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INTEGRAL INEQUALITIES INVARIANT UNDER CONFORMAL TRANSFORMATIONS

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Abstract. Employing the Poincaré metric, we introduce conformally invariant integrals for smooth compactly supported functions defined on domains of hyperbolic type in the extended plane. For these integrals, which involve the hyperbolic radius, a smooth function, and its gradient or Laplacian, we consider conformally invariant analogues of Hardy and Rellich type inequalities with constants depending on the domain. We provide explicit estimates for the constants using numerical characteristics, namely, the maximal moduluses of the domain and a geometric constant involved in the linear hyperbolic isoperimetric inequality.

In the paper we prove several new statements. In particular, we justify a criterion for the positivity of constants for finitely–connected domains of hyperbolic type and prove several integral inequalities universal in the sense that these inequalities involve no unknown constants and are valid in each domain of hyperbolic type.

In the beginning of the paper, we briefly outline the properties of hyperbolic radius and describe several related. In particular, we mention the results by Schmidt, Osserman, Fernández, and Rodríguez on hyperbolic isoperimetric inequalities and their applications, provide the Elstrodt — Patterson — Sullivan formula for the critical exponents of convergence of the Poincaré — Dirichlet series, and present a result by Carleson and Gamelin on the maximal moduli of a domain with uniformly perfect boundary.

Keywords: Poincaré metric, conformal mapping, isoperimetric inequality, Hardy type inequality, Laplace operator.

Mathematics Subject Classification: 26D10, 33C20

1. Introduction

Let $\Omega \subset \overline{\mathbb{C}}$ be a domain of hyperbolic type. In this domain we define the Poincaré metric $ds = \lambda_\Omega(z)|dz|$ with the Gauss curvature $\kappa = -4$. We recall that a domain $\Omega \subset \overline{\mathbb{C}}$ is of hyperbolic type if and only if its boundary $\partial\Omega$ contains at least three points. We briefly describe some information about the hyperbolic radius $R(z,\Omega) := 1/\lambda_\Omega(z)$; it can be found in the monographs by Goluzin [1], Ahlfors [2], Avkhadiev and Wirths [3].

The well–known formula for the Gauss curvature gives

$$R^2(z,\Omega)\,\Delta \ln R(z,\Omega) \equiv -4 (= \kappa). \tag{1.1}$$

The hyperbolic radius $R(z,\Omega)$ is attractive since it is comparable with the Euclidean distance $\operatorname{dist}(z,\partial\Omega)$ from the point $z \in \Omega$ to the boundary $\partial\Omega$ of the domain

$$R(z,\Omega) \geqslant \operatorname{dist}(z,\partial\Omega) := \min_{w\in\partial\Omega} |z-w|, \quad z \in \Omega. \tag{1.2}$$

For a series of domains $\Omega \subset \mathbb{C}$ also the estimate

$$\operatorname{dist}(z,\partial\Omega) \geqslant \alpha(\Omega)\, R(z,\Omega), \qquad z \in \Omega,$$

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holds, where $\alpha(\Omega) = \text{const} \in (0, 1/2]$.

We also recall the known examples: $R(z, \mathbb{D}) = 1 - |z|^2$ for the unit circle $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $R(z, P) = 2\,\mathrm{dist}(z, \partial P)$ for the half–plane, $R(z, \mathbb{D} \setminus \{0\}) = 2\,|z| \ln(1/|z|)$ for the circle $\mathbb{D}$ with the punctured center.

If $\infty \in \Omega$, then near this point we have $R(z, \Omega) \asymp |z|^2$.

Let $f : \mathbb{D} \to \Omega$ be the univercal covering mapping of the circle $\mathbb{D}$ to a domain Ω of hyperbolic type. We note that the function f is holomorphic or meromorphic. If Ω is a simply–connected domain, then f is a univalent conformal mapping, while if Ω is not simply–connected, then the covering mapping is just locally univalent. For each domain $\Omega \subset \overline{\mathbb{C}}$ of hyperbolic type the identity

$$R(z, \Omega) \equiv |f'(\zeta)|(1 - |\zeta|^2), \qquad \zeta \in \mathbb{D}, \quad z = f(\zeta) \in \Omega,$$

holds. As a corollary of this identity we can obtain Equation (1.1) and the equivalent Liouville equation

$$R(z, \Omega)\, \Delta R(z, \Omega) = |\nabla R(z, \Omega)|^2 - 4,$$

where the Euclidean gradient and Euclidean Laplacian of the function $\varphi = R(\,\cdot\,, \Omega)$ are defined by the well–known formulas

$$\nabla \varphi(z) = 2 \frac{\partial \varphi(z)}{\partial \overline{z}}, \qquad \Delta \varphi(z) = 4 \frac{\partial^2 \varphi(z)}{\partial z\, \partial \overline{z}}.$$

As it was proved in the recent paper [4], the identity

$$\frac{R^3(z, \Omega)}{4}\, \Delta^2 R(z, \Omega) \equiv (1 - |\zeta|^2)^4\, |S_f(\zeta)|^2, \quad z = f(\zeta) \in \Omega,$$

holds, where $\Delta^2 R := \Delta(\Delta R)$, the function S_f is the Schwarzian derivative of the universal covering mapping, that is,

$$S_f(\zeta) = \frac{f'''(\zeta)}{f'(\zeta)} - \frac{3}{2} \left(\frac{f''(\zeta)}{f'(\zeta)} \right)^2, \quad \zeta \in \mathbb{D}.$$

This identity implies that for each domain $\Omega \subset \overline{\mathbb{C}}$ of hyperbolic type the function $\psi : \Omega \to \mathbb{R}$ defined by the formula $\psi(z) = \Delta R(z, \Omega)$ is subharmonic.

In the domain $\Omega \setminus \{\infty\}$ the hyperbolic radius $R(z, \Omega)$ is a real analytic function. If $\infty \in \Omega$, then near the infinity the function $|z|^{-2} R(z, \Omega)$ is real analytic and the limit

$$\lim_{z \to \infty} \frac{R(z, \Omega)}{|z|^2} > 0$$

is well–defined. If $z_0 \in (\partial \Omega) \setminus \{\infty\}$, then

$$\lim_{z \to z_0} R(z, \Omega) = \lim_{z \to z_0} \mathrm{dist}(z, \partial \Omega) = 0.$$

We consider an m–connected domain $G_m \subset \Omega$ with a piecewise–smooth boundary $\partial G_m \subset \Omega$, where m stands for the number of boundary components. The hyperbolic area and perimeter of a domain $G_m \subset \Omega$ are defined by the following integrals

$$A(G_m) = \iint\limits_{G_m} \frac{dxdy}{R^2(z, \Omega)}, \qquad L(\partial G_m) = \int\limits_{\partial G_m} \frac{|dz|}{R(z, \Omega)}, \tag{1.3}$$

which are invariant with respect to the conformal mappings Ω.

Under some conditions on the pair (Ω, G_m) the following conformally invariant hyperbolic isoperimetric inequality is true:

$$4\pi\, A(G_m) - \kappa\, A^2(G_m) \leqslant L^2(\partial G_m),$$

where $k = const < 0$ is the curvature of the hyperbolic metrics, the area $A(G_m)$ and the length $L(\partial G_m)$ are defined by the formulas (1.3). Let us provide the exact formulations.

Theorem 1.1 ([5]). *If Ω is a simply–connected domain of hyperbolic type and it is equipped with the Poincaré metrics of the curvature $\kappa = -4$, then for each simply–connected domain G_1 with a piecewise–smooth boundary, which is compactly embedded into the domain Ω the inequality*

$$4\pi\, A(G_1) + 4\, A^2(G_1) \leqslant L^2(\partial G_1)$$

holds, in which the identity is achieved if and only if G_1 is the hyperbolic circle.

It is easy to show that the following theorem holds; it is simultaneously the generalization and corollary of Theorem 1.1.

Theorem 1.2. *Suppose that $\Omega \subset \overline{\mathbb{C}}$ is a domain of hyperbolic type equipped with the Poincaré metrics of the curvature $\kappa = -4$ and let $f : \mathbb{D} \to \Omega$ be the universal covering mapping of the circle $\mathbb{D}$ to the domain Ω. Let G_m be an m–connected domain with a piecewise–smooth boundary and $\overline{G}_m \subset \Omega$. If we additionally suppose that a univalent branch of the analytic function $f^{-1} : \Omega \to \mathbb{D}$ is well–defined on the domain G_m, then the inequality*

$$4\pi\, A(G_m) + 4\, A^2(G_m) \leqslant L^2(\partial G_m) \tag{1.4}$$

holds.

It is obvious that if the domain Ω is simply–connected, then the additional condition holds immediately. We also recall that for each domain Ω of hyperbolic type the existence of the univalent branch of the function f^{-1} in the simply–connected domain $G_1 \subset \Omega$ is satisfied by the monodromy theorem. It is easy to verify that without the additional condition on G_m with $m \geqslant 2$ the inequality (1.4) can be wrong even in the case of doubly–connected domains Ω.

We note that for sufficiently smooth functions $u : \Omega \to \mathbb{R}$ the integrals

$$\iint\limits_{\Omega} |\nabla u(z)|^2 dxdy, \qquad \iint\limits_{\Omega} \Delta u(z) dxdy, \qquad \iint\limits_{\Omega} |\Delta u(z)| dxdy$$

are conformally invariant. We also mention that the Green formula

$$\iint\limits_{\Omega} (u\Delta v + (\nabla u, \nabla v)) dxdy = \int\limits_{\partial\Omega} u\, (\partial v/\partial n) |dz|$$

is a conformally invariant identity.

It is easy to construct a series of other conformally invariant integrals. For $p \in [1, \infty)$ the integrals of form $\iint\limits_{\Omega} |\nabla u(z)|^p R^{p-2}(z, \Omega)\, dxdy$ are conformally invariant. Indeed, if $\Phi : \Pi \to \Omega$ is a univalent conformal mapping of the domain Π of hyperbolic type to the domain Ω, then

$$R(z, \Omega) = |\Phi'(\zeta)| R(\zeta, \Pi), \qquad \zeta = \xi + i\eta \in \Pi, \quad z = x + iy = \Phi(\zeta) \in \Omega.$$

Therefore,

$$\frac{|\nabla_z\, u(z)|^p}{R^{2-p}(z, \Omega)} dxdy = \frac{|\nabla_\zeta\, u(\Phi(\zeta))|^p}{R^{2-p}(\zeta, \Pi)} d\xi d\eta \qquad \left(\frac{dxdy}{R^2(z, \Omega)} = \frac{d\xi d\eta}{R^2(\zeta, \Pi)} \right). \tag{1.5}$$

In each domain of hyperbolic type the Euclidean distance $\operatorname{dist}(z, \partial\Omega)$ satisfies the Lipschitz condition

$$|\operatorname{dist}(z_1, \partial\Omega) - \operatorname{dist}(z_2, \partial\Omega)| \leqslant |z_1 - z_2|, \qquad z_1, z_2 \in \Omega,$$

but it is not a smooth function even for bounded convex domains Ω. In view of the inequality (1.2) and other mentioned properties of the hyperbolic radius, the quantity $R(z, \Omega)$ can be regarded as a smooth replacement for the Euclidean distance $\operatorname{dist}(z, \partial\Omega)$ in known integral inequalities of mathematical physics.

Let $k \in \mathbb{N}$. By the symbol $C_0^k(\Omega)$ we denote the family k times continuously differentiable functions $u : \Omega \to \mathbb{R}$ compactly supported in the domain Ω.

If $\infty \in \Omega$, then the continuity of $u(z)$ and its derivatives at the point $z = \infty$ means the continuity of $u(1/z)$ and its derivatives in the variables x and y $(x + iy = z)$ at the point $z = 0$.

Let $p \in [1, \infty)$. Using the formulas (1.5) and their analogues for the Laplacian, we define and study a series of new conformally invariant numerical characteristics for the domains of hyperbolic type. The basic ones are the following constants

$$c_p(\Omega) = \inf_{u \in C_0^1(\Omega), u \not\equiv 0} \frac{\iint\limits_{\Omega} |\nabla u(z)|^p R^{p-2}(z, \Omega) dxdy}{\iint\limits_{\Omega} |u(z)|^p R^{-2}(z, \Omega) dxdy},$$

$$c_p^{**}(\Omega) = \inf_{u \in C_0^2(\Omega), u \not\equiv 0} \frac{\iint\limits_{\Omega} |\Delta u(z)|^p R^{2p-2}(z, \Omega) dxdy}{\iint\limits_{\Omega} |u(z)|^p R^{-2}(z, \Omega) dxdy}.$$

These characteristics $c_p(\Omega) \in [0, \infty)$ and $c_p^{**}(\Omega) \in [0, \infty)$ are conformally invariant. The quantity $c_p(\Omega)$ is the exact constant in the integral identity

$$\iint\limits_{\Omega} \frac{|\nabla u(z)|^p}{R^{2-p}(z, \Omega)} dxdy \geqslant c_p(\Omega) \iint\limits_{\Omega} \frac{|u(z)|^p}{R^2(z, \Omega)} dxdy \qquad \forall u \in C_0^1(\Omega),$$

while the quantity $c_p^{**}(\Omega)$ is a sharp constant in the inequality

$$\iint\limits_{\Omega} \frac{|\Delta u(z)|^p}{R^{2-2p}(z, \Omega)} dxdy \geqslant c_p^{**}(\Omega) \iint\limits_{\Omega} \frac{|u(z)|^p}{R^2(z, \Omega)} dxdy \qquad \forall u \in C_0^2(\Omega).$$

If $c_p(\Omega) = 0$ or $c_p^{**}(\Omega) = 0$, then the corresponding inequality becomes meaningless. This is why it is important to find the positivity of these constants. In Sections 2 and 3 we describe such conditions and also provide the estimates for these constants via geometric characteristics of the domains.

2. Constants $h(\Omega)$ and $c_2(\Omega)$, maximal moduluses $M(\Omega)$ and $M_0(\Omega)$

The following linear isoperimetric inequalities

$$A(G_m) \leqslant h(\Omega)\, L(\partial G_m), \qquad h(\Omega) := \sup_{G_m} \frac{A(G_m)}{L(\partial G_m)}. \tag{2.1}$$

turned out to be useful in applications. Here the area $A(G_m)$ of the domain G_m and the length $L(\partial G_m)$ of its boundary are defined by the formulas (1.3).

The supremum in the formula (2.1) is taken over the set of finitely–connected domains G_m $(m \in \mathbb{N})$ having piecewise–smooth boundaries and being compactly embedded into the domain Ω. It is obvious that if the isoperimetric inequality (1.4) holds for the pair (Ω, G_m), then

$$A(G_m) \leqslant \frac{1}{2} L(\partial G_m).$$

It is known, see the review by R. Osserman [6]), that the following statement is true.

Proposition 2.1. *If Ω is a simply–connected or doubly–connected domain of hyperbolic type equipped with the Poincaré metrics of curvature $k = -4$, then $h(\Omega) = 1/\sqrt{-\kappa} = 1/2$.*

In [6] Osserman also pointed out that there exist domains of hyperbolic type, for which $h(\Omega) = \infty$, in particular, $h(\mathbb{C} \setminus \{0, 1\}) = \infty$.

We recall that by the symbol $C_0^1(\Omega)$ we denote the set of continuously differentiable functions $u : \Omega \to \mathbb{R}$ compactly supported in the domain $\Omega \subset \overline{\mathbb{C}}$ of hyperbolic type. We consider the following conformally invariant analogue of Hardy type inequality:

$$\iint\limits_{\Omega} |\nabla u(z)|^2 dxdy \geqslant c_2(\Omega) \iint\limits_{\Omega} \frac{|u(z)|^2}{R^2(z,\Omega)} dxdy \qquad \forall u \in C_0^1(\Omega), \tag{2.2}$$

where the constant $c_2(\Omega) \geqslant 0$ is supposed to be the maximal possible. The exact definition of $c_p(\Omega)$ for each fixed $p \in [1,\infty)$ is given above. It is clear that the inequality (2.2) is also a hyperbolic analogue of the classical Poincaré — Friedrichs inequality.

The next statement is known, see [7], [8], [9].

Theorem 2.1. *For each simply–connected or doubly–connected domain we have* $c_2(\Omega) = 1$.

The next statement is also well–known, [10], [6], [11], see also [12].

Theorem 2.2. *For each domain* $\Omega \subset \overline{\mathbb{C}}$ *equipped with the Poincaré metrics of curvature* $k = -4$ *the inequalities*

$$\frac{1}{4\,h^2(\Omega)} \leqslant c_2(\Omega) \leqslant \frac{3}{h(\Omega)} \tag{2.3}$$

hold.

We first mention a simple corollary.

Corollary 2.1. *For each domain* $\Omega \subset \overline{\mathbb{C}}$ *of hyperbolic type*

$$c_2(\Omega) > 0 \Longleftrightarrow h(\Omega) < \infty.$$

The next nontrivial statement complements Proposition 2.1.

Proposition 2.2. *For each domain* $\Omega \subset \overline{\mathbb{C}}$ *equipped with the Poincaré metrics of curvature* $k = -4$ *the inequalities*

$$c_2(\Omega) \leqslant 1, \qquad h(\Omega) \geqslant \frac{1}{2}$$

hold.

Proof. By Elstrodt — Patterson — Sullivan formula [8]

$$c_2(\Omega) = \begin{cases} 1, & 0 \leqslant \beta \leqslant \frac{1}{2}, \\ 4\beta(1-\beta), & \frac{1}{2} \leqslant \beta \leqslant 1. \end{cases}$$

where $\beta = \beta(\Omega) \in [0,1]$ is the critical convergence exponent of the Poincaré — Dirichlet series for the fundamental group of transformations of Ω. This formula implies that $c_2(\Omega) \leqslant 1$ for each domain of hyperbolic type. Combining this fact with the lower bound in the inequalities (2.3), we get $h(\Omega) \geqslant 1/2$. The proof is complete. □

In what follows we shall use the conformal maximal modulus $M(\Omega)$. Let Ω_2 be a doubly–connected domain, which is conformally equivalent to the annulus

$$A(a, z_0, b) = \{z \in \mathbb{C} : a < |z - z_0| < b\}.$$

As usually, the conformal modulus $\mathrm{mod}(\Omega_2)$ of the doubly–connected domain Ω_2 is defined by the identity

$$\mathrm{mod}(\Omega_2) = \mathrm{mod}(A(a, z_0, b)) = \frac{1}{2\pi} \ln \frac{b}{a} \quad (0 \leqslant a < b \leqslant \infty)$$

under the convention that if $a = 0$ and (or) $b = \infty$, then $\mathrm{mod}(A(a, z_0, b)) = \infty$.

Definition 2.1. *The conformal maximal modulus $M(\Omega)$ of non–simply–connected domain $\Omega \subset \overline{\mathbb{C}}$ is defined by the identity*

$$M(\Omega) = \sup_{\Omega_2} \operatorname{mod}(\Omega_2),$$

where the supremum is taken over the set of doubly–connected domains $\Omega_2 \subset \Omega$, each of which partitions the boundary of Ω. In particular, if the domain Ω is doubly–connected, then $M(\Omega) = \operatorname{mod}(\Omega)$. If Ω is a simply–connected domain of hyperbolic type, we let $M(\Omega) = 0$.

Theorem 2.3 ([12])**.** *For each non–simply–connected domain of hyperbolic type we have*

$$M(\Omega) < \infty \Longrightarrow c_2(\Omega) > 0 \tag{2.4}$$

The opposite implication is wrong. In particular, for the unit circle with the punctured center we have $c_2(\mathbb{D} \setminus \{0\}) = 1$, but $M(\mathbb{D} \setminus \{0\}) = \operatorname{mod}(\mathbb{D} \setminus \{0\}) = \infty$. There is an open problem posed in [12]: *in terms of the Euclidean geometry, describe all domains Ω, for which $c_2(\Omega) > 0$.* This problem is not solved yet. There is only the following partial advantage: *the implication (2.4) can be replaced by the equivalent implication $M_0(\Omega) < \infty \Longrightarrow c_2(\Omega) > 0$, where $M_0(\Omega)$ is a constant defined in terms of the Euclidean geometry of the domain Ω without using the conformal mappings.* Let us give the definition of $M_0(\Omega)$.

Definition 2.2. *The Euclidean maximal modulus $M_0(\Omega)$ of a non–simply–connected domain $\Omega \subset \overline{\mathbb{C}}$ is defined by the identity*

$$M_0(\Omega) = \sup_{A(a,z_0,b)} (2\pi)^{-1} \ln(b/a),$$

where the supremum is taken over the set of annuli

$$A(a, z_0, b) = \{z \in \mathbb{C} : a < |z - z_0| < b\},$$

such that $A(a, z_0, b) \subset \Omega$, $A(a, z_0, b)$ partitions the boundary Ω, $z_0 \in \partial\Omega$ and $0 < a < b < \infty$.

If there is no such annuli, we then we let $M_0(\Omega) = 0$. We let that $M_0(\Omega) = 0$ for each simply–connected domain $\Omega \subset \overline{\mathbb{C}}$ of hyperbolic type.

If $M_0(\Omega) < \infty$, then following [13], we say that $\partial\Omega$ is a uniformly perfect set. If the domain Ω is a piecewise–connected, then its boundary $\partial\Omega$ is uniformly perfect if and only if it is perfect. The Euclidean maximal modulus $M_0(\Omega)$ plays the most essential role for finitely–connected domains. And in this case the uniform perfectness of boundary differs substantially from the perfectness of boundary.

Carleson and Gamelin pointed out [14] that *by rescaling and applying normal families one can prove the following statement:*

$$M(\Omega) < \infty \Longleftrightarrow M_0(\Omega) < \infty.$$

It is obvious that the estimate $M_0(\Omega) \leqslant M(\Omega)$ is trivial and the implication

$$M_0(\Omega) < \infty \Longrightarrow M(\Omega) < \infty$$

is an unexpected statement. Because of this, we draw the attention of the reader to the fact that in the paper [12] the domains obeying the condition $M(\Omega) < \infty$, were called modulated domains. Together with modulated domains, in this paper we separately consider the domains with uniformly perfect boundaries obeying the condition $M_0(\Omega) < \infty$. Thus, the appearance of the term «modulated domains» in the paper [12] is related just with the fact that the authors did not know the implication

$$M_0(\Omega) < \infty \Longrightarrow M(\Omega) < \infty.$$

The upper bounds for $M(\Omega)$ via $M_0(\Omega)$ appear in few works. The best known estimates are represented in the following theorem.

Theorem 2.4 ([3] for $\infty \notin \Omega$; [15] for $\infty \in \Omega$). *For each non–simply–connected domain $\Omega \subset \overline{\mathbb{C}}$ the following statements hold:*

1) *If $\infty \notin \Omega$, then*
$$M_0(\Omega) \leqslant M(\Omega) \leqslant M_0(\Omega) + \frac{1}{2}.$$
2) *If $\infty \in \Omega$, then*
$$M_0(\Omega) \leqslant M(\Omega) \leqslant 2\, M_0(\Omega) + 1,$$
where the constant 2 can not be replaced by $2 - \varepsilon$ with any $\varepsilon > 0$.

In the proof of this theorem, namely, in the upper bounds for $M(\Omega)$ via $M_0(\Omega)$, the identity $\Lambda(1) = 1/2$ and the following formula by Ahlfors [2] were employed essentially:
$$t = t(q) = \frac{1}{16q} \prod_{n=1}^{\infty} \left(\frac{1 - q^{2n-1}}{1 + q^{2n}} \right)^8, \qquad q = \exp(-2\pi\Lambda(t)),$$
where $\Lambda(t) = \operatorname{mod}(G_t) > 0$ is the modulus of the Teinchmüller annulus
$$G_t := \overline{\mathbb{C}} \setminus ([-1, 0] \cup [t, \infty]), \qquad 0 < t < \infty.$$
The uniform perfectness of boundary of domain is preserved under univalent conformal or K–quasiconformal transformations of domain since the condition $M_0(\Omega) < \infty$ is equivalent to the condition $M(\Omega) < \infty$, while the invariance of the condition $M(\Omega) < \infty$ under the aforementioned transformations is a classical fact. We also note that the absence of conformal invariance of Euclidean conformal modulus $M_0(\Omega)$ is compensated by the simplicity of its calculation.

3. Main results

We begin with describing an interesting effect demonstrating the important difference of the inequality (2.2) with the sharp constant $c_2(\Omega)$ from the classical Poincaré — Friedrichs inequality
$$\iint\limits_{\Omega} |\nabla u(z)|^2 dxdy \geqslant \lambda_1(\Omega) \iint\limits_{\Omega} |u(z)|^2 dxdy \quad \forall u \in C_0^1(\Omega),$$
where the sharp constant $\lambda_1(\Omega) \in [0, \infty)$ is the lowest eigenvalue of the Dirichlet problem for the Laplace equation. It is well–known that for the domain $\Omega \subset \mathbb{C}$ with a smooth boundary the Poincaré — Friedrichs inequality becomes the identity
$$\iint\limits_{\Omega} |\nabla u(z)|^2 dxdy = \lambda_1(\Omega) \iint\limits_{\Omega} |u(z)|^2 dxdy < \infty$$
for the extremal function $u \not\equiv 0$ with the properties $u \in C^2(\Omega) \cap C(\overline{\Omega})$, $u(z) = 0$ for each $z \in \partial\Omega$, $\Delta u(z) + \lambda_1(\Omega)\, u(z) = 0$ in the domain Ω.

By definition, the sharpness of the constant $c_2(\Omega) \geqslant 0$ in the inequality (2.2) implies that for each $\varepsilon > 0$ we find a function $u_\varepsilon \in C_0^1(\Omega)$ obeying the opposite inequality
$$\iint\limits_{\Omega} |\nabla u_\varepsilon(z)|^2 dxdy < (c_2(\Omega) + \varepsilon) \iint\limits_{\Omega} \frac{|u_\varepsilon(z)|^2}{R^2(z, \Omega)} dxdy.$$
In particular, for simply–connected and doubly–connected domains the sharpness of the constant $c_2(\Omega) = 1$ in the inequality (2.2) just means that the inequality
$$\iint\limits_{\Omega} |\nabla u(z)|^2 dxdy \geqslant \iint\limits_{\Omega} \frac{|u(z)|^2}{R^2(z, \Omega)} dxdy \qquad \forall u \in C_0^1(\Omega) \tag{3.1}$$

is true and for each $\varepsilon > 0$ there exists a function $u_\varepsilon \in C_0^1(\Omega)$ such that

$$\iint_\Omega |\nabla u_\varepsilon(z)|^2 dxdy < (1+\varepsilon) \iint_\Omega \frac{|u_\varepsilon(z)|^2}{R^2(z,\Omega)} dxdy.$$

For a simply–connected or doubly–connected domain $\Omega \subset \overline{\mathbb{C}}$ with a smooth boundary there exists no extremal function with the properties $u \not\equiv 0$, $u \in C^1(\Omega) \cap C(\overline{\Omega})$, $u(z) = 0$ for each $z \in \partial\Omega$, and

$$\iint_\Omega |\nabla u(z)|^2 dxdy = \iint_\Omega \frac{|u(z)|^2}{R^2(z,\Omega)} dxdy < \infty.$$

This fact is related with the presence of singular weight function $R^{-2}(z,\Omega)$. The absence of the extremal function allows us to strengthen the inequality (3.1). Despite $c_2(\Omega) = 1$ for simply–connected and doubly–connected domains $\Omega \subset \overline{\mathbb{C}}$, in the right hand side of the inequality (3.1) we can insert an additional positive term. Namely, we have the following theorem on improving the inequality (3.1) for simply–connected and doubly–connected domains.

Theorem 3.1 ([15])**.** *The following statements hold.*

1) *Let $\Omega \subset \overline{\mathbb{C}}$ be a simply–connected domain of hyperbolic type, g be any of univalent conformal mappings of Ω onto the half–plane $\{\zeta \in \mathbb{C} : \operatorname{Im} \zeta > 0\}$. Then*

$$\iint_\Omega |\nabla u(z)|^2 dxdy \geqslant \iint_\Omega \frac{|u(z)|^2 dxdy}{R^2(z,\Omega)} + \frac{1}{4} \iint_\Omega |u(z)|^2 \left| \frac{g'(z)}{g(z)} \right|^2 dxdy \qquad \forall u \in C_0^1(\Omega).$$

The constant $1/4$ is sharp, that is, it maximal possible.

2) *Let $\Omega \subset \overline{\mathbb{C}}$ be a doubly–connected domain of hyperbolic type, g be any of univalent conformal mappings of Ω onto the annulus $\{\zeta \in \mathbb{C} : q < |\zeta| < 1\}$. Then*

$$\iint_\Omega |\nabla u(z)|^2 dxdy \geqslant \iint_\Omega \frac{|u(z)|^2 dxdy}{R^2(z,\Omega)} + \frac{1/16}{M^2(\Omega)} \iint_\Omega |u(z)|^2 \left| \frac{g'(z)}{g(z)} \right|^2 dxdy \qquad \forall u \in C_0^1(\Omega).$$

The constant $M^{-2}(\Omega)/16$ is sharp, that is, it is maximal possible.

An important ingredient of the proof of this theorem is the following double strengthening of the Hardy inequality on finite segments.

Lemma 3.1 ([15])**.** *Let $v : [0,\pi] \to \mathbb{R}$ be an absolutely continuous function obeying the conditions*

$$v(0) = v(\pi) = 0, \qquad v \not\equiv 0, \qquad v' \in L^2(0,\pi).$$

Then

$$\int_0^\pi v'^2(\theta)\, d\theta > \frac{1}{4} \int_0^\pi \frac{v^2(\theta)}{\sin^2\theta}\, d\theta + \frac{1}{4} \int_0^\pi v^2(\theta)\, d\theta.$$

The constants $1/4$ is sharp in the sense that none of them can be replaced by $(1+\varepsilon)/4$ for any $\varepsilon > 0$.

Under the assumptions of Lemma 3.1 the results by Hardy give only the inequality [16]

$$\int_0^\pi v'^2(\theta)\, d\theta > \frac{1}{4} \int_0^\pi \frac{v^2(\theta)}{\min\{\theta^2, (\pi-\theta)^2\}}\, d\theta,$$

where

$$\frac{1}{\min\{\theta^2, (\pi-\theta)^2\}} < \frac{1}{\sin^2\theta}$$

for all $\theta \in (0, \pi)$ due to the known properties of the sine.

For comparison we provide the following analogue of Lemma 3.1.

Lemma 3.2 ([17]). *Let $v : [0, \pi] \to \mathbb{R}$ be an absolutely continuous function obeying the conditions*

$$v(0) = v(\pi) = 0, \qquad v \not\equiv 0, \qquad v' \in L^2(0, \pi).$$

Then

$$\int_0^\pi v'^2(\theta)\, d\theta > \frac{1}{4} \int_0^\pi \frac{v^2(\theta)}{\min\{\theta^2, (\pi - \theta)^2\}}\, d\theta + \frac{4\lambda_0^2}{\pi^2} \int_0^\pi v^2(\theta)\, d\theta,$$

where $\lambda_0 \approx 0.940$ is the Lambda constant defined as the first positive root of the equation $J_0(x) + 2xJ_0'(x) = 0$ for the Bessel function of zero order. The constants $1/4$ and $4\lambda_0^2/\pi^2$ are sharp.

As it is known, the hyperbolic radius for the annulus $A(q, 0, 1) = \{z \in \mathbb{C} : q < |z| < 1\}$ for $q \in (0, 1)$ is expressed by the following formula

$$R(z, A(q, 0, 1)) = \frac{2|z|\, \ln \frac{1}{q}}{\pi} \sin \frac{\pi \ln |z|}{\ln q}.$$

Let $C = \pi / \ln(1/q)$ and $0 < q < r < 1$. Replacing $\theta = -C \ln r$ in the integrals of Lemma 3.1, we get a statement, which is a base for the proof Theorem 3.1 for doubly–connected domains.

Lemma 3.3 ([15]). *Let $q \in (0, 1)$. For each absolutely continuous function $v : [q, 1] \to \mathbb{R}$ obeying the conditions*

$$v(q) = v(1) = 0, \qquad v \not\equiv 0, \qquad v' \in L^2(q, 1),$$

the inequality

$$\int_q^1 v'^2(r)\, r dr > \int_q^1 \frac{v^2(r)}{\rho^2(x)}\, r dr + \frac{\pi^2}{4 \ln^2 q} \int_q^1 \frac{v^2(r)}{r}\, dr$$

holds, where

$$\rho(r) = \frac{2r \ln q}{\pi} \sin \frac{\pi \ln r}{\ln q}.$$

The constant $\pi^2/4$ is sharp.

We note that by applying Lemmas 3.1, 3.2 and 3.3 we obtain several inequalities similar to ones provided in Theorem 3.1, see [17]–[19].

In what folows we consider the inequalities involving the conformally invariant integrals of form

$$\iint_\Omega \frac{|\nabla u(z)|^p}{R^{2-p}(z, \Omega)} dxdy, \qquad \iint_\Omega \frac{|u(z)|^p}{R^2(z, \Omega)} dxdy, \qquad \iint_\Omega \frac{|\Delta u(z)|^p}{R^{2-2p}(z, \Omega)} dxdy.$$

First of all we consider the L_p–version of the inequality (2.2)

$$\iint_\Omega \frac{|\nabla u(z)|^p}{R^{2-p}(z, \Omega)} dxdy \geqslant c_p(\Omega) \iint_\Omega \frac{|u(z)|^p}{R^2(z, \Omega)} dxdy \qquad \forall u \in C_0^1(\Omega), \tag{3.2}$$

where $p \in [1, \infty)$ is a fixed parameter. The constant $c_p(\Omega) \in [0, \infty)$ in (3.2) is supposed to be the maximal possible.

We shall need the next lemma proved in [20].

Lemma 3.4. *Let $1 \leqslant p \leqslant q < \infty$. Then*

$$p\,(c_p(\Omega))^{\frac{1}{p}} \leqslant q\,(c_q(\Omega))^{\frac{1}{q}}$$

for each domain $\Omega \subset \overline{\mathbb{C}}$ of hyperbolic type. In particular, $c_1(\Omega) \leqslant 2\sqrt{c_2(\Omega)}$.

Theorem 3.2. *Let $p \in [1, \infty)$ and $\Omega \subset \overline{\mathbb{C}}$ be a domain of hyperbolic type. Then the following statements are true:*

1) *If Ω is simply–connected or doubly–connect domain, then $c_p(\Omega) = 2^p/p^p$.*
2) *If Ω is a finitely–connected domain having at least three boundary components, then $c_p(\Omega) > 0$ if and only if at least one of these components is a continuum.*
3) *Let Ω be a finitely–connected or infinitely–connected domain, the boundary of which is a uniformly perfect set and has at least three component. Then*

$$c_p(\Omega) \geqslant \frac{1}{p^p\,\sigma^p(\Omega)} > 0,$$

where

$$\sigma(\Omega) := \pi\, M_0(\Omega) + \frac{\Gamma^4\left(\frac{1}{4}\right)}{4\pi^2}$$

in the case $\infty \notin \Omega$ and

$$\sigma(\Omega) := 2\pi\, M_0(\Omega) + \pi + \frac{\Gamma^4\left(\frac{1}{4}\right)}{4\pi^2}$$

in the case $\infty \in \Omega$, where Γ is the Euler Gamma function, $M_0(\Omega)$ is the Euclidean maximal modulus.

Statements 1 and 3 of this theorem were proved in our paper [15], while Assertion 2 is new and requires the proof. The proof consists of two statements.

Proposition 3.1. *Let $m \geqslant 3$ be a natural number. Suppose that $\Omega \subset \overline{\mathbb{C}}$ is m–connected domain, γ_1, γ_2, …, γ_m are its boundary components, γ_m is the continuum. Then $c_p(\Omega) > 0$ for each $p \in [1, \infty)$.*

Proof. In view of Lemma 3.4 it is sufficient to show that $c_1(\Omega) > 0$. Without loss of generality we can suppose that $\Omega \subset \mathbb{C}$ is a bounded circular domain, each boundary component of which is either a circumference or an isolated point. Suppose that the domain Ω lies in a circle bounded by the circumference $\gamma_m \subset \partial\Omega$. We consider two–connected domains Ω_1, Ω_2, …, Ω_{m-1}, such that

$$\Omega \subset \Omega_k, \qquad \partial\Omega_k = \gamma_k \cup \gamma_m, \qquad k = 1, 2, \ldots, m-1, \qquad \Omega = \bigcap_{k=1}^{m-1} \Omega_k.$$

In view of the local behavior of the hyperbolic radii near the boundary we conclude that there exist positive constants $\sigma_1(\Omega)$ and $\sigma_2(\Omega)$ such that

$$\frac{\sigma_1(\Omega)}{R(z,\Omega)} \leqslant \sum_{k=1}^{m-1} \frac{1}{R(z,\Omega_k)} \leqslant \frac{m-1}{R(z,\Omega)}, \qquad \forall z \in \Omega,$$

$$\frac{\sigma_2(\Omega)}{R^2(z,\Omega)} \leqslant \sum_{k=1}^{m-1} \frac{1}{R^2(z,\Omega_k)} \leqslant \frac{m-1}{R^2(z,\Omega)}, \qquad \forall z \in \Omega.$$

Due to the double–connectedness of the domain Ω_k and Assertion 1 of the theorem we have $c_1(\Omega_k) = 2$ for all $k = 1, 2, \ldots, m-1$. This is why for each function $u \in C_0^1(\Omega)$ continued by zero to the set $\overline{\mathbb{C}} \setminus \Omega$ we have

$$(m-1)\iint\limits_{\Omega} \frac{|\nabla u(z)|}{R(z,\Omega)}dxdy \geqslant \sum_{k=1}^{m-1} \iint\limits_{\Omega_k} \frac{|\nabla u(z)|}{R(z,\Omega_k)}dxdy$$
$$\geqslant 2\sum_{k=1}^{m-1} \iint\limits_{\Omega_k} \frac{|u(z)|}{R^2(z,\Omega_k)}dxdy \geqslant 2\sigma_2(\Omega) \iint\limits_{\Omega} \frac{|u(z)|}{R^2(z,\Omega)}dxdy.$$

Therefore,

$$c_1(\Omega) \geqslant \frac{2\sigma_2(\Omega)}{m-1} > 0,$$

and this completes the proof. □

Proposition 3.2. *Let $m \geqslant 3$ be a natural number. Suppose that $\Omega \subset \overline{\mathbb{C}}$ is an m–connected domain each boundary component of which contains just a single point. Then $c_p(\Omega) = 0$ for each $p \in [1, \infty)$.*

Proof. By Lemma 3.4 it is sufficient to show that the $c_p(\Omega) = 0$ for each $p \in (2, \infty)$. This is why we suppose that $p > 2$.

We shall employ the Bernoulli function, namely, a convex function $g : [0, \infty) \to [0, \infty)$ defined by the identities $g(\varepsilon) = \varepsilon^{\varepsilon}$ as $\varepsilon \in (0, \infty)$ and $g(0) = \lim\limits_{\varepsilon \to 0^+} \varepsilon^{\varepsilon} = 1$. We have

$$g(0) = g(1) = 1, \quad \min_{\varepsilon\in[0,\infty)} g(\varepsilon) = g\left(\frac{1}{e}\right) = \frac{1}{e^{\frac{1}{e}}} > 0.$$

Without loss of generality we can suppose that $\infty \in \Omega$ and the boundary of the domain reads as $\partial\Omega = \{z_1, z_2, \ldots, z_m\} \subset \mathbb{C}$. It is obvious that $\Omega = \overline{\mathbb{C}} \setminus \{z_1, z_2, \ldots, z_m\}$.

We shall also need the small parameter ε such that

$$0 < \varepsilon < \varepsilon_0 := \frac{1}{2} \min_{k \neq j} |z_k - z_j|.$$

We introduce the notation

$$S_k(\varepsilon) = \{z \in \mathbb{C} : |z - z_k| \leqslant \varepsilon\}, \qquad \Omega(\varepsilon) = \overline{\mathbb{C}} \setminus \bigcup_{k=1}^{m} S_k(\varepsilon).$$

We are going to show that there exists a family of real–valued functions $u_\varepsilon \in C_0^1(\Omega)$ such that $u_\varepsilon \not\equiv 0$ and

$$\lim_{\varepsilon\to 0^+} \frac{\iint\limits_{\Omega} R^{p-2}(z,\Omega)\,|\nabla u_\varepsilon(z)|^p dxdy}{\iint\limits_{\Omega} R^{-2}(z,\Omega)\,|u_\varepsilon(z)|^p dxdy} = 0$$

for each fixed $p \in (2, \infty)$. This implies the identity $c_p(\Omega) = 0$ for each $p \in (2, \infty)$.

We first consider continuous piecewise–smooth positive in the domain functions U_ε defined by the identities

$$U_\varepsilon(z) \equiv \varepsilon^{\varepsilon}, \quad z \in \Omega(\varepsilon); \quad U_\varepsilon(z) = |z - z_k|^{\varepsilon}, \quad z \in S_k(\varepsilon), \quad k = 1, 2, \ldots, m.$$

We construct the functions $u_\varepsilon \in C_0^1(\Omega)$ letting $u_\varepsilon(z) \equiv U_\varepsilon(z)$ in the domain $\Omega(\varepsilon)$. For each $z \in S_k(\varepsilon)$ below we define the values $u_\varepsilon(z)$ to satisfy the inequality

$$|\nabla u_\varepsilon(z)| \leqslant 2|\nabla U_\varepsilon(z)|.$$

For each $z \in S_k(\varepsilon)$ we have

$$|\nabla U_\varepsilon(z)| = \varphi_\varepsilon(r) := \varepsilon r^{\varepsilon-1},$$

where $r = |z - z_k| \in [0, \varepsilon]$.

Let $\delta \in (0, \varepsilon/4)$. We define a function $\varphi_{\delta,\varepsilon} : [0, \varepsilon] \to [0, \infty)$ by the identities

$$\begin{aligned}
&\varphi_{\delta,\varepsilon}(r) = 0 && \text{for} \quad r \in [0, \delta] \quad \text{and} \quad r \in [\varepsilon - \delta, \varepsilon],\\
&\varphi_{\delta,\varepsilon}(r) = \frac{(r-\delta)\varphi_\varepsilon(2\delta)}{\delta} && \text{for} \quad r \in (\delta, 2\delta],\\
&\varphi_{\delta,\varepsilon}(r) = \frac{(\varepsilon-\delta-r)\varphi_\varepsilon(\varepsilon-2\delta)}{\delta} && \text{for} \quad r \in (\varepsilon-2\delta, \varepsilon-\delta],\\
&\varphi_{\delta,\varepsilon}(r) = \varphi_\varepsilon(r) && \text{for} \quad \in (2\delta, \varepsilon-2\delta).
\end{aligned}$$

It is easy to see that $\varphi_{\delta,\varepsilon}(r) \leqslant \varphi_\varepsilon(r)$ for each $r \in [0, \varepsilon]$ and the compact support $\varphi_{\delta,\varepsilon}$ is located inside the interval $(0, \varepsilon)$. Let

$$\psi_{\delta,\varepsilon}(r) := \int_0^r \varphi_{\delta,\varepsilon}(t)dt, \quad r \in [0, \varepsilon].$$

For a fixed $\varepsilon \in (0, 1)$ the integral

$$\psi_{\delta,\varepsilon}(\varepsilon) = \int_0^\varepsilon \varphi_{\delta,\varepsilon}(t)dt$$

is a continuous decreasing function of the parameter $\delta \in (0, \varepsilon/4)$. Since

$$\lim_{\delta \to \frac{\varepsilon}{4}} \psi_{\delta,\varepsilon}(\varepsilon) = \frac{\varepsilon^{1+\varepsilon}}{2^{1+\varepsilon}}, \qquad \lim_{\delta \to 0} \psi_{\delta,\varepsilon}(\varepsilon) = \varepsilon^\varepsilon,$$

we have

$$\frac{\varepsilon^{1+\varepsilon}}{2^{1+\varepsilon}} < \psi_{\delta,\varepsilon}(\varepsilon) < \varepsilon^\varepsilon$$

for each $\delta \in (0, \varepsilon/4)$. Therefore, for each $\varepsilon \in (0, 1)$ there exists a number $\delta = \delta(\varepsilon) \in (0, \varepsilon/4)$ such that $2\psi_{\delta,\varepsilon}(\varepsilon) = \varepsilon^\varepsilon$.

We let

$$\psi_\varepsilon(r) := 2\psi_{\delta(\varepsilon),\varepsilon}(r),\ r \in [0, \varepsilon], \quad u_\varepsilon(z) = \psi_\varepsilon(|z - z_k|), \qquad z \in S_k(\varepsilon), \quad k = 1, 2, \ldots, m.$$

Thus, for each $\varepsilon \in (0, 1)$ a real–valued function $u_\varepsilon \in C_0^1(\Omega)$ is well–defined. In what follows we shall employ the following properties of this function:

$$\begin{aligned}
&u_\varepsilon(z) = \varepsilon^\varepsilon, \qquad |\nabla u_\varepsilon(z)| = 0 && \text{for all} \quad z \in \Omega(\varepsilon),\\
&|\nabla u_\varepsilon(z)| \leqslant 2|\nabla U_\varepsilon(z)| = 2\varepsilon|z - z_k|^{\varepsilon-1} && \text{for all} \quad z \in S_k(\varepsilon), \qquad k = 1, 2, \ldots, m.
\end{aligned}$$

Let $\varepsilon \in (0, \min\{\varepsilon_0, 1/e\})$. Let us estimate the integrals

$$X_{p\varepsilon}(\Omega) := \iint_\Omega R^{-2}(z, \Omega)\, |u_\varepsilon(z)|^p dxdy, \qquad Y_{p\varepsilon}(\Omega) := \iint_\Omega R^{p-2}(z, \Omega)\, |\nabla u_\varepsilon(z)|^p dxdy.$$

Taking into account the identity $u_\varepsilon(z) = \varepsilon^\varepsilon$ for all $z \in \Omega(\varepsilon)$ and the identity $\varepsilon^\varepsilon \geqslant e^{\frac{1}{e}}$, we get

$$X_{p\varepsilon}(\Omega) \geqslant \varepsilon^{p\varepsilon} \iint_{\Omega(\varepsilon)} R^{-2}(z, \Omega)\, dxdy \geqslant \frac{1}{e^{\frac{p}{e}}} \iint_{\Omega(\frac{1}{e})} R^{-2}(z, \Omega)\, dxdy > 0 \quad \forall \varepsilon \in (0, \min\{\varepsilon_0, 1/e\}).$$

We have $|\nabla u_\varepsilon(z)| = 0$ for all $z \in \Omega(\varepsilon)$. Moreover,

$$|\nabla u_\varepsilon(z)| \leqslant 2|\nabla U_\varepsilon(z)| = 2\varepsilon|z - z_k|^{\varepsilon-1}$$

for all $z \in S_k(\varepsilon)$ and

$$R(z, \Omega) \asymp |z - z_k| \ln \frac{1}{|z - z_k|}$$

in a sufficiently small neighbourhood of the point z_k, $k=1,2,\dots,m$. This is why for each fixed $p\in(2,\infty)$ and sufficiently small $\varepsilon>0$

$$Y_{p\varepsilon}(\Omega)\leqslant 2\sum_{k=1}^{m}\iint\limits_{S_k(\varepsilon)} R^{p-2}(z,\Omega)\,|\nabla U_\varepsilon(z)|^p dxdy = 2\sum_{k=1}^{m}\iint\limits_{S_k(\varepsilon)} R^{p-2}(z,\Omega)\,\left(\varepsilon|z-z_k|^{\varepsilon-1}\right)^p dxdy$$

$$\asymp 4m\pi\,\varepsilon^p\int\limits_0^\varepsilon r^{p\varepsilon-p}\left(r\ln\frac{1}{r}\right)^{p-2} rdr = 4m\pi\,\varepsilon^p\int\limits_0^\varepsilon r^{2\varepsilon-1}\left(r^\varepsilon\ln\frac{1}{r}\right)^{p-2}dr =: Z_{p\varepsilon}.$$

For $r\in(0,\varepsilon)$ and $p\in(2,\infty)$ we have

$$\left(r^\varepsilon\ln\frac{1}{r}\right)^{p-2}\leqslant\frac{1}{e^{p-2}\varepsilon^{p-2}},$$

and this is why

$$Z_{p\varepsilon}\leqslant\frac{4m\pi\,\varepsilon^2}{e^{p-2}}\int\limits_0^\varepsilon r^{2\varepsilon-1}dr=\frac{2m\pi\,\varepsilon^{1+2\varepsilon}}{e^{p-2}}=O(\varepsilon),\quad \varepsilon\to 0.$$

Thus, we arrive at the identities

$$\lim_{\varepsilon\to 0^+} Y_{p\varepsilon}(\Omega)=\lim_{\varepsilon\to 0^+} Z_{p\varepsilon}=0.$$

We hence obtain

$$\lim_{\varepsilon\to 0^+}\frac{Y_{p\varepsilon}(\Omega)}{X_{p\varepsilon}(\Omega)}=0,$$

and this implies the identity $c_p(\Omega)=0$. This completes the proof of Proposition 3.2 and Assertion 2 of Theorem 3.2. □

In contrast to $M(\Omega)$, the Euclidean maximal modulus $M_0(\Omega)$ can be calculated for a series of domains. Knowing $M_0(\Omega)$, we can apply Assertion 3 of Theorem 3.2 and obtain explicit lower bounds for the constant $c_p(\Omega)$. We provide only one example of applying Assertion 3 of Theorem 3.2 to a multiply–connected domain, the boundary of which consists of uncountably many components.

Example 1. *Let $\mathbb{K}$ be the classical Cantor set on the segment $[0,1]$. We consider the domain $\overline{\mathbb{C}}\setminus\mathbb{K}_1$, where $\mathbb{K}_1$ is the Cantor stakewall defined by the formula*

$$\mathbb{K}_1=\{x+iy\in\mathbb{C}: x\in\mathbb{K},\quad |y|\leqslant 1\}.$$

Then $M_0(\overline{\mathbb{C}}\setminus\mathbb{K}_1)=0$. Therefore, applying Theorem 3.2, for each $p\in[1,\infty)$ we obtain

$$c_p(\overline{\mathbb{C}}\setminus\mathbb{K}_1)\geqslant p^{-p}\left(\pi+\frac{\Gamma^4(\frac{1}{4})}{4\pi^2}\right)^{-p}>0.$$

We return back to the constant $h(\Omega)\in(0,\infty]$, the smallest quantity in the linear hyperbolic isoperimetric inequality of form

$$A(G_m)\leqslant h(\Omega)L(\partial G_m),\qquad \forall G_m\Subset\Omega.$$

By Theorem 2.2 and Assertion 2 of Theorem 3.2 we obtain one more statement.

Corollary 3.1. *If Ω is a finitely–connected domain having at least three boundary components, then $h(\Omega)<\infty$ if and only if at least one of the boundary components of this domain is the continuum.*

We give two simple corollaries.

Corollary 3.2. *Let $p \in [1,\infty)$ and let $\Omega \subset \overline{\mathbb{C}}$ be a domain of hyperbolic type. Then the uniform perfectness of the boundary of domain is a sufficient but not necessary condition for $c_p(\Omega) > 0$.*

Corollary 3.3. *Suppose that $p \in [1,\infty)$ and m is a natural number, $m \geqslant 3$. If $\Omega \subset \mathbb{C}$ is a bounded domain having m boundary components, then $c_p(\Omega) > 0$.*

We note that by Assertion 1 of Theorem 3.2 for each simply–connected or doubly–connected domain $\Omega \subset \overline{\mathbb{C}}$ of hyperbolic type and for all $\varepsilon > 0$ and $p \in [1,\infty)$ there exists a function $u_{p,\varepsilon} \in C_0^1(\Omega)$ such that

$$\iint\limits_{\Omega} |\nabla u_{p,\varepsilon}(z)|^p dxdy < (1+\varepsilon)\frac{2^p}{p^p}\iint\limits_{\Omega} \frac{|u_{p,\varepsilon}(z)|^p}{R^2(z,\Omega)}dxdy.$$

There is the following generalization of Theorem 3.1 proved by Nasibullin [21]. We give this statement with a simpler proof.

Theorem 3.3 ([21])**.** *Suppose that $p \in [2,\infty)$.*

1) *Let $\Omega \subset \overline{\mathbb{C}}$ be a simply–connected domain of hyperbolic type, the function g be any of univalent conformal mappings of Ω onto the half–plane $\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta > 0\}$. Then*

$$\iint\limits_{\Omega} \frac{|\nabla u(z)|^p}{R^{2-p}(z,\Omega)}dxdy \geqslant \frac{2^p}{p^p}\iint\limits_{\Omega} \frac{|u(z)|^p dxdy}{R^2(z,\Omega)} + \frac{2^{p-3}}{p^{p-1}}\iint\limits_{\Omega} |u(z)|^p \left|\frac{g'(z)}{g(z)}\right|^2 dxdy$$

   for each function $u \in C_0^1(\Omega)$.

2) *Let $\Omega \subset \overline{\mathbb{C}}$ be a two–connected domain of hyperbolic type, g be any of univalent conformal mappings of Ω onto the annulus $\{\zeta \in \mathbb{C} : q < |\zeta| < 1\}$. Then*

$$\iint\limits_{\Omega} \frac{|\nabla u(z)|^p}{R^{2-p}(z,\Omega)}dxdy \geqslant \frac{2^p}{p^p}\iint\limits_{\Omega} \frac{|u(z)|^p dxdy}{R^2(z,\Omega)} + \frac{\left(\frac{2}{p}\right)^{p-1}}{16M^2(\Omega)}\iint\limits_{\Omega} |u(z)|^p \left|\frac{g'(z)}{g(z)}\right|^2 dxdy$$

   for each function $u \in C_0^1(\Omega)$.

Proof. For $p = 2$ this theorem coincides with Theorem 3.1, and this is why we suppose that $p \in (2,\infty)$. We are going to show that for $p > 2$, the inequalities in Theorem 3.3 can be obtained from the corresponding inequalities of Theorem 3.1 by means of some transformations of functions and additional estimates.

Let us prove the first statement of the theorem. Suppose that $p > 2$, $\Omega \subset \overline{\mathbb{C}}$ is a simply–connected domain of hyperbolic type, g is any of univalent conformal mappings of Ω onto the half–plane $\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta > 0\}$.

We take an arbitrary real–valued function $u \in C_0^1(\Omega)$. Then the function $v := |u|^{\frac{p}{2}}$ also belongs to the family $C_0^1(\Omega)$ since $p > 2$. We apply the first inequality of Theorem 3.1 to the function v and in view of the identities

$$v^2(z) = |u(z)|^p, \qquad |\nabla v(z)|^2 = \left(\frac{p}{2}\right)^2 |u(z)|^{p-2}|\nabla u(z)|^2,$$

we obtain

$$\iint\limits_{\Omega} \left(\frac{p}{2}\right)^2 |u(z)|^{p-2}|\nabla u(z)|^2 dxdy \geqslant \iint\limits_{\Omega} \frac{|u(z)|^p dxdy}{R^2(z,\Omega)} + \frac{1}{4}\iint\limits_{\Omega} |u(z)|^p \left|\frac{g'(z)}{g(z)}\right|^2 dxdy.$$

Let us estimate from above the integral in left hand side of this inequality. In order to do this, we let

$$a = \frac{p^2|\nabla u(z)|^2}{2^2 R^{\frac{2(2-p)}{p}}(z,\Omega)}, \qquad b = \frac{|u(z)|^{p-2}}{R^{\frac{2(2-p)}{p}}(z,\Omega)}, \quad z\in\Omega,$$

and apply the Young inequality

$$ab \leqslant \frac{2}{p} a^{\frac{p}{2}} + \left(1-\frac{2}{p}\right) b^{\frac{p}{p-2}}.$$

By direct calculations we get the inequality

$$\frac{p^{p-1}}{2^{p-1}} \iint\limits_{\Omega} \frac{|\nabla u(z)|^p}{R^{2-p}(z,\Omega)} dxdy + \left(1-\frac{2}{p}\right) \iint\limits_{\Omega} \frac{|u(z)|^p dxdy}{R^2(z,\Omega)}$$
$$\geqslant \iint\limits_{\Omega} \frac{|u(z)|^p dxdy}{R^2(z,\Omega)} + \frac{1}{4} \iint\limits_{\Omega} |u(z)|^p \left|\frac{g'(z)}{g(z)}\right|^2 dxdy,$$

and this yields

$$\frac{p^{p-1}}{2^{p-1}} \iint\limits_{\Omega} \frac{|\nabla u(z)|^p}{R^{2-p}(z,\Omega)} dxdy \geqslant \frac{2}{p} \iint\limits_{\Omega} \frac{|u(z)|^p dxdy}{R^2(z,\Omega)} + \frac{1}{4} \iint\limits_{\Omega} |u(z)|^p \left|\frac{g'(z)}{g(z)}\right|^2 dxdy.$$

It is easy to see that this inequality is equivalent to the inequality in Assertion 1.

It is clear that the inequality in Assertion 2 can be proved in the same way. This completes the proof. □

We consider the following conformally invariant inequality for real–valued functions $u:\Omega\to\mathbb{R}$:

$$\left(\iint\limits_{\Omega} \frac{|\nabla u(z)|^p dxdy}{R^{2-p}(z,\Omega)}\right)^{\frac{1}{p}} \geqslant c_{p,q}(\Omega) \left(\iint\limits_{\Omega} \frac{|u(z)|^q dxdy}{R^2(z,\Omega)}\right)^{\frac{1}{q}} \qquad \forall u \in C_0^1(\Omega),$$

where the constant $c_{p,q}(\Omega)\in[0,\infty)$ is regarded to be sharp, that is, the maximal possible. Moreover, we define the quantity $h_{p,q}(\Omega)$ by the identity

$$h_{p,q}(\Omega) = \sup_G \left(\iint\limits_{G} R^{-2}(z,\Omega) dxdy\right)^{\frac{1}{q}-\frac{1}{p}+1} \left(\int\limits_{\partial G} R^{-1}(z,\Omega)|dz|\right)^{-1},$$

where the supremum is taken over all the domains G with piecewise–smooth boundaries such that $\overline{G}\subset\Omega$.

Theorem 3.4. *Suppose that $p\in[1,2)$, $q\in[p,2p/(2-p)]$ and $\Omega\subset\overline{\mathbb{C}}$ is a simply–connected domain of hyperbolic type. Let $\mu:[1,2]\to(0,\infty)$ be a function defined by the identity*

$$\mu(\lambda) = \frac{(\lambda-1)^{\lambda-1}(2-\lambda)^{1-\frac{\lambda}{2}}}{2\lambda^{\frac{\lambda}{2}}\pi^{\lambda-1}}, \quad 1<\lambda<2,$$

with a continuous continuation to the boundary points

$$\mu(1) := \lim_{\lambda\to1^+}\mu(\lambda) = \frac{1}{2}, \qquad \mu(2) := \lim_{\lambda\to2^-}\mu(\lambda) = \frac{1}{4\pi}.$$

If $p=1$ and $q\in[1,2]$, then

$$c_{1,q}(\Omega) \geqslant \frac{q^{-\frac{1}{q}}}{(\mu(q))^q}.$$

If $p \in (1,2)$ and $q \in [p, 2p/(2-p)]$, then

$$c_{p,q}(\Omega) \geqslant \frac{q^{\frac{1}{p}-\frac{1}{q}-1}}{\left(\mu\left(\frac{pq}{p-q+pq}\right)\right)^{\frac{pq}{p-q+pq}}} \left(\frac{p(q-1)}{q(p-1)}\right)^{\frac{p-1}{p}}.$$

Proof. The inequality $h_{p,q}(\Omega) < \infty$ is ensured for simply–connected domains $\Omega \subset \overline{\mathbb{C}}$ of hyperbolic type due to (1.4) and restrictions for the parameters p and q. Indeed, it is easy to show that for each $\lambda \in [1,2]$ the hyperbolic isoperimetric inequality (1.4) implies the estimate $A(G_m) \leqslant \mu(\lambda)\, L^{\lambda}(\partial G_m)$. Therefore,

$$h_{p,q}(\Omega) \leqslant (\mu(\lambda(p,q))^{\lambda(p,q)}, \qquad \lambda(p,q) = \frac{pq}{p-q+pq} \in [1,2].$$

On the other hand, in the paper [20] the following estimates were proved:

$$c_{1,q}(\Omega) \geqslant \frac{q^{-\frac{1}{q}}}{h_{1,q}(\Omega)} \qquad \text{if} \quad q \in [1,2],$$

$$c_{p,q}(\Omega) \geqslant \frac{q^{\frac{1}{p}-\frac{1}{q}-1}}{h_{p,q}(\Omega)} \left(\frac{p(q-1)}{q(p-1)}\right)^{\frac{p-1}{p}} \qquad \text{if} \quad p \in (1,2), \quad q \in \left[p, \frac{2p}{2-p}\right].$$

Combining these estimates with the inequality $h_{p,q}(\Omega) \leqslant (\mu(\lambda(p,q))^{\lambda(p,q)}$, we complete the proof. □

Using the Keobe theorem on 1/4, that is, the pointwise inequality

$$\frac{1}{4} R(z,\Omega) \leqslant \operatorname{dist}(z, \partial\Omega),$$

which is valid for simply–connected domains $\Omega \subset \mathbb{C}$ of hyperbolic type, by Theorem 3.4 with $p=1$, $q=2$ we obtain the next corollary.

Corollary 3.4. *Let $\Omega \subset \mathbb{C}$ be a simply–connected domain, $\Omega \neq \mathbb{C}$. Then*

$$\iint_{\Omega} \frac{|\nabla u(z)|\, dx\, dy}{\operatorname{dist}(z,\partial\Omega)} \geqslant \left(\frac{\pi}{8} \iint_{\Omega} \frac{|u(z)|^2\, dx\, dy}{\operatorname{dist}^2(z,\partial\Omega)}\right)^{\frac{1}{2}} \qquad (z = x+iy).$$

for each real–valued function $u \in C_0^1(\Omega)$.

Now we consider integral inequalities related with the Laplacian. Namely, we consider conformally invariant inequalities involving the Laplacian of a function $u \in C_0^2(\Omega)$ and its gradient

$$\iint_{\Omega} \frac{|\Delta u(z)|^p}{R^{2-2p}(z,\Omega)} dxdy \geqslant c_p^*(\Omega) \iint_{\Omega} \frac{|\nabla u(z)|^p}{R^{2-p}(z,\Omega)} dxdy \qquad \forall u \in C_0^2(\Omega),$$

$$\iint_{\Omega} \frac{|\Delta u(z)|^p}{R^{2-2p}(z,\Omega)} dxdy \geqslant c_p^{**}(\Omega) \iint_{\Omega} \frac{|u(z)|^p}{R^{2}(z,\Omega)} dxdy \qquad \forall u \in C_0^2(\Omega),$$

where $p \in [1,\infty)$ is a fixed parameter. The constants $c_p^*(\Omega)$, $c_p^{**}(\Omega)$ in these inequalities are supposed to be sharp, that is, maximal possible. In [18], [19] we proved the following theorem.

Theorem 3.5. *Suppose that $\Omega \subset \overline{\mathbb{C}}$ is a domain of hyperbolic type. If Ω is a simply–connected or doubly–connected domain, then $c_2^*(\Omega) = c_2^{**}(\Omega) = 1$. If Ω is a multiply–connected domain with a uniformly perfect boundary, then $c_2^*(\Omega) > 0$ and $c_2^{**}(\Omega) > 0$. For each domain of hyperbolic type the inequalities*

$$c_2^*(\Omega) \geqslant c_2(\Omega), \qquad c_2^{**}(\Omega) \geqslant c_2(\Omega) c_2^*(\Omega)$$

hold.

Moreover, it was proved in [18] that

$$c_p^{**}(\Omega) \geqslant 4^p(p-1)^p p^{-2p} c_2^p(\Omega)$$

for $p \in [2,\infty)$. This estimate is presented in the next theorem.

Theorem 3.6. *Suppose that $p \in [2,\infty)$ and $\Omega \subset \overline{\mathbb{C}}$ is a domain of hyperbolic type. If $c_2(\Omega) > 0$, then the conformally invariant inequality*

$$\iint\limits_{\Omega} R^{2p-2}(z,\Omega)|\Delta u(z)|^p dx\, dy \geqslant \frac{4^p(p-1)^p\, c_2^p(\Omega)}{p^{2p}} \iint\limits_{\Omega} \frac{|u(z)|^p dx\, dy}{R^2(z,\Omega)} \qquad \forall u \in C_0^2(\Omega)$$

holds.

Theorem 3.2 and 3.6 imply the following corollary.

Corollary 3.5. *Suppose that $p \in [2,\infty)$ and $\Omega \subset \overline{\mathbb{C}}$ is a domain of hyperbolic type . Then $c_p^{**}(\Omega) > 0$ under one of the following two conditions:*

1) *The domain Ω is finitely–connected and at least one of the boundary components of this domain is continuum.*
2) *The domain Ω is multi–connected and has a uniformly perfect boundary.*

Let us mention some unsolved problems.

The next two statements are likely true: *for each domain $\Omega \subset \overline{\mathbb{C}}$ of hyperbolic type the inequalities $c_2^*(\Omega) \leqslant 1$ and $c_2^{**}(\Omega) \leqslant 1$ are true.* At present, the author does not know the proof of these inequalities.

For $p \in [1,2)$ the properties of the constants $c_p^*(\Omega)$, $c_p^{**}(\Omega)$ are not known. We have not succeeded to find appropriate methods and all interesting questions on the properties of the constants $c_p^*(\Omega)$, $c_p^{**}(\Omega)$ for the cases $p \in [1,2)$ remain open.

4. Universal inequalities

An integral inequality in a domain $\Omega \subset \overline{\mathbb{C}}$ of hyperbolic type is called universal if three conditions are satisfied:

1) the inequality is true for smooth functions compactly supported in each domain of hyperbolic type;
2) the inequality is invariant with respect to linear conformal transformations of the domain, that is, to the transformations of form $w = az + b$, where $a \neq 0$;
3) the inequality involves no undefined constants.

Two universal inequalities are given in the next theorem.

Theorem 4.1 ([15])**.** *For each domain $\Omega \subset \overline{\mathbb{C}}$ of hyperbolic type the followings statements are true.*

1) *Let $s \in (2,\infty)$, $p \in [2,\infty)$. Then*

$$\iint\limits_{\Omega} \frac{|\nabla u(z)|^p}{R^{s-p}(z,\Omega)} dxdy \geqslant \frac{4^p(s-2)^{\frac{p}{2}}}{p^p} \iint\limits_{\Omega} \frac{|u(z)|^p}{R^s(z,\Omega)} dxdy \qquad \forall u \in C_0^1(\Omega).$$

For $p = 2$ and each $s \in (3,\infty)$ the constant $4^p(s-2)^{\frac{p}{2}}/p^p$ i sharp for each finitely–connected domain $\Omega \subset \mathbb{C}$ with a smooth boundary.

2) *Let $1 \leqslant p < \infty$, then*

$$\iint\limits_{\Omega} \frac{|(\nabla u(z), \nabla R(z,\Omega))|^p}{R^{2-p}(z,\Omega)}\, dxdy \geqslant \frac{4^p}{p^p} \iint\limits_{\Omega} \frac{|u(z)|^p}{R^2(z,\Omega)}\, dxdy \qquad \forall u \in C_0^1(\Omega),$$

where $(\nabla u, \nabla R)$ *is the scalar product of gradients.*

Assertion 2 of this theorem with $p = 1$ and the estimate [22]

$$|\nabla R(z, \Omega)| \leqslant \frac{2R(z, \Omega)}{\operatorname{dist}(z, \partial\Omega)}$$

give the next corollary.

Corollary 4.1 ([15]). *Let* $\Omega \subset \overline{\mathbb{C}}$ *be a domain of hyperbolic type. Then the universal inequality*

$$\iint_{\Omega} \frac{|\nabla u(z)|}{\operatorname{dist}(z, \partial\Omega)}\, dxdy \geqslant 2 \iint_{\Omega} \frac{|u(z)|}{R^2(z, \Omega)}\, dxdy \quad \forall u \in C_0^1(\Omega)$$

is true.

For simply–connected and doubly–connected domains we have $c_1(\Omega) = 2$ and the latter inequality is also implied by Assertion 1 of Theorem 3.2.

In our recent paper [23] we have constructed several universal conformally invariant integral inequalities. One of them reads

$$\iint_{\Omega} |\Delta u(z)|\, dxdy \geqslant \frac{2}{\pi} \iint_{\Omega} \frac{|\nabla u(z)|^2}{1 + |u(z)|^2}\, dxdy \qquad \forall u \in C_0^2(\Omega), \tag{4.1}$$

where u is a real–valued function. As a corollary of (4.1) we obtain the following statement for smooth compactly supported functions $u : \Omega \to \mathbb{R}$.

Proposition 4.1. *Let* $\Omega \subset \overline{\mathbb{C}}$ *be a domain of hyperbolic type. Then the following conformally invariant inequalities hold:*

$$\iint_{\Omega} |\Delta\, (\sinh\, u(z))|\, dxdy \geqslant \frac{2}{\pi} \iint_{\Omega} |\nabla u(z)|^2\, dxdy \quad \forall u \in C_0^2(\Omega), \tag{4.2}$$

and

$$\iint_{\Omega} |\Delta\, (\sinh u(z))|\, dxdy \geqslant \frac{2\, c_2(\Omega)}{\pi} \iint_{\Omega} \frac{|u(z)|^2}{R^2(z, \Omega)}\, dxdy, \quad \forall u \in C_0^2(\Omega), \tag{4.3}$$

where sinh *is the hyperbolic sine,* $z = x + iy$*,* u *is a real–valued function.*

Proof. For a real–valued function $u \in C_0^2(\Omega)$ the function $\sinh u$ is also real–valued and belongs to the family $C_0^2(\Omega)$. Substituting $\sinh u(z)$ instead of $u(z)$ into the inequality (4.1), we obtain (4.2).

Applying the inequality (2.2) to estimate from below the Dirichlet integral $\iint_{\Omega} |\nabla u(z)|^2\, dxdy$ in the inequality (4.2), we arrive at the inequality (4.3), and this completes the proof. □

It is clear that the inequality (4.3) implies meaningful inequalities in the domains, for which the lower bounds of the constant $c_2(\Omega)$ are known. We recall that $c_2(\Omega) = 1$ for simply–connected and two–connected domains $\Omega \subset \overline{\mathbb{C}}$ of hyperbolic type and

$$c_2(\Omega) \geqslant \left(2\pi\, M_0(\Omega) + \frac{\Gamma^4\left(\frac{1}{4}\right)}{2\pi^2} \right)^{-2}$$

for domains $\Omega \subset \mathbb{C}$ with uniformly perfect boundaries.

Now we are going to prove a universal inequality relative to the linear hyperbolic isoperimetric inequality.

Theorem 4.2. *Let $\Omega \subset \overline{\mathbb{C}}$ be a domain of hyperbolic type. Then for each finitely–connected domain G with a piecewise–smooth boundary compactly embedded into the domain Ω the inequality*

$$\iint\limits_{G} \frac{dxdy}{R^2(z,\Omega)} \leqslant \frac{1}{2} \int\limits_{\partial G} \frac{|dz|}{\operatorname{dist}(z,\partial\Omega)} \tag{4.4}$$

is true.

Proof. We apply the Green formula

$$\iint\limits_{G} \left(u(z)\Delta v(z) + (\nabla u(z), \nabla v(z))\right) dxdy = \int\limits_{\partial G} u(z)\frac{\partial v(z)}{\partial n}\,|dz|$$

in the domain G to the pair of functions

$$u(z) \equiv 1, \qquad v(z) \equiv -\frac{1}{4}\ln R(z,\Omega).$$

Using the formula (1.1) for the Gaussian curvature for $k = -4$, we obtain

$$\iint\limits_{G} \frac{dxdy}{R^2(z,\Omega)} = \frac{1}{4} \int\limits_{\partial G} \frac{\partial \ln R(z,\Omega)}{\partial n}\,|dz|.$$

This implies the inequality (4.4). Indeed, we have

$$\left|\frac{\partial \ln R(z,\Omega)}{\partial n}\right| \leqslant \frac{|\nabla R(z,\Omega)|}{R(z,\Omega)}.$$

Then we apply the estimate [22]

$$|\nabla R(z,\Omega)| \leqslant 2\frac{R(z,\Omega)}{\operatorname{dist}(z,\partial\Omega)}, \quad z \in \Omega,$$

and this completes the proof. □

In what follows we use the number

$$c_0 := \frac{\Gamma^4\left(\frac{1}{4}\right)}{4\pi^2} \approx 4.38.$$

Using Proposition 2.2 and Theorem 4.2, we prove the next proposition.

Proposition 4.2. *Let $\Omega \subset \overline{\mathbb{C}}$ be a domain of hyperbolic type equipped with the Poincaré metric of curvature $k = -4$ and having a uniformly perfect boundary. Let $h(\Omega)$ be the constant in the linear hyperbolic isoperimetric inequality in Ω. Then*

$$\frac{1}{2} \leqslant h(\Omega) \leqslant \pi M(\Omega) + c_0, \tag{4.5}$$

where $M(\Omega)$ is the conformal maximal modulus of the domain Ω.

If $\infty \notin \Omega$, then the specified estimates

$$\frac{1}{2} \leqslant h(\Omega) \leqslant \pi M_0(\Omega) + c_0 \tag{4.6}$$

hold, where $M_0(\Omega)$ is the Euclidean maximal modulus of the domain Ω.

Proof. As it was shown in Proposition 2.2, the inequality $h(\Omega) \geqslant 1/2$ holds for each domain of hyperbolic type. This is why we are going to prove only the upper bounds for $h(\Omega)$.

Since the domain has a uniformly perfect boundary, the maximal moduluses $M_0(\Omega)$ and $M(\Omega)$ are finite quantities. Let $\Omega \subset \mathbb{C}$, that is, $\infty \notin \Omega$. Then the inequality

$$\frac{R(z,\Omega)}{\operatorname{dist}(z,\partial\Omega)} \leqslant 2\pi M_0(\Omega) + 2c_0, \quad \forall z \in \Omega, \tag{4.7}$$

holds. This inequality was justified in the work by Avkhadiev and Wirths [3]. It is a specification of estimates by Beardon and Pommerenke [24].

By (4.4) and (4.7) for each simply–connected domain G with a piecewise smooth boundary compactly supported into the domain Ω the inequality

$$\iint\limits_G \frac{dxdy}{R^2(z,\Omega)} \leqslant (\pi M_0(\Omega)+c_0)\int\limits_{\partial G} \frac{|dz|}{R(z,\Omega)}$$

holds true.

Since the number $h(\Omega)$ is expressed by the formula (2.1) as the best possible constant in the linear hyperbolic isoperimetric inequality for the domain Ω, the latter inequality implies the right estimate in (4.6) and also in (4.5) in the case $\infty \notin \Omega$.

Suppose that $\infty \in \Omega$. We take one of points $z_0 \in (\partial\Omega)\cap\mathbb{C}$ and consider the domain

$$\Omega_0 = \left\{\zeta \in \mathbb{C} : \zeta = \frac{1}{z-z_0},\ z\in\Omega\right\} \subset \mathbb{C}.$$

By (4.6) we have

$$\frac{1}{2} \leqslant h(\Omega_0) \leqslant \pi M_0(\Omega_0) + c_0 \leqslant \pi M(\Omega_0) + c_0.$$

This implies the inequality (4.5) for the case $\infty \in \Omega$, since by the conformal invariance of the quantities $h(\Omega)$ and $M(\Omega)$ the identities $h(\Omega_0) = h(\Omega)$ and $M(\Omega_0) = M(\Omega)$ are true. This completes the proof. □

By Theorem 2.4 we have the estimate $M(\Omega) \leqslant 2M_0(\Omega) + 1$. Using this estimate and the inequality (4.5) for the domain $\Omega \subset \overline{\mathbb{C}}$, we obtain the following corollary.

Corollary 4.2. *Let $\Omega \subset \overline{\mathbb{C}}$ be a domain of hyperbolic type equipped with the Poincaré metric of curvature $k=-4$ and having a uniformly perfect boundary. Then*

$$\frac{1}{2} \leqslant h(\Omega) \leqslant 2\pi M_0(\Omega) + \pi + c_0,$$

where $M_0(\Omega)$ is the Euclidean maximal modulus of the domain Ω.

We attract the attention of the reader to the fact that the inequality (4.7) is not true for all points in the domain Ω containing the infinity. Indeed, if Ω has a uniformly perfect boundary and $\infty \in \Omega$, then $M_0(\Omega) < \infty$ and, at the same time,

$$\lim_{z\to\infty} \frac{R(z,\Omega)}{\operatorname{dist}(z,\partial\Omega)} = \infty.$$

Therefore, the inequalty (4.7) fails for the points $z \in \Omega$ close enough to the infinity.

In conclusion we provide an example of explicit estimate for $M(\Omega)$ and $h(\Omega)$ for a particular multi–connected domain, the boundary of which consists of uncountably many components.

Example 2. *Let*

$$S = \{x+iy \in \mathbb{C} : 0 < x < 1, -\infty < y < \infty\}$$

be a strip and $\mathbb{K}_1$ be the Cantor stakewall defined by the formula

$$\mathbb{K}_1 = \{x+iy \in \mathbb{C} : x \in \mathbb{K}, \quad |y| \leqslant 1\},$$

where $\mathbb{K}$ is the classical Cantor set located in the segment $[0,1]$.

We consider the domain $\Omega = S\setminus\mathbb{K}_1 \subset \mathbb{C}$. It is easy to show that $M(S\setminus\mathbb{K}_1) > 0$ and $M_0(S\setminus\mathbb{K}_1) = 0$. Therefore, applying Theorem 2.4 and Proposition 4.2, we obtain

$$0 < M(S\setminus\mathbb{K}_1) \leqslant \frac{1}{2}, \quad \frac{1}{2} \leqslant h(S\setminus\mathbb{K}_1) \leqslant c_0 \approx 4.38.$$

The exact values of the constants $M(S\setminus\mathbb{K}_1)$ and $h(S\setminus\mathbb{K}_1)$ are unknown.

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