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ON COMMUTANT OF SYSTEM OF INTEGRATION OPERATORS IN MULTIDIMENSIONAL DOMAINS

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Abstract. We describe the commutant of system of integration operators in the Fréchet space $H(\Omega)$ of all functions holomorphic in a domain Ω in $\mathbb{C}^N$, which is polystar with respect to the origin. In particular, among such domains, there are the products of domains $\mathbb{C}$ being star with respect to the origin and complete Reinhardt domains with center at the origin. As in the one–dimensional case, the operators in the commutant are the Duhamel operators. We show that $H(\Omega)$ with the Duhamel product $*$ is an associative and commutative topological algebra. It is topologically isomorphic to the commutant with the product, which is the composition of operators, and with the topology of bounded convergence. We obtain a similar to one–dimensional representation of the product $f * g$ as a sum containing one term being a multiple of f and terms with the integrals at least in one variable of the function independent of the derivatives of f. By means of this representation we prove the criterion of $*$–invertibility of a function in $H(\Omega)$ and the corresponding convolution operator. We establish that the algebra $(H(\Omega), *)$ is local. In the case when the domain Ω is in addition convex, in the dual situation we obtain the criterion for the invertibility of operator from the commutant of system of operators of partial backward shift.

Keywords: holomorphic function, integration operator, commutant, Duhamel product.

Mathematics Subject Classification: 46E10, 47B91

1. Introduction

In the present work we study continuous linear operators in the space $H(\Omega)$ of functions holomorphic in a domain Ω in $\mathbb{C}^N$, which is polystar with respect to the origin; the operators commute with each partial integration operator and the space is equipped with the Duhamel product $*$. The Duhamel product in the spaces of holomorphic functions is rather intensively studied, and it has applications in the theory of differential equations, spectral theory in the generalized sense, in the operator calculus, in problem on spectral multiplicity of a linear continuous operator, see [3]. For $N = 1$, this product was first introduced and studied in the space $H(\Omega)$ by Wigley [11]. The obtained results mostly concern the functions of one variable, the case of many variables is significantly less studied.

The polystar domains Ω were earlier considered in [2], the integration operator J_k, $1 \leqslant k \leqslant N$, in separate variables are well–defined in $H(\Omega)$. The set of all such domains contains all products of domains in $\mathbb{C}$, which are star with respect to the origin and all complete Reinhardt domains with the center at the origin. It is strictly narrower than the set of all domains in $\mathbb{C}^N$, which are star with respect to the origin. The main results of work are Theorem 2.2 on representation of operators in the commutant $\mathcal{K}(\mathcal{J})$ of the system $\{J_k \,|\, 1 \leqslant k \leqslant N\}$ in the algebra of all

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continuous linear operators in $H(\Omega)$, Theorem 3.1 on a topological isomorphism between the spaces $H(\Omega)$ and $\mathcal{K}(\mathcal{J})$ with the topology of bounded convergence and on the isomorphism between the algebras $(H(\Omega), *)$ and $\mathcal{K}(\mathcal{J})$, as well as Theorem 4.1 providing a criterion of the invertibility of an element in the algebra $(H(\Omega), *)$. For $N = 1$ Theorems 2.2 and 4.1 are well–known. Raichinov [6] described the commutant of the integration operator in the space $H(\Omega)$ and established the condition for the invertibility of an operator in the commutant for domains Ω in $\mathbb{C}$, which are star with respect to the origin. Later Kiryutenko [4] obtained similar results for a simply–connected star–shaped domain $\mathbb{C}$ containing the origin. Wigley [11] studied the Duhamel product in $H(\Omega)$ (assuming that Ω is star with respect to the origin) and established a criterion for the invertibility of an element in the algebra $(H(\Omega), *)$.

As in [11], in the present paper the proof of $*$–invertibility of functions from $H(\Omega)$ uses the Neumann series. This is possible owing to Lemma 2.1, which provides the representation for the product $f * g$ as a sum containing one integrated term, which is a multiple of f, and terms with integrals at least in one of variables of the function independent of the derivatives of f. Such representation allows us to obtain estimates in multidimensional situation, which are used in the proof of quasinilpotency of a Volterra operator in Banach spaces. The proven invertibility criterion shows that, as in the one–dimensional case, the algebra $(H(\Omega), *)$, is local and its only maximal ideal is the set of all $*$–non–invertible elements.

If the domain Ω is in addition convex, by means of the Laplace transform the adjoint operators for J_k are realized as the operator $D_{k,0}$ of partial backward shift in some space $P(\Omega)$ of entire functions of exponential type. This gives an opportunity for the dual situation to obtain the criterion of invertibility in $P(\Omega)$ for an operator from the commutant of system $\{D_{k,0} \,|\, 1 \leqslant k \leqslant N\}$.

2. Description of operators commuting with system of integration operators

In what follows we consider a domain Ω in $\mathbb{C}^N$, $N \in \mathbb{N}$, which is polystar with respect to the origin. We recall that a set $Q \subset \mathbb{C}^N$ is called polystar with respect to the origin [2, Sect. 4.3] if for each $z \in Q$ the parallelepiped

$$\Pi(z) := [0, z_1] \times \cdots \times [0, z_N]$$

is contained in Q. Let $P_N := \{1, \ldots, N\}$. The set $Q \subset \mathbb{C}^N$ is polystar with respect to the origin if and only if for all $z \in Q$, $k \in P_N$, $\xi \in [0, z_k]$ the point $(z_1, \ldots, z_{k-1}, \xi, z_{k+1}, \ldots, z_N)$ belongs to Q.

If $\Omega = \Omega_1 \times \cdots \times \Omega_N$, where Ω_k, $1 \leqslant k \leqslant N$, are domains in $\mathbb{C}$, then Ω is polystar with respect to the origin if and only if each domain Ω_k is polystar with respect to the origin. Each complete Reinhart domain with center at the origin is polystar with respect to the origin. For $N \geqslant 2$ not each convex domain in $\mathbb{C}^N$ containing the origin possesses this property.

By the symbol $H(\Omega)$ we denote the space of all holomorphic in Ω functions with the topology of uniform convergence on compact sets in Ω. For $k \in P_N$ we introduce the operators of partial integration

$$J_k(f)(z) := \int\limits_0^{z_k} f(z_1, \ldots, z_{k-1}, \xi, z_{k+1}, \ldots, z_N) d\xi, \quad z \in \Omega, \quad f \in H(\Omega),$$

where the integral is taken over the segment $[0, z_k]$. All of them are linear and continuous in $H(\Omega)$ and mutually commuting.

The Duhamel product in $H(\Omega)$ is defined as follows [11], [10], [2]:

$$(f*g)(z) := \frac{\partial^N}{\partial z_1\cdots\partial z_N}\int\limits_{\Pi(z)} f(t)g(z-t)dt,\quad g\in H(\Omega),\quad z\in\Omega.$$

Here, for $u\in H(\Omega)$, the latter integral is treated as iterated

$$\int\limits_{\Pi(z)} u(t)dt := \int\limits_0^{z_N}\cdots\int\limits_0^{z_1} u(t)dt_1\cdots dt_N,\quad z\in\Omega;$$

it is independent of the integration order.

For $N=1$ the identity

$$(f*g)(z) = g(0)f(z) + \int\limits_0^z g'(z-t)f(t)dt,\quad z\in\Omega \tag{2.1}$$

holds. Our aim is to prove its analogue for $N\geqslant 1$. For a non–empty set $\tau\subset P_N$ and $z\in\mathbb{C}^N$ the symbol $z(\tau)$ denotes a point in $\mathbb{C}^{\operatorname{card}\tau}$ obtained from z by omitting the coordinates z_j, $j\in P_N\setminus\tau$, and preserving the order for others. If $\tau=P_N$, then $z(\tau)=z$. For the multiindex $1=(1,\ldots,1)\in\mathbb{N}_0^N$, $\tau\subset P_N$, $\tau\neq\emptyset$, by $1[\tau]$ we denote the multiindex in $\mathbb{N}_0^N$ such that

$$1[\tau]_k = 1\quad\text{for}\quad k\in\tau,$$

and

$$1[\tau]_k = 0\quad\text{for}\quad k\in P_N\setminus\tau;$$

$z' := (z_1,\ldots,z_{N-1})$; $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$.

We let

$$\partial^\alpha f := \frac{\partial^{|\alpha|}f}{\partial z_1^{\alpha_1}\cdots\partial z_N^{\alpha_N}},\quad \alpha\in\mathbb{N}_0^N;\qquad \partial_k f := \frac{\partial f}{\partial z_k},\quad k\in P_N;$$

here $|\alpha| := \alpha_1+\cdots+\alpha_N$. For $t,z\in\mathbb{C}^N$, $\sigma\subset P_N$ the point $t_{\sigma,z}\in\mathbb{C}^N$ is defined as

$$(t_{\sigma,z})_k := \begin{cases} z_k, & k\in\sigma,\\ t_k, & k\in P_N\setminus\sigma.\end{cases}$$

For $\varepsilon>0$

$$D_N(\varepsilon) := \{z\in\mathbb{C}^N \,|\, |z_k|<\varepsilon,\ k\in P_N\},\qquad \overline{D}_N(\varepsilon) := \{z\in\mathbb{C}^N \,|\, |z_k|\leqslant\varepsilon,\ k\in P_N\}.$$

Lemma 2.1. *For each domain Ω in $\mathbb{C}^N$, which is polystar with respect to the origin, and all $f,g\in H(\Omega)$, $z\in\Omega$ the identity*

$$(f*g)(z) = g(0)f(z) + \sum_{\sigma\subsetneqq P_N}\ \int\limits_{\Pi(z(P_N\setminus\sigma))} \partial^{1[P_N\setminus\sigma]}g(z-t_{\sigma,z})f(t_{\sigma,z})dt(P_N\setminus\sigma) \tag{2.2}$$

holds.

Proof. We first suppose that Ω is a polydisk $D_N(\varepsilon)$, $\varepsilon>0$. The identity (2.2) is this case will be proved by the induction in N. For $N=1$ it holds, see (2.1). Suppose that it holds for $N-1$ with $N\geqslant 2$. Differentiating under the integral, which is possible, see, for instance, [5, Ch. 4, Sect. 1, 1.2], we obtain

$$(f*g)(z) = \frac{\partial}{\partial z_N}\int\limits_0^{z_N}\left(\frac{\partial^{N-1}}{\partial z_1\cdots\partial z_{N-1}}\int\limits_{\Pi(z')} g(z'-t', z_N-t_N)f(t',t_N)dt'\right)dt_N,\quad z\in D_N(\varepsilon).$$

We fix $z_N \in \mathbb{C}$ such that $|z_N| < \varepsilon$ and $t_N \in [0, z_N]$. By the induction assumption applied to the functions $f(\cdot, t_N)$ and $g(\cdot, z_N - t_N)$ we have the identity

$$\frac{\partial^{N-1}}{\partial z_1 \cdots \partial z_{N-1}} \int\limits_{\Pi(z')} g(z'-t', z_N - t_N) f(t', t_N) dt' = g(0', z_N - t_N) f(z', t_N)$$
$$+ \sum_{\nu \subsetneqq P_{N-1}} \int\limits_{\Pi(z'(P_{N-1}\setminus\nu))} \partial^{1'[P_{N-1}\setminus\nu]} g(z' - t'_{\nu,z'}, z_N - t_N) f(t'_{\nu,z'}, t_N) dt'(P_{N-1}\setminus\nu), \quad z' \in D_{N-1}(\varepsilon).$$

Taking into consideration the one–dimensional identity (2.1), for $z \in D_N(\varepsilon)$ we obtain

$$\begin{aligned}(f*g)(z) =& g(0)f(z) + \int\limits_0^{z_N} \partial_N g(0', z_N - t_N) f(z', t_N) dt_N \\ &+ \sum_{\nu \subsetneqq P_{N-1}} \int\limits_{\Pi(z'(P_{N-1}\setminus\nu))} \left(\frac{\partial}{\partial z_N} \int\limits_0^{z_N} \partial^{1'[P_{N-1}\setminus\nu]} g(z' - t'_{\nu,z'}, z_N - t_N) \right. \\ &\qquad \left. \cdot f(t'_{\nu,z'}, t_N) dt_N \right) dt'(P_{N-1}\setminus\nu) \\ =& g(0)f(z) + \int\limits_0^{z_N} \partial_N g(0', z_N - t_N) f(z', t_N) dt_N \\ &+ \sum_{\nu \subsetneqq P_{N-1}} \int\limits_{\Pi(z'(P_{N-1}\setminus\nu))} \left(\partial^{1'[P_{N-1}\setminus\nu]} g(z' - t'_{\nu,z'}, 0) f(t'_{\nu,z'}, z_N) \right. \\ &+ \left. \int\limits_0^{z_N} \partial_N (\partial^{1'[P_{N-1}\setminus\nu]} g)(z' - t'_{\nu,z'}, z_N - t_N) f(t'_{\nu,z'}, t_N) dt_N \right) dt'(P_{N-1}\setminus\nu) \\ =& g(0)f(z) + \sum_{\sigma \subsetneqq P_N} \int\limits_{\Pi(z(P_N\setminus\sigma))} \partial^{1[P_N\setminus\sigma]} g(z - t_{\sigma,z}) f(t_{\sigma,z}) dt(P_N\setminus\sigma).\end{aligned}$$

Now let Ω be an arbitrary domain in $\mathbb{C}^N$, which is polystar with respect to the origin. There exists $\varepsilon > 0$ such that $D_N(\varepsilon) \subset \Omega$. As it has been proved above, the identity (2.2) holds for each $z \in D_N(\varepsilon)$. Since the function $f * g$ and each term in the right hand side of (2.2) are holomorphic in Ω, the uniqueness theorem ensures its validity for all $z \in \Omega$. □

For a compact set $Q \subset \Omega$ we let

$$p_Q(f) := \sup_{z\in Q} |f(z)|, \qquad f \in H(\Omega).$$

Lemma 2.1 implies the following result on the continuity of product $*$.

Lemma 2.2. *Let Ω be a domain $\mathbb{C}^N$, which is polystar with respect to the origin. For each compact set Q in Ω, which is polystar with respect to the origin, and each $\varepsilon > 0$ such that*

$$Q(\varepsilon) := Q + \overline{D}_N(\varepsilon) \subset \Omega,$$

there exists a constant $C > 0$ such that

$$p_Q(f * g) \leqslant C p_Q(f) p_{Q(\varepsilon)}(g)$$

for all functions $f, g \in H(\Omega)$.

For the functions with separated variables their Duhamel product is reduced to one–dimensional product in the following sense. For the functions u and v we define the variables z_k, $k \in P_N$ and one–dimensional products in z_k:

$$(u *_k v)(z_k) := \frac{\partial}{\partial z_k} \int\limits_0^{z_k} u(z_k - t_k)v(t_k)dt_k.$$

The next lemma is obvious.

Lemma 2.3. *Let*

$$f(z) = f_1(z_1)\cdots f_N(z_N), \qquad g(z) = g_1(z_1)\cdots g_N(z_N),$$

where the functions f_k and g_k are holomorphic in polystar with respect to the origin domain $G_k \subset \mathbb{C}$, $k \in P_N$. Then

$$(f * g)(z) = (f_1 *_1 g_1)(z_1)\cdots(f_N *_N g_N)(z_N)$$

for each $z \in G_1 \times \cdots \times G_N$

We let

$$f_\alpha(z) := \frac{1}{\alpha!} z_1^{\alpha_1}\cdots z_N^{\alpha_N}, \qquad J^\alpha := J_1^{\alpha_1}\cdots J_N^{\alpha_N}, \quad z \in \mathbb{C}^N, \quad \alpha \in \mathbb{N}_0^N.$$

We denote by $\mathbf{1}$ the function, which is identically equal to 1, while $\mathbb{C}[z]$ stands for the set of all polynomials of variables $z_1, \dots, z_N$ over $\mathbb{C}$.

For $N = 1$ the identities in the next statement are known, see, for instance, [3], [11].

Lemma 2.4. *The following statements hold.*

(i) *For all $\alpha, \beta \in \mathbb{N}_0^N$ the identity $f_\alpha * f_\beta = f_{\alpha+\beta}$ holds.*

(ii) *Let Ω be a Runge domain in $\mathbb{C}^N$, which is polystar with respect to the origin. For all $\alpha \in \mathbb{N}_0^N$, $f, g \in H(\Omega)$ the identities*

$$J^\alpha(f * g) = J^\alpha(f) * g = f * J^\alpha(g), \quad J^\alpha(f) = f * f_\alpha$$

hold in Ω.

Proof. The validity of (i) is implied by the same identities for $N = 1$ and Lemma 2.3.

By (i), the first relations in (ii) are true if the functions f and g are monomials and hence, by Lemma 2.2 and the density of $\mathbb{C}[z]$ in $H(\Omega)$, they hold for arbitrary functions $f, g \in H(\Omega)$. The second identity in (ii) is a particular case of the first identity with $g = \mathbf{1}$. □

Theorem 2.1. *Let Ω be a Runge domain in $\mathbb{C}^N$, which is polystar with respect to the origin.*

(i) *The space $H(\Omega)$ with the multiplication $*$ is an associative and commutative topological algebra.*

(ii) *The algebra $(H(\Omega), *)$ has no divisors of zero.*

Proof. (i): The mapping $(f, g) \mapsto f * g$ from $H(\Omega) \times H(\Omega)$ into $H(\Omega)$ is bilinear. Lemma 2.2 implies its continuity. It is clear that the multiplication $*$ is commutative. The associativity of multiplication $*$ is implied by Lemma 2.4, its continuity and the density of $\mathbb{C}[z]$ in $H(\Omega)$.

(ii): We choose $r > 0$ such that $D_N(r) \subset \Omega$. Let

$$f, g \in H(\Omega), \qquad f = \sum_{\alpha \in \mathbb{N}_0^N} b_\alpha f_\alpha, g = \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha f_\alpha \quad \text{in} \quad D_N(r).$$

By Lemma 2.4 we have

$$f * g = \sum_{\gamma \in \mathbb{N}_0^N} a_\gamma f_\gamma, \quad \text{where} \quad a_\gamma = \sum_{0 \leqslant \alpha \leqslant \gamma} b_\alpha c_{\gamma-\alpha}, \quad \gamma \in \mathbb{N}_0^N,$$

and the series for $f * g$ converges in $H(D_N(r))$. If $f * g = 0$ in $H(\Omega)$, then all coefficients a_γ vanish. This yields that one of multi–sequences b and c is zero. □

For each function $g \in H(\Omega)$ we introduce the operator

$$S_g(f) := f * g, \qquad f \in H(\Omega).$$

It is linear and due to Lemma 2.2 it is continuous in $H(\Omega)$.

Corollary 2.1. *For each non–zero function $g \in H(\Omega)$ the operator $S_g : H(\Omega) \to H(\Omega)$ is injective.*

By the symbol $\mathcal{K}(\mathcal{J})$ we denote the commutant of set $\mathcal{J} = \{J_k \,|\, k \in P_N\}$ in the algebra of all linear continuous operators $H(\Omega)$ with composition of operators as multiplication; $\mathcal{K}(\mathcal{J})$ is its subalgebra.

Theorem 2.2. *Let Ω be a Runge domain in $\mathbb{C}^N$, which is polystar with respect to the origin.*
(i) *If $A \in \mathcal{K}(\mathcal{J})$, then there exists a unique function $g \in H(\Omega)$, for which $A = S_g$.*
(ii) *$S_g \in \mathcal{K}(\mathcal{J})$ for each function $g \in H(\Omega)$.*

Proof. The second part of theorem is implied by Lemmas 2.2 and 2.4.
(i): Let $A \in \mathcal{K}(\mathcal{J})$. We take $g := A(\mathbf{1})$. It is sufficient to show that $A(f_\alpha) = f_\alpha * g$ for each $\alpha \in \mathbb{N}_0^N$. Since $f_\alpha = J^\alpha(\mathbf{1})$, we have the identities

$$A(f_\alpha) = A(J^\alpha(\mathbf{1})) = J^\alpha(A(\mathbf{1})) = J^\alpha(\mathbf{1} * g) = f_\alpha * g.$$

The uniqueness of function g is due to the identity $\mathbf{1} * h = h$ for all $h \in H(\Omega)$. □

Remark 2.1. *Each domain $G = G_1 \times \cdots \times G_N$, where G_k, $1 \leqslant k \leqslant N$, are simply–connected domains in $\mathbb{C}$, is a Runge domain* [7, Ch. 1, Sect. 3]. *Each complete Reinhardt domain and each convex domain in $\mathbb{C}^N$ are also Runge domains* [1, Ch. IV, Sect. 24, Subsect. 8, 9].

3. Isomorphism between space $H(\Omega)$ and commutant

Let $Co(\Omega)$ be the set of all compact subsets in Ω, and $\mathcal{B}(H(\Omega))$ be the set of all subsets in $H(\Omega)$. By $\mathcal{K}(\mathcal{J})_b$ we denote the space $\mathcal{K}(\mathcal{J})$ equipped with the topology of bounded convergence. It is defined by the family of pre–norms

$$p_{Q,B}(A) := \sup_{f\in B}\sup_{z\in Q}|A(f)(z)|, \quad A \in \mathcal{K}(\mathcal{J}), \quad Q \in Co(\Omega), \quad B \in \mathcal{B}(H(\Omega)).$$

Theorem 3.1. *Let Ω be a Runge domain in $\mathbb{C}^N$, which is polystar with respect to the origin. The mapping $\chi(g) := S_g$ is a topological isomoprhism of the space $H(\Omega)$ onto $\mathcal{K}(\mathcal{J})_b$ and is an isomorphism of the algebra $(H(\Omega), *)$ onto the algebra $\mathcal{K}(\mathcal{J})$.*

Proof. By Theorem 2.2, the mapping $\chi : H(\Omega) \to \mathcal{K}(\mathcal{J})$ is bijective. It is clear that χ is linear. For $Q \in Co(\Omega)$ we choose $\varepsilon > 0$, for which $Q(\varepsilon) = Q + \overline{D}_N(\varepsilon) \subset \Omega$, and $C > 0$ by Lemma 2.2. For $B \in \mathcal{B}(H(\Omega))$, $g \in H(\Omega)$ we have

$$p_{Q,B}(S_g) = \sup_{f\in B}\sup_{z\in Q}|(f*g)(z)| \leqslant C\sup_{f\in B}(p_Q(f)p_{Q(\varepsilon)}(g)) = C_1 p_{Q(\varepsilon)}(g),$$

where $C_1 := C\sup_{f\in B} p_Q(f) < +\infty$. Hence, the linear mapping χ is continuous from $H(\Omega)$ into $\mathcal{K}(\mathcal{J})_b$.

Since for each compact set Q in Ω we have

$$p_Q(g) = \sup_{z\in Q}|g(z)| = \sup_{z\in Q}|(\mathbf{1}*g)(z)| = p_{Q,B}(S_g)$$

for all $g \in H(\Omega)$, where $B = \{\mathbf{1}\} \in \mathcal{B}(H(\Omega))$, the mapping χ^{-1} is contiunous from $\mathcal{K}(\mathcal{J})_b$ into $H(\Omega)$.

Since $\chi(f_\alpha) = J^\alpha$ for each $\alpha \in \mathbb{N}_0^N$, for all polynomials g, h the identity $\chi(g * h) = S_g S_h$ holds. Due to the continuity of multiplication $*$ from $H(\Omega) \times H(\Omega)$ into $H(\Omega)$, the continuity of χ and χ^{-1} and the density of $\mathbb{C}[z]$ in $H(\Omega)$, the identity $\chi(g * h) = S_g S_h$ holds also for all functions $g, h \in H(\Omega)$. □

Let $\mathbb{C}[\mathcal{J}]$ be the set of all polynomials of operators J_k, $k \in P_N$, that is, the set of operators of form

$$\sum_{|\alpha| \leqslant n} b_\alpha J^\alpha, \quad b_\alpha \in \mathbb{C}, \quad n \in \mathbb{N}_0.$$

Corollary 3.1. *Let Ω be a Runge domain in $\mathbb{C}^N$, which is polystar with respect to the origin. The set $\mathbb{C}[\mathcal{J}]$ is dense in $\mathcal{K}(\mathcal{J})_b$.*

In the terminology of paper [9] Corollary 3.1 means that the commutant $\mathcal{K}(\mathcal{J})$ is minimal.

4. Criterion of invertibility of element in algebra $(H(\Omega), *)$ and Duhamel operator

For $z \in \mathbb{C}^N$, $\tau \subset P_N$, $\tau \neq \emptyset$, and $z \in \mathbb{C}^N$ by $|z|(\tau)$ we denote the point $[0, +\infty)^{\operatorname{card} \tau}$ obtained from $(|z_1|, \dots, |z_N|)$ by omitting the coordinates $|z_j|$, $j \in P_N \setminus \tau$, and preserving the order of others.

The next result is a multidimensional analogue of [11, Sect. 1, Thm.].

Lemma 4.1. *Let Ω be a Runge domain in $\mathbb{C}^N$, which is polystar with respect to the origin. If $h \in H(\Omega)$, $h(0) = 0$, then the function $\mathbf{1} - h$ is invertible in the algebra $(H(\Omega), *)$.*

Proof. We let

$$h^{[0]} := \mathbf{1}, \quad h^{[n]} := h^{[n-1]} * h, \quad n \in \mathbb{N}.$$

We fix a compact set $Q \subset \Omega$, which is polystar with respect to the origin. Let

$$M := \max \left\{ \sup_{z \in Q} |\partial^\alpha h(z)| \,\middle|\, 0 \leqslant \alpha_k \leqslant 1,\, k \in P_N,\, \alpha \in \mathbb{N}_0^N \right\}.$$

By Lemma 2.1 for $z \in Q$ we have

$$|h^{[2]}(z)| \leqslant M^2 \sum_{\sigma \subsetneqq P_N} \prod_{k \in P_N \setminus \sigma} |z_k| \leqslant M^2 (2^N - 1) \prod_{k \in P_N} (1 + |z_k|).$$

Let us show that for all $n \geqslant 2$, $z \in Q$ the identity

$$|h^{[n]}(z)| \leqslant \frac{M^n (2^N - 1)^{n-1}}{(n-1)!} \prod_{k \in P_N} (1 + |z_k|)^{n-1} \tag{4.1}$$

holds. For $n = 2$ this estimate holds. Suppose that it holds for some $n \geqslant 2$. Then by Lemma 2.1

$$\begin{aligned} |h^{[n+1]}(z)| &\leqslant \sum_{\sigma \subsetneqq P_N} \left| \int\limits_{\Pi(z(P_N \setminus \sigma))} \partial^{1(P_N \setminus \sigma)} h(z - t_{\sigma,z}) h^{[n]}(t_{\sigma,z}) dt(P_N \setminus \sigma) \right| \\ &\leqslant \frac{M^{n+1} (2^N - 1)^{n-1}}{(n-1)!} \sum_{\sigma \subsetneqq P_N} \int\limits_{\Pi(|z|(P_N \setminus \sigma))} \prod_{k \in P_N} (1 + r_k)^{n-1} dr(P_N \setminus \sigma) \\ &\leqslant \frac{M^{n+1} (2^N - 1)^n}{n!} \prod_{k \in P_N} (1 + |z_k|)^n. \end{aligned}$$

We let $d := \sup\left\{|z_k| \,\middle|\, z \in Q,\, k \in P_N\right\}$. It follows from the estimates (4.1) that for each $n \geqslant 2$

$$\sup_{z\in Q}|h^{[n]}(z)| \leqslant \frac{M^n(2^N-1)^{n-1}(1+d)^{Nn}}{(n-1)!}.$$

By [2, Lm. 3] the family of all polystar with respect to the origin compact sets in Ω forms a fundamental family of compact subsets in Ω. Hence, the series $\sum_{n=0}^{\infty} h^{[n]}$ converges absolutely in $H(\Omega)$ to some function $v \in H(\Omega)$. At the same time, $(\mathbf{1}-h)*v = \mathbf{1}$. □

Theorem 4.1. *Let Ω be a Runge domain $\mathbb{C}^N$, which is polystar with respect to the origin, $g \in H(\Omega)$. The following statements are equivalent.*

(i) *The operator $S_g : H(\Omega) \to H(\Omega)$ is invertible.*
(ii) *The element g is invertible in the algebra $(H(\Omega), *)$.*
(iii) *$g(0) \neq 0$.*

Proof. The implication (iii)⇒(ii) is true by Lemma 4.1.

(ii)⇒(i): Let $g * h = h * g = \mathbf{1}$, $h \in H(\Omega)$. By Theorem 3.1

$$S_{\mathbf{1}} = SgS_h = S_hS_g.$$

Since $S_{\mathbf{1}}$ is the identity operator, we see that - S_h is the inverse operator for S_g.

(i)⇒(iii): Suppose that $g(0) = 0$. By Lemma 2.1

$$S_g(f)(0) = 0$$

for each function $f \in H(\Omega)$. Hence, $S_g : H(\Omega) \to H(\Omega)$ is not surjective and therefore, is not bijective. □

Corollary 4.1. *Let Ω be a Runge domain in $\mathbb{C}^N$, which is polystar with respect to the origin. The algebra $(H(\Omega), *)$ is local. Its only maximal ideal is the set of all its $*$–non–invertible elements.*

A similar result for the Hardy space in a polydisk was obtained by another method in the paper [10, Cor. 3]. In [10] it was proved via considering an appropriate space of functions with values in the same Banach space with the number of variables less by one and subsequent reduction to the one–dimensional situation.

5. Corollaries for dual situation

We consider the dual situation, when the domain Ω is in addition convex. We fix a sequence of convex compact sets Q_n, $n \in \mathbb{N}$, in Ω such that $Q_n \subset \operatorname{int} Q_{n+1}$ for each $n \in \mathbb{N}$ and $\Omega = \bigcup_{n\in\mathbb{N}} Q_n$; here $\operatorname{int} M$ denotes the interior of a set $M \subset \mathbb{C}^N$ in $\mathbb{C}^N$. Let

$$H_n(z) := \sup_{t\in Q_n} \operatorname{Re}\langle z, t\rangle, \quad z \in \mathbb{C}^N,$$

be the complex–valued support function of Q_n, $n \in \mathbb{N}$; here $\langle z, t\rangle := \sum_{k=1}^{N} z_k t_k$. We define the weighted space $P(\Omega)$ of entire in $\mathbb{C}^N$ functions

$$P(\Omega) := \bigcup_{n\in\mathbb{N}} P(Q_n),$$

where

$$P(Q_n) := \left\{ f \in H(\mathbb{C}^N) \,\middle|\, \|f\|_n := \sup_{z\in\mathbb{C}^N} \frac{|f(z)|}{\exp(H_n(z))} < +\infty \right\}, \quad n \in \mathbb{N}.$$

It is equipped with the topology of inductive limit of sequence of Banach spaces $(P(Q_n), \|\cdot\|_n)$, $n \in \mathbb{N}$, with respect to their embeddings into $P(\Omega)$. The space $P(\Omega)$ is contained in $\mathbb{C}[z]$.

The Laplace transform

$$\mathcal{F}(\nu)(z) := \nu_t(e^{\langle z,t\rangle}), \quad \nu \in H(\Omega)', \quad z \in \mathbb{C}^N,$$

is a topological isomorphism of strong dual space for $H(\Omega)$ onto $P(\Omega)$ [8, Thm. 4.5.3]. The bilinear form

$$(h, f) \mapsto \mathcal{F}^{-1}(f)(h), \quad h \in H(\Omega), \quad f \in P(\Omega),$$

establishes the duality between $H(\Omega)$ and $P(\Omega)$. According to [2], the operators of partial backward shift

$$D_{k,0}(f)(z) := \frac{f(z) - f(z_1, \ldots, z_{k-1}, 0, z_{k+1}, \ldots, z_N)}{z_k}, \quad f \in P(\Omega), \quad k \in P_N,$$

linearly and continuously map $P(\Omega)$ into $P(\Omega)$. For a functional $\varphi \in P(\Omega)'$ we let

$$B_\varphi(f)(z) := \varphi(T_z(f)), \quad z \in \mathbb{C}^N, \quad f \in P(\Omega).$$

The operator B_φ is linear and continuous in $P(\Omega)$, the set $\{B_\varphi \,|\, \varphi \in P(\Omega)'\}$ is a commutant of the system $\{D_{k,0} \,|k \in P_N\}$ in the algebra of all linear continuous operators in $P(\Omega)$ [2]; in the present situation the latter also implied by Theorem 2.2. Let $\mathcal{F}' : P(\Omega)' \to H(\Omega)$ be the mapping adjoint to $\mathcal{F} : H(\Omega)' \to P(\Omega)$ with respect to the dual pairs $(H(\Omega)', H(\Omega))$ and $(P(\Omega), P(\Omega)')$. We note that $\varphi(\mathbf{1}) = \mathcal{F}'(\varphi)(0)$ for each functional $\varphi \in P(\Omega)'$.

Lemma 5.1. *Let Ω be a convex domain in $\mathbb{C}^N$, which is polystar with respect to the origin.*

(i) *For $g \in H(\Omega)$, the adjoint operator for the operator S_g with respect to the dual pair $(H(\Omega), P(\Omega))$ is the operator B_φ for $\varphi = (\mathcal{F}')^{-1}(g)$.*

(ii) *For $k \in P_N$, the adjoint operator for the operator*

$$J_k : H(\Omega) \to H(\Omega)$$

with respect to the dual pair $(H(\Omega), P(\Omega))$ is the operator

$$D_{k,0} : P(\Omega) \to P(\Omega).$$

Proof. Statement (i) was proved in [2, Cor. 4], while (ii) is a particular case of (i) for $g(t) = t_k$; then $(\mathcal{F}')^{-1}(g)(f) = \frac{\partial f}{\partial z_k}(0)$. □

Theorem 4.1 and Lemma 5.1 imply the following statement.

Theorem 5.1. *Let Ω be a convex domain in $\mathbb{C}^N$, which is polystar with respect to the origin, $\varphi \in P(\Omega)'$. The operator B_φ is invertible in $P(\Omega)$ if and only if $\varphi(\mathbf{1}) \neq 0$.*

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