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BI–CONTINUOUS SEMIGROUPS OF STOCHASTIC QUANTUM DYNAMICS

A.V. UTKIN

Abstract. This paper is devoted to the aspects of derivation of dynamics equation of quantum system under a stochastic dynamics. We study the conditions, under which a sequence of random variations of wave function can approximate a random diffusion process in a Hilbert space. A random variation $|\psi_0\rangle \mapsto G_{t_N} \dots G_{t_1}|\psi_0\rangle = |\psi_{t_N}\rangle$ is associated with a transform of distribution of vector $|\psi\rangle$, as well as with the variation of its characteristic functional $\varphi(v) = \mathbb{E}\exp(i\,\mathrm{Re}\langle v|\psi\rangle)$. For a continuous random walk we study the approximation of a Markov semigroup by the Markov operators of discrete random walk. We pay a special attention to the cases, when the derivative of random operator $F'(0)$ is an unbounded operator. However, we restrict the consideration to the case when the Markov operators of random walks with the operators G_t mutually commute.

The characteristic functional is transformed by the Markov operator of adjoint process, and in contrast to the dynamics of wave function, it has a deterministic nature that allows us to rely on the developed theory of semigroups in Banach spaces. The most illustrative examples are the process of continuous measurement, that is, the measurement of trajectories of some observable, and the random control.

Keywords: bi–continuous semigroup, random operator–valued function, Markov operator.

Mathematics Subject Classification: 81P20, 60J60, 47D99

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space, on which a family of random operators $\{G_t\}$ act. For each $t \in [0,T]$, $\omega \mapsto G_t(\omega)$ is a measurable mapping from the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ into the space of bounded operators $\mathcal{B}(\mathcal{H})$ on $\mathcal{H}$. We suppose that in some sense, which will be clarified later, as $t \to 0$, the operators G_t tend to the identity operator I. We are going to construct a model of continuous process of random walk on a Hilbert space by using the model of second wonderful limit. We partition the segment $[0,t]$ into N equal parts and on each of them we define a random variation of vectors $|\psi\rangle \mapsto G_{t/N}|\psi\rangle$. We say that the total variation on the segment $[0,t]$ is the sum of composition of independent transforms on each segment in the chronological order, that is, the vector $|\psi\rangle$ is transformed by the rule

$$|\psi\rangle \mapsto G_N \dots G_1|\psi\rangle, \tag{1.1}$$

where the operators $G_N, \dots, G_1$ are independent and identically distributed in accordance with the distribution $G_{\frac{t}{N}}$. The question is whether it is possible, in the limit $N \to \infty$, to approximate a time–continuous stochastic dynamics. The solution is made in accordance with Markov semigroups approach. The above described stochastic process is Markov and for each $t \in [0,T]$, it is assigned with the system of Markov operators $\{\mathbf{F}[G_t]^k, 1 \leqslant k \leqslant N\}$. The operator $\mathbf{F}[G_t]$

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acts on the space of bounded Borel functions $B_B(\mathcal{H})$ on the Hilbert space, and we study the question under which conditions for each $t \geqslant 0$ some Markov semigroup $\{\mathbf{T}_t\}$ can be approximated by the operators $\{\mathbf{F}[G_{\frac{t}{N}}]^N\}$ as $N \to \infty$. All the obtained below results are true under the assumption that the commutation conditions $\Big[\mathbf{F}[G_t], \mathbf{F}[G_s]\Big] = 0$ hold for all $0 \leqslant s, t \leqslant T$. It should be said that while the condition is rather restrictive, it does not imply that the corresponding random operators $G_t(\omega)$ and $G_s(\omega)$ commute almost surely, see Example 3.1.

The main results are Theorem 5.1, which establishes the conditions for approximation of $\mathbf{T}_t$ by Chernoff iterations $\mathbf{F}[G_{\frac{t}{N}}]^N$ on some closed subspace $F_G \subset B_B(\mathcal{H})$, and Theorem 5.2, which admits to continue the limiting semigroup on $B_B(\mathcal{H})$ to a Markov semigroup.

We deal with the following functional spaces: $B_B(\mathcal{H})$ is the space of Borel (with respect with the topology of norm or weak topology, this is the same) bounded functions on the Hilbert space with sup–norm: $\|f\| = \sup\limits_{v\in\mathcal{H}} |f(v)|$, $C_B(\mathcal{H})$ is the space of continuous bounded functions on $\mathcal{H}$ with the sup–norm and an additional topology τ of uniform convergence on bounded sets, $C_{BWS}(\mathcal{H})$ is the space of bounded weakly sequentially continuous functions. For this operator family $\mathbf{F}[G_t]$ obeying $\Big[\mathbf{F}[G_t], \mathbf{F}[G_s]\Big] = 0$, we define the subspace $\mathcal{L}_G \subset C_B(\mathcal{H})$ of functions f, for which the uniformly bounded in norm and uniform on balls in $\mathcal{H}$ convergence of differential inclusions $\dfrac{\mathbf{F}[G_t]f - f}{t}$ holds. The space F_G is defined as the set of functions, which are approximated uniformly on balls by the uniformly bounded sequence of functions in $\mathcal{L}_G$. Then under the condition that for all $r > 0$, $\varepsilon > 0$ there exists $R > 0$, for which

$$\sup\Big\{\mathbb{P}\Big(\|G_t^{\perp\!\!\perp m} v\| > R\Big), \quad \|v\| \leqslant r,\ t \in [0,T],\ m \in \mathbb{N}\Big\} < \varepsilon, \tag{1.2}$$

Theorem 5.1 states that for each $f \in F_G$ the identity $\tau \lim\limits_{N\to\infty} \mathbf{F}[G_{\frac{t}{N}}]^N f = \mathbf{T}_t f$ holds for some semigroup $\mathbf{T}_t$ uniformly in t on the segments of ray $[0, +\infty)$. The symbol $G_t^{\perp\!\!\perp m}$ denotes the product of independent identically distributed random operators G_t, the total amount of which is m.

It turns out that under the assumption that $C_{BWS}(\mathcal{H}) \subset F_G$, the semigroup $\mathbf{T}_t$ can be extended from F_G to the Markov semigroup $\mathbf{Q}_t$ on $B_B(\mathcal{H})$ (Theorem 5.2), and for all $f \in C_B(\mathcal{H})$, $v \in \mathcal{H}$, $t \geqslant 0$ we have $\lim\limits_{N\to\infty} \mathbf{F}[G_{\frac{t}{N}}]^N f(v) = \mathbf{Q}_t f(v)$. Nevertheless, sometimes it is very non–trivial to verify the inclusion $C_{BWS}(\mathcal{H}) \subset F_G$. It is interesting that if $\mathbf{F}[G_t]$ is already a semigroup on $C_{BWS}(\mathcal{H})$, then the conditions

$$\forall \varepsilon > 0, r > 0 \quad \exists R > 0 \quad \text{such that} \quad \sup_{t\in[0,T]} \sup_{\|v\|\leqslant r} \mathbb{P}\Big(\|G_t v\| > R\Big) < \varepsilon,$$
$$\forall \varepsilon > 0, w \in \mathcal{H} \quad \lim_{t\to 0} \mathbb{P}\Big(\|(G_t^* - G_{t_0}^*)w\| > \varepsilon\Big) = 0$$

are sufficient to ensure that $\mathbf{F}[G_t]$ is bi–continuous semigroup (Theorem 4.1).

There exist semigroups represented explicitly (Proposition 4.3). The main attention is paid to the diffusion processes, the generator of which is a second order differential operator. If the homomorphism $\Upsilon : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ is defined on subalgebra $\mathcal{E} \subset C_B(\mathbb{R})$ containing the functions of form $\{g[a,b] := e^{-ax^2+bx}\}$, then the family $\{\mathbf{F}[G_t]\}$ for $G_t(y) = \Upsilon(g[at,\ yb\sqrt{t}])$ with some $a > 0$, $b \in \mathbb{R}$ forms a semigroup on $B_B(\mathcal{H})$ with respect to the measure $d\mathbb{P}(y) = \dfrac{e^{-y^2}dy}{\sqrt{\pi}}$. For instance, for a self–adjoint operator C the family $\{\exp(yb\sqrt{t}C - atC^2)\}$ defines a semigroup. Under some natural assumptions this semigroup is Markov and its restriction to $C_{BWS}(\mathcal{H})$ is bi–continuous. The generator of this semigroup is the operator

$$f(v) \mapsto -\left(a - \frac{b^2}{4}\right) df(v)[Cv] + \frac{b^2}{4} d^2 f(v)[Cv, Cv], \tag{1.3}$$

the domain of which has bi–dense closure containing the space $C_{BWS}(\mathcal{H})$.

The proposed method allows one to construct more complicated semigroups not writing out explicitly the expressions for the operators. For instance, there arises a question whether the operator–valued function obtained by averaging the above proposed semigroups $\{\mathbf{F}[G_t]\}$, $G_t(y,\alpha) = \exp(yb\sqrt{t}C_\alpha - atC_\alpha^2)$, with a random self–adjoint operator $C = \{C_\alpha\}$ satisfies the conditions of Chernoff approximation. More precisely, let $d\nu(\alpha)$ be a probability measure and

$$(\mathbf{F}_t f)(v) = \int d\nu(\alpha) \int \frac{dy}{\sqrt{\pi/\gamma}} e^{-\gamma y^2}\, f\left(e^{yb\sqrt{t}C_\alpha - at^2 C_\alpha^2} v\right). \tag{1.4}$$

However, it is not easy to answer this question in the general case. We introduce the operators

$$A_1 : \mathscr{D}(A_1) \to \mathcal{H},\ \ A_2 : \mathscr{D}(A_2) \to \mathcal{H}^{\otimes 2},$$

appearing in the limits

$$\lim_{t\to 0} \int d\nu(\alpha) \int \frac{dy}{\sqrt{\pi/\gamma}} e^{-\gamma y^2}\, \frac{G_t(y,\alpha)w - w}{t} = A_1 w, \qquad w \in \mathscr{D}(A_1),$$

$$\lim_{t\to 0} \int d\nu(\alpha) \int \frac{dy}{\sqrt{\pi/\gamma}} e^{-\gamma y^2}\, \frac{(G_t(y,\alpha)w - w)^{\otimes 2}}{2t} = A_2 w^{\otimes 2}, \quad w^{\otimes 2} \in \mathscr{D}(A_2).$$

We give the positive answer if the commutation condition for Markov operators hold, A_1 and A_2 are bounded or possess in some sense consistent point spectrum. This example is described in more detail in Section 5.1. There we also provide a similar example of random control characterized by the fact that instead of self–adjoint operators C_α, anti–Hermitian operators iH_α are used. The reason of difficulty is that for unbounded operators A_1, A_2 it is non–trivial to construct the functions $f \in C_B(\mathcal{H})$, which remain in $C_B(\mathcal{H})$ under the action $\frac{d}{dt}F_t\Big|_{t=0}$.

Examples of diffusion and Poisson processes have long been known and are used in physical applications. Methods of white noise theory and stochastic processes are successfully employed to analyze bosonic, fermion, nonlinear and open quantum systems ([1]–[4]). In particular, the behavior of system under the influence of continuous measurement were studied in the works ([5]–[7]) theoretically and experimentally.

The paper consists of four sections. Sections 2 and 3 elucidate some issues of quantum theory, Markov processes, and the theory of strongly continuous and bi–continuous semigroups and their approximation. The study is based on an illustrative example of a continuous process of inaccurate coordinate measurement ([8]–[10]), leading to a stochastic differential Schrödinger — Belavkin equation. Its feature is in the commutativity of operators realizing the random walk, which allows one to decompose the dynamics into independent processes in terms of the components $\psi(x)$ of wave function $|\psi\rangle$. This example is generalized to arbitrary processes in an infinite–dimensional separable Hilbert space (Section 4). Section 5 contains formulations of the main theorems 5.1, 5.2 and justification of examples of processes in the scheme of continuous measurements and random unitary controls.

2. Preliminaries

2.1. Process of quantum measurement. A fundamental feature of quantum mechanics is that the measurement process is probabilistic. In addition, it makes a destructive effect on the system, often called the collapse of the wave function, or the reduction of the wave packet determined by the Lüders — von Neumann projection postulate [3], [11]. The most general (for our purposes) definition of quantum measurement is formulated in terms of the completely positive instrument introduced in [12]. The measurement means obtaining the distribution of measurement result in the measurable space $(\mathcal{Y}, \Sigma)$ for a given state $\rho \in \mathfrak{S}(\mathcal{H})$, and the law of change of statistical ensemble (state) during measurement, if the information about the obtained measurement event is available. For example, the result of a measurement can be an element of the measurable space $(\mathfrak{X}, \mathscr{B})$, then in the case of repeated measurements the result already lies in the measurable space $(\mathcal{Y}, \Sigma) = (\mathfrak{X}^J, \mathscr{B}^{\otimes J})$ of functions $J \to \mathfrak{X}$ (for example, $J \subset \mathbb{R}$ is countable for a discrete measurement process and $J = [0, T]$ for a continuous measurement process on the interval $t \in [0, T]$).

We give the definition of a completely positive instrument [13]–[15]. The set of positive trace–class operators on $\mathcal{H}$ with the trace 1 is denoted by $\mathfrak{S}(\mathcal{H})$ and is called the set of quantum states. The space of trace–class operators on $\mathcal{H}$ is denoted by $\mathfrak{T}(\mathcal{H})$.

Definition 2.1. *The mapping* $\mathbf{M}[B](\rho) : \Sigma \times \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H})$ *is called the completely positive instrument if the conditions hold:*

1. *for each* $B \in \Sigma$ *the mapping* $\rho \mapsto \mathbf{M}[B](\rho)$ *is affine (can be continued to a linear mapping on* $\mathfrak{T}(\mathcal{H})$*);*
2. $\mathbf{M}[B]$ *is a completely positive mapping for all* $B \in \Sigma$*;*
3. $\mathbf{M}[B](\rho)$ *is* σ*–additive in the sense of weak topology on* $\mathfrak{T}(\mathcal{H})$ *for each* $\rho \in \mathfrak{T}(\mathcal{H})$*;*
4. $\mathbf{M}[\mathcal{Y}]$ *preserves the trace:* $\operatorname{Tr} \mathbf{M}[\mathcal{Y}](\rho) = 1$ *for all* $\rho \in \mathfrak{S}(\mathcal{H})$*.*

In what follows a completely positive instrument is sometimes simply called instrument.

In the concept of instruments it is supposed that when measuring by an instrument $\mathbf{M}$, the probability that the result is into the measurable set B for the initial state $\rho \in \mathfrak{S}(\mathcal{H})$ (which is also called a priori) is equal to

$$\mu_\rho(B) = \operatorname{Tr} \mathbf{M}[B](\rho), \tag{2.1}$$

while the part of statistical ensemble for such events is described by the state

$$\rho' = \frac{\mathbf{M}[B](\rho)}{\operatorname{Tr} \mathbf{M}[B](\rho)}. \tag{2.2}$$

According to the above definitions, under a successive measuring by means of the instruments $\mathbf{M}_1, \ldots, \mathbf{M}_n$ at the times $t_1 < \ldots < t_n$ with the space of events $(\mathfrak{X}, \mathscr{B})$, the statistics is defined by the probability distribution on $(\mathfrak{X}^n, \mathscr{B}^{\otimes n})$ associated with the instrument

$$\mathbf{M}_{t_{1,\ldots,t_n}}[B_1 \times \ldots \times B_n](\rho) = \mathbf{M}_{t_n}[B_n](\ldots \mathbf{M}_{t_1}[B_1](\rho) \ldots), \quad B_i \in \mathscr{B}. \tag{2.3}$$

2.2. Random processes corresponding to quantum instrument. For clarity, let us consider an example of an instrument with a discrete set of events. Namely, let $\mathfrak{X} = \{x_k\}$, $\Sigma = 2^{\mathfrak{X}}$ be the σ–algebra containing all subsets and the instrument be defined in the form $\mathbf{M}[B](\rho) = \sum\limits_{x_k \in B} \mathbf{M}_k(\rho)$, where $\mathbf{M}_k$ are completely positive mappings of form

$$\mathbf{M}_k(\rho) = \sum_m G_{km} \rho G_{km}^*.$$

We require $\sum\limits_{m,k} G_{km}^* G_{km} = I$.

There is a state transformation rule for each obtained event x_k:

$$(\rho, k) \mapsto \frac{\mathbf{M}_k(\rho)}{\operatorname{Tr}\ \mathbf{M}_k(\rho)}. \tag{2.4}$$

We follow the interpretation that an ensemble in which a pure state with a unit vector $|\psi_k\rangle$ occurs with probability p_k can be described by a quantum state of form $\rho = \sum\limits_k p_k |\psi_k\rangle\langle\psi_k|$.

Therefore, the action of the instrument $\mathbf{M}$ can be written in terms of a random walk on $\mathcal{H}$:

$$(|\psi\rangle, \omega) \mapsto \frac{G_{k(\omega)m(\omega)}|\psi\rangle}{\|G_{k(\omega)m(\omega)}|\psi\rangle\|}, \tag{2.5}$$

where k and m are random indices and only $k(\omega)$ is the observable fixed by a measuring device. The wave function in the right hand side is normalized and the probability $p_{k_0 m_0}(\psi) = \mathbb{P}(k(\omega) = k_0, m(\omega) = m_0)$ is defined by the formula

$$p_{k_0 m_0}(\psi) = \|G_{k_0 m_0}|\psi\rangle\|^2, \qquad \forall k_0, m_0,$$

which in accordance with the axioms of quantum mechanics, gives a correct expression for the state

$$\mathbf{M}[\mathbb{R}](|\psi\rangle\langle\psi|) = \sum_{k,m} p_{km}(\psi) \frac{G_{km}|\psi\rangle}{\|G_{km}|\psi\rangle\|} \frac{\langle\psi|G^*_{km}}{\|G_{km}|\psi\rangle\|}. \tag{2.6}$$

The transition probabilities p_{km} depend on $|\psi\rangle$, and this fact is due a restriction for the normalization of wave functions. An attempt to remove the dependence of probabilities on the initial state gives rise to a random walk on the Hilbert space, which can be characterized as linear.

We consider the following transformation. Let $\{\pi_{km}\}_{k,m}$ be the probability distribution. Then the mapping

$$(|\psi\rangle, \omega) \mapsto \frac{G_{k(\omega)m(\omega)}|\psi\rangle}{\sqrt{\pi_{k(\omega)m(\omega)}}}, \tag{2.7}$$

where $\pi_{k_0 m_0} = \mathbb{P}\Big(k(\omega) = k_0, m(\omega) = m_0\Big)$, defines the action of the instrument $\mathbf{M}$ in the sense that

$$\mathbf{M}[\mathbb{R}](|\psi\rangle\langle\psi|) = \sum_{k,m} \pi_{km} \frac{G_{km}|\psi\rangle}{\sqrt{\pi_{km}}} \frac{\langle\psi|G^*_{km}}{\sqrt{\pi_{km}}} \tag{2.8}$$

The distribution of random variables k, m are independent of $|\psi\rangle$ and this is why we can choose a probability space with random elements $|\psi\rangle, k, m$, independent in total so that the formulas (2.7) and (2.8) are satisfied.

We have described just a case of the single action of instrument. If several measurements are made successively, the events form a random sequence $(k_1(\omega), k_2(\omega), \ldots)$. At the same time, each outcome is assigned with a random process on the set of quantum states, so–called quantum trajectory $\{\rho_n, k_n\}$ (see, for instance, [5]), which can be described recursively

$$(\rho_n, k_n) \mapsto (\rho_{n+1}, k_{n+1}), \qquad \rho_{n+1}(\omega) = \frac{\sum_m G_{k_{n+1}m}\rho_n G^*_{k_{n+1}m}}{\operatorname{Tr} \sum_m G_{k_{n+1}m}\rho_n G^*_{k_{n+1}m}}. \tag{2.9}$$

In the general case by a linear random walk on a Hilbert space we mean the following random process.

Definition 2.2. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, $\mathcal{H}$ be a Hilbert space. A linear random walk on $\mathcal{H}$ is the set $\{G_{(t|s)}(\omega)\}_{0\leqslant s\leqslant t,\ t,s\in J} \subset \mathcal{B}(\mathcal{H})$ if*

1. *$\omega \mapsto G_{(t|s)}(\omega)$ is measurable in the pair $(\mathscr{F}, \mathscr{B}_{WOT})$, where $\mathscr{B}_{WOT}$ is the Borel σ–algebra of weak operator topology;*
2. *$G_{(r|t)}G_{(t|s)} = G_{(r|s)}$ almost surely for all $0 \leqslant s \leqslant t \leqslant r$ and $G_{(t|t)} = I$ almost surely.*

The additional requirement

$$\mathbb{E}\Big[G^*_{(t|s)}G_{(t|s)}\Big] = I$$

in the sense of the Pettis integral allows us to relate this linear random walk with the system of quantum channels

$$\Phi_{(t|s)}(\rho) = \mathbb{E}\Big[G_{(t|s)}\rho G^*_{(t|s)}\Big], \quad 0 \leqslant s \leqslant t.$$

The above described example of discrete process of measurements is a particular case of linear random walks when $J = \{t_0, t_1, \ldots, t_N\} \subset \mathbb{R}_+$ is discrete,

$$G_{(t_\ell|t_{\ell+1})}(\omega) = \frac{G_{k(\omega)m(\omega)}}{\sqrt{\pi_{k(\omega)m(\omega)}}},$$

and

$$\mathbb{P}\Big(k(\omega) = k, m(\omega) = m\Big) = \pi_{km}.$$

The main object of our study is the passage to the limit as the number of measurements tend to infinite and the influence lessens ([5], [10], [16]). Here we obtain stochastic differential equations for the density operator or for the wave function. Such approach is called unravelling, see [3, Ch. Ⅲ], [4] for more detail.

2.3. Markov processes and semigroups. We should give some ideas about the Markov processes, which we shall deal with ([17]–[19]).

Definition 2.3. *Let $(\mathcal{Y}, \Sigma)$ be a measurable space. A random process $\{X_t : \Omega \to \mathcal{Y},\ t \geqslant 0\}$ is called Markov if the identity $\mathbb{E}(X_t|\mathscr{F}_s) = \mathbb{E}(X_t|\sigma(X_s))$ holds for all $0 \leqslant s \leqslant t$. Here $\mathscr{F}_s = \sigma\left(X_{s'},\ 0 \leqslant s' \leqslant s\right)$ is the natural filtration of process X_t.*

Markov processes are associated with a transition probability that completely characterizes their probabilistic properties (i.e., completely specifies their joint distributions). Note that the existence of a transition probability is not trivial and the condition that $\mathcal{Y}$ is a separable complete metric space is sufficient [17, Vol. 2, Ch. I, Sect. 2].

Definition 2.4. *The system of functions $P(B, t|x, s)$, where $x \in \mathcal{Y}$, $0 \leqslant s \leqslant t$, $B \in \Sigma$ is called the transition probability of the process X_t if*

1. *for fixed $0 \leqslant s \leqslant t$ and $B \in \Sigma$ the function $P(B, t|x, s)$ is measurable in x, and for fixed s, t and $x \in \mathcal{Y}$ it is a probability measure of B;*
2. *$P(B, t|x, t) = \chi_B(x)$;*
3. *$P(B, t|x, s) = \mathbb{P}(X_t \in B \mid X_s = x)$ (for almost each x with respect to the measure $\mathbb{P}_{X_s}$);*
4. *for $\mathbb{P}_{X_s}$–almost each x the Chapman — Kolmogorov equation*

$$P(B, t|x, s) = \int P(B, t|y, r)P(dy, r|x, s), \qquad 0 \leqslant s \leqslant r \leqslant t, \tag{2.10}$$

holds.

By the functions $P(B, t|x, s)$ we construct Markov operators $\mathbf{P}_{(t|s)}$ on the Banach space $B_B(\mathcal{Y})$ of Borel bounded functions with sup–norm

$$(\mathbf{P}_{(t|s)}f)(x) = \int f(y)P_t(dy, t|x, s) = \mathbb{E}(f(X_t)|X_s = x). \tag{2.11}$$

If the transition probability $P(B, t|x, s)$ depends only on the difference of times $t - s$, that is, $P(B, t + h|x, t) = P_h(B, x)$, then the corresponding Markov process X_t is called time homogeneous.

It follows from the Chapman — Kolmogorov equation that for a homogeneous Markov process the family $\{\mathbf{P}_t := \mathbf{P}_{(t|0)},\ t \geqslant 0\}$ forms a Markov semigroup with the generator

$$\mathbf{L} = \lim_{t \to 0} \frac{\mathbf{P}_{t+h} - \mathbf{P}_t}{h}$$

on the domain

$$\mathcal{D}(\mathbf{L}) = \left\{ f \in B_B(\mathcal{Y}) \;\middle|\; \exists \lim_{t\to 0} \frac{\mathbf{P}_{t+h}f - \mathbf{P}_t f}{h} \right\};$$

the limit is taken in the sense of the norm.

To a linear random walk on a Hilbert space, under which a random vector $|\psi\rangle \in \mathcal{H}$ is mapped into the random vector $G_{(t|s)}|\psi\rangle$ under the condition that the operator $G_{(t|s)}$ is independent of $|\psi\rangle$, the Markov operator

$$f(v) \mapsto \int d\mathbb{P}(d\psi_t, t|v,0) f(\psi_t) = \int d\mathbb{P}(\omega)\; f(G_{(t|s)}(\omega)v), \quad v \in \mathcal{H}, \tag{2.12}$$

is related.

In this example, the process is defined on a separable Hilbert space, so the transition probabilities are given.

The characteristic functional of a random vector ξ with a value in an infinite–dimensional Banach space X is defined by the formula $\varphi(v) = \mathbb{E}e^{i\,\mathrm{Re}\,\ell(\xi)}$, $\ell \in X^*$ (see [18, Ch. V, Def. 9], where the definition is given for real–valued random vectors). The complex case is reduced to the real one by the realification procedure $X \to X_{\mathbb{R}}$, in which, as it is easy to see, $(X_{\mathbb{R}})^* = \{x \mapsto \mathrm{Re}\,\ell(x)\}$. The important for us example is $X = \mathcal{H}$, in which any continuous functional is a scalar product with a fixed vector $v \in \mathcal{H}$. In general, the Markov operator $\mathbf{P}_{(t|s)}$ does not map the characteristic functional of the random vector $|\psi_s\rangle$ into the characteristic functional of the random vector $|\psi_t\rangle$. However, for a linear random walk $\{G_{(t|s)}\}$, the variation

$$\varphi_s(v) = \mathbb{E}e^{i\,\mathrm{Re}\langle v|\psi_s\rangle} \mapsto \varphi_t(v) = \mathbb{E}e^{i\,\mathrm{Re}\langle v|\psi_t\rangle}$$

under the evolution on the interval (s,t) occurs under the action of Markov process of adjoint linear random walk $\{G^*_{(t|s)}\}$ as the identity

$$\mathbb{E}e^{i\,\mathrm{Re}\langle v|\psi_t\rangle} = \mathbb{E}e^{i\,\mathrm{Re}\langle G^*_{(t|s)}v|\psi_s\rangle} = \int d\mathbb{P}(\omega)\; \varphi_s(G^*_{(t|s)}v)$$

shows.

We will be interested in random processes being the solutions to stochastic differential equations of a special type on a finite–dimensional space

$$\begin{cases} dX_t = AX_t dt + \sum\limits_k B_k X_t dW_{k,t}, \\ X_0 = \xi, \end{cases} \tag{2.13}$$

where $\{A, B_k\}$ are linear operators, $W_t = (W_{k,t})$ is a multi–dimensional Wiener process, and their generalizations to the infinite–dimensional case. Such processes are called geometric Brownian motions, and they play an important role in financial mathematics (see [18], [20], [21]) and quantum mechanics. These are Markov processes, and in finite–dimensional spaces they are described by Markov semigroups with generators of the form

$$(\mathbf{L}f)(v) = (Av, df(v)) + \frac{1}{2}\sum_{k,l}(B_k v, B_l v)\frac{\partial^2 f(v)}{\partial_k \partial_l}. \tag{2.14}$$

The definition of infinite–dimensional analogues is non–trivial; the works [22]–[24] are devoted to them. This work also touches the issue of constructing processes, the Markov operators of which have a generator similar to (2.14).

The process X_t with the initial condition $X_0 = \xi$ can be obtained by approximating by discrete–time processes $X_t^{(N)}$

$$X_{t_N}^{(N)} = \left(1 + A(t_N - t_{N-1}) + \sum_k B_k (W_{k,t_N} - W_{k,t_{N-1}})\right)$$

$$\dots \left(1 + A(t_1 - t_0) + \sum_k B_k(W_{k,t_1} - W_{k,t_0})\right) \xi.$$

In the next sections we implement the following idea: for a homogeneous linear random walk on a Hilbert space we construct a Markov semigroup, calculate its generator, which will have a form similar to the generator of diffusion process (2.13), and approximate the semigroup by a family of Markov operators of a discrete linear random walk.

2.4. Process of continuous imprecise measurements of coordinates. As usually, the set of states on a separable Hilbert space $\mathcal{H}$ is denoted by $\mathfrak{S}(\mathcal{H})$, and the space of bounded operators is denoted by $\mathcal{B}(\mathcal{H})$. However, we often deal with unbounded operators on a dense domain. These are the coordinate operator $\hat{x}$ and the momentum operator $\hat{p}$. Using the coordinate representation, that is, the identification $\mathcal{H} \simeq \mathbb{L}_2(\mathbb{R}, dx)$, $|\psi\rangle \mapsto \psi(x)$, we can develop a functional calculus for the operator $\hat{x}$. For each Borel bounded function $f(x)$ on the line, the operator $f(\hat{x}) \in \mathcal{B}(\mathbb{L}_2(\mathbb{R}))$ of multiplication by the function f is defined

$$(f(\hat{x})\psi)(x) = f(x)\psi(x).$$

The measurement instrument corresponding to the simplest imprecise measurement of the coordinate, see [25, Ex. 2], [15], reads as follows

$$\rho \mapsto \mathbf{M}[B](\rho) = \int_B \sqrt{p(yI - \hat{x})}\rho\sqrt{p(yI - \hat{x})}\; dy, \quad B \in \mathscr{B}(\mathbb{R}), \quad \rho \in \mathfrak{S}(\mathcal{H}). \tag{2.15}$$

Here I is the identity mapping, $\hat{x}$ is the operator of coordinate, p is the density of some probability measure. The random event of measurement $q(\omega)$ has the distribution

$$\int_B d\mathbb{P}_q = \int_B \operatorname{Tr} \rho p(yI - \hat{x})\; dy = \operatorname{Tr} \mathbf{M}[B](\rho). \tag{2.16}$$

Under an imprecise measurement of the pure state the a posteriori state is also pure. The random walk of the normalized wave function reads

$$(|\psi\rangle, \omega) \mapsto \frac{\sqrt{p(q(\omega)I - \hat{x})}|\psi\rangle}{\|\sqrt{p(q(\omega)I - \hat{x})}|\psi\rangle\|}, \tag{2.17}$$

since the probability density for the random variable $q(\omega)$ depends on the initial wave function

$$d\mathbb{P}_q(y) = \|\sqrt{p(yI - \hat{x})}|\psi\rangle\|^2\; dy.$$

Suppose that some probability measure μ is absolutely continuous with respect to the Lebesgue measure $d\mu(y) = \pi(y)dy$ and $\pi(y) > 0$. Then, in accordance with the arguing in Section 2.1, we can consider a random walk of the form

$$(|\psi\rangle, \omega) \mapsto \frac{\sqrt{p(q(\omega)I - \hat{x})}|\psi\rangle}{\sqrt{\pi(q(\omega))}} \tag{2.18}$$

with the event probability $\mathbb{P}(q \in dy) = \pi(y)dy$. In particular, if the density $p(x)$ is strictly positive, it can serve as $\pi(x)$.

For simplicity, we suppose that $p_N(x) \propto e^{-t\lambda x^2}$, $t = \dfrac{T}{N}$, $\pi(x) = p_N(x)$ for each fixed N (see [9]). Here the constant $\lambda > 0$ is interpreted as the degree of accuracy of measurements made in a series.

The Schrödinger — Belavkin equation, which describes the limit evolution, contains a random component in addition to the Hamiltonian part, and it represents the stochastic differential equation (SDE) given in the paper [9]

$$d|\psi_t\rangle = -iH|\psi_t\rangle dt - \frac{\lambda^2}{4}\hat{x}^2|\psi_t\rangle dt + \sqrt{\frac{\lambda^2}{2}}\hat{x}|\psi_t\rangle dW_t. \tag{2.19}$$

The given example of a continuous quantum measurement process corresponds to a homogeneous linear random walk on a Hilbert space, where all random walk operators mutually commute.

3. Chernoff theorem for bi–continuous semigroups

3.1. Bi–continuous semigroups. The concept of bi–continuous semigroups was formulated and developed in the dissertation by Kühnemund (see [26]) and subsequent works. Bi–continuous semigroups quite naturally generalize strongly continuous semigroups, for which Chernoff approximation theorem [27, Thm. 5.2] is well–known. The possibility of approximating bi–continuous semigroups was studied in the works [28], [29], where an analogue of Chernoff theorem was proved.

Let $(X, \|\cdot\|)$ be a Banach space dual for X^* and equipped also with a locally convex topology τ, which possesses the properties

1. $\|\cdot\|$–bounded τ–closed sets are sequentially τ–complete (each fundamental in τ sequence converges),
2. the topology τ is Hausdorff and coarser than the norm topology;
3. for each $x \in X$

$$\|x\| = \sup\{|\ell(x)|: \quad \ell \in (X,\tau)', \ \|\ell\|_{X^*} \leqslant 1\},$$

where $(X,\tau)'$ is the space topologically dual to (X,τ), on which we consider the norm $\|\cdot\|_{X^*}$ of the dual space X^*.

Remark 3.1. *In what follows such conditions on the topology of τ with respect to the Banach space $(X, \|\cdot\|)$ will be called bi–conditions. In addition, for a mapping $f: M \to X$ and a metric space M we denote*

$$\operatorname{bi}\lim_{a\to a_0} f(a) = f_0 \in X,$$

if $\{f(a),\ a \neq a_0\}$ is $\|\cdot\|$–bounded and $\tau \lim\limits_{a\to a_0} f(a) = f_0$, and we call it the bi–convergence.

We note that since τ is coarser than the norm topology, we can treat $(X,\tau)'$ as a subspace X^* with the induced norm.

As $(X, \|\cdot\|, \tau)$, the spaces of bounded continuous functions on a Banach space E with the sup–norm and the convergence topology on some class of subsets of E are widely used (see examples in [26]).

Example 3.1. *Let E be a Banach space, $X = C_B(E)$ with $\sup$–norm. The topology of uniform convergence on bounded sets τ is generated by the $\sup$–seminorms $\|\cdot\|_B$, where B is a ball in E. Let us verify that such a topology satisfies the bi–conditions.*

Let $M \subset X$ be a uniformly bounded family of functions closed in the topology of uniform convergence on balls, then the set $M|_B$ of functions from M bounded on B is closed in $C_B(B)$. If $\{f_n\}$ is a τ–fundamental sequence of functions, then their restrictions to B converge in $C_B(B)$ to a function, which we denote by f_B. Obviously, the balls B and B' satisfy the compatibility condition $(f_B)|_{B\cap B'} = (f_{B'})_{B\cap B'}$, which allows us to construct a function f_E, which is continuous at each point and bounded.

The second condition is obvious.

Among the functionals from $(X,\tau)'$ there are all δ–functionals of the form

$$\delta_{e_0}(f) = f(e_0), \quad e_0 \in E.$$

Thus, the sup*–norm of f is attained.*

Remark 3.2. *It also suggests itself to consider the topology of pointwise convergence (which plays a large role in working with characteristic functions of random variables). But it does not satisfy the first condition. Even in the one–dimensional case, one can take a countable family of functions*

$$\mathcal{M} = \{f_n(x) = \mathrm{arctg}(nx),\ n \in \mathbb{N}\},$$

which is obviously uniformly bounded and closed in the topology of pointwise convergence. If each neighbourhood of the function $f(x) \in C_B(\mathbb{R})$ contains the elements of $\mathcal{M}$, it is easy to verify that it is also its element. But $\mathcal{M}$ is sequentially incomplete since for all $x \in \mathbb{R}$ the sequence $\{f_n(x)\}$ converges.

Definition 3.1. *We say that a semigroup $\{T_t\}$ on $(X, \|\cdot\|, \tau)$ is bi–continuous if*

1. *there exist $\exists M \geqslant 1$, $w \in \mathbb{R}$ such that $\|T_t\| \leqslant Me^{wt}$ for all $t \geqslant 0$,*
2. *$\tau \lim\limits_{t\to 0} T_t x = x$ for all $x \in X$,*
3. *for each bi–converging sequence $\{x_n\}$ to x the identity $\mathrm{bi}\lim\limits_{n\to\infty} T_t x_n = T_t x$ holds uniformly in t in each segment in $\mathbb{R}_+$.*

Definition 3.2. *A generator of bi–continuous semigroup $\{T_t\}$ is the operator $L : \mathcal{D}(L) \to X$ defined as the limit*

$$Lx = \tau \lim_{t\to 0} \frac{T_t x - x}{t}, \quad \mathcal{D}(L) = \left\{ x \in X : \quad \sup_{t\in(0,1]} \frac{\|T_t x - x\|}{t} < \infty,\ \exists\ \tau \lim_{t\to 0} \frac{T_t x - x}{t} \in X \right\}.$$

We shall also employ the following definitions ([26]).

Definition 3.3. *Let $(L, \mathcal{D}(L))$ be an operator on X.*

1. *The subspace $\mathcal{D} \subset \mathcal{D}(L)$ is called bi–core domain if for all $x \in \mathcal{D}(L)$ there exists $\{x_n\} \subset \mathcal{D}$ such that $\{x_n\}$ and $\{Lx_n\}$ bi–converge to x and Lx, respectively.*
2. *L is bi–closable if it admits a bi–closed extension (the closure $\overline{L}^{\|\cdot\|,\tau}$ is the minimal closed extension).*

Definition 3.4. *The set $M \subset X$ is called bi–dense if for each $x \in X$ there exists a bi–convergent to a point x sequence $\{x_n\} \subset M$.*

Definition 3.5. *The familly $\{S_\alpha\}_{\alpha\in\mathcal{A}}$ of $\|\cdot\|$–continuous operators is called bi–equicontinuous if for each bi–converging to x sequence $\{x_n\}$ the identity $\tau \lim\limits_{n\to\infty} S_\alpha x_n = x$ holds uniformly in α.*

Now we formulate the Chernoff theorem for bi–continous semigroups [29, Thm. 4.1].

Theorem 3.1. *Let $F : [0, +\infty) \to \mathcal{B}(X)$ be a function such that*

1. *$F_0 = I$,*
2. *$\|F_t^m\| \leqslant Me^{mwt}$,*
3. *$\{e^{-mwt}(F_t)^m,\ t \geqslant 0\}$ is locally uniformly bi–equicontinuous in m, that is, for each bi–converging to x sequence $\{x_n\}$ the identity*
$$\tau \lim_{n\to\infty} (e^{-wt} F_t)^m (x - x_n) = 0$$
holds uniformly in m and t from segments in $\mathbb{R}_+$,
4. *$\dfrac{F_t x - x}{t}$ is $\|\cdot\|$–bounded on $(0, T]$ and the limit $\tau \lim\limits_{t\to 0} \dfrac{F_t x - x}{t} = Lx$ is well–defined on bi–dense subspace $\mathcal{D} \subset X$.*

Let $(\lambda_0 I - L)(\mathcal{D})$ be bi–dense for some $\lambda_0 > w$. Then the closure $\overline{L}^{\|\cdot\|,\tau}$ is a generator of bi–continuous semigroups $\{T_t\}$ given by the approximation

$$T_t x = \tau \lim_{N\to\infty} (F_{\frac{t}{N}})^N x$$

uniform on segments in $\mathbb{R}_+$.

3.2. Semigroup related with process of continuous measuring. By the symbol η we denote the normal random variable $\eta \sim \mathcal{N}(0, 1/2)$.

In terms of random walks in the Hilbert space, in the simplest formulation, the wave function, in accordance with the obtained result η (independent of $\psi(x)$), «collapses» into (see the formula (2.18))

$$\psi(x) \mapsto \exp\left(-t\lambda \frac{\left(\frac{\eta}{\sqrt{t\lambda}} - x\right)^2}{2}\right) e^{\frac{\eta^2}{2}} \psi(x) = e^{-\frac{t\lambda x^2}{2} + \eta x\sqrt{t\lambda}} \psi(x). \tag{3.1}$$

Remark 3.3. *The formulas involve the combination $t\lambda$ and this is why in what follows we can let $\lambda = 1$.*

When performing a sequence of measurements, at each step there arises a variation of the same form independent of the previous moments of time. Thus, we can speak about the Markov character of the vector random process $|\psi_t\rangle$, but in this section we consider a one–dimensional real random process $\operatorname{Re}\psi_t(x)$, since the random operators of the linear walk can be represented as a function of the coordinate operator. In this example, we construct an approximation of the Markov semigroup of a one–dimensional random process, and the derivation of the presented result can be taken from the case of a walk on an arbitrary separable Hilbert space.

The single variation (3.1) with parameter t corresponds to the Markov operator

$$(\mathbf{T}_t f)(z) = \int \frac{dy}{\sqrt{\pi}} e^{-y^2} f\left(\exp\left(-\frac{tx^2}{2} + yx\sqrt{t}\right) z\right), \tag{3.2}$$

which is obtained according the formula (2.12).

Moreover, a simple calculation shows (see Proposition 4.3) that $\mathbf{T}_t$ is already a semigroup, but it is not strongly continuous in the norm topology, and it is bi–continuous if the domain is the space $(C_B(\mathbb{R}), \|\cdot\|, \tau)$ of bounded continuous functions with the sup-norm and the uniform convergence topology on compact sets. Indeed, as $t \to 0$, the function $f(z) = \sin(z)$ is immediately approximated in a neighborhood of the origin, while in a neighborhood of infinity the image of $\mathbf{T}_t f$ can be well molified. The bi–continuity will be shown in the general case.

The generator $\mathbf{L}$ of the semigroup $\mathbf{T}_t$ is easily calculated:

$$\mathbf{L}f(z) = -\frac{x^2}{4} z f'(z) + \frac{x^2}{4} z^2 f''(z)$$

(its domain includes the Schwartz space), therefore, the random process $\psi_t(x)$ satisfies the stochastic Schrödinger — Belavkin equation

$$d\psi(x) = -\frac{x^2}{4}\psi(x)dt + \frac{x}{\sqrt{2}}\psi(x)dW_t(x), \tag{3.3}$$

where $W_t(x)$ is the standard Wiener process.

It is interesting that under the choice of other densities $\{p_t\}$ the semigroup property can fail and we need to employ the theorem on approximation of bi–continuous semigroup.

4. Random walks in Hilbert space

Now we obtain some generalization of the established result for an infinite–dimensional non–commutative case.

4.1. On weakly continuous and differentiable functions on Hilbert space. Let $\mathcal{H}$ be a complex Hilbert space with the scalar product $\langle\cdot|\cdot\rangle$ linear in the second argument. To realify $\mathcal{H}_{\mathbb{R}}$ as a vector space over $\mathbb{R}$, the scalar product $\langle u|v\rangle_{\mathbb{R}} = \mathrm{Re}\langle u|v\rangle$ is induced. The weak topology of $\mathcal{H}$ coincides with the weak topology of $\mathcal{H}_{\mathbb{R}}$.

We consider the space $B_B(\mathcal{H})$ of complex bounded Borel functions with respect to the weak topology on $\mathcal{H}$ complete with respect to the norm

$$\|f(v)\| = \sup_{v\in\mathcal{H}} |f(v)|. \tag{4.1}$$

The completeness of $B_B(\mathcal{H})$ is implied by the fact that the pointwise limit of measurable functions is measurable.

Among the closed subspaces $B_B(\mathcal{H})$, we can distinguish the Banach subspaces of bounded continuous $C_B(\mathcal{H})$ and bounded weakly sequentially continuous $C_{BWS}(\mathcal{H})$ functions. There is an embedding $C_{BWS}(\mathcal{H}) \subset C_B(\mathcal{H}) \subset B_B(\mathcal{H})$. By definition, $f \in C_{BWS}(\mathcal{H})$, if for each weakly convergent sequence $\tau_W \lim_{n\to\infty} v_n = v_0$ we have $f(v_n) = f(v_0)$, and if the restriction of f to each ball $B \subset \mathcal{H}$ is uniformly approximated by weakly continuous functions, then $f \in C_{BWS}(\mathcal{H})$. Moreover, it is known that weakly continuous functions on bounded sets are uniformly weakly continuous. For completeness, the proof of these facts and some other properties are given in the Appendix.

Let $F \subset C_B(\mathcal{H})$ be some $\|\cdot\|$–closed subspace. On F, we introduce the topology τ of uniform convergence on bounded sets $\mathcal{H}$. We denote the corresponding seminorms by $\|\cdot\|_B$ if B is a bounded subset. Note that τ is metrizable since there exists a countable system of seminorms generating τ, namely $\left\{\|f\|_n = \sup_{\|v\|\leqslant 2^n} |f(v)|,\ n \in \mathbb{N}\right\}$.

Proposition 4.1. *The space $(F, \|\cdot\|, \tau)$ satisfies the bi–conditions.*

Proof. It was noted (example 3.1) that the bi–conditions are satisfied by the space $(C_B(\mathcal{H}), \|\cdot\|, \tau)$. Then the result is also extended to the $\|\cdot\|$–closed subspace of F. First, $\|\cdot\|$–bounded τ–closed $M \subset F$ is the same in $C_B(\mathcal{H})$, and therefore it is also sequentially τ–complete. Second, the topology τ and τ–continuous linear functionals are induced from $C_B(\mathcal{H})$, preserving the stated properties. □

Remark 4.1. *The bi–conditions are formulated in terms of sequential convergence in the topology τ. For example, the bi–density assumes the existence of a sequence with certain properties. In our case, there is no difference between a topological and a sequential condition, since bounded sets of a Hilbert space are metrizable in the weak topology.*

Definition 4.1. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. The mapping $|\psi\rangle : \Omega \to \mathcal{H}$ is called the random vector if it is measurable in the pair $(\mathscr{F}, \mathscr{B}_W)$. By $\mathscr{B}_W$ we mean a Borel σ–algebra with respect to the weak topology.*

Definition 4.2. *A random (bounded) operator is the mapping $G : \Omega \to \mathcal{B}(\mathcal{H})$ measurable with respect to the pair $(\mathscr{F}, \mathscr{B}_{WOT})$, where $\mathscr{B}_{WOT}$ stands for the σ–algebra generated by the weak operator topology.*

Remark 4.2. *The norm topology and the weak topology on a separable Hilbert space generates the same Borel σ–algebra* [30, Prop. 6.10.64]. *In particular, the sets*

$$\{\omega :\ \|\psi(\omega)\| < R\}, \quad R > 0$$

are measurable, and this is why for each $\varepsilon > 0$ *there exists* $R_\varepsilon > 0$ *such that* $\mathbb{P}(\|\psi(\omega)\| < R_\varepsilon) > 1-\varepsilon$. *The space of bounded Borel functions on* $\mathcal{H}$ *appearing in the definition of Markov operators is denoted by* $B_B(\mathcal{H})$. *The norm topology, SOT and WOT topologies on* $\mathcal{B}(\mathcal{H})$ *also generate the same* σ*–algebra* [31, Prop. 2.11]. *The random operator has a bounded norm with a large probability.*

The characteristic functional of random vector $|\psi\rangle$ is the function

$$\varphi(v) = \mathbb{E}e^{i\langle v|\psi\rangle_{\mathbb{R}}} : \mathcal{H} \to \mathbb{C}. \tag{4.2}$$

For a given $v \in \mathcal{H}$ the mapping $|w\rangle \mapsto e^{i\langle v|w\rangle_{\mathbb{R}}}$ is weakly continuous, and this is why the composition $\omega \mapsto e^{i\langle v|\psi(\omega)\rangle_{\mathbb{R}}}$ is a random variable and the integral $\varphi(v) = \mathbb{E}e^{i\langle v|\psi\rangle_{\mathbb{R}}}$ is well–defined.

In the present work an important example of transformation of functions is the Markov operator, which acts on the space $B_B(\mathcal{H})$ by the rule

$$(\mathbf{F}[G]f)(v) = \int d\mathbb{P}(\omega)\ f(G(\omega)v). \tag{4.3}$$

Proposition 4.2. *For a random operator* $G : \Omega \to \mathcal{B}(\mathcal{H})$ *the Markov operator* $\mathbf{F}[G] : B_B(\mathcal{H}) \to B_B(\mathcal{H})$ *is well–defined by the formula* (4.3). *The subspaces* $C_B(\mathcal{H})$ *and* $C_{BWS}(\mathcal{H})$ *are invariant with respect to the operator* $\mathbf{F}[G]$.

Proof. For each v we define a probability measure $P(B|v)$, where $v \in \mathcal{H}$, $B \in \mathscr{B}_W$, as the distribution of a random vector $G(\omega)v$ on $\mathcal{H}$. Then the action of the operator $\mathbf{F}[G]$ has the form $f(v) \mapsto \int P(d\psi|v)\ f(\psi)$, it is Markov. It is easy to see that bounded weakly sequentially continuous functions map $\mathscr{B}_W$–Borel sets to Borel sets on $\mathbb{C}$, and therefore lie in $B_B(\mathcal{H})$.

Let $\{v_n\}$ be a sequence weakly converging to v_0 and $f \in C_{BWS}(\mathcal{H})$. This yields that $\{G(\omega)v_n\}$ converges almost surely weakly to $G(\omega)v_0$. For $\varepsilon > 0$, the union of measurable sets

$$\begin{aligned}\Omega(\varepsilon, N) =& \{\omega \in \Omega : \quad |f(G(\omega)v_n) - f(G(\omega)v_0)| < \varepsilon,\ \forall n > N\} \\ =& \bigcap_{n \geqslant N} \{\omega \in \Omega : \quad |f(G(\omega)v_n) - f(G(\omega)v_0)| < \varepsilon\}\end{aligned}$$

has the probability 1, this is why for some N_ε the inequality

$$\mathbb{P}\left(\Omega\left(\frac{\varepsilon}{2}, N_\varepsilon\right)\right) > 1 - \frac{\varepsilon}{4\|f\|}$$

holds and

$$|\mathbf{F}[G]f(v_n) - \mathbf{F}[G]f(v_0)| < \frac{\varepsilon}{4\|f\|}2\|f\| + \int\limits_{\Omega(\frac{\varepsilon}{2}, N_\varepsilon)} d\mathbb{P}(\omega)\ |f(G(\omega)v_n) - f(G(\omega)v_0)| < \varepsilon, \quad n > N_\varepsilon,$$

hence, $\mathbf{F}[G]f \in C_{BWS}(\mathcal{H})$.

For $C_B(\mathcal{H})$ the statement can be proved in the same way. □

The scheme of proof of weak sequential continuity for characteristic functionals of random vectors $|\psi\rangle$ is similar. Note that Sazonov theorem establishes necessary and sufficient conditions on the characteristic functional [30, Cor. 7.13.8].

4.2. Semigroups related with linear random walk on $\mathcal{H}$.

Before proceeding to the general case, we present the explicit form of some important semigroups.

Proposition 4.3. *Suppose that a subalgebra* $\mathcal{E} \subset C_B(\mathbb{R})$ *contains the functions*

$$g[a,b](x) = e^{-ax^2+bx}, \quad a > 0, \quad b \in \mathbb{R},$$

and a homomorphism $\Upsilon : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ *is given. Then the family* $\{\mathbf{F}[G_t]\}$ *for* $G_t(y) = \Upsilon(g[at,\ yb\sqrt{t}])$ *with some* $a > 0$, $b \in \mathbb{R}$ *with respect to the measure* $d\mathbb{P}(y) = \dfrac{e^{-y^2}dy}{\sqrt{\pi}}$ *forms a semigroup on* $B_B(\mathcal{H})$.

Proof. The identities

$$\mathbf{F}[G_{t_1}]\mathbf{F}[G_{t_1}]f(v) = \int \frac{dy_1 dy_2}{\pi} e^{-y_1^2} e^{-y_2^2}\ f\Big(\Upsilon(g[at_1,\ y_1 b\sqrt{t_1}])\Upsilon(g[at_2, y_2 b\sqrt{t_2}])v\Big)$$
$$= \int \frac{dy_1 dy_2}{\pi} e^{-y_1^2} e^{-y_2^2}\ f\Big(\Upsilon(g[a(t_1+t_2),\ b(y_1\sqrt{t_1} + y_2\sqrt{t_2})])v\Big)$$

hold.

If $\eta_{1,2} \sim \mathcal{N}(0, 1/2)$ and the random variables η_1, η_2 are independent, then

$$\eta_1\sqrt{t_1} + \eta_2\sqrt{t_2} \sim \mathcal{N}(0, t_1 + t_2),$$

and this is why

$$\mathbf{F}[G_{t_1}]\mathbf{F}[G_{t_1}]f(v) = \int \frac{dy}{\sqrt{\pi}} e^{-y^2}\ f\Big(\Upsilon(g[a(t_1+t_2),\ by\sqrt{t_1+t_2}])v\Big) = \mathbf{F}[G_{t_1+t_2}]f(v).$$

□

Remark 4.3. *If for an operator* $C : \mathscr{D}(C) \to \mathcal{H}$ *the functional calculus is developed, that is, the homomorphism* $\Upsilon_C : \mathbb{L}_\infty(\mathbb{C}) \to \mathcal{B}(\mathcal{H})$ *is defined, then a semigroup is constructed by the operators of form* $G_t(y) = \Upsilon_C(g[at, yb\sqrt{t}]) = \exp(yb\sqrt{t}C - atC^2)$.

The next result provides conditions, under which the operator–valued functions (OVF) $\mathbf{F}[G_t]$, and in particular, the semigroups of such form on $C_{BWS}(\mathcal{H})$ are bi–continuous.

Theorem 4.1. *Suppose that we are given OVF* $\{\mathbf{F}[G_t]\}_{t\in[0,T]}$ *of operators on* $C_B(\mathcal{H})$.

1. *If*

$$\text{for all}\quad \varepsilon > 0,\ r > 0 \quad \text{there exists} \quad R > 0 \quad \text{such that} \quad \sup_{t\in[0,T]} \sup_{\|v\|\leqslant r} \mathbb{P}\Big(\|G_t v\| > R\Big) < \varepsilon, \tag{4.4}$$

then for each bi–converging to f_0 *sequence* $\{f_n\}$ *the sequence* $\mathbf{F}[G_t]f_n$ *bi–converges to* $\mathbf{F}[G_t]f_0$ *uniformly in* $t \in [0, T]$.

2. *If*

$$\text{for all}\quad \varepsilon > 0, \quad w \in \mathcal{H} \quad \lim_{t\to t_0} \mathbb{P}\Big(\|(G_t^* - G_{t_0}^*)w\| > \varepsilon\Big) = 0, \tag{4.5}$$

then for all $f \in C_{BWS}(\mathcal{H})$ *the convergence* $\operatorname{bi}\lim\limits_{t\to t_0} \mathbf{F}[G_t]f = \mathbf{F}[G_{t_0}]f$ *holds.*

Proof. 1) Let f_n bi–converge to f. For all $\varepsilon > 0$ and $r > 0$ we find $R_\varepsilon > 0$ (depending also on r) such that for all $v \in B_r$ and $t \in [0, T]$ the inequality

$$\mathbb{P}\Big(\|G_t(\omega)v\| > R_\varepsilon\Big) < \frac{\varepsilon}{4C}$$

holds, where $C = \sup\limits_{n\in\mathbb{N}} \|f_n\|$. Employing the sets

$$\Omega_\varepsilon^t(v) = \{\omega :\ \|G_t(\omega)v\| \leqslant R_\varepsilon\},$$

we obtain the estimate

$$\|\mathbf{F}[G_t]f_n(v) - \mathbf{F}[G_t]f_0(v)\|_{B_r} \leqslant \frac{\varepsilon}{4C}2C + \int\limits_{\Omega_\varepsilon^t(v)} d\mathbb{P}(\omega)\ |f_n(G_t(\omega)v) - f_0(G_t(\omega)v)|. \tag{4.6}$$

Since $\|f_n - f_0\|_{B_{R_\varepsilon}} \to 0$ as $n \to \infty$, we can estimate the right hand side of the inequality (4.6) by ε for sufficiently large n. The arbitrariness of $\varepsilon > 0$ implies the first condition of bi–continuity at the point t_0.

2) As before, there are $\varepsilon > 0$, $R_\varepsilon > 0$. On the ball B_{R_ε} the function $f \in C_{BWS}(\mathcal{H})$ is uniformly weakly continuous, that is, there exist $\delta > 0$ and $w_1, \dots, w_n \in \mathcal{H}$ such that $|\langle w_l | v' - v'' \rangle| < \delta$ implies $|f(v') - f(v'')| < \varepsilon$. By the assumption, for some neighbourhood $U(t_0)$ and for all $t \in U(t_0)$, $\varepsilon > 0, v \in B_r$ there exist sets $\Omega^t_\varepsilon(v) \subset \Omega$ such that

$$\mathbb{P}(\Omega^t_\varepsilon(v)) > 1 - 2\varepsilon, \quad \Omega^t_\varepsilon(v) = \Big\{ \forall l \quad \|(G^*_t - G^*_{t_0})w_l\| < \frac{\delta}{r}, \quad \|G_t v\|, \|G_{t_0} v\| \leqslant R_\varepsilon \Big\}.$$

Therefore, on $\Omega^t_\varepsilon(v)$ the inequality $|f(G_t(\omega)v) - f(G_{t_0}(y)v)| < \varepsilon$ holds as well as

$$\Big|\Big| (\mathbf{F}[G_t] - \mathbf{F}[G_{t_0}]) f \Big|\Big|_{B_r} \leqslant 4\varepsilon \|f\| + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\mathbf{F}[G_t]f$ converges uniformly on bounded sets to $\mathbf{F}[G_{t_0}]f$ and this completes the proof. □

Corollary 4.1. *If the semigroup $\{\mathbf{T}_t\}$ on $C_{BWS}(\mathcal{H})$ is of form $\mathbf{T}_t = \mathbf{F}[G_t]$, which satisfies the assumptions of Theorem 4.1, then it is a bi–continuous semigroup on $C_{BWS}(\mathcal{H})$.*

Of course, the convergence $\tau \lim\limits_{t \to t_0} \mathbf{F}[G_t]f = \mathbf{F}[G_{t_0}]f$ can hold not only for $f \in C_{BWS}(\mathcal{H})$.

Example 4.1. *We define $\{G_t(\omega) = e^t \mathbf{I}\}$, $f(v) = \exp(-\|v\|^2)$. Then*

$$\mathbf{F}[G_t]f(v) = \exp\Big(-e^{2t}\|v\|^2\Big),$$

and in this case the convergence $\mathbf{F}[G_t]f(v) \to f(v)$ as $t \to 0$ is uniform on $\mathcal{H}$.

Moreover, if a function $f \in C_B(\mathcal{H})$ is such that $\Big|\Big| \frac{\mathbf{F}[G_t]f - f}{t} \Big|\Big| < C$ for all $t \in [0, T]$, then $\mathbf{F}[G_t]f(v) \to f(v)$ as $t \to 0$ uniformly on $\mathcal{H}$.

We recall that the bi–closure $\mathcal{L} \subset C_B(\mathcal{H})$ is the set of functions $f \in C_B(\mathcal{H})$, which are bi–approximated by sequences in $\mathcal{L}$.

Proposition 4.4. *Let OVF $\{\mathbf{F}[G_t]\}$ satisfy the condition (4.4) and*

$$\Big[\mathbf{F}[G_t], \mathbf{F}[G_s]\Big] = 0 \quad \textit{for all} \quad t, s \in [0, T].$$

By F_G we denote the bi–closure of set of functions $f \in C_B(\mathcal{H})$ obeying

$$\sup_{t \in [0,T]} \Big|\Big| \frac{\mathbf{F}[G_t]f - f}{t} \Big|\Big| < \infty \quad \textit{and there exists} \quad \tau \lim_{t \to 0} \frac{\mathbf{F}[G_t]f - f}{t}.$$

Then F_G is invariant with respect to all operators in OVF $\{\mathbf{F}[G_t]\}$.

Proof. Let for $f \in C_B(\mathcal{H})$ and all $t \in [0, T]$ the relations

$$\Big|\Big| \frac{\mathbf{F}[G_t]f - f}{t} \Big|\Big| \leqslant C, \qquad \tau \lim_{t \to 0} \frac{\mathbf{F}[G_t]f - f}{t} = \theta$$

hold. Then for an arbitrary operator $A \in \mathcal{B}(\mathcal{H})$ the estimate

$$\Big|\Big| \int d\mathbb{P}(\omega) \; \frac{f(G_t(\omega)Av) - f(Av)}{t} \Big|\Big| \leqslant C,$$

holds and this implies

$$\Big|\Big| \int d\mathbb{P}(\omega) \; \int d\tilde{\mathbb{P}}(\tilde{\omega}) \; \frac{f(G_t(\omega)\tilde{G}(\tilde{\omega})v) - f(\tilde{G}(\tilde{\omega})v)}{t} \Big|\Big| \leqslant C$$

for the random operator $(\tilde{G}, \tilde{\mathbb{P}})$.

Let $\varepsilon > 0$, B_r be the ball of radius r centered at zero, $(\tilde{G}, \tilde{\mathbb{P}})$ be an arbitrary random operator. As it has already been mentioned, there exists $R_\varepsilon > 0$ such that

$$\mathbb{P}\Big(\|\tilde{G}\| > R_\varepsilon\Big) < \frac{\varepsilon}{4C}.$$

Suppose that

$$\Big|\Big|\int d\mathbb{P}(\omega)\ \frac{f(G_t(\omega)v) - f(v)}{t} - \theta(v)\Big|\Big|_{B_{rR_\varepsilon}} \leqslant \frac{\varepsilon}{2}.$$

We observe that the boundedness in norm of the differential relations implies the estimate $\|\theta\| \leqslant C$. Then

$$\Big|\Big|\int d\mathbb{P}(\omega)\ \int d\tilde{\mathbb{P}}(\tilde{\omega})\ \Big(\frac{f(G_t(\omega)\tilde{G}(\tilde{\omega})v) - f(\tilde{G}(\tilde{\omega})v)}{t} - \theta(\tilde{G}(\tilde{\omega})v)\Big)\Big|\Big|_{B_r} \leqslant \frac{2C\varepsilon}{4C} + \frac{\varepsilon}{2} \leqslant \varepsilon.$$

Thus, the subspace

$$\mathcal{L}_G = \Big\{f \in C_B(\mathcal{H}) \ \Big|\ \sup_{t\in[0,T]}\Big|\Big|\frac{\mathbf{F}[G_t]f - f}{t}\Big|\Big| < \infty,\ \exists \tau \lim_{t\to 0}\frac{\mathbf{F}[G_t]f - f}{t}\Big\}$$

is invariant with respect to the actions of each operator $\mathbf{F}[\tilde{G}]$ if $\Big[\mathbf{F}[\tilde{G}], \mathbf{F}[G_t]\Big] = 0$ for all $t \in [0, T]$, and in particular, $\{\mathbf{F}[G_t]\}$. We note that

$$\tau \lim_{t\to 0}\frac{\mathbf{F}[G_t]f - f}{t} \in C_B(\mathcal{H}),$$

as the bi–limit of a sequence of functions in $C_B(\mathcal{H})$.

Now we take $f \in F_G$ and an approximating it bi–sequence $\{f_n\} \subset \mathcal{L}_G$. We suppose that $\sup\limits_{n\in\mathbb{N}} \|f_n\| = C$. Then for a random $(\tilde{G}, \tilde{\mathbb{P}})$ we have

$$\|\mathbf{F}[\tilde{G}]f - \mathbf{F}[\tilde{G}]f_n\| \leqslant C,$$

and the τ–convergence $\mathbf{F}[\tilde{G}]f_n \in \mathcal{L}_G$ to $\mathbf{F}[\tilde{G}]f$ is implied by the first assertion of Theorem 4.1 applied to the constant OVF $\{\mathbf{F}[\tilde{G}]\}$. Thus, F_G is invariant with respect to the action of operator $\mathbf{F}[\tilde{G}]$. □

In what follows we shall employ the subspace

$$\mathcal{L}_G = \Big\{f \in C_B(\mathcal{H}) \ \Big|\ \sup_{t\in[0,T]}\Big|\Big|\frac{\mathbf{F}[G_t]f - f}{t}\Big|\Big| < \infty,\ \exists \tau \lim_{t\to 0}\frac{\mathbf{F}[G_t]f - f}{t}\Big\} \subset F_G. \tag{4.7}$$

We introduce the linear operator

$$\mathbf{L}f = \lim_{t\to 0}\frac{\mathbf{F}[G_t] - \mathbf{I}}{t}f$$

on a bi–dense subspace $\mathcal{L}_G \subset F_G$.

4.3. Approximations of bi–continuous semigroups by operators $\mathbf{F}[G_{\frac{t}{N}}]^N$. Chernoff Theorem 3.1 for bi–continuous semigroups allows one not only to prove the possibility of approximation of semigroups, but also to construct a bi–continuous semigroup if it is not given explicitly. This section is devoted to verifying the assumptions of Theorem 3.1 for a given OVF $\mathbf{F}[G_t]\Big|_{F_G}$. Namely, we need to verify that $\mathbf{F}[G_t]$ is locally uniformly bi–equicontinuous, and for the derivative $\mathbf{L} = \dfrac{d}{dt}\mathbf{F}[G_t]\Big|_{t=0}$ with the domain $\mathcal{L}_G$ the subspace $(\lambda\mathbf{I} - \mathbf{L})(\mathcal{L}_G) \subset F_G$ is bi–dense for some $\lambda > 0$.

We denote the products of independent random operators as

$$(G_t^{\perp\!\!\!\perp m}, \mathbb{P}^{\otimes m}) = (\{G_t(\omega_1)\ldots G_t(\omega_m)\},\ d\mathbb{P}(\omega_1)\ldots d\mathbb{P}(\omega_m)).$$

Proposition 4.5. *The assertions hold.*

1. *Assume that we are given an operator–valued function* $\{G_t\}$ *and for all* $r > 0$, $\varepsilon > 0$ *there exists* $R > 0$ *such that*

$$\sup\Big\{\mathbb{P}\Big(\|G_t^{\perp\!\!\!\perp m} v\| > R\Big),\quad \|v\| \leqslant r,\ t \in [0,T],\ m \in \mathbb{N}\Big\} < \varepsilon. \tag{4.8}$$

Then the OVF $\mathbf{F}[G_t]$ *on* $B_B(\mathcal{H})$ *is locally uniformly bi–equicontinuous.*

2. *The condition*

$$\sup_{t\in[0,T]} \|G_t\|_1 < \infty, \tag{4.9}$$

where $\|G_t\|_1 = \mathbb{E}\|G_t\|$, *is sufficient for the assumptions of previous assertion.*

Proof. We begin with showing the sufficiency of (4.9) for ensuring the inequalities (4.8), and then we shall prove the uniform bi–equicontinuity. Let B_r be a ball in $\mathcal{H}$, $\varepsilon > 0$ and $M = \sup\limits_{t\in[0,T]} \|G_t\|_1$. We introduce the function

$$p(R) = 1 - \inf_{t\in[0,T]} \mathbb{P}(\Omega_R^t) \leqslant \frac{M}{R},$$

where $\Omega_R^t = \{\omega :\ \|G_t(\omega)\| \leqslant R\}$. For $\varepsilon \in (0,1)$ we let $R_\varepsilon = \max(1, (M/\varepsilon)^{1/\varepsilon})$, then for all $R > R_\varepsilon$, on one hand, for $s \in [0,\varepsilon]$ by the Bernoulli inequality

$$(1 - p(R^s))^s \geqslant \left(1 - \frac{M}{R^s}\right)^s \geqslant 1 - s\frac{M}{R^s} \geqslant 1 - \varepsilon,$$

and on the other hand, for $s \in [\varepsilon, 1]$,

$$(1 - p(R^s))^s \geqslant 1 - \frac{M}{R^s} \geqslant 1 - \varepsilon.$$

Therefore, for all $R > R_\varepsilon$, $t \in [0,T]$, $m \in \mathbb{N}$, $v \in B_r$ the inequality

$$\|G_t(\omega_1)\dots G_t(\omega_m)v\| \leqslant rR_\varepsilon$$

holds with a probability at least $1 - \varepsilon$.

We proceed to verifying the condition. We suppose that the sequence $\{f_n\}$ in uniformly bounded in the norm by a constant $C > 0$ and converges uniformly to zero on each bounded set. We denote

$$f_{nm}^t(v) = (\mathbf{F}[G_t])^m f_n(v) = \int d\mathbb{P}(\omega_1)\dots d\mathbb{P}(\omega_m) f_n(G_t(\omega_m)\dots G_t(\omega_1)v).$$

Partitioning the events in «far» and «close», we estimate

$$|f_{nm}^t(v)| \leqslant \varepsilon C + \sup_{v\in B_\varepsilon} |f_n(v)|.$$

But on the bounded set $B_\varepsilon = B_{rR_\varepsilon}$ the sequence $\{f_n\}$ uniformly converges to zero and this is why by the arbitrariness of $\varepsilon > 0$ the sequence $\{f_{nm}^t(v)\}$ converges to zero as $n \to \infty$ uniformly in $t \in [0,T]$, $m \in \mathbb{N}$ and $v \in B_r$. □

Remark 4.4. *The property of locally uniform bi–equicontinuity of* $\{\mathbf{F}[G_t]\}$ *is preserved under the restriction to each subspace* $B_B(\mathcal{H})$.

Remark 4.5. *The finiteness of* $\sup\limits_{t\in[0,T]} \|G_t\|_1$ *implies the first condition of Theorem 4.1 uniformly for all compositions of independent operators* $G_t^{\perp\!\!\!\perp m}$, $m \in \mathbb{N}$, $t \in [0,T]$.

We introduce the operators acting on $B_B(\mathcal{H})$ by the formula

$$\mathbf{F}_{t,\lambda} = \sum_{m=0}^{\infty} te^{-\lambda mt}\mathbf{F}[G_t]^m, \quad t \in [0,T], \quad \lambda > 0. \tag{4.10}$$

The general fact is that if $\{\mathbf{F}_{\alpha,k},\ k \in \mathbb{N},\ \alpha \in \mathcal{A}\}$ is bi–equicontinuous operator family on a closed subspace $F \subset B_B(\mathcal{H})$, then for the sequences $\{p_\alpha = (p_{\alpha,k})_k\}_\alpha$ satisfying the condition $\sum\limits_k |p_\alpha| = 1$ the family $\{\mathbf{F}_\alpha\}_\alpha$, where $\mathbf{F}_\alpha f = \sum\limits_k p_{\alpha,k}\mathbf{F}_{\alpha,k}f$, is also bi–equicontinuous for all $f \in F$. Indeed, for a given $\varepsilon > 0$ by the definition of bi–equicontinuity, for each ball $B \subset \mathcal{H}$, a sequence f_n bi–convergent to f_0, the convergence $\sup\limits_{\alpha,k} \|\mathbf{F}_{\alpha,k}(f_n - f_0)\|_B \to 0$ holds as $n \to \infty$. This inequality is obviously preserved while passing to the operators $\{\mathbf{F}_\alpha\}$.

In our case the locally uniform bi–equicontinuity of $\{\mathbf{F}[G_t]\}$ implies the bi–equicontinuity of the family $\{\mathbf{F}_{t,\lambda}\}$ for a fixed $\lambda > 0$.

Proposition 4.6. *Let a random OVF $\{G_t\}$ satisfies the conditions of first assertion of Proposition 4.5 and the commutation condition $\Big[\mathbf{F}[G_t], \mathbf{F}[G_s]\Big] = 0$ for all $t, s \in [0,T]$. The subspaces $C_B(\mathcal{H})$, $C_{BWS}(\mathcal{H})$, F_G and $\mathcal{L}_G$ are invariant with respect to $\mathbf{F}_{t,\lambda}$.*

Proof. Using the invariance of $C_B(\mathcal{H})$, $C_{BWS}(\mathcal{H})$ and F_G with respect to the Markov operators $\mathbf{F}[G_t]$, $t \in [0,T]$ (Proposition 4.4), their closedness in norm and the absolute convergence of series in the definition of $\mathbf{F}_{t,\lambda}$, we obtain the invariance of the subspaces $C_B(\mathcal{H})$, $C_{BWS}(\mathcal{H})$, F_G with respect to $\mathbf{F}_{t,\lambda}$.

Then for $\varepsilon > 0$ we employ the inequalities (4.8) to find $R_\varepsilon > 0$, such that with the probability at least $1 - \varepsilon$ a vector from the ball B_r is mapped into the ball B_{R_ε} for all random operators of the family $\{G_t^{\perp\!\!\perp m},\ t \in [0,T],\ m \in \mathbb{N}\}$. Suppose that $f \in \mathcal{L}_G$ obeys the conditions

$$\Big\|\frac{\mathbf{F}[G_s] - \mathbf{I}}{s} f - \mathbf{L}f\Big\|_{B_{R_\varepsilon}} < \varepsilon, \qquad \Big\|\frac{\mathbf{F}[G_s] - \mathbf{I}}{s} f\Big\| \leqslant C.$$

Then

$$\Big\|\frac{\mathbf{F}[G_s] - \mathbf{I}}{s}\mathbf{F}[G_t]^m f\Big\| \leqslant C,$$

for all t, m since

$$\Big|\frac{\mathbf{F}[G_s] - \mathbf{I}}{s} f(G_t^{\perp\!\!\perp m} v)\Big| \leqslant C$$

almost surely and, using the commutation condition,

$$\Big\|\frac{\mathbf{F}[G_s] - \mathbf{I}}{s}\mathbf{F}[G_t]^m f - \mathbf{L}\mathbf{F}[G_t]^m f\Big\|_{B_r} < \varepsilon(C+1)$$

by the same reason. We note that the estimate is uniform in m and $t \in [0,T]$. We have

$$\Big\|\frac{\mathbf{F}[G_s] - \mathbf{I}}{s}\mathbf{F}_{t,\lambda} f\Big\| \leqslant C\sum_{m=0}^{\infty} te^{-\lambda mt} \leqslant \frac{CT}{1 - e^{-\lambda T}}, \quad \Big\|\frac{\mathbf{F}[G_s] - \mathbf{I}}{s}\mathbf{F}_{t,\lambda} f - \mathbf{L}\mathbf{F}_{t,\lambda} f\Big\|_{B_r} < \frac{\varepsilon(C+1)T}{1 - e^{-\lambda T}}.$$

As a result we obtain that $\mathbf{F}_{t,\lambda} f \in \mathcal{L}_G$ for all $t \in [0,T]$. □

Proposition 4.7. *Let a random OVF $\{G_t\}$ satisfy the conditions in the first assertion of Proposition 4.5, the commutation condition and $\lambda > 0$. Then the space $(\lambda\mathbf{I} - \mathbf{L})(\mathcal{L}_G)$ is bi–dense in F_G.*

Proof. We observe an elementary relation

$$\frac{\mathbf{F}[G_t] - \mathbf{I}}{t}\mathbf{F}_{t,\lambda} = \frac{e^{\lambda t} - 1}{t}\mathbf{F}_{t,\lambda} - \mathbf{I}. \tag{4.11}$$

Let $\varepsilon > 0$, $f \in \mathcal{L}_G$. There exists $s_\varepsilon > 0$ such that for all $s \in [0, s_\varepsilon]$

$$\Big\|\frac{\mathbf{F}[G_s]-\mathbf{I}}{s}f - \mathbf{L}f\Big\|_{B_{R_\varepsilon}} < \varepsilon, \qquad \Big\|\frac{\mathbf{F}[G_s]-\mathbf{I}}{s}f\Big\| \leqslant C,$$

where $R_\varepsilon > 0$ is fixed by the requirement

$$\mathbb{P}\Big(\|G_t^{\perp\!\!\!\perp m}v\| > R_\varepsilon, \quad \forall v : \|v\| \leqslant r\Big) < \varepsilon, \quad \forall t \in [0,T],\ m \in \mathbb{N}. \tag{4.12}$$

Then, according the estimates from the proof of Proposition 4.6, for all $t, s \in [0, s_\varepsilon]$ we have

$$\Big\|\frac{\mathbf{F}[G_s]-\mathbf{I}}{s}\mathbf{F}_{t,\lambda}f - \Big(\frac{e^{\lambda t}-1}{t}\mathbf{F}_{t,\lambda}f - f\Big)\Big\|$$
$$\leqslant \Big\|\frac{\mathbf{F}[G_s]-\mathbf{I}}{s}\mathbf{F}_{t,\lambda}f\Big\| + \Big\|\frac{\mathbf{F}[G_t]-\mathbf{I}}{t}\mathbf{F}_{t,\lambda}f\Big\| \leqslant\ 2\frac{CT}{1-e^{-\lambda T}},$$
$$\Big\|\frac{\mathbf{F}[G_s]-\mathbf{I}}{s}\mathbf{F}_{t,\lambda}f - \Big(\frac{e^{\lambda t}-1}{t}\mathbf{F}_{t,\lambda}f - f\Big)\Big\|_{B_r} \leqslant \Big\|\frac{\mathbf{F}[G_s]-\mathbf{I}}{s}\mathbf{F}_{t,\lambda}f - \mathbf{L}\mathbf{F}_{t,\lambda}f\Big\|_{B_r}$$
$$+ \Big\|\frac{\mathbf{F}[G_t]-\mathbf{I}}{t}\mathbf{F}_{t,\lambda}f - \mathbf{L}\mathbf{F}_{t,\lambda}f\Big\|_{B_r} <\ 2\frac{\varepsilon(C+1)T}{1-e^{-\lambda T}}.$$

Hence, the bi–closure $(\lambda\mathbf{I} - \mathbf{L})(\mathcal{L}_G)$ contains a function f and it can be bi–approximated by the functions $\{(\lambda\mathbf{I} - \mathbf{L})\mathbf{F}_{t_k,\lambda}f\}$ for a sequence $\{t_k\}$ converging to zero. □

4.4. On one class of differential operators. Some important examples of semigroups and operator–valued functions are related with differential operators on the space of functions from $\mathcal{H}$ to $\mathbb{C}$, so we provide the necessary information about them.

The classical approach to differential operators in general is as follows. Let X, Y be linear normed spaces, and $F(X,Y)$ be the space of all functions from X to Y. A function $f \in F(X,Y)$ is Fréchet differentiable at $x \in X$ if the limit

$$\lim_{t\to 0}\frac{f(x+th)-f(x)}{t} = df(x)\,h$$

is well–defined uniformly in h from a neighbourhood of $0 \in X$. At the same time, the Fréchet derivative $df(x)$ is an element of the space $\mathcal{B}(X,Y)$ of continuous linear maps from X to Y. If $f \in F(X,Y)$ is Fréchet differentiable at each point, then its derivative df belongs to the space $F(X, \mathcal{B}(X,Y))$. Higher–order Fréchet derivatives are defined by induction. Let $\mathcal{B}_1(X,Y) = \mathcal{B}(X,Y)$, and the spaces $\mathcal{B}_{n+1}(X,Y) = \mathcal{B}(X, \mathcal{B}_n(X,Y))$, defined for all $n \in \mathbb{N}$, be normed spaces with respect to the operator norm; we denote $d^1 f := df$. If f has derivatives $df, d^2 f, \ldots, d^{n-1} f$ in some neighborhood of a point $x \in X$, and the mapping $x \mapsto d^{n-1}f(x)$ is Fréchet differentiable at x, then the derivative of order $n > 1$, $d^n f(x)$, at x is the Fréchet derivative $d(d^{n-1}f)(x)$ of the function $d^{n-1}f$ acting from the space X to $\mathcal{B}_{n-1}(X,Y)$. The space $\mathcal{B}_n(X,Y)$ is naturally embedded into the space $\mathcal{L}_n(X,Y)$ of n–linear mappings $X \times \ldots \times X \to Y$, where the values of derivatives define symmetric multilinear mappings.

Then differential operators are introduced.

Definition 4.3. [32, Def. 7.1.1] *Let Z be a topological vector space and F_n be some space consisting of n times Fréchet differentiable functions $X \to Y$ at each point $x \in X$. A mapping $\mathcal{D} : F_n \to F(X,Z)$ is called the nth order differential operator if there exists a mapping*

$$\theta : X \to \mathcal{L}(\mathcal{B}_n(X,Y), Z), \quad x \mapsto \theta_x, \tag{4.13}$$

such that for all $x \in X$, $f \in F_n$ the identity

$$(\mathbf{D}f)(x) = \theta_x(d^n f(x)) \tag{4.14}$$

holds.

In what follows we consider the case, when $X = \mathcal{H}_{\mathbb{R}}$ is the realification of the complex Hilbert space $\mathcal{H}$, $Y = \mathbb{C}$, or $Y = \mathbb{R}$. On $\mathcal{H}_{\mathbb{R}}$ the scalar product is induced by the scalar product in $\mathcal{H}$: $\langle w|v\rangle_{\mathbb{R}} = \mathrm{Re}\langle w|v\rangle$. However, despite the difference between the spaces $\mathcal{H}$ and $\mathcal{H}_{\mathbb{R}}$, the spaces of functions on them $F(\mathcal{H}, Y)$ and $F(\mathcal{H}_{\mathbb{R}}, Y)$ are naturally isomorphic for any set Y. The scalar product on the complexification of $\mathcal{H}_{\mathbb{R}}$ and its tensor powers are denoted by the brackets $\langle\cdot|\cdot\rangle_{\mathbb{C}}$.

Example 4.2. *Let $A_n : \mathscr{D}(A_n) \to \mathcal{H}_{\mathbb{R}}^{\otimes n}$ be a (possibly unbounded) operator on the Hilbert tensor product $\mathcal{H}_{\mathbb{R}}^{\otimes n}$, and let F be the space of real n times differentiable functions on $\mathcal{H}_{\mathbb{R}}$, for which the derivative $d^n f$ at each point $v \in \mathcal{H}$ as an n–linear function admits a continuous extension to $\mathcal{H}_{\mathbb{R}}^{\otimes n}$, and $d^n f(v) \in \mathscr{D}(A_n)$. The functional $\theta_v : F \to \mathbb{R}$ defined by the identity*

$$\theta_v d^n f(v) = \langle A_n d^n f(v)|v^{\otimes n}\rangle_{\mathbb{R}}, \tag{4.15}$$

defines the action of a differential operator $\mathbf{D}(A_n) : F \to F(\mathcal{H}, \mathbb{R})$.

Let $A = (A_1, \ldots, A_n)$ be a family of operators $A_k : \mathscr{D}(A_k) \to \mathcal{H}_{\mathbb{R}}^{\otimes k}$, $1 \leqslant k \leqslant n$. Let F be a subspace of functions $\mathcal{H} \to \mathbb{R}$ having Fréchet derivatives of nth order at all points $v \in \mathcal{H}$, each of which admits an extension to a linear continuous functional $d^n f(v) \in (\mathcal{H}_{\mathbb{R}}^{\otimes n})^*$, $\forall v$ and $d^k f(v) \in \mathscr{D}(A_k)$ for all $1 \leqslant k \leqslant n, v \in \mathcal{H}$. We define the differential operator by the identity $\mathbf{D}(A)f = \sum_{k=1}^{n} \mathbf{D}(A_k)f$, which maps F into $F(\mathcal{H}, \mathbb{R})$.

This scheme allows us to extend the differential operator $\mathbf{D}(A)$ to the case of complex functions. If for $f \in F(\mathcal{H}, \mathbb{C})$ the real and imaginary parts lie in the subspace F specified in Example 4.2, then we let

$$(\mathbf{D}(A)f)(v) = (\mathbf{D}(A)\,\mathrm{Re}\, f)(v) + i(\mathbf{D}(A)\,\mathrm{Im}\, f)(v). \tag{4.16}$$

Let us demonstrate the action of the above differential operator $\mathbf{D}(A_n)$ using polynomials as the example. Let the subspace $F \subset F(\mathcal{H}, \mathbb{C})$ contain monomials

$$p : v \mapsto \langle w_1|v\rangle_{\mathbb{R}} \ldots \langle w_m|v\rangle_{\mathbb{R}},$$

which obey the condition $d^n p(v) \in \mathscr{D}(A_n)$. It sufficient to suppose that for all injective mappings $j : \{1, \ldots, n\} \to \{1, \ldots, m\}$ we have

$$w_{j(1)} \otimes \ldots \otimes w_{j(n)} \in \mathscr{D}(A_n).$$

In this case it is easy to find that

$$(\mathbf{D}(A_n)p)(v) = \sum_{\sigma \in S(m)} \langle A_n(w_{\sigma_1} \otimes \ldots \otimes w_{\sigma_n})|v^{\otimes n}\rangle_{\mathbb{R}} \frac{\langle w_{\sigma_{n+1}}|v\rangle_{\mathbb{R}} \ldots \langle w_{\sigma_m}|v\rangle_{\mathbb{R}}}{(m-n)!}. \tag{4.17}$$

Let us clarify the reasons why second–order differential operators will appear in what follows. In general, we could consider averaging over any complex measures of finite variation. For a measure ν ($\nu(\Omega) = 1$) on a measurable space $(\Omega, \mathscr{F})$ with finite total variation, the structure of the probability space $(\Omega, \mathscr{F})$ is naturally defined by taking $\mathbb{P} = \dfrac{|\nu|}{|\nu|(\Omega)}$. As we shall see below, by averaging over the measure ν we can obtain a probability function

$$\mathbf{F}_\nu[G_t] : f(v) \mapsto \int d\nu(\omega)\ f(G_t(\omega)v),$$

the derivative of which can be a differential operator of any order. But if ν is positive, then under sufficiently weak technical constraints only differential operators of order at most two are possible.

Let $A = (A_1, \ldots, A_n)$ be a set of densely defined operators

$$A_k : \mathscr{D}(A_k) \to \mathcal{H}_{\mathbb{R}}^{\otimes k}, \qquad 1 \leqslant k \leqslant n,$$

$F \subset F(\mathcal{H}, \mathbb{C})$ be the space of $n+1$ times Fréchet differentiable functions f, the derivatives $\{d^k f(v)\}$ of which are continued by the continuation to the complex space $\mathbb{C}(\mathcal{H}_{\mathbb{R}}^{\otimes k})$ under the condition $d^k f(v) \in \mathbb{C}\mathscr{D}(A_k)$ for all $1 \leqslant k \leqslant n$, $v \in \mathcal{H}$, and $\sup_{v \in \mathcal{H}} \|d^{n+1} f(v)\| < \infty$.

Proposition 4.8. *Assume that we are give a finite measure ν ($\nu(\Omega) = 1$) on a measurable space $(\Omega, \mathscr{F})$ and an operator–valued function $\mathbf{F}_\nu[G_t]$, for which*

1. *the limiting relations*
$$\lim_{t \to 0} \int d\nu(\omega)\ \frac{\langle w|(G_t(\omega)v - v)^{\otimes k}\rangle_{\mathbb{R}}}{k!t} = \langle A_k w|v^{\otimes k}\rangle_{\mathbb{R}}, \quad \forall w \in \mathscr{D}(A_k) \subset \mathcal{H}_{\mathbb{R}}^{\otimes k},\ 1 \leqslant k \leqslant n$$
hold;
2. *for each $v \in \mathcal{H}$ the convergence*
$$\int d|\nu|(\omega)\ \frac{\|(G_t(\omega) - I)v\|^{n+1}}{t} \to 0$$
holds as $t \to 0$.

Then for $f \in F$ for each vector $v \in \mathcal{H}$ the derivative $\frac{d}{dt}(\mathbf{F}_\nu[G_t]f(v))\Big|_{t=0}$ is equal to $(\mathbf{D}(A)f)(v)$.

If the convergences of Bochner integrals
$$\lim_{t \to 0} \int d\nu(\omega)\ \frac{(G_t^*(\omega) - I)^{\otimes k} w}{k!t} = A_k w, \quad \forall w \in \mathscr{D}(A_k),\ 1 \leqslant k \leqslant n; \tag{4.18}$$
hold in the sense of norm, then $\frac{\mathbf{F}[G_t] - \mathbf{I}}{t} f$, $f \in F$, tends to $\mathbf{D}(A)f$ as $t \to 0$ uniformly on bounded sets.

Proof. By the Taylor formula, see [30, Thm. 12.4.4]) we have
$$f(G_t(\omega)v) = f(v) + \sum_{k=1}^{n} \frac{1}{k!} \langle d^k f(v)|(G_t(\omega) - I)^{\otimes k} v^{\otimes k}\rangle_{\mathbb{C}} + r_n(t, \omega, v),$$
ν–almost everywhere and
$$r_n(t, \omega, v) = \int_0^1 ds\ \frac{(1-s)^n}{n!} \langle d^{n+1} f(v)|(G_t(\omega)v - v)^{\otimes(n+1)}\rangle_{\mathbb{C}}.$$
The estimate
$$\frac{|r_n(t, \omega, v)|}{t} \leqslant \frac{M_{n+1}}{(n+1)!} \frac{\|(G_t(\omega) - I)v\|^{n+1}}{t}$$
holds; averaging this estimate over $|\nu|$ gives the uniform on balls estimate for the error term.

Applying the condition to the real and imaginary parts of f, we obtain that the difference
$$\sum_{k=1}^{n} \int d\nu(\omega)\ \frac{\langle d^k f(v)|(G_t(\omega)v - v)^{\otimes k}\rangle_{\mathbb{C}}}{k!t} - \sum_{k=1}^{n} \int d\nu(\omega)\ \langle A_k d^k f(v)|v^{\otimes k}\rangle_{\mathbb{C}}$$
tends to zero as $t \to 0$ for each $v \in \mathcal{H}$. If this convergence is uniform in v on balls, then $\frac{\mathbf{F}[G_t] - \mathbf{I}}{t} f \to \mathbf{D}(A)f$, $f \in F$ as $t \to 0$ uniformly on balls and this completes the proof. □

Let us consider the case when a random OVF $\{G_t\}$ has the analytic form
$$G_t(\omega) = I + \sum_{k=1}^{m} t^{\frac{1}{k}} G_{k,t}(\omega), \tag{4.19}$$

and the convergence $\lim_{t\to 0}\langle w|G_{k,t}(\omega)v\rangle = \langle L_k(\omega)w|v\rangle$ holds ν–almost everywhere for all w in the dense subspace $\mathscr{D}\subset\mathcal{H}$ and all $v\in\mathcal{H}$. Suppose also that the conditions hold

1. $\lim\limits_{t\to 0}\int d\nu(\omega)\ (\langle L_k(\omega)w|v\rangle)^j = 0, \quad j<k\leqslant m,\ w\in\mathscr{D},\ v\in\mathcal{H};$
2. $\lim\limits_{t\to 0}\int d\nu(\omega)\ (\langle L_k(\omega)w|v\rangle)^k = \langle A_k w^{\otimes k}|v^{\otimes k}\rangle, \quad w\in\mathscr{D},\ v\in\mathcal{H}.$

Then it follows from 4.8 that for $f\in F$ the identity

$$\frac{d}{dt}\mathbf{F}_\nu[G_t]f(v)\Big|_{t=0} = \mathbf{D}(A)f(v)$$

holds for all $v\in\mathcal{H}$.

If $\nu\geqslant 0$, then the conditions of the previous point are satisfied only in the case $L_k=0$ for all $k>2$ ν–almost everywhere. Indeed, if $k>2$, $w\in\mathscr{D}$, then

$$\int d\nu(\omega)\ |\langle L_k(\omega)w|v\rangle|^2 = 0$$

is equivalent to

$$\langle L_k(\omega)|v\rangle = 0$$

ν–almost everywhere, whence

$$\int d\nu(\omega)\ (\langle L_k(\omega)w|v\rangle)^k = 0.$$

This argument illustrates the fact that among the generators of Markov semigroups, differential operators of order higher than two do not arise.

5. Main result

We proceed to the formulation of main result. For the brevity by the symbol (#) we denote the condition of the first assertion of Proposition 4.5. The condition $\Big[\mathbf{F}[G_t],\mathbf{F}[G_s]\Big]=0$ for all $t,s\in[0,T]$ will be called the commutation condition.

The closed subspace $F_G\subset C_B(\mathcal{H})$ is chosen as the bi–closure of the subspace $\mathcal{L}_G$ defined in accordance with (4.7). It is the domain of the operator $\mathbf{L}$.

The operators $\{\mathbf{F}_{t,\lambda},\ t\in[0,T]\}$, $\lambda>0$, are the discrete analogue of the resolvent and have $\mathcal{L}_G$ as an invariant subspace under the condition (#).

Under these conditions, the main theorem is as follows.

Theorem 5.1. *Let OVF $\{\mathbf{F}[G_t]\}$ satisfy the condition (#) and the commutation condition. Then for each function $f\in F_G$ the action of bi–continuous semigroup $\{\mathbf{T}_t\}_{t\geqslant 0}$ is defined in the space F_G and*

$$\mathbf{T}_t f = \tau\lim_{N\to\infty}\mathbf{F}[G_{\frac{t}{N}}]^N f \tag{5.1}$$

uniformly on the segments in $\mathbb{R}_+$ in the variable $t>0$.

The choice of the space $(F_G,\|\cdot\|,\tau)$ already provides the desired properties of the differential relation $\dfrac{\mathbf{F}[G_t]-\mathbf{I}}{t}f$ on a bi–dense subspace, and if we want to work with the dynamics of characteristic functionals, we need to understand whether $C_{BWS}(\mathcal{H})$ is embedded into F_G. However, one can weaken the functional properties of the semigroup by extending its action to Borel bounded functions.

Theorem 5.2. *Let $\{\mathbf{T}_t\}$ be a bi–continuous semigroup on $C_{BWS}(\mathcal{H})$ and be approximated by the Chernoff iterates of an OVF $\{\mathbf{F}[G_t]\}$ satisfying (#). Then it has a unique extending semigroup $\{\mathbf{Q}_t\}$ on $B_B(\mathcal{H})$, which is Markov. Moreover, for each $f \in C_B(\mathcal{H})$ we have*

$$\lim_{N\to\infty} \mathbf{F}[G_{\frac{t}{N}}]^N f(v) = \mathbf{Q}_t f(v)$$

for all $v \in \mathcal{H}$, $t \geqslant 0$.

Proof. We recall that a family of random elements $\{x_\alpha\}_\alpha$ on a metric space (S,ρ) is called tight if for all $\varepsilon > 0$ one can find a compact $K_\varepsilon \subset S$ such that $\mathbb{P}(x_\alpha \notin K_\varepsilon) < \varepsilon$ for all α, see [18, Ch. V, Def. 4]. In our case, the family of random vectors $\mathcal{M}_v = \{G_t^{\perp\!\!\!\perp m} v\}$ in the topology of weak sequential convergence τ_{WS} is tight for fixed $v \in \mathcal{H}$ according to (#) (this topology is generated by intersections of τ_W–open sets with concentric balls in $\mathcal{H}$ with integer radii; τ_{WS} is metrizable, and the balls are compact).

By the Prokhorov theorem [18, Ch. V, Thm. 5]) the set of measures of random vectors $\mathcal{M}_v$ in $(\mathcal{H}, \mathscr{B}_W)$ is weakly relatively compact, that is, in each sequence in $\mathcal{M}_v$ one can choose a sequence of random vectors converging in distribution. For example, there exists a sequence $\{N_k\} \subset \mathbb{N}$ such that $G_{\frac{t}{N_k}}^{\perp\!\!\!\perp N_k} v$ converges in distribution to some random vector $\psi_{v,t}$ with distribution (probability measure on $\mathcal{H}$) $\mathbb{P}_{v,t}$ (this choice is not unique). Moreover, $\{N_k\}$ can be chosen as a subsequence from any prescribed sequence. Then we can define operators $\{\mathbf{Q}_t\}$ of norm 1 from $B_B(\mathcal{H})$ into the space of all functions,

$$(\mathbf{Q}_t f)(v) = \int d\mathbb{P}_{v,t}(\psi)\ f(\psi), \quad f \in B_B(\mathcal{H}). \tag{5.2}$$

However, we have not proved that the Borel function are mapped into the Borel ones. Let the sequence $\{f_n\} \subset C_{BWS}(\mathcal{H})$ π–approximate $f_0 \in B_B(\mathcal{H})$ (that is, the sequence $\{f_n\}$ is uniformly bounded and converges to f_0 pointwise; this type of convergence was introduced by Priola in his paper [33] devoted to π–continuous semigroups). By the Lebesgue theorem

$$\int d\mathbb{P}_{v,t}(\psi)\ f_n(\psi) \to \int d\mathbb{P}_{v,t}(\psi)\ f_0(\psi),$$

that is, $\mathbf{T}_t f_n$ π–converges to $\mathbf{Q}_t f_0$ (the uniform boundedness is implied by fact that the measure $\mathbb{P}_{v,t}$ is probabilistic). The space of bounded Borel functions is the π–closure $C_{BWS}(\mathcal{H})$ (the indicator functions of sets of form $\{v :\ \langle w|v\rangle \in [a,b]\}$ are π–approximated by weakly continuous functions), this is why the functions $v \mapsto \mathbf{Q}_t f$, $f \in B_B(\mathcal{H})$ are approximated pointwise by Borel functions, and hence, $B_B(\mathcal{H})$ is invariant with respect to the operators $\mathbf{Q}_t$. Thus, the operators $\mathbf{Q}_t$ are Markov and extend $\mathbf{T}_t$, moreover, uniquely since they are approximated pointwise by the actions of operator $\mathbf{T}_t$.

Let us show that the system $\{\mathbf{Q}_t\}_{t\geqslant 0}$ is a semigroup on $B_B(\mathcal{H})$. For an arbitrary $f_0 \in B_B(\mathcal{H})$ we find a sequence $\{f_n\} \subset C_{BWS}$, which π–approximates f_0. Then

$$\mathbf{Q}_t\mathbf{Q}_s f_0 = \pi \lim_{n\to\infty} \mathbf{T}_t\mathbf{T}_s f_n = \pi \lim_{n\to\infty} \mathbf{T}_{t+s} f_n = \mathbf{Q}_{t+s} f_0.$$

By construction, $\mathbf{Q}_0 = \mathbf{I}$.

We choose an arbitrary sequence $\{n_k\} \subset \mathbb{N}$. As it has been mentioned, we can choose a subsequence $\{N_k\}$, for which the distributions of random vectors $\{G_{\frac{t}{N_k}}^{\perp\!\!\!\perp N_k} v\}$ weakly converge to some measure $\tilde{\mathbb{P}}_{v,t}$. However, due to the coincidence

$$\int d\mathbb{P}_{v,t}(\psi)\ f(\psi) = \int d\tilde{\mathbb{P}}_{v,t}(\psi)\ f(\psi)$$

on $C_{BWS}(\mathcal{H})$, we obtain the identities $\tilde{\mathbb{P}}_{v,t} = \mathbb{P}_{v,t}$ and

$$\lim_{k\to\infty} \mathbf{F}[G_{\frac{t}{N_k}}]^{N_k} f(v) = \mathbf{Q}_t f(v).$$

Therefore, $\lim\limits_{N\to\infty} \mathbf{F}[G_{\frac{t}{N}}]^N f(v) = \mathbf{Q}_t f(v)$. □

Example 5.1. *A random operator–valued function* $\{G_t(y) = e^{by\sqrt{t}C - atC^2}\}$ *with respect to the measure* $d\mathbb{P}(y) = \dfrac{dy}{\sqrt{\pi/\gamma}} e^{-\gamma y^2}$ *satisfies the condition (#) for each self–adjoint operator* C *on* $\mathcal{H}$, $a > 0$, $\gamma > \dfrac{b^2}{4a}$. *It can be easily verified taking into account the representation*

$$G_t(y) = e^{\frac{y^2 b^2}{4a}} \exp\Big(-\Big(C\sqrt{at} - \frac{by}{2\sqrt{a}}\Big)^2\Big),$$

which implies

$$\frac{d\mathbb{P}(y)}{dy}\ \|G_t(y)\| \leqslant \frac{1}{\sqrt{\pi}} e^{-\left(\gamma - \frac{b^2}{4a}\right)y^2}.$$

Using some class of OVF described in 5.1, there is a possibility to construct the Chernoff approximations of bi–continuous semigroups. However, we separately need to verify the commutation condition, which can be violated by the following operations. Namely, we can

1. average avaliable OVF: if for each α OVF $\{\mathbf{F}[G_{\alpha,t}]\}$ satisfies (#) and a quasiprobability measure $d\nu(\alpha)$ of bounded measure is given, then $\{\mathbf{F}_\nu[G_t] := \int d\nu(\alpha)\ \mathbf{F}[G_{\alpha,t}]\}$ satisfies (#);
2. reparametrize the family: if the OVF $\{\mathbf{F}[G_t]\}$ satisfies (#) on the segment $[0,T]$, and $s : [0,T] \to [0,S]$, then $\{\mathbf{F}[G_{s(t)}]\}$ also satisfies (#), but on the segment $[0,S]$.

The first option corresponds to the replacement of probability space $(\Omega, \mathscr{F}, \mathbb{P})$ on $\Big(\mathcal{A}\times\Omega, \Sigma\otimes\mathscr{F}, \tilde{\mathbb{P}} := \dfrac{|\nu|}{|\nu|(\mathcal{A})}\otimes\mathbb{P}\Big)$. It is supposed that the necessary measurability conditions with respect to $(\Sigma\otimes\mathscr{F}, \mathscr{B}_{WOT})$ are satisfied for the random operators $\{G_{\alpha,t}(\omega)\}$ and

$$\zeta(\alpha) := \Big\{ \sup_{t\in[0,T],\ \|v\|\leqslant r} \mathbb{P}\Big(\|G_{\alpha,t}(\omega)v\| > R\Big)\Big\}.$$

The mapping $\alpha \mapsto \zeta(\alpha)$ is measurable and this is why for all $\varepsilon > 0$ there exist $R_\varepsilon > 0$ and a measurable $\mathcal{A}_\varepsilon \subset \mathcal{A}$ such that

$$\frac{|\nu|(\mathcal{A}_\varepsilon)}{|\nu|(\mathcal{A})} > 1 - \varepsilon$$

and $|\zeta(\alpha)| \leqslant R_\varepsilon$ for all $\alpha \in \mathcal{A}_\varepsilon$. At the same time,

$$\tilde{\mathbb{P}}\Big(\alpha\in\mathcal{A}_\varepsilon,\ \|G_{\alpha,t}(\omega)v\| \leqslant R_\varepsilon\Big) \geqslant 1 - 2\varepsilon.$$

The second option is obvious.

5.1. Examples.

5.1.1. Quantum measurements. We consider a particular example of a quantum measurement scheme that, in the limit of the interval $t > 0$ between measurements with identical instruments tending to zero, gives a continuous process in the state space admitting the description in terms of linear random walks. Suppose that a measurable space with measure $(\mathcal{A}, \Sigma, \nu)$ and measurable mappings $\alpha \mapsto \langle w|C_\alpha v\rangle$ for all $w \in \mathcal{H}$, $v \in \mathscr{D}$ (the subspace $\mathscr{D}$ is dense in $\mathcal{H}$) and some self–adjoint operators $\{C_\alpha\}$ are given. We consider a completely positive instrument

$$\mathbf{M}[B](\rho) = \int_B d\nu(\alpha) \int \frac{dy}{\sqrt{\pi/\gamma}} e^{-\gamma y^2}\ G_{\alpha,t}(y)\rho G^*_{\alpha,t}(y), \tag{5.3}$$

where $B \in \Sigma$, $G_{\alpha,t}(y) = e^{by\sqrt{t}C_\alpha - atC_\alpha^2}$, $a > 0$, $b \in \mathbb{R}$.

A linear random walk with a parameter t is defined by the family of random operators $\{G_{\alpha,t}(y)\}$. As it has been already noted, the condition (#) is satisfied for $\gamma > \frac{b^2}{4a}$, so it remains to verify that on some space $\mathcal{D}_0 \subset C_B(\mathcal{H})$, the bi–closure of which contains $C_{BWS}(\mathcal{H})$, we have

$$\lim_{t\to 0}\Big|\Big|\frac{\mathbf{F}[G_t]-\mathbf{I}}{t}f-\mathbf{L}f\Big|\Big|=0, \quad f\in\mathcal{D}_0. \tag{5.4}$$

Let

$$\lim_{t\to 0}\int d\nu(\alpha)\int \frac{dy}{\sqrt{\pi/\gamma}}e^{-\gamma y^2}\ \frac{G_{\alpha,t}(y)w-w}{t} = A_1 w, \quad w\in\mathscr{D},$$
$$\lim_{t\to 0}\int d\nu(\alpha)\int \frac{dy}{\sqrt{\pi/\gamma}}e^{-\gamma y^2}\ \frac{(G_{\alpha,t}w-w)^{\otimes 2}}{2t} = A_2 w^{\otimes 2}, w\in\mathscr{D},$$

denote $A=(A_1,A_2)$ and let $A_1\in\mathcal{B}(\mathcal{H})$, $A_2\in\mathcal{B}(\mathcal{H}^{\otimes 2})$.

Formally the relation of the operators A_1, A_2 with the random operator C_α can be expressed by the formulas

$$-\Big(a-\frac{b^2}{4\gamma}\Big)\int d\nu(\alpha)\ \langle w|C_\alpha^2 v\rangle = \langle A_1 w|v\rangle, \tag{5.5}$$
$$\frac{b^2}{4\gamma}\int d\nu(\alpha)\ \langle w|C_\alpha v\rangle^2 = \langle A_2 w^{\otimes 2}|v^{\otimes 2}\rangle. \tag{5.6}$$

We form $\mathcal{D}_0$ as a $\mathbb{C}$–linear span of functions of form

$$f(v)=p(\langle w_1|v\rangle_{\mathbb{R}},\dots,\langle w_n|w\rangle_{\mathbb{R}})h(\delta\|v\|^2),$$

where $p\in\mathbb{R}[x_1,\dots,x_n]$, $n\in\mathbb{N}$, $\delta>0$, and the function $h(x):\mathbb{R}\to\mathbb{R}$ obey the following conditions

1. $h(x)\in\mathcal{S}(\mathbb{R})$;
2. $\|h\|=1$;
3. $h(x)=1$ for all $x\in[-1,1]$, and $h(x)=0$ for all $x\in\mathbb{R}\setminus[-2,2]$.

Under these conditions the function $f(v)$ has bounded derivatives of all orders, which are non–zero only inside some bounded set. We restrict ourselves by the case, when the operators A_1 and A_2 are continuous and hence, at each point $v\in\mathcal{H}$ the action $(\mathbf{D}(A)f)(v)$ is well–defined for $A=(A_1,A_2)$ and $\mathbf{D}(A)f\in C_B(\mathcal{H})$. Since the condition (4.18) of Proposition 4.8 is satisfied, on the bounded sets the convergence $\dfrac{\mathbf{F}[G_t]-\mathbf{I}}{t}f-\mathbf{L}f$ is uniform and therefore, $f\in\mathcal{L}_G$. It remains to note that each weakly sequentially continuous function can be uniformly approximated by polynomials on each bounded subset of $\mathcal{H}$, and hence, for each $f\in C_{BWS}(\mathcal{H})$ there exists a sequence from $\mathcal{D}_0$, which bi–converges to this function.

Thus, the Chernoff iterations of the constructed OVF $\{\mathbf{F}[G_t]\}$ bi–approximate some bi–continuous semigroup on $C_{BWS}(\mathcal{H})$, which can be extended to a Markov semigroup on $B_B(\mathcal{H})$ if the commutation condition is satisfied; this conditions is to be verified separately.

The same conclusion can be made in the case, when A_1, A_2 are self–adjoint (not necessarily bounded) operators, where $\{e_k\}$ is an orthonormal basis of eigenvectors of A_1, and $\{e_j\otimes e_k\}$ is an orthonormal basis of eigenvectors of A_2. Then, as $\mathcal{D}_0$, we can take $\mathbb{C}$, the linear span of functions of the form $f(v)=g(\langle e_k|v\rangle_{\mathbb{R}})$, where $g\in\mathcal{S}(\mathbb{R})$. It is easy to see that $\mathbf{D}(A)f\in C_B(\mathcal{H})$ and the bi–closure of $\mathcal{D}_0$ contains $C_{BWS}(\mathcal{H})$. That is, $\mathbf{D}(A)$ can potentially serve as a generator of a bi–continuous semigroup. If we additionally require that (5.4) hold for the OVF $\mathbf{F}[G_t]$, then the Chernoff iterations of the constructed OVF $\{\mathbf{F}[G_t]\}$ also bi–approximate some bi–continuous semigroup according to Theorems 5.1 and 5.2.

5.1.2. Random control. Let the random Hamiltonian (H_α,ν) be bounded by the norm, that is, the inequality $\|H_\alpha\| \leqslant M$ holds ν–almost everywhere for some constant $M>0$. We construct the random OVF $\{G_t\}$, $G_{\alpha,t}(y) = \exp\Big(iby\sqrt{t}H_\alpha + atH_\alpha\Big)$ with respect to the measure $\nu \otimes \dfrac{dy}{\sqrt{\pi/\gamma}}e^{-\gamma y^2}$, for which the commutation condition is supposed to hold.

The condition (#) is obviously implied by the estimate for the norm $\|G_{\alpha,t}(y)\| \leqslant e^{atM}$, the boundedness of the Hamiltonians is employed essentially. The differential operators are similarly defined on the family $\mathcal{D}_0$ and they read $\mathbf{D}(A)$, where $A=(A_1,A_2)$,

$$\begin{aligned} A_1 w &= \Big(a - \frac{b^2}{4\gamma}\Big)\int d\nu(\alpha)\ H_\alpha^2 w,\\ A_2 w^{\otimes 2} &= -\frac{b^2}{4\gamma}\int d\nu(\alpha)\ H_\alpha^{\otimes 2} w^{\otimes 2}, \quad v\in\mathcal{H}, \end{aligned}$$

and they generate the bi–continuous semigroup $\mathbf{T}_t$ on C_{BWS}.

5.1.3. Validity of commutation condition for non–commuting random OVF. It has been noted more than once that although the commutation condition is essentially restrictive, it does not require almost sure commutation $[G_t(\omega),G_s(\omega)]=0$, $t,s\in[0,T]$. Let us give an example when the Markov operators corresponding to the random operators

$$G = \{(p_j,G_j)\}_{j=1}^n, \qquad \tilde{G} = \{(\tilde{p}_j,\tilde{G}_j)\}_{j=1}^n$$

commute, but $[G_j,\tilde{G}_k]\neq 0$ for $j\neq k$.

We let $n=2$ ($\dim\mathcal{H}=2$),

$$\begin{aligned} &p_1 = p_2 = \tilde{p}_1 = \tilde{p}_2 = \frac{1}{2},\\ &G_1 = \hat{\sigma}_x, \qquad G_2 = \hat{\sigma}_z, \qquad \tilde{G}_1 = i\hat{\sigma}_y, \qquad \tilde{G}_2 = -i\hat{\sigma}_y, \end{aligned}$$

where $\{\hat{\sigma}_x,\hat{\sigma}_y,\hat{\sigma}_z\}$ are the Pauli matrices. Then the random operators $G\tilde{G}$ and $\tilde{G}G$ have the same distribution

$$\{(p_ip_j,G_i\tilde{G}_j)\} = \{(p_ip_j,\tilde{G}_jG_i)\} \sim \left\{\left(\frac{1}{4},\hat{\sigma}_x\right),\ \left(\frac{1}{4},-\hat{\sigma}_x\right),\ \left(\frac{1}{4},\hat{\sigma}_z\right),\ \left(\frac{1}{4},-\hat{\sigma}_z\right)\right\}$$

Therefore, the commutation $\Big[\mathbf{F}[G],\mathbf{F}[\tilde{G}]\Big]=0$ holds.

This example allows us to construct OVF $\{\mathbf{F}[G_t]\}$ of mutually commuting Markov operators on the base of non–commuting random operators: $G_t(\omega)=I$ with the probability$1-t$, and $G_t(\omega)$ is equal to one of the operators $\{\hat{\sigma}_x,\hat{\sigma}_z,i\hat{\sigma}_y,-i\hat{\sigma}_y\}$ with the probability $t/4$.

Conclusion

It has been established that under the commutation condition and under the natural technical assumption (#) on the bounded behavior of random operators, a discrete in time random walk approximates a continuous Markov homogeneous process described by a semigroup, the restriction of which to the space of bounded weakly sequentially continuous functions has the bi–continuity property. The importance of such a space is that each bounded Borel function can be approximated pointwise by its elements. Processes important for applications in quantum mechanics can be approximated by discrete processes with the above property. This result not only indicates a method of approximation that is certainly useful in modeling processes, but also implicitly indicates the existence of a semigroup with a certain type of generator. In the general case, the question of the existence of a semigroup with some unbounded operator as a generator was not resolved completely. The commutation condition of Markov operators of the family $\{\mathbf{F}[G_t]\}$ turns out to be the most restrictive in our analysis, but, as it is shown in Example 3, is not equivalent to the commutation of random operators almost surely. In the framework of this approach we have not succeeded to overcome the necessity of this requirement.

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Appendix. Some known properties of functions on Hilbert space

The weak topology τ_W on $\mathcal{H}$ is generated by the intersections of the neighbourhoods of form

$$U(w,a,\varepsilon) = \{v \in \mathcal{H} \mid \langle w|v\rangle \in (a-\varepsilon, a+\varepsilon)\}$$

and it coincides with the weak topology. On the set $K \subset \mathcal{H}$ the weak topology is induced by the weak topology on $\mathcal{H}$. A weakly continuous function is one continuous with respect to the weak topology. The each closed ball $B \subset \mathcal{H}$ the weak topology is metrizable, and B is a weakly compact set. We recall that a function $f : \mathcal{H} \to \mathbb{C}$ is called weakly sequentially continuous if $\tau_W \lim\limits_{n\to\infty} v_n = v_0$ implies $\lim\limits_{n\to\infty} f(v_n) = f(v_0)$.

Proposition 5.1. *A function $f : \mathcal{H} \to \mathbb{C}$ is weakly sequentially continuous at a point $v \in \mathcal{H}$ if and only if its restriction to each bounded neighborhood U of v is weakly continuous.*

Proof. $\Rightarrow$ Let $U_r(v)$ be a ball centered at v. The weak topology on this ball is metrizable. Therefore, the weak continuity is equivalent to the weak sequential continuity.

$\Leftarrow$ A weakly convergent sequence $\tau_W \lim\limits_{n\to\infty} v_n = v_0$ is bounded, so it can be placed into a spherical neighborhood of U. The weak continuity of f on U then implies $\lim\limits_{n\to\infty} f(v_n) = f(v_0)$. □

The spaces of bounded Borel functions ($B_B(\mathcal{H})$), bounded continuous ($C_B(\mathcal{H})$), bounded weakly sequentially continuous ($C_{BWS}(\mathcal{H})$) and bounded weakly continuous functions ($C_{BW}(\mathcal{H})$) satisfy the following chain of inclusions: $C_{BW}(\mathcal{H}) \subset C_{BWS}(\mathcal{H}) \subset C_B(\mathcal{H}) \subset B_B(\mathcal{H})$. Each of these subspaces is Banach with respect to the sup–norm. We shall verify this property for the space $C_{BWS}(\mathcal{H})$ we are interesting in.

Proposition 5.2. *The space $C_{BWS}(\mathcal{H})$ with the* sup*–norm is Banach.*

Proof. It is sufficient to show that it is closed in $C_B(\mathcal{H})$. We take

$$f_1 \in C_B(\mathcal{H}) \setminus C_{BWS}(\mathcal{H}).$$

Then there exists a sequence $\{v_n\} \subset \mathcal{H}$ weakly converging to v_0, but

$$|f_1(v_n) - f_1(v_0)| > \varepsilon$$

for some $\varepsilon > 0$. Then it is sufficient to consider the neighbourhood $U_{\frac{\varepsilon}{3}}(f_1)$ of the radius $\varepsilon/3$ of element f_1. For each $f_2 \in U_{\varepsilon/3}(f_1)$ and for all $v \in \mathcal{H}$ we have $|f_1(v) - f_2(v)| < \varepsilon/3$ and

$$|f_2(v_n) - f_2(v_0)| \geqslant |f_1(v_n) - f_1(v_0)| - |f_1(v_n) - f_2(v_n)| - |f_1(v_0) - f_2(v_0)| > \frac{\varepsilon}{3}, \tag{5.7}$$

and this implies the absence of weak sequential continuity of f_2. □

The weak uniform continuity of a function $f : K \to \mathbb{C}$ is expressed by the fact that for each $\varepsilon > 0$ there exist $\delta > 0$, $w_1, \dots w_n \in \mathcal{H}$ such that the inequality $|\langle w_k|v' - v''\rangle| < \delta$ for all $k = 1, \dots, n$, $v', v'' \in K$ implies $|f(v') - f(v'')| < \varepsilon$. On balls (or on other weakly compact set) the weakly continuous functions are uniformly continuous.

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Andrey Vladimirovich Utkin,
Steklov Mathematical Institute
of Russian Academy of Sciences,
Gubkin str. 8,
119991, Moscow, Russia
E-mail: `utkin.av@phystech.edu`