ISSN 2304-0122 Ufa Mathematical Journal. Vol. 16. No 4 (2024). P. 94-115.

doi:10.13108/2024-16-4-94

SERIES OF BOUNDARY VALUE PROBLEMS FOR EULER — DARBOUX EQUATION WITH TWO DEGENERACY LINES

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Abstract. In this paper we introduce a new partial differential equation, which is an extension of the known Euler — Darboux equation. On the base of the proven properties of solution to introduced equation we explicitly find general solutions for various values of parameters, prove existence and uniqueness theorems. Basing on the general solutions of introduced equation, we solve the Cauchy problems and modified Cauchy problems in an upper right triangle. We find explicit solutions and prove the existence and uniqueness theorems for the posed problems.

Key words: Euler — Darboux problem, Cauchy problem, general solution, solvability.

Mathematics Subject Classification: 35L35

1. Introduction

We consider the equation

$$L(u) \equiv (\xi \operatorname{sgn} \xi - \eta)\, u_{\xi\eta} - b u_\xi + a \operatorname{sgn} \xi u_\eta = 0. \tag{1.1}$$

For positive values of the variable ξ this equation becomes the Euler — Darboux equation

$$L(u) \equiv (\xi - \eta)\, u_{\xi\eta} - b u_\xi + a u_\eta = 0, \tag{1.2}$$

while for $\xi < 0$ it reads

$$L(u) \equiv (\xi + \eta)\, u_{\xi\eta} + b u_\xi + a u_\eta = 0. \tag{1.3}$$

We note that the Euler — Poisson — Darboux equation is a traditional name for the equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\gamma}{t}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

where $u = u(x,t)$, $x \in \mathbb{R}$, $t > 0$, $\gamma \in \mathbb{R}$. In the characteristic coordinates

$$\xi = x + t, \qquad \eta = x - t$$

this equation becomes

$$(\xi - \eta)\, u_{\xi\eta} - \frac{\gamma}{2} u_\xi + \frac{\gamma}{2} u_\eta = 0.$$

This equation coincides with Equation (1.2) for

$$a = b = \frac{\gamma}{2}.$$

Thus, Equation (1.1) can be treated as the union of generalization of Euler — Darboux equation (1.2) and its image (1.3) with respect to the η axis. Because of this we call it Euler — Darboux equation with two degeneracy lines.

S.V. Podkletnova, Series of boundary value problems for Euler — Darboux equation with two degeneracy lines.

Submitted December 23, 2023.

Equation (1.2) written in form

$$u_{xy} + \frac{b}{x-y}u_x + \frac{a}{x-y}u_y + \frac{c}{(x-y)^2}u = 0 \tag{1.4}$$

was first considered by Euler for $a = b = m$, $c = n$ with natural m and n in connection with the study of air motion in pipes of constant and variable cross–sections, as well as vibrations of strings of variable thickness. The most general solution of Equation (1.4) in the case considered by Euler was derived by Riemann. In addition, in his article "On the Propagation of Plane Waves of Finite Amplitude" [1] Riemann showed that the exact equation of adiabatic and reversible motion of a gas in a pipe of constant cross–section reduces to Equation (1.4) in characteristic coordinates if the entropy of a unit mass in all particles is the same. As it turned out later, Equation (1.4) has wide application in gas and hydrodynamics [2], shell theory, and various fields in the continuum mechanics. For different parameters a, b, and c, Equation (1.4) was studied by Poisson and Darboux, so historically this equation was called the Euler — Poisson— Darboux equation.

It should be noted that in the hyperbolicity domains many equations of mixed type are reduced to the Euler — Darboux equation. As an example we can cite the Tricomi, Carol, Euler — Poisson — Darboux equations of mixed type, a number of equations of mixed type with degeneration of type and order. For this reason, further study of the specified equation seems relevant.

In this paper we consider a series of boundary value problems for the Euler — Darboux equation with two degeneration lines (1.1) in triangular domains. Earlier similar problems were considered by Volkodavov (see, for example, [3], [4]), Aristova [5] and Khairullin [6] for other parameters and domains. Let us give an overview of the Cauchy problems and modified Cauchy problems solved by the mentioned authors for Equation (1.2). Taking into consideration that Equation (1.2) is a part of Equation (1.1), for the sake of completeness of study we shall give all definitions and theorems for Equation (1.1).

Theorem 1.1. *A solution to the Cauchy problem with data*

$$\begin{aligned} &u(\xi,\xi) = \tau_+(\xi), \qquad \xi \in [0,h], \\ &\lim_{\eta-\xi\to+0} (\eta-\xi)^{\alpha+\beta}(u_\xi - u_\eta) = \nu_+(\xi), \qquad \xi \in (0,h) \end{aligned} \tag{1.5}$$

for

$$a = \alpha, \qquad b = \beta, \qquad 0 < \alpha, \beta, \alpha+\beta < 1$$

for Equation (1.1) in the domain

$$G_+ = \{(\xi,\eta) | \, 0 < \xi < \eta < h\} \tag{1.6}$$

is expressed by the formula

$$\begin{aligned} u(\xi,\eta) =& \gamma_2 \int_\xi^\eta \nu_+(t)(t-\xi)^{-\alpha}(\eta-t)^{-\beta}dt \\ &+ \gamma_1(\eta-\xi)^{1-\alpha-\beta}\int_\xi^\eta \tau_+(t)(t-\xi)^{\beta-1}(\eta-t)^{\alpha-1}dt, \end{aligned}$$

where

$$\gamma_1 = \frac{1}{\mathrm{B}(\alpha,\beta)}, \tag{1.7}$$

$$\gamma_2 = -\frac{1}{2(1-\alpha-\beta)\,\mathrm{B}(1-\alpha,1-\beta)}, \tag{1.8}$$

and it is unique if $\tau_+ \in C^1_{[0,h]}$, $\nu_+ \in C^1_{[0,h]}$.

Theorem 1.2. *A solution to the Cauchy problem with data (1.5) and*

$$\lim_{\eta-\xi\to+0} (\eta-\xi)^{\alpha-\beta}(u_\xi - u_\eta) = \nu_+(\xi), \qquad \xi \in (0,h),$$

for

$$a = \alpha, \qquad b = -\beta, \qquad 0 < \alpha, \beta, \alpha-\beta < 1$$

for Equation (1.1) in the domain G_+ *(1.6) is expressed by the formula*

$$u(\xi,\eta) = \gamma_4 \int_\xi^\eta \nu_+(t)(t-\xi)^{-\alpha}(\eta-t)^{\beta}dt + \gamma_3(\eta-\xi)^{\beta-\alpha}\int_\xi^\eta \tau'_+(t)(t-\xi)^{-\beta}(\eta-t)^{\alpha}dt$$

$$+ (\beta-\alpha)\gamma_3(\eta-\xi)^{\beta-\alpha}\int_\xi^\eta \tau_+(t)(t-\xi)^{-\beta}(\eta-t)^{\alpha-1}dt,$$

where

$$\gamma_3 = \frac{1}{(\beta-\alpha)\,\mathrm{B}(\alpha, 1-\beta)}, \tag{1.9}$$

$$\gamma_4 = -\frac{1}{2(\alpha-\beta-1)\,\mathrm{B}(1-\alpha, 1+\beta)}, \tag{1.10}$$

and it is unique if $\tau_+ \in C^2_{[0,h]}$, $\nu_+ \in C^1_{[0,h]}$.

Theorem 1.3. *A solution to the modified Cauchy problem with data (1.5) and*

$$\lim_{\eta-\xi\to+0} \frac{1}{(\alpha+\beta)^2}(\eta-\xi)^{-\alpha-\beta}(\beta u_\xi - \alpha u_\eta) = \nu_+(\xi), \qquad \xi \in (0,h),$$

for

$$a = -\alpha, \qquad b = -\beta, \qquad 0 < \alpha, \beta, \alpha+\beta < 1$$

for Equation (1.1) in the domain G_+ *(1.6) is expressed by the formula*

$$u(\xi,\eta) = -\mu_3 \int_\xi^\eta \nu_+(t)(t-\xi)^{\alpha}(\eta-t)^{\beta}dt$$

$$+ \mu_1(\eta-\xi)^{\alpha+\beta-1}\int_\xi^\eta \tau_+(t)(t-\xi)^{-\beta}(\eta-t)^{-\alpha}dt$$

$$- \mu_2(\eta-\xi)^{\alpha+\beta-1}\int_\xi^\eta \tau'_+(t)(t-\xi)^{-\beta}(\eta-t)^{-\alpha}\left[(\alpha+\beta)(t-\xi) + \alpha(\xi-\eta)\right]dt,$$

where

$$\mu_1 = \frac{1}{\mathrm{B}(1-\alpha, 1-\beta)},$$

$$\mu_2 = -\frac{1}{(\alpha+\beta)(1-\alpha-\beta)\,\mathrm{B}(1-\alpha, 1-\beta)}, \tag{1.11}$$

$$\mu_3 = -\frac{\alpha+\beta}{(1+\alpha+\beta)\,\mathrm{B}(1+\alpha, 1+\beta)}, \tag{1.12}$$

and it is unique if $\tau_+ \in C^2_{[0,h]}$, $\nu_+ \in C_{[0,h]}$.

Theorem 1.4. *A solution to the modified Cauchy problem with data (1.5) and*

$$\lim_{\eta-\xi\to+0}\frac{1}{\alpha-\beta}(\eta-\xi)^{\alpha-\beta}\left(\beta u_\xi+\alpha u_\eta\right)=\nu_+\left(\xi\right),\qquad \xi\in\left(0,h\right),$$

for

$$a=\alpha,\qquad b=-\beta,\qquad 0<\alpha,\beta,\beta-\alpha<1$$

for Equation (1.1) in the domain G_+ (1.6) is expressed by the formula

$$u\left(\xi,\eta\right)=-\,2\gamma_4\int\limits_\xi^\eta\nu_+\left(t\right)\left(t-\xi\right)^{-\alpha}(\eta-t)^\beta dt$$

$$+\left(\beta-\alpha\right)\gamma_3(\eta-\xi)^{\beta-\alpha}\int\limits_\xi^\eta\tau_+\left(t\right)\left(t-\xi\right)^{-\beta}(\eta-t)^{\alpha-1}dt$$

$$+\gamma_3(\eta-\xi)^{\beta-\alpha}\int\limits_\xi^\eta{\tau'}_+\left(t\right)\left(t-\xi\right)^{-\beta}(\eta-t)^{\alpha}dt,$$

where γ_3 is defined by the formula (1.9), γ_4 is defined by the formula (1.10), and it is unique if $\tau_+\in C^2_{[0,h]}$, $\nu_+\in C^1_{[0,h]}$.

2. General solutions for some values of parameters in Euler — Darboux equation with two degeneracy lines

We represent a solution to Equation (1.3) as

$$u\left(\xi,\eta\right)=\left(\xi+\eta\right)^{1-a-b}v\left(\xi,\eta\right).$$

Then the function $v\left(\xi,\eta\right)$ solves the equation

$$\left(\xi+\eta\right)v_{\xi\eta}+\left(1-a\right)v_\xi+\left(1-b\right)v_\eta=0.$$

Thus, if $u\left(\xi,\eta\right)$ is a solution to Equation (1.3) and

$$u\left(a,b\right)=\left(\xi+\eta\right)^{1-a-b}u\left(1-b,1-a\right),\tag{2.1}$$

then $u\left(1-b,1-a\right)$ is a solution to Equation (1.3).

Let us provide some properties of Equation (1.3).

1. *Homogeneous solutions to Equation (1.3) are expressed via the hypergeometric functions.* Indeed, by means of change of variables

$$t=-\frac{\eta}{\xi},\qquad u=\xi^\mu\cdot\varphi\left(t\right)\tag{2.2}$$

Equation (1.3) is reduced to the ordinary differential equation

$$t\left(1-t\right)\varphi''\left(t\right)+\left[\left(1-a-\mu\right)-\left(1+b-\mu\right)t\right]\varphi'\left(t\right)+b\mu\varphi\left(t\right)=0.$$

This is the Gauss equation [7], which has two linearly independent solutions, which are regular at the point $t=0$

$$\varphi_1\left(t\right)=F\left(-\mu,b;1-\mu-a;t\right)$$

and

$$\varphi_2\left(t\right)=t^{\mu+a}F\left(a,a+b+\mu;1+a+\mu;t\right).$$

Then by (2.2) Equation (1.3) has homogeneous solutions of arbitrary degree μ

$$u_1\left(\xi,\eta\right)=\xi^\mu F\left(-\mu,b;1-\mu-a;-\frac{\eta}{\xi}\right),$$

$$u_2(\xi,\eta) = \xi^{2\mu+a}(-\eta)^{-a-\mu}F\left(a, a+b+\mu; 1+\mu+a; -\frac{\eta}{\xi}\right).$$

2. *If $\varphi(\xi,\eta)$ is an arbitrary solution to Equation (1.1), then the function*

$$(B-A\xi)^{-a}(A\eta+B)^{-b}\varphi\left(-\frac{D+C\xi}{B-A\xi}; \frac{C\eta+D}{A\eta+B}\right),$$

where A, B, C and D are arbitrary constants and $BC-AD\neq 0$, is also a solution to Equation (1.3).

3. *Denote*

$$L(a,b) \equiv (\xi+\eta)\,u_{\xi\eta} + bu_\xi + au_\eta = 0$$

and $u(a,b)$ be an arbitrary solution to the equation $L(a,b)\equiv 0$. Then the functions

$$u(1+a,b) = \frac{\partial u(a,b)}{\partial\xi}, \tag{2.3}$$

$$u(a,1+b) = \frac{\partial u(a,b)}{\partial\eta} \tag{2.4}$$

are solutions to the equations

$$L(a+1,b)\equiv 0,$$
$$L(a,b+1)\equiv 0.$$

And

$$u(a+m-1,b+n-1) = \frac{\partial^{m+n-2}u(a,b)}{\partial\xi^{m-1}\partial\eta^{n-1}} \tag{2.5}$$

is a solution to the equation

$$L(a+m-1,b+n-1)\equiv 0.$$

By (2.1) the formula (2.5) can be written as

$$(\xi+\eta)^{3-a-b-n-m}u(2-b-n,2-a-m) = \frac{\partial^{m+n-2}}{\partial\xi^{m-1}\partial\eta^{n-1}}\left[\frac{u(1-b,1-a)}{(\xi+\eta)^{a+b-1}}\right].$$

We replace a, b, $n-1$, $m-1$ respectively by $1-b$, $1-a$, m, n, and we obtain

$$u(a-m,b-n) = (\xi+\eta)^{m+n-1-a-b}\frac{\partial^{m+n}}{\partial\xi^n\partial\eta^m}\left[\frac{u(a,b)}{(\xi+\eta)^{1-a-b}}\right].$$

Let us find a general solution to Equation (1.3) for

$$0<a<1, \qquad 0<b<1, \qquad 0<a+b<1.$$

First we are going to prove that the function

$$u_1(\xi,\eta) = \int_{-\xi}^{\eta}\Phi_-(t)\,(t+\xi)^{-a}(\eta-t)^{-b}dt,$$

where $\Phi_-(t)$ is an arbitrary continuously differentiable function is a solution to Equation (1.3).

In order to find partial derivatives of the function $u_1(\xi,\eta)$, we integrate by parts with

$$u = \Phi_-(t), \qquad v = \int_{-\xi}^{t}(s+\xi)^{-a}(\eta-s)^{-b}ds$$

and we find

$$u_1(\xi,\eta) = \mathrm{B}(1-\alpha,1-\beta)\,\Phi_-(\eta)\,(\xi+\eta)^{1-a-b}$$

$$-\frac{(\xi+\eta)^{-b}}{1-a}\int_{-\xi}^{\eta}\Phi'_{-}(t)\,(t+\xi)^{1-a}F\left(1-a,b;2-a;\frac{t+\xi}{\eta+\xi}\right)dt.$$

Using the formula of short differentiation [7]

$$az^{a-1}F\left(a+1,b;c;z\right)=\frac{d}{dz}\left[z^{a}F\left(a,b;c;z\right)\right],$$
$$(c-1)\,z^{c-2}(1-z)^{b-c}F\left(a-1,b;c-1;z\right)=\frac{d}{dz}\left[z^{c-1}(1-z)^{b-c+1}F\left(a,b;c;z\right)\right],$$

we find

$$\frac{\partial u_1(\xi,\eta)}{\partial\xi}=(1-a-b)\,\mathrm{B}\,(1-a,1-b)\,\Phi_{-}(\eta)\,(\xi+\eta)^{-a-b}$$
$$-(\xi+\eta)^{-b}\int_{-\xi}^{\eta}\Phi'_{-}(t)\,(t+\xi)^{-a}F\left(-a,b;1-a;\frac{t+\xi}{\eta+\xi}\right)dt,$$

$$\frac{\partial u_1(\xi,\eta)}{\partial\eta}=(1-a-b)\,\mathrm{B}\,(1-a,1-b)\,\Phi_{-}(\eta)\,(\xi+\eta)^{-a-b}$$
$$+\frac{b}{1-a}(\xi+\eta)^{-b-1}\int_{-\xi}^{\eta}\Phi'_{-}(t)\,(t+\xi)^{1-a}F\left(1-a,b+1;2-a;\frac{t+\xi}{\eta+\xi}\right)dt,$$

$$\frac{\partial^2 u_1(\xi,\eta)}{\partial\xi\partial\eta}=-(a+b)\,(1-a-b)\,\mathrm{B}\,(1-a,1-b)\,\Phi_{-}(\eta)\,(\xi+\eta)^{-a-b-1}$$
$$+b(\xi+\eta)^{-b-1}\int_{-\xi}^{\eta}\Phi'_{-}(t)\,(t+\xi)^{-a}F\left(-a,b+1;1-a;\frac{t+\xi}{\eta+\xi}\right)dt.$$

Substituting the found partial derivatives into Equation (1.3), we arrive at

$$L(u_1)\equiv(\xi+\eta)\Big[-(a+b)\,(1-a-b)\,\mathrm{B}\,(1-a,1-b)\,\Phi_{-}(\eta)\,(\xi+\eta)^{-a-b-1}$$
$$+b(\xi+\eta)^{-b-1}\int_{-\xi}^{\eta}\Phi'_{-}(t)\,(t+\xi)^{-a}F\left(-a,b+1;1-a;\frac{t+\xi}{\eta+\xi}\right)dt\Big]$$
$$+b\Big[(1-a-b)\,\mathrm{B}\,(1-a,1-b)\,\Phi_{-}(\eta)\,(\xi+\eta)^{-a-b}$$
$$-(\xi+\eta)^{-b}\int_{-\xi}^{\eta}\Phi'_{-}(t)\,(t+\xi)^{-a}F\left(-a,b;1-a;\frac{t+\xi}{\eta+\xi}\right)dt\Big]$$
$$+a\Big[(1-a-b)\,\mathrm{B}\,(1-a,1-b)\,\Phi_{-}(\eta)\,(\xi+\eta)^{-a-b}$$
$$+\frac{b}{1-a}(\xi+\eta)^{-b-1}\int_{-\xi}^{\eta}\Phi'_{-}(t)\,(t+\xi)^{1-a}F\left(1-a,b+1;2-a;\frac{t+\xi}{\eta+\xi}\right)dt\Big],$$

or

$$L(u_1)\equiv\frac{b}{1-a}(\xi+\eta)^{-b}\int_{-\xi}^{\eta}\Phi'_{-}(t)\,(t+\xi)^{-a}S\left(a,b;\frac{t+\xi}{\eta+\xi}\right)dt,$$

where

$$S(a,b;z) = (1-a)\,F(-a,b+1;1-a;z) - (1-a)\,F(-a,b;1-a;z) + azF(1-a,b+1;2-a;z).$$

Using the Gauss recurrent formula [8]

$$c\,F(a,b;c;z) - c\,F(a,b+1;c;z) + a\,z\,F(a+1,b+1;c+1;z) = 0,$$

we find

$$S\left(a,b;\frac{t+\xi}{\eta+\xi}\right) = 0.$$

But then $L(u_1) \equiv 0$ and hence, $u_1(\xi,\eta)$ is a solution to equation (1.3). In view of (2.1) we conclude that the other linearly independent solution to equation (1.3) is of the form

$$u_2(\xi,\eta) = (\eta+\xi)^{1-\alpha-\beta}\int\limits_{-\xi}^{\eta}\Psi_-(t)\,(t+\xi)^{b-1}(\eta-t)^{a-1}dt,$$

where $\Psi_-(t)$ is an arbitrary continuously differentiable function. Therefore, a general solution to Equation (1.3) reads as

$$\begin{aligned} u(\xi,\eta) = &\int\limits_{-\xi}^{\eta}\Phi_-(t)\,(t+\xi)^{-a}(\eta-t)^{-b}dt \\ &+ (\xi+\eta)^{1-\alpha-\beta}\int\limits_{-\xi}^{\eta}\Psi_-(t)\,(t+\xi)^{b-1}(\eta-t)^{a-1}dt, \end{aligned} \tag{2.6}$$

where $\Psi_-(t)$ are continuously differentiable functions, or, after the change $t = -\xi + (\xi+\eta)\,s$

$$\begin{aligned} u(\xi,\eta) = &(\xi+\eta)^{1-\alpha-\beta}\int\limits_0^1\Phi_-(-\xi+(\xi+\eta)\,s)\,s^{-a}(1-s)^{-b}ds \\ &+ \int\limits_0^1\Psi_-(-\xi+(\xi+\eta)\,s)\,s^{b-1}(1-s)^{a-1}ds. \end{aligned} \tag{2.7}$$

We consider Equation (1.3) for

$$a = -\alpha, \qquad b = -\beta, \qquad 0 < \alpha, \beta, \alpha+\beta < 1.$$

In this case Equation (1.3) becomes

$$L(-\alpha,-\beta) \equiv (\xi+\eta)\,u_{\xi\eta} - \beta u_\xi - \alpha u_\eta = 0. \tag{2.8}$$

Let us find the general solution to Equation (2.8). According to (2.1), for $a=-\alpha$, $b=-\beta$,

$$u(-\alpha,-\beta) = (\xi+\eta)^{1+\alpha+\beta}u(1+\beta,1+\alpha). \tag{2.9}$$

We substitute $n = m = 2$, $a = \beta$, $b = \alpha$ into the formula (2.5) and we get

$$u(\beta+1,\alpha+1) = \frac{\partial^2 u(\beta,\alpha)}{\partial\xi\partial\eta}. \tag{2.10}$$

Substituting the value of the function $u(\beta+1,\alpha+1)$ defined by the formula (2.10) into the identity (2.9), we get

$$u(-\alpha,-\beta) = (\xi+\eta)^{1+\alpha+\beta}\frac{\partial^2 u(\beta,\alpha)}{\partial\xi\partial\eta}. \tag{2.11}$$

Differentiating in ξ and η the function defined by the formula (2.6) for $a=\beta$, $b=\alpha$, in view of the identity (2.11) we find the general solution to Equation (2.8)

$$\begin{aligned} u(\xi,\eta) = &-(\alpha+\beta)(1-\alpha-\beta)(\xi+\eta)^{\alpha+\beta-1} \\ &\cdot\int\limits_{-\xi}^{\eta}\Phi_-(t)(t+\xi)^{-\beta}(\eta-t)^{-\alpha}dt-(\xi+\eta)^{\alpha+\beta-1} \\ &\cdot\int\limits_{-\xi}^{\eta}\Phi'_-(t)(t+\xi)^{-\beta}(\eta-t)^{-\alpha}\left[(\alpha+\beta)(t+\xi)-\alpha(\xi+\eta)\right]dt \\ &-\int\limits_{-\xi}^{\eta}\Psi_-(t)(t+\xi)^{\alpha}(\eta-t)^{\beta}dt, \end{aligned} \tag{2.12}$$

where $\Phi_-\in C^3$, $\Psi_-\in C$. The proof that the function (2.12) indeed solves Equation (2.8) can be done in the same way as for Equation (1.3) with positive parameters a and b by straightforward substitution and application of the Gauss recurrent formulas [8].

Making the change $t=-\xi+(\xi+\eta)s$ in the formula (2.12), we arrive at the following solution to Equation (2.8):

$$\begin{aligned} u(\xi,\eta) = &-(\alpha+\beta)(1-\alpha-\beta)\int\limits_0^1\Phi_-(-\xi+(\xi+\eta)s)s^{-\beta}(1-s)^{-\alpha}ds \\ &-(\xi+\eta)\int\limits_0^1\Phi'_-(-\xi+(\xi+\eta)s)s^{-\beta}(1-s)^{-\alpha}\left[(\alpha+\beta)s-\alpha\right]ds \\ &-(\xi+\eta)^{\alpha+\beta+1}\int\limits_0^1\Psi_-(-\xi+(\xi+\eta)s)s^{\alpha}(1-s)^{\beta}ds. \end{aligned} \tag{2.13}$$

Now we consider Equation (1.3) for

$$a=\alpha,\qquad b=-\beta,\qquad 0<\alpha,\beta<1.$$

For the mentioned parameters a and b Equation (1.3) becomes

$$L(\alpha,-\beta)\equiv(\xi+\eta)u_{\xi\eta}-\beta u_\xi+\alpha u_\eta=0. \tag{2.14}$$

Let us find a general solution to Equation (2.14). For $a=\alpha$, $b=-\beta$ the formula (2.1) is written as

$$u(\alpha,-\beta)=(\xi+\eta)^{1-\alpha+\beta}u(1+\beta,1-\alpha).$$

For $a=\beta$, $b=1-\alpha$ the formula (2.3) becomes

$$u(1+\beta,1-\alpha)=\frac{\partial u(\beta,1-\alpha)}{\partial\xi}.$$

By two latter expressions we find

$$u(\alpha,-\beta)=(\xi+\eta)^{1-\alpha+\beta}\frac{\partial u(\beta,1-\alpha)}{\partial\xi}.$$

After differentiation in ξ of the function defined by the formula (2.6) for $a=\beta$, $b=1-\alpha$ we obtain the general solution to Equation (2.14)

$$\begin{aligned} u(\xi,\eta) = &(\alpha-\beta)(\xi+\eta)^{\beta-\alpha}\int\limits_{-\xi}^{\eta}\Phi_-(t)(t+\xi)^{-\beta}(\eta-t)^{\alpha-1}dt \\ &-(\xi+\eta)^{\beta-\alpha}\int\limits_{-\xi}^{\eta}\Phi'_-(t)(t+\xi)^{-\beta}(\eta-t)^{\alpha}dt - \int\limits_{-\xi}^{\eta}\Psi_-(t)(t+\xi)^{-\alpha}(\eta-t)^{\beta}dt, \end{aligned} \tag{2.15}$$

where $\Phi_-\in C^2$, $\Psi_-\in C^1$. The proof that the function (2.15) indeed solves Equation (2.14) can be done in the same way as for Equation (1.3) with positive parameters a and b by straightforward substitution and application of the Gauss recurrent formulas [8].

Making the change $t=-\xi+(\xi+\eta)s$ in the formula, we arrive at the following form of solution to Equation (2.14)

$$\begin{aligned} u(\xi,\eta) = &(\alpha-\beta)\int\limits_{0}^{1}\Phi_-(-\xi+(\xi+\eta)s)s^{-\beta}(1-s)^{\alpha-1}ds \\ &-(\xi+\eta)\int\limits_{0}^{1}\Phi'_-(-\xi+(\xi+\eta)s)s^{-\beta}(1-s)^{\alpha}ds \\ &-(\xi+\eta)^{1-\alpha+\beta}\int\limits_{0}^{1}\Psi_-(-\xi+(\xi+\eta)s)s^{-\alpha}(1-s)^{\beta}ds. \end{aligned} \tag{2.16}$$

For

$$a=-\alpha, \qquad b=\beta, \qquad 0<\alpha,\beta<1$$

Equation (1.2) reads

$$L(-\alpha,\beta)\equiv(\xi-\eta)u_{\xi\eta}-\beta u_\xi-\alpha u_\eta=0. \tag{2.17}$$

Let us find the general solution to Equation (2.17). In order to do this, we employ the following properties of the Euler — Darboux equation (1.2) [9]:

$$u(a,b)=(\eta-\xi)^{1-a-b}u(1-b,1-a), \tag{2.18}$$

$$u(a,1+b)=\frac{\partial u(a,b)}{\partial\eta}. \tag{2.19}$$

For $a=-\alpha$, $b=\beta$ the formula (2.18) becomes

$$u(-\alpha,\beta)=(\eta-\xi)^{1+\alpha-\beta}u(1-\beta,1+\alpha).$$

For $a=1-\beta$, $b=\alpha$ the formula (2.19) reads as

$$u(1-\beta,1+\alpha)=\frac{\partial u(1-\beta,\alpha)}{\partial\eta}.$$

By the two latter expressions we find

$$u(-\alpha,\beta)=(\xi+\eta)^{1+\alpha-\beta}\frac{\partial u(1-\beta,\alpha)}{\partial\eta}.$$

The form of general solution to the Euler — Darboux equation (1.2) is known for $0 < \alpha, \beta < 1$, $0 < \alpha + \beta < 1$ [9]:

$$u(\xi, \eta) = \int_{\xi}^{\eta} \Phi_{+}(t)(t-\xi)^{-a}(\eta - t)^{-b} dt + (\eta - \xi)^{1-\alpha-\beta} \int_{\xi}^{\eta} \Psi_{+}(t)(t-\xi)^{b-1}(\eta - t)^{a-1} dt,$$

if $\Phi_{+}, \Psi_{+} \in C^2$. After differentiation in η of the function defined by the latter formula for $a = 1 - \beta$, $b = \alpha$ we obtain

$$\begin{aligned} u(\xi, \eta) &= (\beta - \alpha)(\eta - \xi)^{\alpha-\beta} \int_{\xi}^{\eta} \Phi_{+}(t)(t-\xi)^{\beta-1}(\eta - t)^{-\alpha} dt \\ &+ (\eta - \xi)^{\alpha-\beta} \int_{\xi}^{\eta} \Phi'_{+}(t)(t-\xi)^{\beta}(\eta - t)^{-\alpha} dt + \int_{\xi}^{\eta} \Psi_{+}(t)(t-\xi)^{\alpha}(\eta - t)^{-\beta} dt, \end{aligned} \tag{2.20}$$

where $\Phi_{+} \in C^2$, $\Psi_{+} \in C^1$, or, after the change $t = \xi + (\eta - \xi)s$

$$\begin{aligned} u(\xi, \eta) &= (\beta - \alpha) \int_{0}^{1} \Phi_{+}(\xi + (\eta - \xi)s)\, s^{\beta-1}(1-s)^{-\alpha} ds \\ &+ (\eta - \xi) \int_{0}^{1} \Phi'_{+}(\xi + (\eta - \xi)s)\, s^{\beta}(1-s)^{-\alpha} ds \\ &+ (\eta - \xi)^{1+\alpha-\beta} \int_{0}^{1} \Psi_{+}(\xi + (\eta - \xi)s)\, s^{\alpha}(1-s)^{-\beta} ds. \end{aligned} \tag{2.21}$$

The proof that the found function indeed solves Equation (2.17) can be done in the same way as for Equation (1.3) with positive parameters a and b by straightforward substitution and application of the Gauss recurrent formulas [8].

For

$$a = -\alpha, \qquad b = \beta, \qquad 0 < \alpha, \beta < 1$$

Equation (1.3) reads as

$$L(-\alpha, \beta) \equiv (\xi + \eta) u_{\xi\eta} + \beta u_{\xi} - \alpha u_{\eta} = 0. \tag{2.22}$$

Let us find a general solution to Equation (2.22). For $a = -\alpha$, $b = \beta$ the formula (2.1) is written as

$$u(-\alpha, \beta) = (\xi + \eta)^{1+\alpha-\beta} u(1-\beta, 1+\alpha).$$

For $a = 1 - \beta$, $b = \alpha$ the formula (2.4) becomes

$$u(1-\beta, 1+\alpha) = \frac{\partial u(1-\beta, \alpha)}{\partial \eta}.$$

Two latter expressions imply

$$u(-\alpha, \beta) = (\xi + \eta)^{1+\alpha-\beta} \frac{\partial u(1-\beta, \alpha)}{\partial \eta}.$$

After the differentiation in η of the function defined by the formula (2.6) for $a = 1-\beta$, $b = \alpha$, we obtain the general solution to Equation (2.22)

$$\begin{aligned} u(\xi,\eta) = {} & (\beta-\alpha)(\xi+\eta)^{\alpha-\beta}\int_{-\xi}^{\eta}\Phi_-(t)(t+\xi)^{\beta-1}(\eta-t)^{-\alpha}dt \\ & -(\xi+\eta)^{\alpha-\beta}\int_{-\xi}^{\eta}\Phi'_-(t)(t+\xi)^{\beta}(\eta-t)^{-\alpha}dt - \int_{-\xi}^{\eta}\Psi_-(t)(t+\xi)^{\alpha}(\eta-t)^{-\beta}dt, \end{aligned} \tag{2.23}$$

where $\Phi_- \in C^2$, $\Psi_- \in C^1$. The proof that this function solves Equation (2.22) can be done in the same way as above.

Making the change $t = -\xi + (\xi+\eta)s$ in the formula (2.23), we arrive at the following form of Equation (2.22):

$$\begin{aligned} u(\xi,\eta) = {} & (\beta-\alpha)\int_0^1 \Phi_-(-\xi+(\xi+\eta)s)\,s^{\beta-1}(1-s)^{-\alpha}ds \\ & -(\xi+\eta)\int_0^1 \Phi'_-(-\xi+(\xi+\eta)s)\,s^{\beta}(1-s)^{-\alpha}ds \\ & -(\xi+\eta)^{1+\alpha-\beta}\int_0^1 \Psi_-(-\xi+(\xi+\eta)s)\,s^{\alpha}(1-s)^{-\beta}ds. \end{aligned} \tag{2.24}$$

3. Cauchy problems and modified Cauchy problems

We let $a = \alpha$, $b = \beta$ in Equation (1.3).

Problem 3.1. *For Equation (1.3) with*

$$a = \alpha, \qquad b = \beta, \qquad 0 < \alpha, \beta, \alpha+\beta < 1$$

in the domain

$$G_- = \{(\xi,\eta) \mid 0 < -\xi < \eta < h\} \tag{3.1}$$

find a solution to the Cauchy problem with the data

$$u(\xi,-\xi) = \tau_-(\xi), \qquad \xi \in [-h,0], \tag{3.2}$$

$$\lim_{\xi+\eta\to+0}(\xi+\eta)^{\alpha+\beta}(u_\xi+u_\eta) = \nu_-(\xi), \qquad \xi \in (-h,0). \tag{3.3}$$

We apply the condition (3.2) to the function defined by Formula (2.6) for $a = \alpha$, $b = \beta$. We obtain

$$u(\xi,-\xi) = \mathrm{B}(\alpha,\beta)\,\Psi_-(-\xi) = \tau_-(\xi).$$

This yields

$$\Psi_-(\xi) = \gamma_1\,\tau_-(-\xi), \tag{3.4}$$

where γ_1 is defined by the formula (1.7).

We calculate the partial derivatives of the function defined by the formula (2.7)

$$\begin{aligned} u_\xi(\xi,\eta) = {} & (1-\alpha-\beta)(\xi+\eta)^{-\alpha-\beta}\int_0^1 \Phi_-\left(-\xi+(\xi+\eta)s\right)s^{-\alpha}(1-s)^{-\beta}ds \\ & -(\xi+\eta)^{1-\alpha-\beta}\int_0^1 \Phi'_-\left(-\xi+(\xi+\eta)s\right)s^{-\alpha}(1-s)^{1-\beta}ds \\ & -\int_0^1 \Psi'_-\left(-\xi+(\xi+\eta)s\right)s^{\beta-1}(1-s)^{\alpha}ds, \end{aligned} \tag{3.5}$$

$$\begin{aligned} u_\eta(\xi,\eta) = {} & (1-\alpha-\beta)(\xi+\eta)^{-\alpha-\beta}\int_0^1 \Phi_-\left(-\xi+(\xi+\eta)s\right)s^{-\alpha}(1-s)^{-\beta}ds \\ & +(\xi+\eta)^{1-\alpha-\beta}\int_0^1 \Phi'_-\left(-\xi+(\xi+\eta)s\right)s^{1-\alpha}(1-s)^{-\beta}ds \\ & +\int_0^1 \Psi'_-\left(-\xi+(\xi+\eta)s\right)s^{\beta}(1-s)^{\alpha-1}ds. \end{aligned} \tag{3.6}$$

We apply the condition (3.3) to the function defined by the formula (2.7) using the expressions (3.5) and (3.6) and we obtain

$$\nu_-(\xi) = 2(1-\alpha-\beta)\,\mathrm{B}(1-\alpha,1-\beta)\,\Phi_-(-\xi).$$

This yields

$$\Phi_-(-\xi) = -\gamma_2\,\nu_-(-\xi), \tag{3.7}$$

where γ_2 is defined by the formula (1.8). We substitute the expressions (3.4) and (3.7) into the function defined by the formula (2.6) and we find

$$\begin{aligned} u(\xi,\eta) = {} & -\gamma_2\int_{-\xi}^{\eta} \nu_-(-t)(t+\xi)^{-\alpha}(\eta-t)^{-\beta}dt \\ & +\gamma_1(\xi+\eta)^{1-\alpha-\beta}\int_{-\xi}^{\eta} \tau_-(-t)(t+\xi)^{\beta-1}(\eta-t)^{\alpha-1}dt. \end{aligned} \tag{3.8}$$

Theorem 3.1. *A solution to the Cauchy problem with data (3.2) and (3.3) for Equation (1.1) in the domain G_- (3.1) is expressed by the formula (3.8), and it is unique if*

$$\tau_- \in C^2_{[-h,0]}, \qquad \nu_- \in C^2_{[-h,0]}, \qquad a=\alpha, \qquad b=\beta, \qquad 0<\alpha,\beta,\alpha+\beta<1.$$

The uniqueness of solution is implied by the unique solvability of all equations involved in the procedure of finding it. In order to prove the existence, we substitute the function defined by the formula (3.8) into Equation (1.1) for $a=\alpha$, $b=\beta$ and values of ξ and η belonging to the domain G_-. This leads us to the identity.

Problem 3.2. *For Equation (1.1) with*

$$a=-\alpha, \qquad b=-\beta, \qquad 0<\alpha,\beta,\alpha+\beta<1$$

in the domain G_- (3.1) find a solution to the modified Cauchy problem with data (3.2) and

$$\lim_{\xi+\eta\to+0} \frac{1}{(\alpha+\beta)^2}(\xi+\eta)^{-\alpha-\beta}\left(\beta u_\xi+\alpha u_\eta\right)=\nu_-\left(\xi\right), \qquad \xi\in(-h,0)\,. \tag{3.9}$$

We apply the condition (3.2) to the function defined by the formula (2.13). We have

$$\tau_-\left(\xi\right)=-\left(\alpha+\beta\right)\left(1-\alpha-\beta\right)\Phi_-\left(-\xi\right)\mathrm{B}\left(1-\alpha,1-\beta\right).$$

This yields

$$\Phi_-\left(\xi\right)=-\mu_2\tau_-\left(-\xi\right), \tag{3.10}$$

where μ_2 is defined by the formula (1.11).

We find first partial derivatives of the function defined by the formula (2.13)

$$\begin{aligned} u_\xi\left(\xi,\eta\right)=&\left(\alpha+\beta\right)\left(1-\alpha-\beta\right)\int\limits_0^1 \Phi'_-\left(-\xi+\left(\xi+\eta\right)s\right)s^{-\beta}(1-s)^{1-\alpha}ds\\ &-\int\limits_0^1 \Phi'_-\left(-\xi+\left(\xi+\eta\right)s\right)s^{-\beta}(1-s)^{-\alpha}\left[\left(\alpha+\beta\right)s-\alpha\right]ds\\ &+\left(\xi+\eta\right)\int\limits_0^1 \Phi''_-\left(-\xi+\left(\xi+\eta\right)s\right)s^{-\beta}(1-s)^{1-\alpha}\left[\left(\alpha+\beta\right)s-\alpha\right]ds\\ &-\alpha(\xi+\eta)^{\alpha+\beta}\int\limits_0^1 \Psi_-\left(-\xi+\left(\xi+\eta\right)s\right)s^{\alpha-1}(1-s)^{\beta}ds, \end{aligned} \tag{3.11}$$

$$\begin{aligned} u_\eta\left(\xi,\eta\right)=&-\left(\alpha+\beta\right)\left(1-\alpha-\beta\right)\int\limits_0^1 \Phi'_-\left(-\xi+\left(\xi+\eta\right)s\right)s^{1-\beta}(1-s)^{-\alpha}ds\\ &-\int\limits_0^1 \Phi'_-\left(-\xi+\left(\xi+\eta\right)s\right)s^{-\beta}(1-s)^{-\alpha}\left[\left(\alpha+\beta\right)s-\alpha\right]ds\\ &-\left(\xi+\eta\right)\int\limits_0^1 \Phi''_-\left(-\xi+\left(\xi+\eta\right)s\right)s^{1-\beta}(1-s)^{-\alpha}\left[\left(\alpha+\beta\right)s-\alpha\right]ds\\ &-\beta(\xi+\eta)^{\alpha+\beta}\int\limits_0^1 \Psi_-\left(-\xi+\left(\xi+\eta\right)s\right)s^{\alpha}(1-s)^{\beta-1}ds. \end{aligned} \tag{3.12}$$

We apply the condition (3.9) to the function defined by the formula (2.13) using the expressions (3.11) and (3.12). In order to do this, we find

$$\begin{aligned} \beta u_\xi+\alpha u_\eta=&(\alpha+\beta)^2\int\limits_0^1 \Phi'_-\left(-\xi+\left(\xi+\eta\right)s\right)s^{-\beta}(1-s)^{-\alpha}\left[1-\beta-\left(2-\alpha-\beta\right)s\right]ds\\ &+(\alpha+\beta)^2\left(\xi+\eta\right)\int\limits_0^1 \Phi''_-\left(-\xi+\left(\xi+\eta\right)s\right)s^{1-\beta}(1-s)^{1-\alpha}ds \end{aligned} \tag{3.13}$$

$$-\alpha\beta\left(\xi+\eta\right)\int\limits_0^1 \Phi''_-\left(-\xi+\left(\xi+\eta\right)s\right)s^{-\beta}(1-s)^{-\alpha}ds$$

$$-\alpha\beta(\xi+\eta)^{\alpha+\beta}\int\limits_0^1 \Psi_-\left(-\xi+\left(\xi+\eta\right)s\right)s^{\alpha-1}(1-s)^{\beta-1}ds.$$

We integrate by parts in the second term in the right hand side taking $u=s^{1-\beta}(1-s)^{-\alpha}$. This gives

$$v=\frac{1}{\xi+\eta}\Phi'_-\left(-\xi+\left(\xi+\eta\right)s\right).$$

Substituting the obtained expression into the identity (3.13), we have

$$\begin{aligned}\beta u_\xi+\alpha u_\eta=&-\alpha\beta\left(\xi+\eta\right)\int\limits_0^1 \Phi''_-\left(-\xi+\left(\xi+\eta\right)s\right)s^{-\beta}(1-s)^{-\alpha}ds\\&-\alpha\beta(\xi+\eta)^{\alpha+\beta}\int\limits_0^1 \Psi_-\left(-\xi+\left(\xi+\eta\right)s\right)s^{\alpha-1}(1-s)^{\beta-1}ds.\end{aligned}$$

Then

$$\lim_{\xi+\eta\to+0}\frac{1}{(\alpha+\beta)^2}(\xi+\eta)^{-\alpha-\beta}\left(\beta u_\xi+\alpha u_\eta\right)=-\frac{\alpha\beta}{(\alpha+\beta)^2}\,\mathrm{B}\left(\alpha,\beta\right)\Psi_-\left(-\xi\right)=\nu_-\left(\xi\right)$$

and hence,

$$\Psi_-\left(\xi\right)=-\mu_3\nu_-\left(-\xi\right),\tag{3.14}$$

where μ_3 is determined by the formula (1.12).

We substitute the expressions (3.10) and (3.14) into the formula (2.12) and we get

$$\begin{aligned}u\left(\xi,\eta\right)=&\mu_3\int\limits_{-\xi}^{\eta}\nu_-\left(-t\right)\left(t+\xi\right)^{\alpha}(\eta-t)^{\beta}dt\\&+\mu_1(\xi+\eta)^{\alpha+\beta-1}\int\limits_{-\xi}^{\eta}\tau_-\left(-t\right)\left(t+\xi\right)^{-\beta}(\eta-t)^{-\alpha}dt\\&-\mu_2(\xi+\eta)^{\alpha+\beta-1}\int\limits_{-\xi}^{\eta}\tau'_-\left(-t\right)\left(t+\xi\right)^{-\beta}(\eta-t)^{-\alpha}\\&\cdot\left[\left(\alpha+\beta\right)\left(t+\xi\right)-\alpha\left(\xi+\eta\right)\right]dt.\end{aligned}\tag{3.15}$$

Theorem 3.2. *A solution to the modified Cauchy problem with data (3.2) and (3.9) for Equation (1.1) in the domain G_- (3.1) is expressed by the formula (3.15) and it is unique if*

$$\tau_-\in C^3_{[-h,0]},\quad \nu_-\in C_{[-h,0]},\quad a=-\alpha,\quad b=-\beta,\quad 0<\alpha,\beta<1,\quad 0<\alpha+\beta<1.$$

The uniqueness of solution is implied by the unique solvability of all equations involved in the procedure of finding it. In order to prove the existence, we substitute the function defined by the formula (3.15) into Equation (1.1) for $a=-\alpha$, $b=-\beta$ and values of ξ and η belonging to the domain G_-. This leads us to the identity.

Problem 3.3. *For Equation (1.1) for*

$$a=\alpha,\qquad b=-\beta,\qquad 0<\alpha,\beta,\alpha-\beta<1$$

in the domain G_- find a solution to the Cauchy problem with data (3.2) and

$$\lim_{\xi+\eta\to+0}(\xi+\eta)^{\alpha-\beta}(u_\xi+u_\eta)=\nu_-(\xi),\qquad \xi\in(-h,0). \tag{3.16}$$

We apply the condition (3.2) to the formula (2.16) and we get

$$\tau_-(\xi)=(\alpha-\beta)\,\mathrm{B}(\alpha,1-\beta)\,\Phi_-(-\xi).$$

This yields

$$\Phi_-(\xi)=-\gamma_3\,\tau_-(-\xi), \tag{3.17}$$

where γ_3 is defined by the expression (1.9).

We calculate the first partial derivatives of the function defined by the formula (2.16):

$$\begin{aligned}
u_\xi(\xi,\eta)=&-(\alpha-\beta)\int_0^1\Phi'_-(-\xi+(\xi+\eta)s)\,s^{-\beta}(1-s)^{\alpha}ds\\
&-\int_0^1\Phi'_-(-\xi+(\xi+\eta)s)\,s^{-\beta}(1-s)^{\alpha}ds\\
&+(\xi+\eta)\int_0^1\Phi''_-(-\xi+(\xi+\eta)s)\,s^{-\beta}(1-s)^{1+\alpha}ds\\
&-(1+\beta-\alpha)(\xi+\eta)^{\beta-\alpha}\int_0^1\Psi_-(-\xi+(\xi+\eta)s)\,s^{-\alpha}(1-s)^{\beta}ds\\
&+(\xi+\eta)^{1+\beta-\alpha}\int_0^1\Psi'_-(-\xi+(\xi+\eta)s)\,s^{-\alpha}(1-s)^{\beta+1}ds,
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
u_\eta(\xi,\eta)=&(\alpha-\beta)\int_0^1\Phi'_-(-\xi+(\xi+\eta)s)\,s^{1-\beta}(1-s)^{\alpha-1}ds\\
&-\int_0^1\Phi'_-(-\xi+(\xi+\eta)s)\,s^{-\beta}(1-s)^{\alpha}ds\\
&-(\xi+\eta)\int_0^1\Phi''_-(-\xi+(\xi+\eta)s)\,s^{1-\beta}(1-s)^{\alpha}ds\\
&-(1+\beta-\alpha)(\xi+\eta)^{\beta-\alpha}\int_0^1\Psi_-(-\xi+(\xi+\eta)s)\,s^{-\alpha}(1-s)^{\beta}ds\\
&-(\xi+\eta)^{\beta-\alpha}\int_0^1\Psi'_-(-\xi+(\xi+\eta)s)\,s^{1-\alpha}(1-s)^{\beta}ds.
\end{aligned} \tag{3.19}$$

In the same way how this was done while solving Problem 3.2, we apply the condition (3.16) to the function defined by the formula (2.16) using the expressions (3.18) and (3.19). We obtain

$$\nu_-(\xi)=-2(1+\beta-\alpha)\,\mathrm{B}(1-\alpha,1+\beta)\,\Psi_-(-\xi),$$

and we hence find

$$\psi_-(\xi)=\gamma_4\,\nu_-(-\xi), \tag{3.20}$$

where γ_4 is defined by the formula (1.10).

We substitute the expressions (3.17) and (3.20) into the formula (2.15) and we have

$$\begin{aligned} u(\xi,\eta) = &-\gamma_4 \int_{-\xi}^{\eta} \nu_-(-t)(t+\xi)^{-\alpha}(\eta-t)^{\beta}dt \\ &-(\alpha-\beta)\gamma_3(\xi+\eta)^{\beta-\alpha}\int_{-\xi}^{\eta} \tau_-(-t)(t+\xi)^{-\beta}(\eta-t)^{\alpha-1}dt \\ &+\gamma_3(\xi+\eta)^{\beta-\alpha}\int_{-\xi}^{\eta} \tau'_-(-t)(t+\xi)^{-\beta}(\eta-t)^{\alpha}dt. \end{aligned} \tag{3.21}$$

Theorem 3.3. *A solution to the Cauchy problem with data (3.2) and (3.16) for Equation (1.1) in the domain G_- (3.1) is expressed by the formula (3.21), and it is unique if*

$$\tau_- \in C^2_{[-h,0]}, \qquad \nu_- \in C^1_{[-h,0]}, \qquad a=\alpha, \qquad b=-\beta, \qquad 0<\alpha,\beta,\alpha-\beta<1.$$

The uniqueness of solution is implied by the unique solvability of all equations involved in the procedure of finding it. In order to prove the existence, we substitute the function defined by the formula (3.21) into Equation (1.1) for $a=\alpha$, $b=-\beta$ and values of ξ and η belonging to the domain G_-. This leads us to the identity.

Problem 3.4. *For Equation (1.1) for $a=\alpha$, $b=-\beta$, $0<\alpha,\beta,\beta-\alpha<1$ in the domain G_- (3.1) find a solution to the modified Cauchy problem with data (3.2) and*

$$\lim_{\xi+\eta\to+0}\frac{1}{\alpha-\beta}(\xi+\eta)^{\alpha-\beta}(\beta u_\xi-\alpha u_\eta)=\nu_-(\xi), \qquad \xi\in(-h,0). \tag{3.22}$$

We apply the condition (3.22) to the function defined by the formula (2.16) using the expressions (3.18) and (3.19). We obtain

$$\nu_-(\xi)=(1+\beta-\alpha)\,\mathrm{B}(1-\alpha,1+\beta)\,\Psi_-(-\xi).$$

This yields

$$\psi_-(\xi)=-2\gamma_4\,\nu_-(-\xi), \tag{3.23}$$

where γ_4 is defined by the formula (1.10).

Substituting the expressions (3.17) and (3.23) into the formula (2.15), we arrive at a solution to Problem3.4

$$\begin{aligned} u(\xi,\eta) = &2\gamma_4 \int_{-\xi}^{\eta} \nu_-(-t)(t+\xi)^{-\alpha}(\eta-t)^{\beta}dt \\ &-(\alpha-\beta)\gamma_3(\xi+\eta)^{\beta-\alpha}\int_{-\xi}^{\eta} \tau_-(-t)(t+\xi)^{-\beta}(\eta-t)^{\alpha-1}dt \\ &+\gamma_3(\xi+\eta)^{\beta-\alpha}\int_{-\xi}^{\eta} \tau'_-(-t)(t+\xi)^{-\beta}(\eta-t)^{\alpha}dt. \end{aligned} \tag{3.24}$$

Theorem 3.4. *A solution to the modified Cauchy problem with data (3.2) and (3.22) for Equation (1.1) in the domain G_- (3.1) is expressed by the formula (3.24), and it is unique if*

$$\tau_-(\xi)\in C^2_{[-h,0]}, \quad \nu_-(\xi)\in C^1_{[-h,0]}, \quad a=\alpha, \quad b=-\beta, \quad 0<\alpha,\beta<1, \quad 0<\beta-\alpha<1.$$

The uniqueness of solution is implied by the unique solvability of all equations involved in the procedure of finding it. In order to prove the existence, we substitute the function defined by the formula (3.24) into Equation (1.1) for $a = \alpha$, $b = -\beta$ and values of ξ and η belonging to the domain G_-. This leads us to the identity.

Problem 3.5. *For Equation (1.1) for*

$$a = -\alpha, \qquad b = \beta, \qquad 0 < \alpha, \beta, \beta - \alpha < 1,$$

in the domain

$$G_+ = \{ (\xi, \eta) |\, 0 < \xi < \eta < h\} \tag{3.25}$$

find a solution to the Cauchy problem with data (1.5) and

$$\lim_{\eta - \xi \to +0} (\eta - \xi)^{\beta - \alpha} (u_\xi - u_\eta) = \nu_+ (\xi), \qquad \xi \in (0, h). \tag{3.26}$$

We apply the condition (1.5) to the function defined by the formula (2.21) and we get

$$\tau_+ (\xi) = (\beta - \alpha)\, \Phi_+ (\xi)\, \mathrm{B} (\beta, 1 - \alpha).$$

This yields

$$\Phi_+ (\xi) = \gamma_5\, \tau_+ (\xi), \tag{3.27}$$

where

$$\gamma_5 = \frac{1}{(\beta - \alpha)\, \mathrm{B} (1 - \alpha, \beta)}. \tag{3.28}$$

We find the partial derivatives of the function defined by the formula (2.21)

$$\begin{aligned} u_\xi (\xi, \eta) = {} & (\beta - \alpha) \int_0^1 \Phi'_+ (\xi + (\eta - \xi)\, s)\, s^{\beta - 1} (1 - s)^{1 - \alpha} ds \\ & - \int_0^1 \Phi'_+ (\xi + (\eta - \xi)\, s)\, s^{\beta} (1 - s)^{-\alpha} ds \\ & + (\eta - \xi) \int_0^1 \Phi''_+ (\xi + (\eta - \xi)\, s)\, s^{\beta} (1 - s)^{1 - \alpha} ds \\ & - (1 + \alpha - \beta)\, (\eta - \xi)^{\alpha - \beta} \int_0^1 \Psi_+ (\xi + (\eta - \xi)\, s)\, s^{\alpha} (1 - s)^{-\beta} ds \\ & + (\eta - \xi)^{1 + \alpha - \beta} \int_0^1 \Psi'_+ (\xi + (\eta - \xi)\, s)\, s^{\alpha} (1 - s)^{1 - \beta} ds, \end{aligned} \tag{3.29}$$

$$\begin{aligned} u_\eta(\xi,\eta) = &(\beta-\alpha)\int\limits_0^1 \Phi'_+(\xi+(\eta-\xi)s)\,s^{\beta}(1-s)^{-\alpha}ds \\ &+\int\limits_0^1 \Phi'_+(\xi+(\eta-\xi)s)\,s^{\beta}(1-s)^{-\alpha}ds \\ &+(\eta-\xi)\int\limits_0^1 \Phi''_+(\xi+(\eta-\xi)s)\,s^{\beta+1}(1-s)^{-\alpha}ds \\ &+(1+\alpha-\beta)(\eta-\xi)^{\alpha-\beta}\int\limits_0^1 \Psi_+(\xi+(\eta-\xi)s)\,s^{\alpha}(1-s)^{-\beta}ds \\ &+(\eta-\xi)^{1+\alpha-\beta}\int\limits_0^1 \Psi'_+(\xi+(\eta-\xi)s)\,s^{\alpha+1}(1-s)^{-\beta}ds. \end{aligned} \tag{3.30}$$

In the same way how this was done in Problem 3.2, we apply the condition (3.26) to the function defined by the formula (2.21) using the expressions (3.29) and (3.30). We get

$$\nu_+(\xi) = -2(1+\alpha-\beta)\,\Psi_+(\xi)\,\mathrm{B}(1+\alpha,1-\beta).$$

This yields

$$\Psi_+(\xi) = -\gamma_6\,\nu_+(\xi), \tag{3.31}$$

where

$$\gamma_6 = \frac{1}{2(1+\alpha-\beta)\,\mathrm{B}(1+\alpha,1-\beta)}. \tag{3.32}$$

We substitute the expressions (3.27) and (3.31) into the formula (2.20) and we get

$$\begin{aligned} u(\xi,\eta) = &-\gamma_6\int\limits_\xi^\eta \nu_+(t)(t-\xi)^{\alpha}(\eta-t)^{-\beta}dt \\ &+(\beta-\alpha)\gamma_5(\eta-\xi)^{\alpha-\beta}\int\limits_\xi^\eta \tau_+(t)(t-\xi)^{\beta-1}(\eta-t)^{-\alpha}dt \\ &+\gamma_5(\eta-\xi)^{\alpha-\beta}\int\limits_\xi^\eta \tau'_+(t)(t-\xi)^{\beta}(\eta-t)^{-\alpha}dt. \end{aligned} \tag{3.33}$$

Theorem 3.5. *A solution to the Cauchy problem with data (1.5) and (3.26) for Equation (1.1) in the domain G_+ (3.25) is expressed by the formula (3.33), and it is unique if*

$$\tau_+ \in C^2_{[0,h]}, \qquad \nu_+ \in C^1_{[0,h]}, \qquad a=-\alpha, \qquad b=\beta, \qquad 0<\alpha,\beta,\beta-\alpha<1.$$

The uniqueness of solution is implied by the unique solvability of all equations involved in the procedure of finding it. In order to prove the existence, we substitute the function defined by the formula (3.33) into Equation (1.1) for $a=-\alpha$, $b=\beta$ and values of ξ and η belonging to the domain G_-. This leads us to the identity.

Problem 3.6. *For Equation (1.1) with*

$$a=-\alpha, \qquad b=\beta, \qquad 0<\alpha,\beta,\alpha-\beta<1$$

in the domain G_+ (3.25) find a solution to the modified Cauchy problem with data (1.5) and

$$\lim_{\eta-\xi\to+0}\frac{1}{\alpha-\beta}(\eta-\xi)^{\beta-\alpha}\left(\beta u_\xi+\alpha u_\eta\right)=\nu_+\left(\xi\right), \qquad \xi\in(0,h). \tag{3.34}$$

We apply the condition (3.34) to the function defined by the formula (2.21) using the expressions (3.29) and (3.30) and we obtain

$$\nu_+\left(\xi\right)=\left(1+\alpha-\beta\right)\Psi_+\left(\xi\right)\mathrm{B}\left(1+\alpha,1-\beta\right).$$

This yields

$$\Psi_+\left(\xi\right)=2\gamma_6\,\nu_+\left(\xi\right), \tag{3.35}$$

where γ_6 is defined by the formula (3.32).

We substitute the expressions (3.27) and (3.35) into the formula (2.20) and we find

$$\begin{aligned} u\left(\xi,\eta\right)=&2\gamma_6\int\limits_\xi^\eta\nu_+\left(t\right)\left(t-\xi\right)^\alpha(\eta-t)^{-\beta}dt\\ &+\left(\beta-\alpha\right)\gamma_5(\eta-\xi)^{\alpha-\beta}\int\limits_\xi^\eta\tau_+\left(t\right)\left(t-\xi\right)^{\beta-1}(\eta-t)^{-\alpha}dt\\ &+\gamma_5(\eta-\xi)^{\alpha-\beta}\int\limits_\xi^\eta{\tau'}_+\left(t\right)\left(t-\xi\right)^{\beta}(\eta-t)^{-\alpha}dt. \end{aligned} \tag{3.36}$$

Theorem 3.6. *A solution to the Cauchy problem with data (1.5) and (3.34) for Equation (1.1) in the domain G_+ (3.25) is expressed by the formula (3.36), and it is unique if*

$$\tau_+\in C^2_{[0,h]},\quad \nu_+\in C^1_{[0,h]},\quad a=-\alpha,\quad b=\beta,\quad 0<\alpha,\beta<1,\quad 0<\alpha-\beta<1.$$

The uniqueness of solution is implied by the unique solvability of all equations involved in the procedure of finding it. In order to prove the existence, we substitute the function defined by the formula (3.36) into Equation (1.1) for $a=-\alpha$, $b=\beta$ and values of ξ and η belonging to the domain G_-. This leads us to the identity.

Problem 3.7. *For equation (1.1) for*

$$a=-\alpha,\qquad b=\beta,\qquad 0<\alpha,\beta,\beta-\alpha<1$$

in the domain G_- (3.1) find a solution to the Cauchy problem with data (3.2) and

$$\lim_{\xi+\eta\to+0}\left(\xi+\eta\right)^{\beta-\alpha}\left(u_\xi+u_\eta\right)=\nu_-\left(\xi\right),\xi\in\left(-h,0\right). \tag{3.37}$$

We apply the condition (3.2) to the function defined by the formula (2.24) and we get

$$\tau_-\left(\xi\right)=\left(\beta-\alpha\right)\mathrm{B}\left(\beta,1-\alpha\right)\Phi_-\left(-\xi\right).$$

This yields

$$\Phi_-\left(\xi\right)=\gamma_5\,\tau_-\left(-\xi\right), \tag{3.38}$$

where γ_5 is defined by the formula (3.28).

We calculate the first partial derivatives of the function defined by the formula (2.24)

$$\begin{aligned} u_\xi(\xi,\eta) = &-(\beta-\alpha)\int_0^1 \Phi'_-\left(-\xi+(\xi+\eta)s\right)s^{\beta-1}(1-s)^{1-\alpha}ds \\ &-\int_0^1 \Phi'_-\left(-\xi+(\xi+\eta)s\right)s^{\beta}(1-s)^{-\alpha}ds \\ &+(\xi+\eta)\int_0^1 \Phi''_-\left(-\xi+(\xi+\eta)s\right)s^{\beta}(1-s)^{1-\alpha}ds \\ &-(1+\alpha-\beta)(\xi+\eta)^{\alpha-\beta}\int_0^1 \Psi_-\left(-\xi+(\xi+\eta)s\right)s^{\alpha}(1-s)^{-\beta}ds \\ &+(\xi+\eta)^{1+\alpha-\beta}\int_0^1 \Psi'_-\left(-\xi+(\xi+\eta)s\right)s^{\alpha}(1-s)^{1-\beta}ds, \end{aligned} \tag{3.39}$$

$$\begin{aligned} u_\eta(\xi,\eta) = &(\beta-\alpha)\int_0^1 \Phi'_-\left(-\xi+(\xi+\eta)s\right)s^{\beta}(1-s)^{-\alpha}ds \\ &-\int_0^1 \Phi'_-\left(-\xi+(\xi+\eta)s\right)s^{\beta}(1-s)^{-\alpha}ds \\ &-(\xi+\eta)\int_0^1 \Phi''_-\left(-\xi+(\xi+\eta)s\right)s^{1+\beta}(1-s)^{-\alpha}ds \\ &-(1+\alpha-\beta)(\xi+\eta)^{\alpha-}\beta\int_0^1 \Psi_-\left(-\xi+(\xi+\eta)s\right)s^{\alpha}(1-s)^{-\beta}ds \\ &-(\xi+\eta)^{1+\alpha-\beta}\int_0^1 \Psi'_-\left(-\xi+(\xi+\eta)s\right)s^{1+\alpha}(1-s)^{-\beta}ds. \end{aligned} \tag{3.40}$$

We apply the condition (3.37) to the function defined by the formula (2.24) using the expressions (3.39) and (3.40) and we obtain

$$\nu_-(\xi) = -2(1+\alpha-\beta)\,\mathrm{B}(1+\alpha,1-\beta)\,\Psi_-(-\xi).$$

This yields

$$\Psi_-(\xi) = -\gamma_6\,\nu_-(-\xi), \tag{3.41}$$

where γ_6 is defined by the formula (3.32).

We substitute the expressions (3.38) and (3.41) into the formula (2.23) and we get

$$
\begin{aligned}
u(\xi,\eta) =&\gamma_6 \int\limits_{-\xi}^{\eta} \nu_-(-t)\,(t+\xi)^{\alpha}(\eta-t)^{-\beta}dt \\
&+(\beta-\alpha)\,\gamma_5(\xi+\eta)^{\alpha-\beta}\int\limits_{-\xi}^{\eta} \tau_-(-t)\,(t+\xi)^{\beta-1}(\eta-t)^{-\alpha}dt \\
&+\gamma_5(\xi+\eta)^{\alpha-\beta}\int\limits_{-\xi}^{\eta} \tau'_-(-t)\,(t+\xi)^{\beta}(\eta-t)^{-\alpha}dt.
\end{aligned}
\tag{3.42}
$$

Theorem 3.7. *A solution to the Cauchy problem with data (3.2) and (3.37) for Equation (1.1) in the domain G_- (3.1) is expressed by the formula (3.42) and it is unique if*

$$\tau_- \in C^2_{[-h,0]}, \qquad \nu_- \in C^1_{[-h,0]}, \qquad a=-\alpha, \qquad b=\beta, \qquad 0<\alpha,\beta,\beta-\alpha<1.$$

The uniqueness of solution is implied by the unique solvability of all equations involved in the procedure of finding it. In order to prove the existence, we substitute the function defined by the formula (3.42) into Equation (1.1) for $a=-\alpha$, $b=\beta$ and values of ξ and η belonging to the domain G_-. This leads us to the identity.

Problem 3.8. *For Equation (1.1) for*

$$a=-\alpha, \qquad b=\beta, \qquad 0<\alpha,\beta,\alpha-\beta<1$$

in the domain G_- (3.1) find a solution to the modified Cauchy problem with data (3.2) and

$$\lim_{\xi+\eta\to+0}\frac{1}{\alpha-\beta}(\eta-\xi)^{\beta-\alpha}\left(\beta u_\xi-\alpha u_\eta\right)=\nu_-(\xi), \qquad \xi\in(-h,0). \tag{3.43}$$

We apply the condition (3.43) to the function defined by the formula (2.24) and we find

$$\nu_-(\xi)=(1+\alpha-\beta)\,\mathrm{B}\,(1+\alpha,1-\beta)\,\Psi_-(-\xi).$$

This yields

$$\Psi_-(\xi)=2\gamma_6\,\nu_-(-\xi), \tag{3.44}$$

where γ_6 is determined by the formula (3.32).

We substitute the expressions (3.38) and (3.44) into the formula (2.23) and we obtain

$$
\begin{aligned}
u(\xi,\eta) =&-2\gamma_6 \int\limits_{-\xi}^{\eta} \nu_-(-t)\,(t+\xi)^{\alpha}(\eta-t)^{-\beta}dt \\
&+(\beta-\alpha)\,\gamma_5(\xi+\eta)^{\alpha-\beta}\int\limits_{-\xi}^{\eta} \tau_-(-t)\,(t+\xi)^{\beta-1}(\eta-t)^{-\alpha}dt \\
&+\gamma_5(\xi+\eta)^{\alpha-\beta}\int\limits_{-\xi}^{\eta} \tau'_-(-t)\,(t+\xi)^{\beta}(\eta-t)^{-\alpha}dt.
\end{aligned}
\tag{3.45}
$$

Theorem 3.8. *A solutio to the modified Cauchy problem with data (3.2) and (3.43) for Equation (1.1) in the domain G_- (3.1) is expressed by the formula (3.45) and it is unique if*

$$\tau_-(\xi)\in C^2_{[-h,0]}, \quad \nu_-(\xi)\in C^1_{[-h,0]}, \quad a=-\alpha, \quad b=\beta, \quad 0<\alpha,\beta<1, \quad 0<\alpha-\beta<1.$$

The uniqueness of solution is implied by the unique solvability of all equations involved in the procedure of finding it. In order to prove the existence, we substitute the function defined by the formula (3.45) into Equation (1.1) for $a = -\alpha$, $b = \beta$ and values of ξ and η belonging to the domain G_-. This leads us to the identity.

Conclusion

In the present paper we solve a series of problems for the Euler — Darboux equation with two degeneracy lines (1.1) in the characteristic coordinates in the domains G_+ (3.25) and G_- (3.1). In the domain G_+ (3.25), Equation (1.1) is equivalent to Euler — Darboux equation. The first four problems for the Euler — Darboux equation are similar to ones solved in this paper we solved by Volkodavov and Nikolaev [3]. Their conditions and corresponding existence and uniqueness theorems are given in the Introduction.

In the first theoretical part of the work we find general solutions for the Euler — Darboux equation in cases not considered earlier, as well as for the equation, which is the image of the Euler — Darboux equation with respect to the η axis. The latter equation is equivalent to the Euler — Darboux equation in the domain G_- (3.1).

The second theoretical part is devoted to solving the Cauchy problems and modified Cauchy problems for the parameter values, for which the general solution of Equation (1.1) is found in the previous part.

We note that the given domains are bounded by the line $\eta = h$, where h is an arbitrary real number. As h tends to infinity, the triangles G_+ (3.25) and G_- (3.1) become infinite (unbounded from above).

BIBLIOGRAPHY

1. B. Riemann. *Workds.* Gostekhizdat, Moscow (1948). (in Russian).
2. V.V. Sokolovskij. *Plastisity theory.* Vysshaya shkola (1969). (in Russian).
3. V.F. Volkodovav, N.Ya. Nikolaev. *Boundary value problems for Euler — Poisson — Darboux equation: learning guide to course "Equations of mathematical physics".* Kujbyshev State Pedagogical Institute, Kujbyshev (1984). (in Russian).
4. V.F. Volkodovav, N.Ya. Nikolaev. *Boundary value problems for Euler — Darboux equation with negative parameters in unbounded domain. Part II. Theorem on existence of solution to problem.* // In: *Differential equations.* Ryazan State Pedagogical Institute named after S.A. Esenin, Ryazan, 20 (1981). (in Russian).
5. O.G. Aristova. *Boundary value problems for one hyperbolic equation with degeneration on non–characteristic curve* // In.: *Third International Conference dedicated to academician M. Kravchuk "Integral operators and their applications to boundary value problems"*, Institute of Mathematics of Academy of Sciences of Ukraine, Kiev, 3 (1994). (in Ukrainian).
6. R.S. Khairullin . *On the Euler — Poisson — Darboux equation theory* // Izv. Vyssh. Uchebn. Zaved., Mat. **1993**:11, 69–76 (1993). [Russ. Math. **37**:11, 67–74 (1993).]
7. A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, H. Bateman. *Higher transcendental functions. Vol. I.* McGraw–Hill Book Co., New York (1953).
8. I.S. Gradshteyn, I.M. Ryzhik. *Table of integrals, series, and products.* Fizmatgiz, Moscow (1962). [Elsevier/Academic Press, Amsterdam (2015).]
9. V.F. Volkodovav, N.Ya. Nikolaev. *On new problem with tranlsation in unbounded domain for Euler — Darboux equation with positive parameters* // In: *Mathematical Physics.* Kyjbyshev Polytechnical Institute, Kujbyshev, 3–9 (1979). (in Russian).

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