$ and recalling the definition of the auxiliary norm, we get the needed estimate (6.2). □

Theorem 6.2. *Let v be the solution of Equation* (5.9) *on the segment* $[0,T]$, $T > 0$, *and the coefficients $\varkappa$, ν, α_i, β_i, $i = \overline{1,L}$, satisfy the conditions* (6.1). *Then the estimates*

$$\varepsilon e^{-\gamma t}\|v'(t)\|_{V^5} \leqslant C_8 + C_9 e^{-\gamma t}\left(C_4\|v(0)\|^2_{V^2} + \varepsilon\|v(0)\|^2_{V^3} + \varepsilon\varkappa\|v(0)\|^2_{V^4}\right); \quad (6.7)$$

$$\|v'(t)\|_{V^1} \leqslant C_{10} + C_{11} e^{-\gamma t}\left(C_4\|v(0)\|^2_{V^2} + \varepsilon\|v(0)\|^2_{V^3} + \varepsilon\varkappa\|v(0)\|^2_{V^4}\right); \quad (6.8)$$

$$e^{-\gamma t}\|v(t)\|_{V^5} \leqslant e^{-\gamma t}\|v(0)\|_{V^5} + \frac{1}{\varepsilon\gamma}\left(C_8 + C_9\left(C_4\|v(0)\|^2_{V^2} + \varepsilon\|v(0)\|^2_{V^3} + \varepsilon\varkappa\|v(0)\|^2_{V^4}\right)\right) \quad (6.9)$$

hold. The constants C_8, C_9, C_{10} and C_{11} depend on f, γ, $\varkappa$, ν, α_i, β_i, $i = \overline{1,L}$, and are independent of ε.

Proof. Since v is the solution of Equation (5.9), for almost all $t \in (0,T)$ we have the equality

$$\|(J + \varepsilon e^{-\gamma t}A^3 + \varkappa A)v'(t)\|_{V^{-1}} = \| - \nu Av(t) + B_1(v)(t) - \varkappa B_2(v)(t) + C(v)(t) + f\|_{V^{-1}}.$$

Then by the estimates (5.10), (5.13), (5.14), (5.15), the continuity of embedding $V^2 \subset V^1$, $V^0 \subset V^{-1}$, the elementary inequality $a \leqslant 1 + a^2$ and the above obtained estimate (6.2) we have

$$\begin{aligned}\|(J + \varepsilon e^{-\gamma t}A^3 + \varkappa A)v'\|_{V^{-1}} \leqslant & \nu\|v(t)\|_{V^1} + C_1\|v(t)\|^2_{V^1} + \varkappa C_2\|v(t)\|^2_{V^2} \\ & + C_3\left(\int_0^t e^{-\gamma(t-s)}\|g(s)\|^2_{V^2}dt\right)^{\frac{1}{2}} + \|f\|_{V^{-1}} \\ \leqslant & C_6\|v(t)\|^2_{V^2} + C_3\left(\int_0^t e^{-\gamma(t-s)}\|g(s)\|^2_{V^2}dt\right)^{\frac{1}{2}} + C_7\|f\|_{V^0} \\ \leqslant & C_8 + C_9 e^{-\gamma t}\left(C_4\|v(0)\|^2_{V^2} + \varepsilon\|v(0)\|^2_{V^3} + \varepsilon\varkappa\|v(0)\|^2_{V^4}\right).\end{aligned}$$

By the inequality (5.12) this gives the desired estimate (6.7).

The estimate (6.8) can be obtained in a similar way. Namely, since v solves Equation (5.9), using the estimate (5.12) for almost all $t \in (0,T)$ we have

$$\begin{aligned}\|(J + \varkappa A)v'(t)\|_{V^{-1}} = & \| - \varepsilon e^{-\gamma t}A^3v'(t) - \nu Av(t) + B_1(v)(t) - \varkappa B_2(v)(t) + C(v)(t) + f\|_{V^{-1}} \\ \leqslant & \| - \varepsilon e^{-\gamma t}A^3v'(t)\|_{V^{-1}} \\ & + \| - \nu Av(t) + B_1(v)(t) - \varkappa B_2(v)(t) + C(v)(t) + f\|_{V^{-1}} \\ \leqslant & 2\| - \nu Av(t) + B_1(v)(t) - \varkappa B_2(v)(t) + C(v)(t) + f\|_{V^{-1}} \\ \leqslant & 2C_8 + 2C_9 e^{-\gamma t}\left(C_4\|v(0)\|^2_{V^2} + \varepsilon\|v(0)\|^2_{V^3} + \varepsilon\varkappa\|v(0)\|^2_{V^4}\right).\end{aligned}$$

As above, by the estimate (6.7) and (5.11) we obtain the desired estimate (6.8).

In order to obtain the estimate (6.9), we observe that for all $t \in [0,T]$ the identity

$$v(t) = v(0) + \int_0^t v'(s)ds$$

holds. We multiply both sides of the identity by $e^{-\gamma t}$ and by (6.7) we find

$$\|e^{-\gamma t}v(t)\|_{V^5} = \left\|e^{-\gamma t}\left(v(0) + \int_0^t e^{-\gamma s}e^{\gamma s}v'(s)ds\right)\right\|_{V^5}$$

$$
\begin{aligned}
&\leqslant e^{-\gamma t}\|v(0)\|_{V^5} + \int_0^t e^{-\gamma(t-s)} e^{-\gamma s}\|v'(s)\|_{V^5} ds \\
&\leqslant e^{-\gamma t}\|v(0)\|_{V^5} \\
&\quad + \frac{1}{\varepsilon}\int_0^t e^{-\gamma(t-s)}\left(C_8 + C_9 e^{-\gamma s}\left(C_4\|v(0)\|^2_{V^2} + \varepsilon\|v(0)\|^2_{V^3} + \varepsilon\varkappa\|v(0)\|^2_{V^4}\right)\right) ds \\
&\leqslant e^{-\gamma t}\|v(0)\|_{V^5} + \frac{C_8}{\varepsilon\gamma}(1-e^{-\gamma t}) \\
&\quad + \frac{C_9}{\varepsilon\gamma}(1-e^{-\gamma t})\left(C_4\|v(0)\|^2_{V^2} + \varepsilon\|v(0)\|^2_{V^3} + \varepsilon\varkappa\|v(0)\|^2_{V^4}\right) \\
&\leqslant e^{-\gamma t}\|v(0)\|_{V^5} \\
&\quad + \frac{1}{\varepsilon\gamma}\left(C_8 + C_9\left(C_4\|v(0)\|^2_{V^2} + \varepsilon\|v(0)\|^2_{V^3} + \varepsilon\varkappa\|v(0)\|^2_{V^4}\right)\right).
\end{aligned}
$$

This yield the desired estimate (6.9). The proof is complete. □

As a direct corollary of the obtained inequality we have the next lemma.

Lemma 6.1. *Let v be the solution of Equation* (5.9) *on the segment* $[0, T]$, $T > 0$, *and the coefficients* $\varkappa$, ν, α_i, β_i, $i = \overline{1, L}$, *obey the conditions* (6.1). *Then for almost all* $t \in (0, T)$ *the estimates*

$$\|v(t)\|_{V^2} + \|v'(t)\|_{V^1} \leqslant C_{12} + C_{13}e^{-\gamma t}\left(C_4\|v(0)\|^2_{V^2} + \varepsilon\|v(0)\|^2_{V^3} + \varepsilon\varkappa\|v(0)\|^2_{V^4}\right); \tag{6.10}$$

$$e^{-\gamma t}(\|v(t)\|_{V^5} + \|v'(t)\|_{V^5}) \leqslant C_{14}, \tag{6.11}$$

hold, where the constant C_{14} *is independent of* ε, *and the constants* C_{12} *and* C_{13} *depend on* f, γ, $\varkappa$, ν, α_i, β_i, $i = \overline{1, L}$, *and are independent of* ε.

7. Existence theorems for solutions to approximation problem

The following theorems on existence of solutions to approximation and original problems on the segment.

Theorem 7.1. *Let the coefficients* $\varkappa$, ν, α_i, β_i, $i = \overline{1, L}$, *obey the conditions* (6.1). *Then for each segment* $[0, T]$ *there exists a solution of Equation* (5.9) *obeying the initial condition* (5.8), *and this solution satisfies the estimates* (6.10), (6.11).

Proof. The proof follows the lines of proof of Theorem 7 in [9] and because of its large volume we provide it schematically. First for a fixed function u belonging to the space $C([0, T], V^3)$ and obeying the inequality $\|u\|_{C([0,T],V^3)} \leqslant M$ (here M is a constant, the exact value of which is given below), we prove the existence of the unique solution Z_u to the Cauchy problem

$$z(\tau; t, x) = x + \int_t^\tau u(s, z(s; t, x)) ds.$$

Then for Z_u and the same function u we prove the existence of a function $w \in W_2[0, T]$ obeying the integral identity

$$\int_\Omega w'\varphi dx - \xi\int_\Omega \sum_{i,j=1}^n u_i w_j \frac{\partial \varphi_j}{\partial x_i} dx + \xi\nu\int_\Omega \nabla w : \nabla\varphi dx$$

$$+\varkappa\int\limits_\Omega \nabla w' : \nabla\varphi dx + \varepsilon\int\limits_\Omega \nabla(\Delta^2 w') : \nabla\varphi dx + \xi\varkappa\int\limits_\Omega \sum_{i,j=1}^n u_i\Delta w_j \frac{\partial\varphi_j}{\partial x_i}dx$$

$$-\xi\int\limits_0^t\sum_{i=1}^L \beta_i e^{-\alpha_i(t-s)}\int\limits_\Omega \Delta w(s, Z_u(s;t,x))\varphi dxds = \xi\int\limits_\Omega f\varphi dx$$

for each test function $\varphi \in V^1$ for almost all $t \in (0,T)$ and satisfying the initial condition

$$w(0) = \xi b, \qquad \xi \in [0,1].$$

Thus, we obtain the family of mappings Ψ, which maps the number $\xi \in [0,1]$ and the function $u \in C([0,T],V^3)$ into the function $w \in W_2[0,T]$. After that we directly establish the continuity of this mapping $\Psi : [0,1]\times\overline{B_M} \to W_2[0,T]$ with respect to its variables. Here B_M is the ball of radius M centered at zero in the space $C([0,T],V^3)$. After that by means of Theorem 3.1 we prove the compactness of the mapping $\Psi : [0,1]\times\overline{B_M} \to C([0,T],V^3)$.

By the estimates (6.7) and (6.9) for fixed points of Ψ we have the inequality

$$\|v\|_{W_2[0,T]} = \|v\|_{C([0,T],V^5)} + \|v'\|_{L_\infty(0,T;V^5)} \leqslant M_1. \tag{7.1}$$

Then by the continuity of the embedding $W_2[0,T] \subset C([0,T],V^3)$ we have the inequality

$$\|v\|_{C([0,T],V^3)} \leqslant M_2\|v\|_{W_2[0,T]}.$$

This directly implies that the fixed points of the mapping Ψ satisfy the inequality

$$\|v\|_{C([0,T],V^3)} \leqslant M_1M_2.$$

Letting $M = M_1M_2 + 1$, we conclude that the mapping Ψ has no fixed points at the boundary of the ball B_M, which is centered at zero.

Since $\Psi(0,\cdot) \equiv 0$ and $0 \in B_M$, all assumptions of the Leray — Schauder theorem are satisfied, see Theorem 3.2. Therefore, the mapping $\Psi(1,\,\cdot\,)$ has at least one fixed point. This is why there exists at least one solution to the approximation problem (5.4)–(5.6). Namely, there exists a solution $v \in W_2[0,T]$ of the operator equation (5.9) satisfying the initial condition (5.8). And by (6.10) and (6.11) this solution obeys the desired inequalities. The proof is complete. □

We note that it is possible to prove the solvability of the approximation problem on an arbitrary finite segment without the conditions (6.1) for the coefficients. But in the general situation the solution does not necessarily satisfy the inequality (6.10), which is needed to use the attractor theory.

In what follows we shall employ the next technical lemma.

Lemma 7.1. *Let a sequence $\{v_m\}$ be bounded in $L_\infty(0,T;V^2)$, and a sequence $\{v'_m\}$ be bounded in $L_\infty(0,T;V^1)$. Then the following statements are true.*

1) *There exists a subsequence $\{v_{m_k}\}$ converging strongly to a limiting function v_* in the space $C([0,T];V^1)$ and the limiting relations*

$$Jv'_{m_k} \rightharpoonup Jv'_* \quad \text{weakly in} \quad L_2(0,T;V^{-1}); \tag{7.2}$$

$$Av'_{m_k} \rightharpoonup Av'_* \quad \text{weakly in} \quad L_2(0,T;V^{-1}); \tag{7.3}$$

$$Av_{m_k} \rightharpoonup Av_* \quad \text{weakly in} \quad L_2(0,T;V^{-1}); \tag{7.4}$$

$$B_1(v_{m_k}) \to B_1(v_*) \quad \text{strongly in} \quad L_\infty(0,T;V^{-1}); \tag{7.5}$$

$$B_2(v_{m_k}) \rightharpoonup B_2(v_*) \quad \text{weakly in} \quad L_2(0,T;V^{-1}); \tag{7.6}$$

$$C(v_{m_k}) \rightharpoonup C(v_*) \quad \text{weakly in} \quad L_2(0,T;V^{-1}) \tag{7.7}$$

hold.

2) *Let $\varepsilon_m \to 0$ be a scalar sequence and the sequence $\{\varepsilon_m v'_m\}$ be bounded in the space $L_\infty(0,T,V^5)$, then without loss of generality $\varepsilon_{m_k} e^{-\gamma t} A^3 v'_{m_k} \rightharpoonup 0$ weakly in $L_2(0,T;V^{-1})$.*
3) *Let the sequence $\{v_m\}$ be bounded in $L_\infty(0,T;V^5)$, then without loss of generality*

$$(J+\varkappa A+\varepsilon e^{-\gamma t}A^3)v'_{m_k} \rightharpoonup (J+\varkappa A+\varepsilon e^{-\gamma t}A^3)v'_*$$

weakly in $L_2(0,T;V^{-1})$.

Proof. 1) The embedding $V^2 \subset V^1$ is compact and this is why the assumptions of Theorem 3.2 are satisfied and the embedding $W_1[0,T] \subset C([0,T],V^1)$ is compact. Since $\{v_m\}$ is bounded in $W_1[0,T]$, it is relatively compact in $C([0,T],V^1)$ and there exists a subsequence $\{v_{m_k}\}$ strongly converging in $C([0,T],V^1)$ to some function v_*.

We pass from non–reflexive spaces L_∞ to the reflexive spaces L_p, $1<p<\infty$, in order to use the weak compactness of bounded sets. Since the space L_∞ is continuously embedded into L_p, the sequences $\{v_m\}$ and $\{v'_m\}$ are bounded in $L_2(0,T;V^2)$ and $L_2(0,T;V^1)$, respectively. Therefore, without loss of generality we can suppose that

$$v_{m_k} \rightharpoonup v_* \quad \text{weakly in} \quad L_2(0,T;V^2), \tag{7.8}$$

$$v'_{m_k} \rightharpoonup v'_* \quad \text{weakly in} \quad L_2(0,T;V^1). \tag{7.9}$$

Thus, the convergence (7.9) implies (7.2). By Lemma 5.1 the linear operator A is continuous. This is why by (7.8) and (7.9) we respectively get the desired convergences (7.4) and (7.3).

Since $V^1 \subset L_4(\Omega)^n$, the strong convergence $v_{m_k} \to v_*$ in $C([0,T],L_4(\Omega)^n)$ and (7.5) are implied by the continuity of the operator B_1.

By Theorem 3.1 due to the compactness of the embedding $V^2 \subset C(\overline{\Omega})^n$ for $n=2,3$ we have the compact embedding $W_1[0,T] \subset C([0,T],C(\overline{\Omega})^n)$. Without loss of generality this yields the strong convergence $v_{m_k} \to v_*$ in $C([0,T],C(\overline{\Omega})^n)$. Together with (7.8) this yields the weak convergence $v_{m_k}\Delta v_{m_k} \rightharpoonup v_*\Delta v_*$ in $L_2(0,T;L_2(\Omega)^n)$. This is why (7.6) is implied by the definition of the operator B_2.

To prove the weak convergence (7.7), we are going to show that for each $i=\overline{1,L}$ in the space $L_2(0,T;L_2(\Omega)^n)$ the weak convergence

$$\int_0^t e^{-\alpha_i(t-s)}\Delta v_{m_k}(s,z_{m_k}(s;t,x))ds \rightharpoonup \int_0^t e^{-\alpha_i(t-s)}\Delta v_*(s,z_*(s;t,x))ds \tag{7.10}$$

holds. Here z_{m_k} and z_* are regular Lagrangian flows corresponding to v_{m_k} and v_*, respectively.

By the Hölder inequality

$$\left\| \int_0^t e^{-\alpha_i(t-s)}\Delta v_{m_k}(s,z_{m_k}(s;t,x))ds \right\|^2_{L_2(0,T;L_2(\Omega)^n)}$$
$$= \int_0^T \int_\Omega \left| \int_0^t e^{-\alpha_i(t-s)}\Delta v_{m_k}(s,z_{m_k}(s;t,x))ds \right|^2 dxdt$$
$$\leqslant \int_0^T \int_\Omega \left(\int_0^t |\Delta v_{m_k}(s,z_{m_k}(s;t,x))|\, ds \right)^2 dxdt$$
$$\leqslant \int_0^T \int_\Omega \left(\sqrt{t} \left(\int_0^t |\Delta v_{m_k}(s,z_{m_k}(s;t,x))|^2\, ds \right)^{\frac12} \right)^2 dxdt$$

$$\leqslant T\int_0^T\int_0^t\int_\Omega |\Delta v_{m_k}(s,z_{m_k}(s;t,x))|^2\,dxdsdt.$$

Similarly to the proof of Lemma 5.2, by making the change $x=z_{m_k}(t;s,y)$ in the latter integral we obtain

$$\left\|\int_0^t e^{-\alpha_i(t-s)}\Delta v_{m_k}(s,z_{m_k}(s;t,x))ds\right\|^2_{L_2(0,T;L_2(\Omega)^n)}\leqslant T^2\|v_{m_k}\|^2_{L_2(0,T;V^2)}.$$

The boundedness of $\{v_{m_k}\}$ in the space $L_\infty(0,T;V^2)$ implies the boundedness

$$\int_0^t e^{-\alpha_i(t-s)}\Delta v_{m_k}(s,z_{m_k}(s;t,x))ds$$

in $L_2(0,T;L_2(\Omega)^n)$.

Therefore, without loss of generality, there exists $w\in L_2(0,T;L_2(\Omega)^n)$ such that

$$\int_0^t e^{-\alpha_i(t-s)}\Delta v_{m_k}(s,z_{m_k}(s;t,x))ds$$

converges weakly to w in the space $L_2(0,T;L_2(\Omega)^n)$ as $m_k\to\infty$. But in the sense of distributions this sequence converges to

$$\int_0^t e^{-\alpha_i(t-s)}\Delta v_*(s,z_*(s;t,x))ds.$$

Indeed, for each $\varphi\in\mathcal{V}$, $\chi\in C_0^\infty(0,T)$ we make the change of variable $x=z_{m_k}(t;s,y)$ and interchange the integration order; this gives

$$\int_0^T\int_\Omega\int_0^t e^{-\alpha_i(t-s)}\Delta v_{m_k}(s,z_{m_k}(s;t,x))ds\varphi(x)dx\chi(t)dt$$
$$=\int_0^T\int_\Omega\int_0^t e^{-\alpha_i(t-s)}\Delta v_{m_k}(s,y)ds\varphi(z_{m_k}(t;s,y))dyds\chi(t)dt$$
$$=\int_0^T\int_\Omega \Delta v_{m_k}(s,y)\int_s^T e^{-\alpha_i(t-s)}\varphi(z_{m_k}(t;s,y))\chi(t)dtdyds$$
$$=\int_0^T\int_\Omega \Delta v^m(s,y)H^m(s,y)dyds,$$

where

$$H_{m_k}(s,y)=\int_s^T e^{-\alpha_i(t-s)}\varphi(z_{m_k}(t;s,y))\chi(t)dt.$$

By Theorem 3.4, the sequence z_{m_k} converges to z_* in the Lebesgue measure on $[0,T]\times\Omega$ uniformly in $t\in[0,T]$. By the smoothness the function $\varphi(z_{m_k}(t;s,y))$ converges to the function $\varphi(z_*(t;s,y))$ almost everywhere on Q_T as $m_k\to\infty$. By the Lebesgue theorem on the dominated convergence the uniformly bounded sequence $H_{m_k}(s,y)$ converges almost everywhere on Q_T to a bounded function

$$H(s,y) = \int_s^T e^{-\alpha_i(t-s)}\varphi(z_*(s;t,y))\chi(t)dt.$$

We thus obtain

$$\int_0^T\int_\Omega \Delta v_{m_k}(s,y)H_{m_k}(s,y)dyds \to \int_0^T\int_\Omega \Delta v_*(s,y)H(s,y)dyds$$

as $m_k \to \infty$. Here the first factor converges weakly in $L_2(Q_T)^n$, while the second factor converges almost everywhere on Q_T. In the obtained integral we interchange the integration order and make the change $y = z_*(s;t,x)$

$$\int_0^T\int_\Omega \Delta v_*(s,y)H(s,y)dyds = \int_0^T\int_\Omega \Delta v_*(s,y)\int_s^T e^{-\alpha_i(t-s)}\varphi(z_*(t;s,y))\chi(t)dtdyds$$

$$= \int_0^T\int_\Omega\int_0^t e^{-\alpha_i(t-s)}\Delta v_*(s,y)ds\varphi(z_*(t;s,y))dsdy\chi(t)dt$$

$$= \int_0^T\int_\Omega\int_0^t e^{-\alpha_i(t-s)}\Delta v_*(z_*(s;t,x))ds\varphi(t,x)dx\chi(t)dt.$$

By the uniqueness of the limit

$$w = \int_0^t e^{-\alpha_i(t-s)}\Delta v_*(z_*(s;t,x))ds,$$

and the weak convergence (7.10) holds. The required weak convergence (7.7) follows from (7.10) and the definition of the operator C.

2) Without loss of generality, the sequence $\{\varepsilon_{m_k}e^{-\gamma t}v'_{m_k}\}$ converges weakly to some function w in $L_2(0,T;V^5)$. But in the sense of distributions on the segment $[0,T]$ with the values in V^{-5} the sequence $\varepsilon_{m_k}e^{-\gamma t}A^3v'_{m_k}$ converges to zero.

Indeed, for all $\chi \in C_0^\infty(0,T)$, $\varphi \in V^5$, using the Green formula and the weak convergence (7.9), we obtain

$$\lim_{m_k\to\infty}\left|\varepsilon_{m_k}\int_0^T\int_\Omega \nabla\left(\Delta^2 v'_{m_k}\right) : \nabla\varphi dx\chi(t)dt\right|$$

$$= \lim_{m_k\to\infty}\varepsilon_{m_k}\lim_{m_k\to\infty}\left|\int_0^T\int_\Omega \nabla v'_{m_k}(t) : \nabla\left(\Delta^2\varphi\right)\,dx\chi(t)dt\right|$$

$$= \left|\int_0^T\int_\Omega \nabla v'_*(t) : \nabla\left(\Delta^2\varphi\right)\,dx\chi(t)dt\right|\lim_{m_k\to\infty}\varepsilon_{m_k} = 0.$$

By the uniqueness of the weak limit

$$\varepsilon_{m_k}\int_\Omega \nabla\left(\Delta^2 v'_{m_k}\right) : \nabla\varphi dx \to 0$$

as $m_k \to +\infty$.

3) Without loss of generality, $\{v'_{m_k}\}$ converges to v'_* weakly in $L_2(0,T,V^5)$. This is why the desired convergence is implied by the continuity of the linear operator $J + \varkappa A + \varepsilon e^{-\gamma t} A^3$. The proof is complete. □

Theorem 7.2. *Let the coefficients* $\varkappa$, ν, α_i, β_i, $i = \overline{1,L}$, *satisfy the conditions* (6.1). *Then for each* $b \in V^5$ *the problem* (5.9),(5.8) *has a solution on the semi–axis* $\mathbb{R}_+$, *which obeys the following inequality*

$$\|v(t)\|_{V^2} + \|v'(t)\|_{V^1} \leqslant C_{16} + C_{17}e^{-\gamma t}\left(C_4\|b\|^2_{V^2} + \varepsilon\|b\|^2_{V^3} + \varepsilon\varkappa\|b\|^2_{V^4}\right); \tag{7.11}$$

$$\varepsilon e^{-\gamma t}\|v'(t)\|_{V^5} \leqslant C_8 + C_9 e^{-\gamma t}\left(C_4\|b\|^2_{V^2} + \varepsilon\|b\|^2_{V^3} + \varepsilon\varkappa\|b\|^2_{V^4}\right), \tag{7.12}$$

where the constants C_{16} *and* C_{17} *are independent of* f, γ, $\varkappa$, ν, α_i, β_i, $i = \overline{1,L}$, *are independent of* ε.

Proof. Let v_m be the solution to the problem (5.9), (5.8) on the segment $[0, m]$, $m = 1, 2, \dots$, which exists due to Theorem 7.1. We continue the functions v_m to the semi–axis $\mathbb{R}_+$ as follows

$$\hat{v}_m(t) = \begin{cases} v_m(t), & 0 \leqslant t \leqslant m, \\ v_m(m), & t \geqslant m. \end{cases}$$

By our assumption, the functions $\hat{v}_m$ belong to the space $W_2^{\mathrm{loc}}(\mathbb{R}_+)$. We are going to show that the sequence $\{\hat{v}_m\}$ is relatively compact in $C(\mathbb{R}_+, V^1)$. According to Lemma 2.1, in order to do this, it is sufficient to establish that for each $T > 0$ the sequence of restrictions $\{\Pi_T\hat{v}_m\}$ is relatively compact in the space $C([0,T], V^1)$.

We choose an arbitrary $T > 0$. Neglecting possibly several first terms of the sequence, we can suppose that the functions $\{\Pi_T\hat{v}_m\}$ are solutions to the problem (5.9), (5.8) on the segment $[0,T]$. Since the functions $\Pi_T\hat{v}_m$ have the same value for $t = 0$, it follows from Lemma 6.1 that for almost all $t \in [0,T]$ they satisfy the estimate

$$e^{-\gamma t}(\|\Pi_T\hat{v}_m(t)\|_{V^5} + \|\Pi_T\hat{v}'_m(t)\|_{V^5}) \leqslant C_{14}.$$

Therefore,

$$\|\Pi_T\hat{v}_m(t)\|_{L_\infty(0,T;V^5)} + \|\Pi_T\hat{v}'_m(t)\|_{L_\infty(0,T;V^5)} \leqslant C_{15} \tag{7.13}$$

with a constant C_{15}, which depends on T and $\frac{1}{\varepsilon}$ and is independent of m.

Thus, the sequence $\{\Pi_T\hat{v}_m\}$ is bounded in $L_\infty(0,T;V^5)$, while the sequence of derivatives $\{\Pi_T\hat{v}'_m\}$ is bounded in $L_\infty(0,T;V^5)$. It follows from the compactness of embedding $V^5 \subset V^1$ and Theorem 3.1 that the sequence $\{\Pi_T\hat{v}_m\}$ is relatively compact in $C([0,T], V^1)$.

By the arbitrariness of choice of T the sequence $\{\hat{v}_m\}$ contains a subsequence $\{\hat{v}_{m_k}\}$ converging in $C(\mathbb{R}_+, V^1)$ to some function v_*. We are going to show that this limiting function is the solution of problem (5.9),(5.8) on $\mathbb{R}_+$.

Let us verify that the function v_* belongs to space $W_2^{\mathrm{loc}}(\mathbb{R}_+)$. It follows from the estimate (7.13) that for each $T > 0$ the sequences $\{\Pi_T\hat{v}_{m_k}\}$ and $\{\Pi_T\hat{v}'_{m_k}\}$ are bounded in $L_\infty(0,T;V^5)$, this is why without loss of generality we can suppose that they converge $*$–weakly in $L_\infty(0,T;V^5)$ respectively to v_* and some function $u \in L_\infty(0,T;V^5)$. However in the sense of distributions on $(0,T)$ with the values in V^5 the derivatives $\{\Pi_T\hat{v}'_{m_k}\}$ converge to v'_* and this is why $u = \Pi_T v'_*$. Thus, the function $\Pi_T v_*$ belongs to the space $L_\infty(0,T;V^5)$ together with its derivative. This implies that the function $\Pi_T v_*$ can be represented as the integral with the varying upper limit and this is why it is continuous as a function with the values in V^5. Therefore, $\Pi_T v_*$ belongs to $W_2[0,T]$. This is true for each T and hence, v_* belongs to $W_2^{\mathrm{loc}}(\mathbb{R}_+)$.

The convergence in $C([0,T], V^1)$ implies the pointwise convergence. Since all functions $\{\hat{v}_{m_k}\}$ satisfy the same initial condition and the sequence $\{\Pi_T\hat{v}_{m_k}\}$ converges pointwise, the function

v_* also satisfies the initial condition (5.8). It remains to verify that this function solves Equation (5.9). In order to do this, we need establish that the restriction of v_* to each segment $[0,T]$ $(T>0)$ solves Equation (5.9) on this segment.

The convergence of the sequence $\{\hat{v}_{m_k}\}$ to v_* in $C(\mathbb{R}_+,V^1)$ implies the convergence of restrictions $\{\Pi_T\hat{v}_{m_k}\}$ to $\Pi_T v_*$ in $C([0,T],V^1)$. Starting from some index, the function $\Pi_T\hat{v}_{m_k}$ solve Equation (5.9), that is, they obey the identity

$$(J+\varepsilon e^{-\gamma t}A^3+\varkappa A)\Pi_T\hat{v}'_{m_k}+\nu A\Pi_T\hat{v}_{m_k}-B_1(\Pi_T\hat{v}_{m_k})+\varkappa B_2(\Pi_T\hat{v}_{m_k})-C(\Pi_T\hat{v}_{m_k})=f. \quad (7.14)$$

It follows from the inequality (7.13) that the sequence $\{\Pi_T\hat{v}_{m_k}\}$ is bounded in $L_\infty(0,T;V^5)$, and the sequence of derivatives $\{\Pi_T\hat{v}'_{m_k}\}$ is bounded in $L_\infty(0,T;V^5)$. This is why the assumptions of Lemma 7.1 hold. According to this lemma, passing in (7.14) to the weak limit in $L_2(0,T;V^{-1})$, we obtain that the limiting function satisfies the relation

$$(J+\varepsilon e^{-\gamma t}A^3+\varkappa A)\Pi_T v'_*+\nu A\Pi_T v_*-B_1(\Pi_T v_*)+\varkappa B_2(\Pi_T v_*)-C(\Pi_T v_*)=f.$$

This means that $\Pi_T v_*$ solves Equation (5.9) on $[0,T]$. In view of the arbitrary choice of T this gives that v_* solves Equation (5.9) on the semi–axis.

We proceed to proving the estimate (7.11). By Lemma 6.1 we have the inequality

$$\|v_{m_k}(t)\|_{V^2}+\|v'_{m_k}(t)\|_{V^1}\leqslant C_{12}+C_{13}e^{-\gamma t}\left(C_4\|b\|^2_{V^2}+\varepsilon\|b\|^2_{V^3}+\varepsilon\varkappa\|b\|^2_{V^4}\right). \quad (7.15)$$

For each m_k this inequality holds for all $t\in\mathbb{R}_+\setminus Q_{m_k}$, where Q_{m_k} is some set of zero measure. This is why for all $t\in\mathbb{R}_+\setminus Q$, where $Q=\cup_{m_k}Q_{m_k}$ is a set of zero measure, this inequality holds for each m_k.

For each t belonging to the set of complete measure $\mathbb{R}_+\setminus Q$, in view of the aforementioned strong convergence $v_{m_k}\to v_*$ in $C(\mathbb{R}_+,V^1)$ we obtain that $v_{m_k}(t)\to v_*(t)$ in V^1. By the inequality (7.15) the sequence $\{v_{m_k}\}$ is bounded in V^2, and $\{v'_{m_k}\}$ is bounded in V^1. This is why there exist subsequences $\tilde{v}_l(t)$ and $\tilde{v}'_l(t)$, which converge weakly in V^2 to $v_*(t)$ and in V^1 to $v'_*(t)$, respectively. This is why

$$\|v_*(t)\|_{V^2}\leqslant\varliminf_{l\to\infty}\|\tilde{v}_l(t)\|_{V^2}\leqslant C_{12}+C_{13}e^{-\gamma t}\left(C_4\|b\|^2_{V^2}+\varepsilon\|b\|^2_{V^3}+\varepsilon\varkappa\|b\|^2_{V^4}\right);$$
$$\|v'_*(t)\|_{V^1}\leqslant\varliminf_{l\to\infty}\|\tilde{v}'_l(t)\|_{V^1}\leqslant C_{12}+C_{13}e^{-\gamma t}\left(C_4\|b\|^2_{V^2}+\varepsilon\|b\|^2_{V^3}+\varepsilon\varkappa\|b\|^2_{V^4}\right).$$

Summing up these estimates, we obtain the desired estimate (7.11). The estimate (7.12) can be established in the same way. The proof is complete. □

8. Solvability of problem (1.1)–(1.4) on semi–axis

The notion of weak solution of initial boundary value problem (1.1)–(1.4) on a finite segment and a semi–axis can be rewritten in the following equivalent form.

Definition 8.1. *The weak solution of problem* (1.1)–(1.4) *on the segment* $[0,T]$ *is a function* $v\in W_1[0,T]$ *obeying the operator equation*

$$(J+\varkappa A)v'(t)+\nu Av(t)-B_1(v)(t)+\varkappa B_2(v)(t)-C(v)(t)=f \quad (8.1)$$

for almost all $t\in[0,T]$ *and the initial condition* (4.2).

Definition 8.2. *The weak solution of problem* (1.1)–(1.4) *on the semi–axis* $\mathbb{R}_+$ *is a function* $v\in W_1^{loc}(\mathbb{R}_+)$ *such that for each* $T>0$ *the restriction of* v *to the segment* $[0,T]$ *solves the operator equation* (8.1) *on this segment and satisfies the initial condition* (4.2).

Theorem 8.1. *Let the coefficients $\varkappa$, ν, α_i, β_i, $i = \overline{1, L}$, obey the conditions* (6.1)*. Then for each $a \in V^2$ the problem* (8.1)*,* (4.2) *has a solution on the semi–axis $\mathbb{R}_+$, which satisfies the inequality*

$$\|v(t)\|_{V^2} + \|v'(t)\|_{V^1} \leqslant C_{19}(1 + e^{-\gamma t}\|a\|^2_{V^2}) \tag{8.2}$$

for almost all $t \geqslant 0$. Here the constant C_{19} is independent of f, γ, $\varkappa$, ν, α_i, β_i, $i = \overline{1, L}$.

Proof. Since the space V^5 is dense in V^2, for each $a \in V^2$ there exists a sequence $\{b_m\} \subset V^5$ such that $\|b_m - a\|_{V^2} \to 0$.

We let

$$\varepsilon_m = \frac{1}{m(1 + \|b_m\|^2_{V^4})}.$$

In this case we have $\varepsilon_m \to 0$ and

$$\varepsilon_m \|b_m\|^2_{V^4} \leqslant 1. \tag{8.3}$$

By Theorem 7.2, for each $b_m \in V^5$ there exists a solution v_m of Equation (5.9) with $\varepsilon = \varepsilon_m$ on $\mathbb{R}_+$, which obeys the initial condition

$$v_m(0) = b_m.$$

By Theorem 7.2, the inequality (8.3) and the compactness of embedding $V^4 \subset V^3$ the estimates

$$\|v_m(t)\|_{V^2} + \|v'_m(t)\|_{V^1} \leqslant C_{16} + C_{17}e^{-\gamma t}\left(C_4\|b_m\|^2_{V^2} + \varkappa + C_{18}\right); \tag{8.4}$$

$$\varepsilon_m e^{-\gamma t}\|v'_m(t)\|_{V^5} \leqslant C_8 + C_9 e^{-\gamma t}\left(C_4\|b_m\|^2_{V^2} + \varkappa + C_{18}\right) \tag{8.5}$$

hold. For each m this inequality is satisfied for all $t \in \mathbb{R}_+ \setminus Q_m$, where Q_m is some set of zero measure. This is why for all $t \in \mathbb{R}_+ \setminus Q$, where $Q = \cup_m Q_m$ is a set of zero measure, this inequality holds for each m.

We are going to show that the sequence $\{v_m\}$ is relatively compact in the space $C(\mathbb{R}_+, V^1)$. According to Lemma 2.1, it is sufficient to establish that for each $T > 0$ the sequence of restrictions $\{\Pi_T v_m\}$ is relatively compact in $C([0,T], V^1)$. However, this is implied by the first assertion of Lemma 7.1 since the estimate (8.4) yields that the sequence $\{\Pi_T v_m\}$ is bounded in $L_\infty(0,T;V^2)$, while the sequence of the derivatives $\{\Pi_T v'_m\}$ is bounded in the space $L_\infty(0,T;V^1)$.

Since the sequence $\{v_m\}$ is relatively compact, it contains a subsequence $\{v_{m_k}\}$ converging to some function v_* in $C(\mathbb{R}_+, V^1)$. Let us show that v_* is the sought solution.

We first note that v_* belongs to the space $W_1^{\mathrm{loc}}(\mathbb{R}_+)$. Indeed, it follows from the estimate (8.4) that for an arbitrary $T > 0$ the sequences $\{\Pi_T v_{m_k}\}$ and $\{\Pi_T v'_{m_k}\}$ are bounded in $L_\infty(0,T;V^2)$ and $L_\infty(0,T;V^1)$, respectively. This is why without loss generality, by the uniqueness of the limit we can suppose that the sequence $\{\Pi_T v_{m_k}\}$ converges $*$–weakly in $L_\infty(0,T;V^2)$ to v_*. Similarly, without loss of generality we can suppose that the sequence $\{\Pi_T v'_{m_k}\}$ converges $*$–weakly in $L_\infty(0,T;V^1)$ to some function $u \in L_\infty(0,T;V^1)$. However, in the sense of distributions the sequence $\{\Pi_T v'_{m_k}\}$ converges to v'_* and hence $u = \Pi_T v'_*$. Thus, the function $\Pi_T v_*$ belongs to the space $L_\infty(0,T;V^2)$, while its derivative is an element of the space $L_\infty(0,T;V^1)$, that is, $\Pi_T v_* \in W_1[0,T]$. Since this is true for each T, the function v_* belongs to $W_1^{\mathrm{loc}}(\mathbb{R}_+)$.

We are going to verify that the function v_* solves Equation (8.1) on $\mathbb{R}_+$. In order to do this, we need to establish that the restriction of $\Pi_T v_*$ to each segment $[0,T]$ $(T > 0)$ is a solution of Equation (8.1) on this segment.

The strong convergence of $\{\hat{v}_{m_k}\}$ to v_* in $C(\mathbb{R}_+, V^1)$ implies the convergence of restrictions $\{\Pi_T v_{m_k}\}$ to $\Pi_T v_*$ in $C([0,T], V^1)$. The functions $\Pi_T v_{m_k}$ are solutions of Equation (5.9), that is,

$$(J + \varepsilon_{m_k} e^{-\gamma t} A^3 + \varkappa A)\Pi_T v'_{m_k} + \nu A \Pi_T v_{m_k} - B_1(\Pi_T v_{m_k}) + \varkappa B_2(\Pi_T v_{m_k}) - C(\Pi_T v_{m_k}) = f. \tag{8.6}$$

It follows from the inequality (8.4) that the sequence $\{\Pi_T v_{m_k}\}$ is bounded in $L_\infty(0,T;V^2)$, and the sequence of derivatives $\{\Pi_T v'_{m_k}\}$ is bounded in $L_\infty(0,T;V^1)$. The estimate (8.5) yields that the sequence $\varepsilon_{m_k} v'_{m_k}$ is bounded in $L_\infty(0,T;V^5)$ and by our choice $\varepsilon_{m_k}\to 0$. This is why by Lemma 7.1 without loss of generality we can suppose that the left hand side of (8.6) converges to

$$(J+\varkappa A)\Pi_T v'_* + \nu A\Pi_T v_* - B_1(\Pi_T v_*) + \varkappa B_2(\Pi_T v_*) - C(\Pi_T v_*),$$

for instance, weakly in $L_2(0,T;V^{-1})$. By the arbitrariness of the choice of T we find that v_* satisfies the following identity

$$(J+\varkappa A)v'_* + \nu A v_* - B_1(v_*) + \varkappa B_2(v_*) - C(v_*) = f. \tag{8.7}$$

This yields that the function v_* solves the problem (8.1), (4.2).

Let us show that the function v_* obeys the initial condition (4.2). The convergence in $C(\mathbb{R}_+,V^1)$ implies the pointwise convergence and hence

$$b_{m_k} = v_{m_k}(0) \to v_*(0) \quad \text{in} \quad V^1.$$

However, by our choice of the sequence $\{b_m\}$ we have that $b_{m_k}\to a$ in V^2. By the uniqueness of the limit $v_*(0)=a$ and hence the initial condition is satisfied.

It remains to prove the inequality (8.2). The inequality (8.4) holds for each k and all t, which belong to some independent of k subset $\mathbb{R}_+$ of a complete measure. We choose such t. It follows from the inequality (8.4) that the sequences $\{v_{m_k}(t)\}$ and $\{v'_{m_k}(t)\}$ are bounded in V^2 and V^1, respectively. Therefore, each of them contains subsequences $\tilde v_l(t)$ and $\tilde v'_l(t)$, which converge weakly in V^2 to $v_*(t)$ and in V^1 to $v'_*(t)$. This is why

$$\|v_*(t)\|_{V^2} \leqslant \varliminf_{l\to\infty} \|\tilde v_l(t)\|_{V^2} \leqslant C_{16} + C_{17}e^{-\gamma t}\left(C_4\|a\|^2_{V^2} + \varkappa + C_{18}\right);$$
$$\|v'_*(t)\|_{V^1} \leqslant \varliminf_{l\to\infty} \|\tilde v'_l(t)\|_{V^1} \leqslant C_{16} + C_{17}e^{-\gamma t}\left(C_4\|a\|^2_{V^2} + \varkappa + C_{18}\right).$$

Summing these estimates, we obtain the estimate, which can be written in the form (8.2). □

9. Trajectory space and attractors

In the present case $E = V^2$ and $E_0 = V^1$. The trajectory space $\mathcal{H}^+$ of Equation (8.1) is introduced as the set of solutions of this equation defined on $\mathbb{R}_+$ essentially bounded as functions with values in V^2 and obeying the estimate

$$\|v(t)\|_{V^2} + \|v'(t)\|_{V^1} \leqslant C_{19}(1 + e^{-\gamma t}\|v\|^2_{L_\infty(\mathbb{R}_+,V^2)}) \tag{9.1}$$

for almost all $t\geqslant 0$.

It is necessary to show that the inclusion

$$\mathcal{H}^+ \subset C(\mathbb{R}_+;V^1)\cap L_\infty(\mathbb{R}_+;V^2)$$

holds.

The inclusion $\mathcal{H}^+\subset L_\infty(\mathbb{R}_+;V^2)$ is immediately implied by the definition of trajectory space. The inequality (9.1) yields that if v is some trajectory, then $\Pi_T v' \in L_\infty(0,T;V^1)$ for an arbitrary segment $[0,T]$. This is why $\Pi_T v$ belongs to the space $C([0,T],V^1)$ as the integral with varying upper limit. This is true for each T and therefore $v\in C(\mathbb{R}_+;V^1)$.

The non–emptiness of the space $\mathcal{H}^+$ is due to the next theorem.

Theorem 9.1. *Let the coefficients $\varkappa$, ν, α_i, β_i, $i=\overline{1,L}$, obey the conditions (6.1). Then for each $a\in V^2$ there exists a trajectory $v\in\mathcal{H}^+$ such that $v(0)=a$.*

Proof. Theorem 8.1 states there exists a solution $v \in W_1^{loc}(\mathbb{R}_+)$ to the problem (8.1), (4.2) on $\mathbb{R}_+$. We are going to show that this function is the sought trajectory. In order to do this, it is sufficient to verify the estimate (9.1). Since v satisfies (8.2), it is sufficient to obtain the inequality

$$\|v(0)\|_{V^2} \leqslant \|v\|_{L_\infty(\mathbb{R}_+;V^2)}. \tag{9.2}$$

It follows from the estimate (8.2) that the function v belongs to $L_\infty(\mathbb{R}_+;V^2)$, while its derivative is an element of $L_\infty(\mathbb{R}_+;V^1)$. This implies that $v \in C(\mathbb{R}_+;V^1)$ (by reproducing the arguing from Theorem 8.1). Thus, $v \in C(\mathbb{R}_+;V^1) \cap L_\infty(\mathbb{R}_+;V^2)$, and by Theorem 2.1 the function v belongs to the space $C_w(\mathbb{R}_+;V^2)$. This is why for each $t \in \mathbb{R}_+$ the value $v(t) \in V^2$ is well–defined. By the definition of norm in $L_\infty(\mathbb{R}_+,V^2)$ this gives the desired inequality (9.2). The proof is complete. □

The main result of work is the following theorem on existence of the minimal and global attractor.

Theorem 9.2. *Let the coefficients $\varkappa$, ν, α_i, β_i, $i=\overline{1,L}$, satisfy the conditions* (6.1). *Then there exists the minimal trajectory attractor $\mathcal{U}$ and the global attractor $\mathcal{A}$ of trajectory space $\mathcal{H}^+$.*

Proof. According to Theorem 2.2, it is sufficient to establish the existence of trajectory semi–attractor.

We consider the set

$$P = \big\{v \in C(\mathbb{R}_+;V^1) \cap L_\infty(\mathbb{R}_+;V^2) \colon v' \in L_\infty(\mathbb{R}_+,V^1),$$
$$\|v(t)\|_{V^2} + \|v'(t)\|_{V^1} \leqslant 2C_{19} \quad \text{for almost all} \quad t \in \mathbb{R}_+\big\}.$$

The definition of P implies immediately that this set is bounded in $L_\infty(\mathbb{R}_+;V^2)$ and translationally invariant, that is, $T(h)P \subset P$, $h \geqslant 0$.

Now we are going to show that the set P is relatively compact in $C(\mathbb{R}_+;V^1)$. In view of Lemma 2.1, in order to do this, it is sufficient to show that for each $T > 0$ the set $\Pi_T P$ is relatively compact in $C([0,T],V^1)$. Indeed, by the definition of P we have that for each $T > 0$ the set $\Pi_T P$ is bounded in $L_\infty(0,T;V^2)$, and the set $\{v' \colon v \in \Pi_T P\}$ is bounded in $L_\infty(0,T;V^1)$. As above, by Theorem 3.1 this implies that the set $\Pi_T P$ is relatively compact in $C([0,T],V^1)$. By the arbitrariness of T we get the relative compactness of set P in $C(\mathbb{R}_+;V^1)$.

Let us show that the set P is absorbing for the trajectory space $\mathcal{H}^+$. We consider an arbitrary set $B \subset \mathcal{H}^+$ bounded in $L_\infty(\mathbb{R}_+;V^2)$. For the sake of definiteness,

$$\|v\|_{L_\infty(\mathbb{R}_+;V^2)} \leqslant R$$

for all $v \in B$.

We choose $h_0 \geqslant 0$ such that $R^2 e^{-\gamma h_0} \leqslant 1$. Let v be an arbitrary function in B. Since v satisfies the inequality (9.1), for $h \geqslant h_0$ we have

$$\|T(h)v(t)\|_{V^2} + \|T(h)v'(t)\|_{V^1} = \|v(t+h)\|_{V^2} + \|v'(t+h)\|_{V^1}$$
$$\leqslant C_{19}(1 + e^{-\gamma(t+h)}R^2) \leqslant C_{19}(1 + e^{-\gamma t}) \leqslant 2C_{19}.$$

Thus, $T(h)v \in P$.

By the arbitrariness of v we obtain that $T(h)B \subset P$ for all $h \geqslant h_0$. Therefore, P is an absorbing set. Then by Lemma 2.2 we see that $\overline{P}$ is the trajectory semi–attractor. Then by Theorem 2.2 there exist a trajectory attractor $\mathcal{U}$ and global attractor $\mathcal{A}$ of trajectory space $\mathcal{H}^+$. The proof is complete. □

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Mikhail Vyacheslavovich Turbin,
Voronezh State University,
Universitetskaya sq. 1,
394018, Voronezh, Russia
E-mail: mrmike@mail.ru

Anastasia Sergeevna Ustiuzhaninova,
Voronezh State University,
Universitetskaya sq. 1,
394018, Voronezh, Russia
E-mail: nastyzhka@gmail.com