doi:10.13108/2025-17-1-138

SPECTRAL AND FUNCTIONAL INEQUALITIES ON ANTISYMMETRIC FUNCTIONS

A. LAPTEV, I.A. SHCHERBAKOV

Abstract. We obtain a number of spectral and functional inequalities related to Schrödinger operators defined on antisymmetric functions. Among them are Lieb — Thirring and CLR inequalities. Besides, we find new constants for the Sobolev and the Gagliardo — Nirenberg inequalities restricted to antisymmetric functions.

Keywords: Schrödinger operators, Lieb — Thirring and Cwikel — Lieb — Rozenblum inequalities, Sobolev and Hardy inequatilies, antisymmetric functions.

Mathematics Subject Classification: 35P15, 81Q10

1. Introduction

Lieb — Thirring (LTh) and Cwikel — Lieb — Rozenblum (CLR) inequalities have important applications in mathematical physics, analysis, dynamical systems and attractors, to mention a few. A current state of art and many aspects of the theory of such inequalities is presented in the recent book [8]. We mention here the celebrated paper by Lieb and Thirring [14], where such inequalities were studied for the questions of stability of matter. The famous CLR inequalities were obtained in the papers [1], [13], [15] by Cwikel, Lieb and Rosenblum.

Let $\mathcal{H}$ be a Schrödinger operator in $L^2(\mathbb{R}^n)$

$$\mathcal{H} = -\Delta - V, \qquad V \geqslant 0, \tag{1.1}$$

and let $\{-\lambda_k\}$ be the negative eigenvalues of the operator $\mathcal{H}$ counted with their multiplicities. Assuming that the electric potential V is an element of $L^{\gamma+n/2}(\mathbb{R}^n)$, $\gamma \geqslant 0$, we have

$$\operatorname{Tr} \mathcal{H}_-^\gamma = \operatorname{Tr}(-\Delta - V)_-^\gamma = \sum_k \lambda_k^\gamma \leqslant L_{\gamma,n} \int_{\mathbb{R}^n} V^{\gamma+n/2}\, dx, \tag{1.2}$$

where $s_- = -s$ if $s < 0$ and $s_- = 0$ if $s \geqslant 0$.

If $\gamma = 0$, the inequality (1.2) is known as CLR inequality and for $\gamma > 0$ as LTh inequality. It is known that the constants in (1.2) are finite for

$$\begin{cases} \gamma \geqslant \dfrac{1}{2} & \text{if} \quad n = 1, \\ \gamma > 0 & \text{if} \quad n = 2, \\ \gamma \geqslant 0 & \text{if} \quad n \geqslant 3. \end{cases} \tag{1.3}$$

A. Laptev, I.A. Shcherbakov, Spectral and functional inequalities on antisymmetric functions.

IS was supported by Moscow Center for Fundamental and Applied Mathematics, Agreement with the Ministry of Science and Higher Education of the Russian Federation, No. 075-15-2022-283 and Keldysh Institute of Applied Mathematics.

Submitted November 21, 2024.

The constants $L_{\gamma,n}$ are usually compared with the so–called semiclassical constant

$$L^{\rm cl}_{\gamma,n} = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}(1-|\xi|^2)^\gamma_+\,d\xi = \frac{\Gamma(\gamma+1)}{(4\pi)^{\frac{n}{2}}\Gamma(\gamma+\frac{n}{2}+1)}. \tag{1.4}$$

Sharp constant in (1.2) are known only for $\gamma=\frac12$, $d=1$, where $L_{\frac12,1}=\frac12=2L^{\rm cl}_{\frac12,1}$ and $\gamma\geqslant\frac32$, $n\geqslant 1$, where $L_{\gamma,n}=L^{\rm cl}_{\gamma,n}$.

Obviously the inequalities (1.2) imply the inequalities on the lowest eigenvalue and thus there are constants $L^1_{\gamma,n}$ such that

$$(\inf\operatorname{spec}(-\Delta-V))^\gamma_- \leqslant L^1_{\gamma,n}\int\limits_{\mathbb{R}^n}V^{\gamma+\frac{n}{2}}\,dx. \tag{1.5}$$

Clearly we have $L^1_{\gamma,n}\leqslant L_{\gamma,n}$. In [14] the authors made a conjecture (that is still open in the most important cases) that the sharp values of $L_{\gamma,n}$ coincide with the values $L^1_{\gamma,n}$. In the cases where the sharp constants are found, this conjecture holds true.

One of the aims of this paper is to obtain better constants in the inequality (1.2) for Schrödinger operators restricted to antisymmetric functions. In this case one assumes that the potential V is symmetric. It is expected that such constants will be a lot better since the operator (1.1) is defined in a smaller class of functions. In the paper [9] the authors obtained the inequality (1.2) for $\gamma=0$ with better constant for antisymmetric functions.

Let N and d be natural numbers. We consider $x=(x_1,\dots,x_N)\in\mathbb{R}^{dN}$, where $x_i=(x_{i1},\dots,x_{id})\in\mathbb{R}^d$ for all $1\leqslant i\leqslant N$. Every function u defined on $\mathbb{R}^{dN}$ we call *antisymmetric* hereafter, if for all $1\leqslant i,j\leqslant N$ and $x_1,\dots,x_N\in\mathbb{R}^d$

$$u(x_1,\dots,x_i,\dots,x_j,\dots,x_N) = -u(x_1,\dots,x_j,\dots,x_i,\dots,x_N) \tag{1.6}$$

and *symmetric* if

$$u(x_1,\dots,x_i,\dots,x_j,\dots,x_N) = u(x_1,\dots,x_j,\dots,x_i,\dots,x_N). \tag{1.7}$$

We shall use the notations $L^2_A(\mathbb{R}^{dN})$, $H^1_A(\mathbb{R}^{dN})$ and $H^2_A(\mathbb{R}^{dN})$ for subclasses of antisymmetric functions from $L^2(\mathbb{R}^{dN})$ and Sobolev classes $H^1(\mathbb{R}^{dN})$ and $H^2(\mathbb{R}^{dN})$ respectively.

We denote by $-\Delta_{as}$ the Laplacian in $L^2(\mathbb{R}^{dN})$ restricted to the antisymmetric functions and let

$$\mathcal{H}_{as} = -\Delta_{as} - V$$

be the respected Schrödinger operator.

Theorem 1.1. *Let $d=1$ and $V\geqslant 0$ be a symmetric potential such that $V\in L^{\gamma+\frac{N}{2}}(\mathbb{R}^N)$. Then for any γ satisfying the conditions* (1.3) *we have*

$$\operatorname{Tr}(-\Delta-V)^\gamma_- \leqslant \frac{L_{\gamma,N}}{N!}\int\limits_{\mathbb{R}^N}V^{\gamma+\frac{N}{2}}\,dx. \tag{1.8}$$

The proof of this result is given in Section 2 and it is based on the recents papers [11], [12], [16], where the authors considered properties of antisymmetric functions via properties of Vandermond determinants. In particular, the authors also obtained some Hardy inequalities which allow us in Section 3 to study the so–called Hardy — Lieb — Thirring inequalities, see [5], [6], [7], [10].

It was obtained in [12] that for any $u\in H^1_A(\mathbb{R}^{dN})$ we have

$$\int\limits_{\mathbb{R}^{dN}}|\nabla u(x)|^2dx \geqslant H_A(dN)\int\limits_{\mathbb{R}^{dN}}\frac{|u(x)|^2}{|x|^2}dx. \tag{1.9}$$

The constant $H_A(dN)$ is much larger than the classical Hardy constant $\frac{(dN-2)^2}{4}$ and is defined in (3.1). Besides, the inequality (1.9) holds even for $dN = 2$.

Let us denote by $\mathfrak{H}_{as}$ the Laplacian $-\Delta_{as}$ with subtracted Hardy term

$$\mathfrak{H}_{as} = -\Delta_{as} - H_A(dN)\,\frac{1}{|x|^2}. \tag{1.10}$$

Theorem 1.2. *Let $V \geqslant 0$ be a symmetric potential such that $V \in L^{\gamma+\frac{dN+\alpha}{2}}(\mathbb{R}^{dN})$, where $dN \geqslant 3$, $\gamma > 0$, and $\alpha \geqslant 0$. Then there is a constant $C_{\gamma,dN,\alpha}$ independent of V such that*

$$\operatorname{Tr}\,(\mathfrak{H}_{as} - V)_-^\gamma \leqslant C_{\gamma,dN,\alpha} \int\limits_{\mathbb{R}^{dN}} V(x)^{\gamma+\frac{dN+\alpha}{2}}\,|x|^\alpha\,dx. \tag{1.11}$$

In the special case $\alpha = 0$ we have

Corollary 1.1. *Let $dN \geqslant 3$ and $\gamma > 0$. Then we have the Hardy — Lieb — Thirring inequality*

$$\operatorname{Tr}\,(\mathfrak{H}_{as} - V)_-^\gamma \leqslant C_{\gamma,dN,0} \int\limits_{\mathbb{R}^n} V(x)^{\gamma+\frac{dN}{2}}\,dx.$$

We note that the sharp values of the constant $C_{\gamma,dN,\alpha}$ are unknown even in the general case. In our case there is an additional complication finding them for $d > 1$ due to the multiplicity of the minimal eigenvalue of the Laplace — Beltrami operator in $L^2(\mathbb{R}^{dN})$ restricted to antisymmetric functions.

Finally in Section 4 we shall obtain versions of Sobolev and Gagliardo — Nirenberg inequalities for antisymmetric function with applications to bounds of the lowest eigenvalues of Schrödinger operators.

2. CLR and LTh inequalities for antisymmetric functions

We begin with presenting some results obtained in [11], [12], [16].

2.1. Vandermonde determinant. Let us consider the unitary monomials of d variables lexicographically, i.e. for $t \in \mathbb{R}^d$ denote $\varphi_1^{(d)}(t) = 1$, $\varphi_2^{(d)}(t) = t_1$ and $\varphi_{d+1}^{(d)}(t) = t_d$, $\varphi_{d+2}^{(d)}(t) = t_1^2$ and so on.

Consider the determinant

$$\psi_N^{(d)}(x_1, \ldots, x_N) = \begin{vmatrix} \varphi_1^{(d)}(x_1) & \cdots & \varphi_1^{(d)}(x_N) \\ \vdots & \ddots & \vdots \\ \varphi_N^{(d)}(x_1) & \cdots & \varphi_N^{(d)}(x_N) \end{vmatrix}. \tag{2.1}$$

We denote by $\mathcal{V}_d(N)$ the degree of $\psi_N^{(d)}$ that is the minimal degree of the antisymmetric polynomial. Clearly the total degree of $\psi_N^{(d)}$ equals the sum of degrees in every row. Note that the function $\psi_N^{(d)}(x_1, \ldots, x_N)$ is an antisymmetric homogeneous harmonic polynomial. The restriction of such polynomial to $\mathbb{S}^{dN}$ is the eigenfunction of the Laplace — Beltrami operator in $L^2(\mathbb{S}^{dN})$ whose respective eigenvalue equals

$$\mu_d(N) = \mathcal{V}_d(N)(\mathcal{V}_d(N) + dN - 2). \tag{2.2}$$

Note, that finding the multiplicity of the eigenvalue $\mu_d(N)$ is not an easy task for $d > 1$. In [16] the author was able to find an algorithm for their calculation and in [12] the asymptotic behaviour for the value $\mathcal{V}_d(N)$ as $N \to \infty$ for a fixed d was obtained, namely,

$$\mathcal{V}_d(N) = \frac{d}{d+1}\sqrt[d]{d!}\,N^{1+\frac{1}{d}} - \frac{d}{2}N + O(N^{1-\frac{1}{d}}) \quad \text{as} \quad N \to \infty.$$

The easier case $d = 1$ was considered in [11]. Then

$$\psi_N^{(1)}(x_1, \dots, x_N) = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_N \\ x_1^2 & x_2^2 & x_3^2 & \dots & x_N^2 \\ \dots & \dots & \dots & \dots & \dots \\ x_1^{N-1} & x_2^{N-1} & x_3^{N-1} & \dots & x_N^{N-1} \end{vmatrix}. \tag{2.3}$$

The degree of the polynomial $\psi_N^{(1)}$ equals $\mathcal{V}_1(N) = \frac{N(N-1)}{2}$ and therefore

$$\mu = \mu_1(N) = \frac{N(N-1)(N^2+N-4)}{4}, \tag{2.4}$$

the multiplicity of which equals 1.

If u is antisymmetric, one can show that the equality $x_i = x_j$ implies $u(x) = 0$. Thus for $d = 1$ the space $\mathbb{R}^{dN}$ can be split into $N!$ cones with zero boundary condition for every antisymmetric function on it. But for arbitrary d the analogous proposition holds. When studying zeros of antisymmetric functions it was proved in [12] that for every antisymmetric u the space $\mathbb{R}^{dN}$ can be split into $N!$ parts, where u vanishes on the boundary.

Let us consider the action of symmetric group $\mathcal{S}_N$ on the space $\mathbb{R}^{dN}$. For an arbitrary $x = (x_1, \dots, x_N) \in \mathbb{R}^{dN}$ $(x_j \in \mathbb{R}^d)$ and $\sigma \in \mathcal{S}_N$ denote by $\sigma x = (x_{\sigma(1)}, \dots, x_{\sigma(N)})$ the permutation of elements x.

The important fact, see [12], is the following statement.

Proposition 2.1. *Let u be an antisymmetric function on $\mathbb{R}^{dN}$. For an arbitrary $x \in \mathbb{R}^{dN}$ and $\sigma \in \mathcal{S}_N$, $\sigma \neq id$, there is no continuous path $\Gamma : [0,1] \to \mathbb{R}^{dN}$, such that $\Gamma(0) = x$, $\Gamma(1) = \sigma x$ and $u(\Gamma(t)) \neq 0$ for all $t \in [0,1]$.*

This proposition implies

Corollary 2.1. *The support of each function $u \in H_A^1(\mathbb{R}^{dN})$ can be split into $N!$ parts such that each part can be obtained from another by action of $\mathcal{S}_N$ and u satisfies the Dirichlet boundary conditions on every part.*

For $d = 1$ we can consider $E_N = \{x \in \mathbb{R}^N \ : \ x_1 < x_2 < \dots < x_N\}$. In this case the parts from Corollary 2.1 are $\overline{\sigma E_N \cap \operatorname{supp} u}$, $\sigma \in \mathcal{S}_N$, because if $x_i = x_j$ for some i and j, then $u = 0$.

2.2. Proof of Theorem 1.1. The proof of Theorem 1.1 is based on the inequality (1.2) and the geometrical properties of the nodal sets of antisymmetric functions given in Proposition 2.1.

Each $u \in H_A^1(\mathbb{R}^N)$ satisfies the Dirichlet boundary conditions at the boundary of each cone σE_N. Let $V \geqslant 0$ be a symmetric function satisfying the conditions of Theorem 1.1 and let us consider the Schrödinger operator $\mathcal{H}(E_N)$, the quadratic form of which equals

$$\int_{E_N} \left(|\nabla u|^2 - V|u|^2\right) dx, \qquad u \in H_0^1(E_N).$$

The class of functions $H_A^1(\mathbb{R}^N)$ restricted to E_N coincides with $H_0^1(E_N)$ and can be extended by zero outside E. At the same time each function $u \in H_A^1(\mathbb{R}^N)$ can be defined by its values on E_N in the unique way. Using the classical CLR and LTh inequalities for each γ and $N = n$ satisfying (1.3), we obtain

$$\operatorname{Tr}(\Delta_{as} - V)_-^\gamma = \operatorname{Tr}\mathcal{H}_{as}(E_N)_-^\gamma \leqslant L_{\gamma,N} \int_{E_N} V^{\gamma+\frac{N}{2}} dx = \frac{L_{\gamma,N}}{N!} \int_{\mathbb{R}^N} V^{\gamma+\frac{N}{2}} dx,$$

where the constants $L_{\gamma,N}$ are defined in (1.2).

3. Lieb — Thirring inequalities with subtracted Hardy term

In order to proof Theorem 1.2 we need to remind some auxiliary results.

3.1. Hardy inequality for antisymmetric functions. We first notice that the properties of Vandermonde type determinants (2.1) in [12] are used in the proof of the following Hardy inequality.

Proposition 3.1. *Let $u \in H^1_A(\mathbb{R}^{dN})$. Then*

$$\int\limits_{\mathbb{R}^{dN}} |\nabla u(x)|^2 dx \geqslant H_A(dN) \int\limits_{\mathbb{R}^{dN}} \frac{|u(x)|^2}{|x|^2} dx,$$

where the constant $H_A(dN)$ is sharp and equals

$$H_A(dN) = \frac{(dN-2)^2}{4} + \mathcal{V}_d(N)(\mathcal{V}_d(N) + dN - 2). \tag{3.1}$$

Thus

$$H_A(dN) = \frac{(dN-2)^2}{4} + \mu_d(N),$$

where $\mu_d(N)$ is defined in (2.2) and $(dN-2)^2/4$ is the standard Hardy constant that comes from the radial part of the Laplacian in $L^2(\mathbb{R}^{dN})$.

In the case $d = 1$ and $N \geqslant 2$ the above formula (3.1) becomes explicit, see [11],

$$\int\limits_{\mathbb{R}^N} |\nabla u|^2\, dx \geqslant \frac{(N^2-2)^2}{4} \int\limits_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx, \quad u \in H^1_A(\mathbb{R}^N). \tag{3.2}$$

Remark 1. *Note that the standard constant in the Hardy inequality for functions in $H^1(\mathbb{R}^{dN})$ equals $(dN-2)^2/4 \sim (dN)^2/4$ as $N \to \infty$, while $H_A(dN) \sim \left(\frac{d\sqrt[d]{d!}}{d+1}\right)^2 N^{2+\frac{2}{d}}/4$ according to* [12].

3.2. Ekholm — Frank inequalities. In [3], [4] Egorov and Kondrat'ev proved the so–called weighted LTh inequalities, which say that for any $\gamma > 0$, $N \geqslant 2$, $\alpha \geqslant 0$, and any $V \geqslant 0$ satisfying $V \in L^{\gamma+\frac{N+\alpha}{2}}(\mathbb{R}^N, |x|^\alpha dx)$, there is a constant $C^{EK}_{\gamma,N,\alpha}$ independent of V such that

$$\operatorname{Tr}(-\Delta - V)^\gamma_- \leqslant C^{EK}_{\gamma,N,\alpha} \int\limits_{\mathbb{R}^N} V^{\gamma+\frac{N+\alpha}{2}} |x|^\alpha dx.$$

In [5] and [6] Ekholm and Frank obtained a stronger result, where the authors were able to subtract from the Laplacian the sharp Hardy term.

Proposition 3.2. *Let $\gamma > 0$, $N \geqslant 3$, $\alpha \geqslant 0$, then for any $V \geqslant 0$ satisfying $V \in L^{\gamma+\frac{N+\alpha}{2}}(\mathbb{R}^N, |x|^\alpha dx)$ there is a constant $C^{EF}_{\gamma,N,\alpha}$ independent of V such that*

$$\operatorname{Tr}\left(-\Delta - \frac{(N-2)^2}{4} - V\right)^\gamma_- \leqslant C^{EF}_{\gamma,N,\alpha} \int\limits_{\mathbb{R}^N} V^{\gamma+\frac{N+\alpha}{2}} |x|^\alpha dx. \tag{3.3}$$

One of important ingredients of the proof of the inequality (3.3) was its one–dimensional version.

Let us consider the operator in $L^2(\mathbb{R}_+)$, $\mathbb{R}_+ = (0,\infty)$, with the Dirichlet boundary condition at zero

$$-\frac{d^2}{dr^2} - \frac{1}{4r^2} - V(r).$$

Assume $\gamma > 0$ and $\alpha \geqslant 0$ such that $\gamma + \frac{1+\alpha}{2} \geqslant 1$. Then there is a constant $C_{\gamma,\alpha}$ independent of the potential $V \geqslant 0$, such that

$$\operatorname{Tr}\left(-\frac{d^2}{dr^2} - \frac{1}{4r^2} - V(r)\right)_-^\gamma \leqslant C_{\gamma,\alpha} \int\limits_0^\infty V^{\gamma+\frac{1+\alpha}{2}} r^\alpha\, dr. \tag{3.4}$$

3.3. Proof of Theorem 1.2. We give a proof that generalises the approach suggested in [5, 6]. Some other proofs of Hardy — Lieb — Thirring inequalities for fractional Schrödinger operators were given in [7], [10].

We introduce the orthogonal projection P in $L^2_A(\mathbb{S}^{dN-1})$ on the subspace

$$\mathcal{Y}_\mu = \{Y_{\mu,\ell}\}_{\ell=1}^{\varkappa}$$

defined by the spherical functions $Y_{\mu,\ell}$ associated with the eigenvalue $\mu_d(N)$ given by (2.2), the multiplicity of which equals $\varkappa$, see [16]. Then

$$Pu(x) = \sum_{\ell=1}^{\varkappa} u_\ell\, Y_{\mu,\ell}\,, \quad x \in \mathbb{R}^{dN},$$

where u_ℓ are the Fourier coefficients

$$u_\ell = (u, Y_{\mu,\ell})_{L^2(\mathbb{S}^{dN-1})}, \quad u \in L^2_A(\mathbb{R}^{dN}).$$

By Q we respectively denote its orthogonal complement $Q = \mathbb{I} - P$ in $L^2_A(\mathbb{R}^{dN})$. For any smooth function u we have

$$2\operatorname{Re}(PVQu, u) \leqslant 2\|V^{\frac12}Qu\|\|V^{\frac12}Pu\| \leqslant (PVPu, u) + (QVQu, u).$$

This implies

$$PVQ + QVP \leqslant PVP + QVQ. \tag{3.5}$$

Let $\mathfrak{H}_{as}$ be difined in (1.10). Using (3.5) we obtain

$$\begin{aligned}\mathfrak{H}_{as} - V =& P(\mathfrak{H}_{as} - V)P + Q(\mathfrak{H}_{as} - V)Q - PVQ - QVP\\ \geqslant& P(\mathfrak{H}_{as} - 2V)P + Q\left(\mathfrak{H}_{as} - 2V\right)Q\end{aligned}$$

and thus

$$\operatorname{Tr}(\mathfrak{H}_{as} - V)_-^\gamma \leqslant \operatorname{Tr}(P(\mathfrak{H}_{as} - 2V)P)_-^\gamma + \operatorname{Tr}(Q(\mathfrak{H}_{as} - 2V)Q)_-^\gamma.$$

After introducing polar coordinates the quadratic form of the operator $P\mathfrak{H}_{as}P$ becomes equal to

$$\begin{aligned}(\mathfrak{H}_{as}Pu, Pu) &= \int\limits_{\mathbb{R}^{dN}} \left(|\nabla Pu|^2 - H_A(dN)\,\frac{1}{|x|^2}\,|Pu|^2\right)\, dx\\ &= \sum_{\ell=1}^{\varkappa}\int\limits_0^\infty \left(|(u_\ell)'_r|^2 - \frac{(dN-2)^2}{4\,r^2}\,|u_\ell|^2\right)\, r^{dN-1}\, dr,\end{aligned} \tag{3.6}$$

where we used the action of the Laplace — Beltrami operator in $L^2(\mathbb{S}^{dN-1})$ and the identity

$$H_A(dN) - \frac{(dN-2)^2}{4} = \mu_d(N).$$

Changing in (3.6) the functions $u_\ell = v_\ell r^{\frac{1-dN}{2}}$, we find

$$(\mathfrak{H}_{as}Pu, Pu) = \sum_{\ell=1}^{\varkappa}\int\limits_0^\infty \left(|(v_\ell)'_r|^2 - \frac{1}{4\,r^2}\,|v_\ell|^2\right)\, dr \tag{3.7}$$

implying that for each ℓ the operator

$$-\frac{d^2}{dr^2}-\frac{(dN-2)^2}{4\,r^2}\quad\text{in}\quad L^2(\mathbb{R}_+,r^{dN-1}dr)$$

is unitary equivalent to

$$-\frac{d^2}{dr^2}-\frac{1}{4\,r^2}\quad\text{in}\quad L^2(\mathbb{R}_+).$$

We define

$$\widetilde{V}(r,\vartheta)=\sum_{m=1}^{\varkappa}V_m\,Y_{\mu,m}=\sum_{m=1}^{\varkappa}(V,Y_{\mu,m})_{L^2(\mathbb{S}^{dN-1})}\,Y_{\mu,m}.$$

Then

$$\int\limits_{\mathbb{R}^{dN}}V|Pu|^2\leqslant\int\limits_0^\infty\left(\sum_{m=1}^{\varkappa}V_m^2\right)^{\frac12}\sum_{\ell=1}^{\varkappa}|u_\ell|^2\,r^{dN-1}\,dr=\int\limits_0^\infty\left(\sum_{m=1}^{\varkappa}V_m^2\right)^{\frac12}\sum_{\ell=1}^{\varkappa}|v_\ell|^2\,dr.\tag{3.8}$$

Combining (3.6) and (3.8), we obtain

$$P(\mathfrak{H}_{as}-V)P\geqslant\bigoplus_{\ell=1}^{\varkappa}\left(-\frac{d^2}{dr^2}-\frac{1}{4\,r^2}-\left(\sum_{m=1}^{\varkappa}V_m^2\right)^{\frac12}\right).$$

Due to (3.4) with $\alpha+dN-1$ instead of α and Hölder's inequality we arrive at

$$\begin{aligned}\operatorname{Tr}(P(\mathfrak{H}_{as}-2V)P)_-^\gamma&\leqslant C\int\limits_0^\infty\left(\sum_{m=1}^{\varkappa}V_m^2\right)^{\frac{\gamma}{2}+\frac{dN+\alpha}{4}}r^{\alpha+dN-1}\,dr\\&\leqslant C\int\limits_0^\infty\left(\int\limits_{\mathbb{S}^{dN-1}}V^2\,d\vartheta\right)^{\frac{\gamma}{2}+\frac{dN+\alpha}{4}}r^{\alpha+dN-1}\,dr\\&\leqslant\frac{C}{|\mathbb{S}^{N-1}|^{\frac{2\gamma+dN+\alpha-4}{2\gamma+dN+\alpha}}}\int\limits_0^\infty\int\limits_{\mathbb{S}^{dN-1}}V^{\gamma+\frac{dN+\alpha}{2}}\,r^{\alpha+dN-1}\,d\vartheta dr\\&=\frac{C}{|\mathbb{S}^{dN-1}|^{\frac{2\gamma+dN+\alpha-4}{2\gamma+dN+\alpha}}}\int\limits_{\mathbb{R}^{dN}}V^{\gamma+\frac{dN+\alpha}{2}}\,|x|^\alpha\,dx.\end{aligned}\tag{3.9}$$

Treating the operator $Q(\mathfrak{H}_{as}-2V)Q$ we consider $\tilde{\mu}>\mu_d(N)$ to be the minimal eigenvalue of the antisymmetric Laplace — Beltrami operator restricted to $Q\,(L^2_A(\mathbb{S}^{dN-1}))$. If $0<t<1$, then due to the Hardy inequality

$$-Q\Delta_{as}Q=-t\,Q\Delta_{as}Q\,-(1-t)Q\Delta_{as}Q\geqslant-t\,Q\Delta_{as}Q+(1-t)\left(\frac{(dN-2)^2}{4}+\widetilde{\mu}\right)Q\frac{1}{|x|^2}Q.$$

Therefore

$$\begin{aligned}Q(\mathfrak{H}_{as}-2V)Q\geqslant{}&tQ\left(-\Delta_{as}-t^{-1}2V\right)Q\\&+\left((1-t)\left(\frac{(dN-2)^2}{4}+\widetilde{\mu}\right)-\frac{(dN-2)^2}{4}-\mu_d(N)\right)Q\frac{1}{|x|^2}Q\end{aligned}$$

and we now choose t such that

$$(1-t)\left(\frac{(dN-2)^2}{4}+\widetilde{\mu}\right)-\frac{(dN-2)^2}{4}-\mu_d(N)=0.$$

By using Proposition 3.2 and the variational principle we have

$$\begin{aligned}\mathrm{Tr}\,(Q(\mathfrak{H}_{as}-2V)Q)_-^\gamma &\leqslant t^\gamma\ \mathrm{Tr}\left(Q(-\Delta_{as}-t^{-1}2V)Q\right)_-^\gamma\\ &\leqslant t^\gamma\ \mathrm{Tr}\left((-\Delta_{as}-t^{-1}2V)\right)_-^\gamma\\ &\leqslant t^{-\frac{dN+\alpha}{2}}2^{\gamma+\frac{dN+\alpha}{2}}C\int\limits_{\mathbb{R}^{dN}}V(x)^{\gamma+\frac{dN+\alpha}{2}}\,|x|^\alpha\,dx.\end{aligned}\tag{3.10}$$

Adding together the inequalities (3.9) and (3.10), we complete the proof.

4. Sobolev and Gagliardo — Nirenberg inequalities

In this section we obtain versions of two classical inequalities for functions from $H^1_A(\mathbb{R}^{dN})$.

4.1. Sobolev inequalities. The classical Sobolev inequality (see [17]) states that for any $u\in H^1(\mathbb{R}^n)$, $n\geqslant 3$, we have

$$\int\limits_{\mathbb{R}^n}|\nabla u|^2\,dx\geqslant S(n)\left(\int\limits_{\mathbb{R}^n}|u|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}},\tag{4.1}$$

where $S(n)$ is the Sobolev constant

$$S(n)=\pi n\,(n-2)\left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)}\right)^{\frac{2}{n}}.$$

Remark 2. *Note that the Sobolev inequality* (4.1) *holds for any* $u\in H^1_0(\Omega)$, $\Omega\subset\mathbb{R}^n$, *with the same constant* $S(n)$.

Clearly when considering the class of functions from $H^1_A(\mathbb{R}^{dN})$ we expect an improvement of the constant $S_A(dN)$ in (4.1). Indeed, in the paper [12] the authors obtained the following result.

Theorem 4.1. *For any* $u\in H^1_A(\mathbb{R}^{dN})$, $dN\geqslant 3$, *we have*

$$\int\limits_{\mathbb{R}^{dN}}|\nabla u|^2\,dx\geqslant S_A(dN)\left(\int\limits_{\mathbb{R}^{dN}}|u|^{\frac{2dN}{dN-2}}\,dx\right)^{\frac{dN-2}{dN}},\tag{4.2}$$

where the constant $S_A(dN)$ *equals*

$$S_A(dN)=(N!)^{\frac{2}{dN}}\,S(dN).\tag{4.3}$$

Proof. The proof is simple and we present it here for completeness. Let E be one of the cones defined in Corollary 2.1. Then applying the classical Sobolev inequality to $u\in H^1_0(\sigma E)$, where σ is a permutation from $\mathcal{S}_N$, we obtain

$$\int\limits_{\sigma E}|\nabla u|^2\,dx\geqslant S(dN)\left(\int\limits_{\sigma E}|u|^{\frac{2dN}{dN-2}}\,dx\right)^{\frac{dN-2}{dN}}.$$

Note that due to the antisymmetry the integrals in the above inequalities are independent of $N!$ cones σE. Therefore

$$\int\limits_{\mathbb{R}^{dN}}|\nabla u|^2\,dx=N!\int\limits_E|\nabla u|^2\,dx\geqslant N!\,S(dN)\left(\int\limits_E|u|^{\frac{2dN}{dN-2}}\,dx\right)^{\frac{dN-2}{dN}}$$

$$= N!\, S(dN) \left((N!)^{-1} \int\limits_{\mathbb{R}^{dN}} |u|^{\frac{2dN}{dN-2}}\, dx \right)^{\frac{dN-2}{dN}}$$

$$= (N!)^{\frac{2}{dN}}\, S(dN) \left(\int\limits_{\mathbb{R}^{dN}} |u|^{\frac{2dN}{dN-2}}\, dx \right)^{\frac{dN-2}{dN}}.$$

□

Corollary 4.1. *Let $V \in L^{\frac{dN}{2}}(\mathbb{R}^{dN})$ be a symmetric function, $V \geqslant 0$. Assume that*

$$\left(\int\limits_{\mathbb{R}^{dN}} V^{\frac{dN}{2}}\, dx \right)^{\frac{2}{dN}} \leqslant S_A(dN).$$

Then

$$\operatorname{spec}(-\Delta_{as} - V)_- = \varnothing.$$

Proof. We consider the quadratic form of the operator $-\Delta_{as} - V$, applying Hölder's inequality and using the Sobolev inequality (4.2) to find

$$\int\limits_{\mathbb{R}^{dN}} \left(|\nabla u|^2 - V|u|^2\right)\, dx \geqslant \left(S_A(dN) - \left(\int\limits_{\mathbb{R}^{dN}} V^{\frac{dN}{2}}\, dx \right)^{\frac{2}{dN}} \right) \left(\int\limits_{\mathbb{R}^{dN}} |u|^{\frac{2dN}{dN-2}}\, dx \right)^{\frac{dN-2}{dN}} \geqslant 0.$$

□

4.2. Gagliardo — Nirenberg interpolation inequality. Similarly we can find a better constant in the classical Gagliardo — Nirenberg interpolation inequality that states

$$\left(\int\limits_{\mathbb{R}^n} |\nabla u|^2 dx \right)^{\theta} \left(\int\limits_{\mathbb{R}^n} |u|^2 dx \right)^{1-\theta} \geqslant S(q,n) \left(\int\limits_{\mathbb{R}^n} |u|^q dx \right)^{\frac{2}{q}}. \tag{4.4}$$

Here $n \geqslant 2$, $q \in \left(2, \frac{2n}{n-2}\right)$ and $\theta \in (0,1)$ satisfies the identity $\theta(n-2) + (1-\theta)n = \frac{2n}{q}$. The constant $S(q,n)$ in (4.4) equals

$$S(q,n) = \inf_{0 \neq u \in H^1(\mathbb{R}^n)} \frac{\left(\int\limits_{\mathbb{R}^n} |\nabla u|^2 dx \right)^{\theta} \left(\int\limits_{\mathbb{R}^n} |u|^2 dx \right)^{1-\theta}}{\left(\int\limits_{\mathbb{R}^n} |u|^q dx \right)^{\frac{2}{q}}}. \tag{4.5}$$

Remark 3. *The sharp constant $S(q,n)$ are known that for some values (q,n), see for example* [2]*. However, it is known that the infimum in* (4.5) *is achieved and up to translation and dilation such an infimum is a solution of the non–linear Euler — Lagrange equation*

$$-\Delta u - \lambda |u|^{q-2} u = -\mu u \quad in \quad \mathbb{R}^n.$$

It is also known that the minimiser in (4.4) *is spherical symmetric. Therefore, the constant $S(q,n)$ can be computed by using numerical methods applied to solutions of one–dimensional differential operator.*

Remark 4. *The inequality* (4.4) *holds for any $u \in H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^n$, with the same constant $S(q,n)$.*

We now can obtain a version of the Gagliardo — Nirenberg inequality.

Theorem 4.2. *Let $d, N \in \mathbb{N}$, $N \geqslant 2$ and $q \in \left(2, \frac{2dN}{dN-2}\right)$. Then for any $u \in H^1_A(\mathbb{R}^{dN})$ we have*

$$\left(\int\limits_{\mathbb{R}^{dN}} |\nabla u|^2 dx\right)^{\theta} \left(\int\limits_{\mathbb{R}^{dN}} |u|^2 dx\right)^{1-\theta} \geqslant S_A(q, dN) \left(\int\limits_{\mathbb{R}^{dN}} |u|^q dx\right)^{\frac{2}{q}}, \tag{4.6}$$

where $\theta \in (0,1)$ such that $\theta(dN-2) + (1-\theta)dN = \frac{2dN}{q}$ and

$$S_A(q, dN) = N!^{1-\frac{2}{q}} S(q, dN).$$

Proof. We use similar arguments as in the proof of Theorem 4.1. Let E is the set from Corollary 2.1. Then for any $u \in H^1(\mathbb{R}^{dN})$ we consider its restriction to E and use the property $u \in H^1_0(E)$. Therefore we can apply the Gagliardo — Nirenberg inequality on E and obtain

$$\begin{aligned}\left(\int\limits_{\mathbb{R}^{dN}} |\nabla u|^2 dx\right)^{\theta} \left(\int\limits_{\mathbb{R}^{dN}} |u|^2\, dx\right)^{1-\theta} &= N! \left(\int\limits_{E} |\nabla u|^2 dx\right)^{\theta} \left(\int\limits_{E} |u|^2 dx\right)^{1-\theta} \\ &\geqslant N! S(q, dN) \left(\int\limits_{E} |u|^q\, dx\right)^{\frac{2}{q}} \\ &= N!^{1-\frac{2}{q}}\, S(q, dN) \left(\int\limits_{\mathbb{R}^{dN}} |u|^q\, dx\right)^{\frac{2}{q}}.\end{aligned}$$

The proof is complete. □

4.3. Inequalities on the lowest eigenvalue.

In this subsection we obtain some bounds on the bottom of the spectrum of the Schrödinger operator $-\Delta_{as} - V$.

Theorem 4.3. *Let $d, N \in \mathbb{N}$, $N \geqslant 2$, and $\gamma > 0$. Assume that $V_+ \in L^{\gamma+\frac{dN}{2}}(\mathbb{R}^{dN})$, $V \geqslant 0$ and that V is symmetric (see (1.7)). Then*

$$\inf \operatorname{spec}(-\Delta_{as} - V) \geqslant -\Big(L^{(1)}_A(\gamma, dN) \int\limits_{\mathbb{R}^{dN}} V_+^{\frac{dN}{2}+\gamma} dx\Big)^{\frac{1}{\gamma}}, \tag{4.7}$$

where

$$L^{(1)}_A(\gamma, dN) = S_A(q, dN)^{-\frac{dN}{2}-\gamma} \frac{(2\gamma)^{\gamma} (dN)^{\frac{dN}{2}}}{(dN+2\gamma)^{\frac{N}{2}+\gamma}}, \qquad q = \frac{2(dN+2\gamma)}{dN-2+2\gamma}.$$

Proof. Let $u \in H^1_A(\mathbb{R}^{dN})$. Then by using Hölder's inequality we find

$$\int\limits_{\mathbb{R}^{dN}} V|u|^2\, dx \leqslant \left(\int\limits_{\mathbb{R}^{dN}} |u|^{\frac{2dN+4\gamma}{dN-2+2\gamma}} dx\right)^{\frac{dN-2+2\gamma}{dN+2\gamma}} \left(\int\limits_{\mathbb{R}^{dN}} V_+^{\frac{dN}{2}+\gamma} dx\right)^{\frac{2}{dN+2\gamma}}.$$

We denote

$$q = \frac{2(dN+2\gamma)}{dN-2+2\gamma}$$

and choose

$$\theta = \frac{dN}{dN+2\gamma} \in (0,1).$$

Then

$$\theta(dN-2)+(1-\theta)dN=\frac{2dN}{q}$$

and we can apply the inequality (4.6)

$$\int\limits_{\mathbb{R}^{dN}} V|u|^2dx \leqslant \frac{1}{S_A(q,dN)}\Bigg(\int\limits_{\mathbb{R}^{dN}} V_+^{\frac{dN}{2}+\gamma}dx\Bigg)^{\frac{2}{dN+2\gamma}}\Bigg(\int\limits_{\mathbb{R}^{dN}}|\nabla u|^2dx\Bigg)^{\theta}\Bigg(\int\limits_{\mathbb{R}^{dN}}|u|^2dx\Bigg)^{1-\theta}.$$

Let us introduce the notation

$$Z=\frac{1}{S_A(q,dN)}\Bigg(\int\limits_{\mathbb{R}^{dN}} V_+^{\frac{dN}{2}+\gamma}dx\Bigg)^{\frac{2}{dN+2\gamma}}.$$

Then

$$\int\limits_{\mathbb{R}^{dN}}(|\nabla u|^2-V|u|^2)dx \geqslant \int\limits_{\mathbb{R}^{dN}}|u|^2dx\cdot(t-Zt^{\theta}),$$

where

$$t=\frac{\int\limits_{\mathbb{R}^{dN}}|\nabla u|^2\,dx}{\int\limits_{\mathbb{R}^{dN}}|u|^2dx.}$$

The minimum of the function $f(t)=t-Zt^{\theta}$ is achieved on the positive semiaxis at $t=(\theta Z)^{\frac{1}{1-\theta}}$. Then

$$\begin{aligned}\int\limits_{\mathbb{R}^{dN}}(|\nabla u|^2-V|u|^2)dx &\geqslant \int\limits_{\mathbb{R}^{dN}}|u|^2dx\cdot\big((\theta Z)^{\frac{1}{1-\theta}}-Z(\theta Z)^{\frac{\theta}{1-\theta}}\big)\\ &= Z^{\frac{1}{1-\theta}}\theta^{\frac{\theta}{1-\theta}}(\theta-1)\int\limits_{\mathbb{R}^{dN}}|u|^2dx.\end{aligned}$$

Due to the variational principle we obtain

$$\begin{aligned}\inf\operatorname{spec}(-\Delta_{as}-V) &\geqslant -\frac{2\gamma}{dN+2\gamma}\cdot\left(\frac{dN}{dN+2\gamma}\right)^{\frac{dN}{2\gamma}} S_A(q,dN)^{-\frac{dN+2\gamma}{2\gamma}}\Bigg(\int\limits_{\mathbb{R}^{dN}} V_+^{\frac{dN}{2}+\gamma}dx\Bigg)^{\frac{1}{\gamma}}\\ &= -\Bigg(S_A(q,dN)^{-\frac{dN}{2}-\gamma}\frac{(2\gamma)^{\gamma}(dN)^{\frac{dN}{2}}}{(dN+2\gamma)^{\frac{dN}{2}+\gamma}}\cdot\int\limits_{\mathbb{R}^{dN}} V_+^{\frac{dN}{2}+\gamma}dx\Bigg)^{\frac{1}{\gamma}}\end{aligned}$$

and thus we complete the proof of Theorem 4.3. □

BIBLIOGRAPHY

1. M. Cwikel. *Weak type estimates for singular values and the number of bound states of Schrödinger operators* // Ann. Math. (2) **106**, 93 – 100 (1977).
2. M. Del Pino and J. Dolbeault. *Best constants for Gagliardo–Nirenberg inequalities and applications to nonlinear diffusions* // J. Math. Pures Appl. **81**:9, 847 – 875 (2002).
3. Yu.V. Egorov, V.A. Kondrat'ev. *On Spectral Theory of Elliptic Operators* Birkhäuser, Basel (1996).

4. Yu.V. Egorov, V.A. Kondrat'ev. *On an estimate of the number of points of the negative spectrum of the Schrödinger operator* // Mat. Sb., Nov. Ser. 134(176):4(12), 556–570 (1987). [Math. USSR–Sb. **62**:2, 551 – 566 (1989).]
5. T. Ekholm and R.L. Frank. *On Lieb — Thirring inequalities for Schrödinger operators with virtual level* // Commun. Math. Phys. **264**:3, 725–740 (2006).
6. T. Ekholm and R.L. Frank. *Lieb — Thirring inequalities on the half–line with critical exponent* // J. Eur. Math. Soc. **10**:3, 739–755 (2008).
7. R.L. Frank. *A simple proof of Hardy — Lieb — Thirring inequalities* // Commun. Math. Phys. **290**:2, 789–800 (2009).
8. R.L. Frank, A. Laptev and T. Weidl *Schrödinger Operators: Eigenvalues and Lieb — Thirring Inequalities.* Cambridge University Press, Cambridge (2023).
9. R.L. Frank, A.Laptev and L. Read. *Weighted CLR type bounds in two dimensions* // Trans. Am. Math. Soc. **377**:5, 3357–3371 (2024).
10. R.L. Frank, E.H. Lieb and R. Seiringer. *Hardy — Lieb — Thirring inequalities for fractional Schrödinger operators* // J. Amer. Math. Soc. **21**:4, 925–950 (2008).
11. Th. Hoffmann–Ostenhof and A. Laptev. *Hardy inequality for antisymmetric functions* // Funkts. Anal. Prilozh. **55**:2, 55–64 (2021). [Funct. Anal. Appl. **55**:2, 122–129 (2021).]
12. T. Hoffmann–Ostenhof, A. Laptev and I. Shcherbakov. *Hardy and Sobolev inequalities on antisymmetric functions* // Bull. Math. Sci. **14**:1, 2350010 (2023)
13. E. H. Lieb. *Bounds on the eigenvalues of the Laplace and Schrödinger operators* // Bull. Amer. Math. Soc. **82**:5, 751 – 753 (1976).
14. E. H. Lieb and W. E. Thirring. *Inequalities for the moments of the eigenvalues of the Schrodinger hamiltonian and their relation to Sobolev inequalities* // in "Studies in Mathematical Physics", Essays Honor Valentine Bargmann, 269–303 (1976).
15. G. V. Rozenblum. *Distribution of the discrete spectrum of singular differential operators* // Dokl. Akad. Nauk SSSR **202**, 1012–1015 (1972). [Sov. Math., Dokl. **13**, 245–249 (1972).]
16. I. Shcherbakov. *Spectrum of the Laplace — Beltrami operator on antisymmetric functions* // Probl. Mat. Anal. **126**, 99–106 (2024). [J. Math. Sci., New York **279**:4, 563–572 (2024).]
17. G. Talenti. *Best constant in Sobolev inequality* // Ann. Mat. Pura Appl., IV. Ser. **110**, 353–372 (1976).

Ari Laptev,
Imperial College London
180 Queen's Gate
London SW7 2AZ, UK
Sirius Mathematics Center
Sirius University of Science and Technology
1 Olympic Ave, 354340, Sochi, Russia
E-mail: a.laptev@imperial.ac.uk

Ilia Alexandrovich Shcherbakov,
Sirius Mathematics Center
Sirius University of Science and Technology,
1 Olympic Ave, 354340, Sochi, Russia
E-mail: stscherbakov99@yandex.ru