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OPERATOR ESTIMATES FOR NON–PERIODIC PERFORATION ALONG BOUNDARY: HOMOGENIZED DIRICHLET CONDITION

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Abstract. We consider a boundary value problem for a second–order elliptic equation with variable coefficients in a multidimensional domain perforated by small cavities along the boundary. We suppose that the sizes of all cavities are of the same order, and their shape and distribution along the boundary can be arbitrary. The cavities are arbitrarily divided into two sets. The Dirichlet condition is imposed on the boundaries of cavities in the first set, and a nonlinear Robin boundary condition is imposed on the boundaries of cavities in the second set. The Neumann condition is imposed on the boundary along which the perforation is arranged. It is assumed that the cavities with the Dirichlet condition are not too small and are located fairly closely. We shown that under such assumptions, the cavities disappear under the homogenization, and the Dirichlet condition arises on the boundary. Our main result is estimates for the difference between the solutions of the homogenized and perturbed problems in the W_2^1–norm uniformly in the L_2–norm of the right hand side.

Key words: perforation along boundary, elliptic operator, operator estimate.

Mathematics Subject Classification: 35B25, 35B27, 35B40

1. Introduction

Boundary value problems in domains perforated along a surface were studied in many papers, see, for example, [1]–[7]. The perforation was described by small cavities located along a given manifold or along the boundary of domain. The sizes of the cavities and the distances between them were governed by one or several small parameters. The studies were aimed on describing the behavior of the solutions to problems as the small parameters tend to zero. The main obtained results were the convergence of the solutions to considered problems in the norms of spaces L_2 or W_2^1 to the solutions of some homogenized problems for fixed right hand sides in the equation and boundary conditions.

In one of interesting formulations of problems with the perforation along the boundary the Dirichlet condition is imposed on the boundaries of cavities, and the Neumann condition on the outer boundary. It is assumed that the cavities are large enough and locate quite frequently. In this case, the condition on the outer boundary changes under the homogenization, namely, the Dirichlet condition arises instead of the Neumann condition. Such problems were considered in [8]–[15], where there was proved the convergence of solutions of perturbed problems to ones of the homogenized in W_2^1–norms for given right hand sides in the equation.

The aforementioned results on the convergence of solutions mean the presence of strong or weak resolvent convergence. A stronger result is the proof of the norm resolvent convergence

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and the corresponding operator estimates. Operator estimates were first obtained for equations with fast oscillating coefficients; the history of this issue is well presented in the surveys [16] and [17]. These works stimulated similar studies for problems of boundary homogenization theory, namely, problems with frequently changing boundary conditions, problems with a fast oscillating boundary, problems with perforation along a given manifold, see [18]–[29]. We also note the works [30]–[32], where operator estimates were obtained for problems with non–periodic perforation over the entire domain.

In the present work, we consider a boundary value problem for a second–order elliptic equation with variable coefficients in a multidimensional domain perforated along the boundary. It is assumed that the sizes of all cavities are of the same order, and their shape and distribution along the boundary are arbitrary. The cavities are arbitrarily divided into two sets. The Dirichlet condition is imposed on the boundaries of cavities in the first set, and a nonlinear Robin boundary condition is imposed on the boundaries of cavities in the second set. The Neumann condition is imposed on the boundary along which the perforation is arranged. Under the homogenization the cavities disappear and the condition on the outer boundary changes, instead of the Neumann condition, the Dirichlet condition arises. The main result of the work is an estimate of the difference between the solutions of the homogenized and perturbed problems in the W_2^1–norm uniformly in the L_2–norm of the right hand side.

2. Formulation of problem and main results

Let $x = (x_1, x_2, \dots, x_n)$ be Cartesian coordinates in $\mathbb{R}^n$, $n \geqslant 3$, and Ω be an arbitrary bounded or unbounded domain in $\mathbb{R}^n$ with a boundary of the class C^2. By S we denote a connected component of boundary Ω. Let ε be a small positive parameter, $\eta = \eta(\varepsilon)$ be some function obeying the inequality $0 < \eta(\varepsilon) \leqslant 1$.

In the domain Ω along S we arbitrarily choose points M_k^ε, $k \in \mathbb{M}^\varepsilon$, where $\mathbb{M}^\varepsilon$ is some at most countable set of indices. We suppose that the chosen points obey the condition

$$\operatorname{dist}(M_k^\varepsilon, S) \leqslant R_0\varepsilon,$$

where R_0 is some positive constant independent of k and ε. We denote by $\omega_{k,\varepsilon}$, $k \in \mathbb{M}^\varepsilon$ bounded domains in $\mathbb{R}^n$ with boundaries of the class C^2 and we let

$$\omega_k^\varepsilon := \big\{x : (x - M_k^\varepsilon)\varepsilon^{-1}\eta^{-1}(\varepsilon) \in \omega_{k,\varepsilon}\big\}, \qquad \theta^\varepsilon := \bigcup_{k\in\mathbb{M}^\varepsilon} \omega_k^\varepsilon, \qquad \Omega^\varepsilon := \Omega \setminus \theta^\varepsilon.$$

We partition the cavities θ^ε into two subsets

$$\theta^\varepsilon = \theta^\varepsilon_{\mathrm{D}} \cup \theta^\varepsilon_{\mathrm{R}}, \qquad \theta^\varepsilon_\natural = \bigcup_{k\in\mathbb{M}^\varepsilon_\natural} \omega_k^\varepsilon, \qquad \natural \in \{\mathrm{D}, \mathrm{R}\},$$

where $\mathbb{M}^\varepsilon_{\mathrm{D}} \cap \mathbb{M}^\varepsilon_{\mathrm{R}} = \emptyset$, $\mathbb{M}^\varepsilon_{\mathrm{D}} \cup \mathbb{M}^\varepsilon_{\mathrm{R}} = \mathbb{M}^\varepsilon$.

Let $A_{ij} = A_{ij}(x)$, $A_i = A_i(x)$, $A_0 = A_0(x)$ be functions defined on Ω and obeying the conditions

$$\begin{gathered} A_{ij} \in W^1_\infty(\Omega), \qquad A_j, A_0 \in L_\infty(\Omega), \qquad A_{ij} = A_{ji}, \qquad i,j = 1,\dots,n, \\ \sum_{i,j=1}^n A_{ij}(x) z_i \overline{z_j} \geqslant c_0 |z|^2, \qquad x \in \Omega, \qquad z = (z_1 \dots, z_n) \in \mathbb{C}^n, \end{gathered}$$

where c_0 is some positive constant independent of x and z. We suppose that the functions A_{ij} are real–valued, while the functions A_j, A_0 are complex–valued. By $a = a(x,u)$ we denote a complex–valued function defined for $u \in \mathbb{C}$ and $x \in \Sigma$, where $\Sigma := \{x : \operatorname{dist}(x, S) \leqslant \tau_0\}$, $\tau_0 > 0$ is some fixed number. We suppose that the function a is piecewise continuous in $(x,u) \in \Sigma \times \mathbb{C}$

and satisfies the conditions

$$|a(x,u_1)-a(x,u_2)|\leqslant a_0|u_1-u_2|,\qquad a(x,0)=0, \tag{2.1}$$

where a_0 is some constant independent of $x\in\Sigma$ and $u_1,u_2\in\mathbb{C}$.

We consider the boundary value problem

$$\begin{aligned}
&\left(-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}A_{ij}\frac{\partial}{\partial x_j}+\sum_{j=1}^{n}A_j\frac{\partial}{\partial x_j}+A_0-\lambda\right)u_\varepsilon=f\quad\text{in}\quad\Omega^\varepsilon,\\
&u_\varepsilon=0\quad\text{on}\quad\partial\Omega\setminus S,\qquad u_\varepsilon=0\quad\text{on}\quad\partial\theta^\varepsilon_{\mathrm{D}},\\
&\frac{\partial u_\varepsilon}{\partial\mathrm{n}}+a(\,\cdot\,,u_\varepsilon)=0\quad\text{on}\quad\partial\theta^\varepsilon_{\mathrm{R}},\qquad\frac{\partial u_\varepsilon}{\partial\mathrm{n}}=0\quad\text{on}\quad S
\end{aligned}\tag{2.2}$$

where f is an arbitrary function in $L_2(\Omega)$, λ is a real number. The conormal derivative is defined by the formula

$$\frac{\partial}{\partial\mathrm{n}}=\sum_{i,j=1}^{n}A_{ij}\nu_i\frac{\partial}{\partial x_j},$$

ν_i is the ith component of the unit normal ν to $\partial\theta^\varepsilon\cup S$ directed outside the domain Ω^ε.

The aim of the work is to study the asymptotic behavior of solution to problem (2.2) as $\varepsilon\to0$.

To formulate main results, we need auxiliary notation and assumption. By τ we denote the distance from a point to S measured along the normal, while by s we denote local variables on S. We make the following assumptions on S and cavities θ^ε.

A1. The variables (τ,s) are well–defined at least on the set Σ. On the same set the Jacobians of the passage from the variables x and the variables (τ,s) and back as well as the derivatives of x in (τ,s) and the derivatives of (τ,s) in x up to the second order are uniformly bounded.

Let $B_r(M)$ be an open ball in $\mathbb{R}^n$ of a radius r centered at a point M.

A2. There exist points $M_{k,\varepsilon}\in\omega_{k,\varepsilon}$, $k\in\mathbb{M}^\varepsilon$, and numbers $0<R_1<R_2$, $b>1$, independent of ε such that for sufficiently small ε the relations

$$\begin{aligned}
&B_{R_1}(M_{k,\varepsilon})\subset\omega_{k,\varepsilon}\subset B_{R_2}(0),\qquad B_{bR_2\varepsilon}(M_k^\varepsilon)\subset\Omega,\qquad k\in\mathbb{M}^\varepsilon,\\
&B_{bR_2\varepsilon}(M_k^\varepsilon)\cap B_{bR_2\varepsilon}(M_i^\varepsilon)=\emptyset,\qquad i,k\in\mathbb{M}^\varepsilon,\quad i\neq k,
\end{aligned}$$

hold. For all k and ε the sets $B_{R_2}(0)\setminus\omega_{k,\varepsilon}$ are connected.

Let ρ be the distance from a point to the boundary $\partial\omega_{k,\varepsilon}$ measured along the outward normal.

A3. There exist fixed constants $\rho_0>0$ and local variables ς on $\partial\omega_{k,\varepsilon}$ such that the variables (ρ,ς) are well–defined at least on the sets

$$\{x:\,\mathrm{dist}(x,\partial\omega_{k,\varepsilon})\leqslant\rho_0\}\setminus\omega_{k,\varepsilon}\subseteq B_{b_*R_2}(0),\qquad b_*:=\frac{b+1}{2},$$

simultaneously for all $k\in\mathbb{M}^\varepsilon$ and on these sets the Jacobians of the passage from the variables x and the variables (ρ,ς) and back as well as the derivatives of x in (ρ,ς) and the derivatives of (ρ,ς) in x up to the second order are uniformly bounded.

A4. There exist numbers $R_3>bR_2$, $0<R_4<R_5$, $R_3<R_5$ such that

$$\theta^\varepsilon\subset\Xi^\varepsilon\subset\bigcup_{k\in\mathbb{M}^\varepsilon_{\mathrm{D}}}B_{R_3\varepsilon}(M_k^\varepsilon)\subset\Omega^\varepsilon,\qquad\Xi^\varepsilon:=\{x:\,R_4\varepsilon<\tau<R_5\varepsilon\}.$$

By $\mathring{W}_2^1(\Omega^\varepsilon,\partial\theta^\varepsilon_{\mathrm{D}}\cup\partial\Omega\setminus S)$ we denote the subspace of the functions in $W_2^1(\Omega)$ vanishing on $\partial\Omega\setminus S$ and $\partial\theta^\varepsilon_{\mathrm{D}}$. A solution to boundary value problem (2.2) is understood in the general sense.

Namely, a solution to boundary value problem (2.2) is a function $u_\varepsilon \in W_2^1(\Omega^\varepsilon)$ satisfying the integral identity

$$\mathfrak{h}_a(u_\varepsilon, v) - \lambda(u_\varepsilon, v)_{L_2(\Omega^\varepsilon)} = (f, v)_{L_2(\Omega^\varepsilon)}$$

for all $v \in \mathring{W}_2^1(\Omega^\varepsilon, \partial\theta_{\mathrm{D}}^\varepsilon \cup \partial\Omega \setminus S)$, where

$$\mathfrak{h}_a(u_\varepsilon, v) := \mathfrak{h}_0(u_\varepsilon, v) + (a(\,\cdot\,, u_\varepsilon), v)_{L_2(\partial\theta_{\mathrm{R}}^\varepsilon)},$$
$$\mathfrak{h}_0(u_\varepsilon, v) := \sum_{i,j=1}^{n} \left(A_{ij}\frac{\partial u_\varepsilon}{\partial x_j}, \frac{\partial v}{\partial x_i}\right)_{L_2(\Omega^\varepsilon)} + \sum_{j=1}^{n}\left(A_j\frac{\partial u_\varepsilon}{\partial x_j}, v\right)_{L_2(\Omega^\varepsilon)} + (A_0 u_\varepsilon, v)_{L_2(\Omega^\varepsilon)}.$$

The integral over the boundary $\partial\theta_{\mathrm{R}}^\varepsilon$ is treated in the sense of the traces. In what follows we shall show that the trace of the function $a(\,\cdot\,, u_\varepsilon)$ on $\partial\theta_{\mathrm{R}}^\varepsilon$ is well–defined, see Lemma 3.8.

We suppose that ε and η are related by the convergence

$$\frac{\varepsilon}{\eta^{n-2}(\varepsilon)} \to +0, \qquad \varepsilon \to +0. \tag{2.3}$$

We consider one more boundary value problem

$$\left(-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}A_{ij}\frac{\partial}{\partial x_j} + \sum_{j=1}^{n}A_j\frac{\partial}{\partial x_j} + A_0 - \lambda\right)u_0 = f \quad \text{in} \quad \Omega, \qquad u_0 = 0 \quad \text{on} \quad \partial\Omega. \tag{2.4}$$

This problem is homogenized for the problem (2.2) under Assumption A4 and the condition (2.3). Its solution is also treated in the generalized sense. A generalized solution to the problem (2.4) is a function $u_0 \in \mathring{W}_2^1(\Omega, \partial\Omega)$ obeying the integral identity

$$\mathfrak{h}_0(u_0, v) - \lambda(u_0, v)_{L_2(\Omega)} = (f, v)_{L_2(\Omega)}$$

for all $v \in \mathring{W}_2^1(\Omega, \partial\Omega)$.

The main result of this work is the following theorem.

Theorem 2.1. *Let Assumptions A1, A2, A3, A4 and the condition (2.3) be satisfied. Then there exists λ_0 independent of ε, η and f such that for $\lambda < \lambda_0$ the problems (2.2), (2.4) are uniquely solvable for all $f \in L_2(\Omega)$ and the inequality*

$$\|u_\varepsilon - u_0\|_{W_2^1(\Omega^\varepsilon)} \leqslant C\left(\frac{\varepsilon}{\eta^{n-2}(\varepsilon)}\right)^{\frac{1}{2}}\|f\|_{L_2(\Omega)} \tag{2.5}$$

holds, which C is a constant independent of ε, η and f but depending on λ.

Let us briefly discuss the problem and results. The equation in the problem (2.2) is a general second order linear elliptic equation with variable coefficients. The perforation is made along the connected component S of the boundary, this component is to be regular enough. A rigorous notion of the regularity is given by Assumption A1.

The perforation along S is made by cavities of arbitrary shapes, their distribution is also arbitrary. This is why this perforation is of a general form and is essentially non–periodic. The requirements for the shapes and the distribution are formulated in Assumptions A2 and A3. Assumption A2 means that the cavities are of approximately same size and are located inside the domain Ω. Assumption A3 states certain uniform regularity of the shapes of cavities, namely, it excludes increasing in k oscillations of their boundaries.

On the boundaries of the cavities we impose the Dirichlet condition or nonlinear Robin boundary condition. The choice of a particular boundary condition for each cavity is arbitrary. The only requirement is the validity of Assumption A4, which means that the cavities with the Dirichlet condition are to be located rather frequently.

The main feature of the problem is that on the boundary S we impose the Neumann condition. Then under the condition (2.3), it turns out that the homogenized problem for (2.2) is the problem (2.4) with the Dirichlet condition on the boundary S instead of the Neumann condition. This is a known phenomenon, which was earlier found in the works [8]–[15], where the perforation was periodic or locally periodic. In this work we show that this phenomenon is preserved also in the case of a non–periodic perforation with the conditions of different types on different cavities. At the same time we succeed to strengthen essential the result on the convergence by proving the uniform in the right hand side f convergence and establishing the operator estimate (2.5).

3. Auxiliary statement

In the present section we provide lemmas, which will be employed in the proof of Theorem 2.1. The first lemma was proved in [30], see Lemma 3.2 in the cited paper.

Lemma 3.1. *Let Assumption A2 be satisfied. Then for all functions*

$$u\in \mathring{W}_2^1(B_{b_*R_2}(0)\setminus\omega_{k,\eta},\partial B_{b_*R_2}(0))$$

the estimate

$$\|u\|_{L_2(B_{b_*R_2}(0)\setminus\omega_{k,\eta})}\leqslant C\|\nabla u\|_{L_2(B_{b_*R_2}(0)\setminus\omega_{k,\eta})},$$

holds, where C is some fixed constant independent of u, k, η and the shapes of cavities $\omega_{k,\eta}$.

Lemma 3.2. *Under Assumptions A2, A3, for all $k\in\mathbb{M}^\varepsilon_{\mathrm{R}}$ and all*

$$u\in \mathring{W}_2^1\big(B_{b_*R_2}(0)\setminus\omega_{k,\eta},\partial B_{b_*R_2}(0)\big)$$

the estimate

$$\|u\|^2_{L_2(\partial\omega_{k,\eta})}\leqslant C\|\nabla u\|^2_{L_2(B_{b_*R_2}(0)\setminus\omega_{k,\eta})}$$

holds, where C is a positive constant independent of the parameters k, ε, η and function u.

The proof of this was given in the work [28], see Lemma 3.2 in the cited work.

We denote $b_\dagger:=(3b+1)/4$.

Lemma 3.3. *Under Assumptions A2, A3, for all $k\in\mathbb{M}^\varepsilon_{\mathrm{R}}$ and all $u\in W_2^1\big(B_{bR_2\varepsilon}(M_k^\varepsilon)\setminus\omega_k^\varepsilon\big)$ the estimate*

$$\|u\|^2_{L_2(\partial\omega_k^\varepsilon)}\leqslant C\Big(\varepsilon\eta\|\nabla u\|^2_{L_2(B_{bR_2\varepsilon}(M_k^\varepsilon)\setminus\omega_k^\varepsilon)}+\varepsilon^{-1}\eta^{n-1}\|u\|^2_{L_2(B_{bR_2\varepsilon}(M_k^\varepsilon)\setminus B_{b_\dagger R_2\varepsilon}(M_k^\varepsilon))}\Big)$$

holds, where C is a positive constant independent of the parameters k, ε, η and function u.

This lemma was proved in work [28], see Lemma 3.3 in the cited work.

Lemma 3.4. *Under Assumptions A1, A2, A3, for each function $u\in\mathring{W}_2^1(\Omega^\varepsilon,\partial\theta^\varepsilon_{\mathrm{D}})$ the estimate*

$$\|u\|^2_{L_2(\partial\theta^\varepsilon_{\mathrm{R}})}\leqslant(C\varepsilon\eta+\delta\eta^{n-1})\|\nabla u\|^2_{L_2(\Omega^\varepsilon)}+C(\delta)\eta^{n-1}\|u\|^2_{L_2(\Omega^\varepsilon)}$$

holds, where $\delta>0$ is an arbitrary constant, while the constants C and $C(\delta)$ are independent of the parameters ε, η, function u, as well as of the shapes and distribution of cavities ω_k^ε, $k\in\mathbb{M}^\varepsilon$.

This lemma was proved in work [28], see Lemma 3.4 in the cited work.

Lemma 3.5. *Under Assumption A1 for each function $u\in W_2^2(\Omega)$ and $|\tau|\leqslant\frac{\tau_0}{3}$ the estimate*

$$|u|^2\leqslant C\tau^2\|u\|^2_{W_2^2(-\frac{\tau_0}{2},\frac{\tau_0}{2})},\qquad |\nabla u|^2\leqslant C\|\nabla u\|^2_{W_2^1(-\frac{\tau_0}{2},\frac{\tau_0}{2})}$$

holds.

This lemma can be proved similarly to Lemma 4.1 in [27].

Lemma 3.6. *Under Assumptions A2, A4, for each point x in Ξ^ε the total number of balls $B_{R_5\varepsilon}(M_k^\varepsilon)$, $R_5:=R_3+(b+1)R_2$, containing this point, does not exceed some absolute constant independent of the point x and parameter ε.*

This lemma was proved in work [28], see Lemma 4.2 in the cited work.

Let $\Pi_\varepsilon := \{x : 0 < \tau < 2R_6\varepsilon\}$, where $R_6 > 0$ is some constant.

Lemma 3.7. *Under Assumptions A1, A2, A4 for each function $u \in \mathring{W}_2^1(\Omega^\varepsilon, \partial\Omega \setminus S \cup \theta_{\mathrm{D}}^\varepsilon)$ the estimate*

$$\|u\|^2_{L_2(\Pi_\varepsilon)} \leqslant C\varepsilon^2\eta^{-n+2}\|\nabla u\|^2_{L_2(\Omega^\varepsilon)}$$

holds, where C is a constant independent of the function u, parameters ε and η, the shapes and location of cavities ω_k^ε, $k \in \mathbb{M}^\varepsilon$.

Proof. Throughout the proof by C we denote various inessential constants independent of u, ε, η, the shapes and distribution of cavities ω_k^ε. We continue the function u be zero inside the cavities $\theta_{\mathrm{D}}^\varepsilon$. By $\mathbb{M}_k^\varepsilon$ we denote the set of indices $j \in \mathbb{M}_{\mathrm{R}}^\varepsilon$ such that

$$\overline{B_{R_3\varepsilon}(M_k^\varepsilon)} \cap \overline{B_{R_2}(M_j^\varepsilon)} \neq \emptyset.$$

It was shown in [30, Sect. 3.2] that under Assumotions A1 and A2 the function u can be continued inside the cavities $\theta_{\mathrm{R}}^\varepsilon$ and the estimates

$$\|u\|^2_{L_2(\omega_{k,\varepsilon})} \leqslant C\|u\|^2_{L_2(B_{R_3\varepsilon\eta}(M_k^\varepsilon)\setminus\omega_{k,\varepsilon})}, \qquad \|\nabla u\|^2_{L_2(\omega_{k,\varepsilon})} \leqslant C\|\nabla u\|^2_{L_2(B_{R_3\varepsilon\eta}(M_k^\varepsilon)\setminus\omega_{k,\varepsilon})}$$

hold, where C is some constant independent of u, ε, η and k.

According to Assumption A4, the balls $B_{R_3\varepsilon}(M_k)$, $k \in \mathbb{M}_{\mathrm{D}}^\varepsilon$ cover the layer Ξ^ε. By Lemma 3.6, each point of the layer Ξ^ε is contained in finitely many sets $B_{R_3\varepsilon}(M_k)$ and their total number is bounded by some absolute constant uniformly in ε, η and the points in the layer. We also note that the dilatation of the introduced set by ε^{-1} with respect to the points M_k^ε gives the ball $B_{R_3}(0)$. Then by means of the change of variables corresponding to such dilatation we obtain the estimate

$$\|u\|^2_{L_2(B_{R_3\varepsilon\eta}(M_k))} \leqslant C\varepsilon^2\eta^{-n+2}\|\nabla v\|^2_{L_2(B_{R_3\varepsilon\eta}(M_k))}.$$

Summing up the obtained inequalities over all $k \in \mathbb{M}_{\mathrm{D}}^\varepsilon$ and taking into consideration the aforementioned properties of the covering of layer Ξ^ε by the sets $B_{R_3\varepsilon\eta}(M_k)$, $k \in \mathbb{M}_{\mathrm{D}}^\varepsilon$, we arrive at the estimate

$$\|u\|^2_{L_2(\Xi^\varepsilon\setminus\theta^\varepsilon)} \leqslant C\varepsilon^2\eta^{-n+2}\|\nabla u\|^2_{L_2(\Omega^\varepsilon)}. \tag{3.1}$$

Let $\chi = \chi(t)$ be an infinitely differentiable cut off function, which is equal to one for $t < R_7$ and vanishes for $t > R_5$, where R_7 is some constant, and $R_4 < R_7 < R_5$. The identity

$$u(x) = \int\limits_{R_7\varepsilon}^{\tau} \frac{\partial}{\partial t}u(t,s)\chi\left(\frac{t}{\varepsilon}\right)\,dt$$

holds. By the Cauchy — Schwarz inequality this identity implies

$$|u(x)|^2 \leqslant C\left(\varepsilon^{-1}\int\limits_{R_7\varepsilon}^{R_5\varepsilon}|u(\tau,s)|^2\,d\tau + \varepsilon\int\limits_0^{R_6\varepsilon}\left|\frac{\partial u}{\partial\tau}(\tau,s)\right|^2\,dt\right).$$

Integrating this estimate over Π^ε and taking into consideration the inequality (3.1), we complete the proof. $\square$

Lemma 3.8. *For an arbitrary function $u \in \mathring{W}_2^1(\Omega, \partial\Omega \setminus S)$ the function $a(x, u(x))$ possesses a trace on θ^ε, which is an element of $L_2(\partial\theta^\varepsilon)$.*

Proof. Since $u \in W_2^1(\Omega)$, there exists a sequence of functions $u_n \in C^\infty(\overline{\Omega})$, $n = 1, 2, 3\ldots$, which converges to the function u in $W_2^1(\Omega)$–norm. The estimate

$$\|u_n - u_m\|_{L_2(S)} \leqslant C\|u_n - u_m\|_{W_2^1(\Omega)} \tag{3.2}$$

holds, where the constant C is independent of n and m. The condition (2.1) implies the inequalities

$$|a(x,u_n)| \leqslant C|u_n|, \qquad |a(x,u_n)-a(x,u_m)|^2 \leqslant C|u_n-u_m|^2, \tag{3.3}$$

where a constant C is independent of n and m. Since the function u_n is integrable on S, the first inequality in (3.3) and piecewise continuity of $a(x,u))$ implies that the function $a(x,u_n(x))$ is also integrable and belongs to $L_2(S)$. Integrating the second estimate in (3.3) over S and taking into consideration the inequality (3.2), we get

$$\|a(\,\cdot\,,u_n)-a(\,\cdot\,,u_m)\|^2_{L_2(S)} \leqslant C\|u_n-u_m\|^2_{W_2^1(\Omega)},$$

where a constant C is independent of n and m. The right hand side of the latter inequality tends to zero. This means that the sequence $a(x,u_n(x))$ is fundamental in $L_2(S)$. Since the space $L_2(S)$ is complete, the sequence $a(x,u_n(x))$ converges to some limit in $L_2(S)$.

In a standard way, see [33, Sect. 5, Subsect. 1], we show that this limit is independent on the choice of the sequence u_n and exactly this limit is called the trace of the function $a(x,u(x))$ on S. The proof is complete. $\square$

4. Convergence of solutions

In this section we prove Theorem 2.1.

Lemma 4.1. *There exists λ_0 such that for $\lambda<\lambda_0$ the problem (2.2) possess a unique solution $u_\varepsilon \in W_2^1(\Omega^\varepsilon)$ for all ε and $f\in L_2(\Omega)$.*

The proof of this lemma is similar to that of [28, Lm. 5.1], [29, Lm. 9].

Lemma 4.2. *The estimate*

$$\|u_0\|_{W_2^2(\Omega)} \leqslant C\|f\|_{L_2(\Omega)} \tag{4.1}$$

holds, where C is a constant independent of f.

Proof. The problem (2.4) is uniquely solvable in $\mathring{W}_2^1(\Omega,\partial\Omega\setminus S)$ for an arbitrary right hand f of the equation, and the equation is linear. This is why the estimate

$$\|u_0\|_{W_2^1(\Omega)} \leqslant C\|f\|_{L_2(\Omega)}$$

holds, where C is a constant independent of f. Using standard theorems on smoothness improving of solutions to elliptic boundary value problems, we obtain the estimate (4.1). The proof is complete. $\square$

Let $\chi_1=\chi_1(t)$ be an infinitely differentiable cut off function, which is equal to one for $t<1$ and vanishes for $t>2$. We consider the function $v_\varepsilon = u_\varepsilon-(1-\chi^\varepsilon)u_0$, where χ^ε is defined as

$$\chi^\varepsilon(x)=\begin{cases}\chi_1\left(\frac{|\tau|}{R_6\varepsilon}\right) & \text{for} \quad x\in\Sigma,\\ 0 & \text{for} \quad x\in\Omega\setminus\Sigma.\end{cases}$$

This function belongs to the space $\mathring{W}_2^1(\Omega^\varepsilon,\partial\theta^\varepsilon_{\mathrm{D}}\cup\partial\Omega\setminus S)$ and the identity

$$v_\varepsilon = u_\varepsilon \qquad \text{on} \qquad \partial\theta^\varepsilon \tag{4.2}$$

holds.

We write the integral identity for the problem (2.2) with the test function v_ε

$$\begin{aligned}&\sum_{i,j=1}^{n}\left(A_{ij}\frac{\partial u_\varepsilon}{\partial x_j},\frac{\partial v_\varepsilon}{\partial x_i}\right)_{L_2(\Omega^\varepsilon)}+\sum_{j=1}^{n}\left(A_j\frac{\partial u_\varepsilon}{\partial x_j},v_\varepsilon\right)_{L_2(\Omega^\varepsilon)}\\&+\sum_{j=1}^{n}\left(u_\varepsilon,A_j\frac{\partial v_\varepsilon}{\partial x_j}\right)_{L_2(\Omega^\varepsilon)}+(A_0u_\varepsilon,v_\varepsilon)_{L_2(\Omega^\varepsilon)}\\&-\lambda(u_\varepsilon,v_\varepsilon)_{L_2(\Omega^\varepsilon)}+(a(\cdot,u_\varepsilon),v_\varepsilon)_{L_2(\partial\theta^\varepsilon_{\mathrm{R}})}=(f,v_\varepsilon)_{L_2(\Omega^\varepsilon)}.\end{aligned}\tag{4.3}$$

It follows from the identity (4.2) that the boundary term in the left hand side of the above identity can be rewritten as

$$(a(\cdot,u_\varepsilon),v_\varepsilon)_{L_2(\partial\theta^\varepsilon_{\mathrm{R}})}=(a(\cdot,v_\varepsilon),v_\varepsilon)_{L_2(\partial\theta^\varepsilon_{\mathrm{R}})}.$$

We continue the function $(1-\chi^\varepsilon)v_\varepsilon$ by zero inside the set θ^ε. We write the integral identity for the problem (2.4) taking $(1-\chi^\varepsilon)v_\varepsilon\in\mathring{W}^1_2(\Omega,\partial\Omega\setminus S)$ as the test function. As a result, we obtain

$$\begin{aligned}&\sum_{i,j=1}^{n}\left(A_{ij}\frac{\partial(1-\chi^\varepsilon)u_0}{\partial x_j},\frac{\partial v_\varepsilon}{\partial x_i}\right)_{L_2(\Omega)}+\sum_{j=1}^{n}\left(A_j\frac{\partial(1-\chi^\varepsilon)u_0}{\partial x_j},v_\varepsilon\right)_{L_2(\Omega)}\\&+\sum_{j=1}^{n}\left((1-\chi^\varepsilon)u_0,A_j\frac{\partial v_\varepsilon}{\partial x_j}\right)_{L_2(\Omega)}+(A_0u_0(1-\chi^\varepsilon),v_\varepsilon)_{L_2(\Omega)}\\&+\lambda(u_0(1-\chi^\varepsilon),v_\varepsilon)_{L_2(\Omega)}=(f(1-\chi^\varepsilon),v_\varepsilon)_{L_2(\Omega)}-K_\varepsilon,\end{aligned}\tag{4.4}$$

where we have denoted

$$\begin{aligned}K_\varepsilon:=&-\sum_{i,j=1}^{n}\left(A_{ij}\frac{\partial u_0}{\partial x_j}\frac{\partial\chi^\varepsilon}{\partial x_i},v_\varepsilon\right)_{L_2(\Omega^\varepsilon)}+\sum_{i,j=1}^{n}\left(A_{ij}u_0\frac{\partial\chi^\varepsilon}{\partial x_j},\frac{\partial v_\varepsilon}{\partial x_i}\right)_{L_2(\Omega^\varepsilon)}\\&+\sum_{j=1}^{n}\left(A_ju_0\frac{\partial\chi^\varepsilon}{\partial x_j},v_\varepsilon\right)_{L_2(\Omega^\varepsilon)}-\sum_{j=1}^{n}\left(u_0\frac{\partial\chi^\varepsilon}{\partial x_j},A_jv_\varepsilon\right)_{L_2(\Omega^\varepsilon)}.\end{aligned}$$

We calculate the difference of (4.3) and (4.4)

$$\begin{aligned}&\sum_{i,j=1}^{n}\left(A_{ij}\frac{\partial v_\varepsilon}{\partial x_j},\frac{\partial v_\varepsilon}{\partial x_i}\right)_{L_2(\Omega^\varepsilon)}+\sum_{j=1}^{n}\left(A_j\frac{\partial v_\varepsilon}{\partial x_j},v_\varepsilon\right)_{L_2(\Omega^\varepsilon)}\\&+\sum_{j=1}^{n}\left(v_\varepsilon,A_j\frac{\partial v_\varepsilon}{\partial x_j}\right)_{L_2(\Omega^\varepsilon)}+(A_0v_\varepsilon,v_\varepsilon)_{L_2(\Omega^\varepsilon)}\\&+(a(\cdot,v_\varepsilon),v_\varepsilon)_{L_2(\partial\theta^\varepsilon_{\mathrm{R}})}+\lambda(v_\varepsilon,v_\varepsilon)_{L_2(\Omega^\varepsilon)}=(\chi^\varepsilon f,v_\varepsilon)_{L_2(\Omega^\varepsilon)}+K_\varepsilon.\end{aligned}\tag{4.5}$$

Similarly to [28, Ineq. (2.7)] we get the estimate

$$\begin{aligned}&\sum_{i,j=1}^{n}\left(A_{ij}\frac{\partial v_\varepsilon}{\partial x_j},\frac{\partial v_\varepsilon}{\partial x_i}\right)_{L_2(\Omega^\varepsilon)}+\sum_{j=1}^{n}\left(A_j\frac{\partial v_\varepsilon}{\partial x_j},v_\varepsilon\right)_{L_2(\Omega^\varepsilon)}\\&+\sum_{j=1}^{n}\left(v_\varepsilon,A_j\frac{\partial v_\varepsilon}{\partial x_j}\right)_{L_2(\Omega^\varepsilon)}+(A_0v_\varepsilon,v_\varepsilon)_{L_2(\Omega^\varepsilon)}\\&+(a(\cdot,v_\varepsilon),v_\varepsilon)_{L_2(\partial\theta^\varepsilon_{\mathrm{R}})}+\lambda(v_\varepsilon,v_\varepsilon)_{L_2(\Omega^\varepsilon)}\geqslant C\|v_\varepsilon\|^2_{W^1_2(\Omega^\varepsilon)},\end{aligned}\tag{4.6}$$

where C is a constant independent of v_ε.

Our further aim is to estimate the right hand side of the identity (4.5). All further calculation follow the scheme of the proof of Theorem 2.1 in [28]. We reproduce the main milestones of these calculations.

Applying Lemma 3.7, we estimate the first term in the right hand side of the identity (4.5)

$$|(\chi_1^\varepsilon f, v_\varepsilon)_{L_2(\Omega^\varepsilon)}| \leqslant C\frac{\varepsilon}{\eta^{\frac{n-2}{2}}}\|f\|_{L_2(\Omega)}\|v_\varepsilon\|_{W_2^1(\Omega^\varepsilon)}. \tag{4.7}$$

Lemmas 3.5, 3.7 and inequalities (4.1) imply the estimates

$$\left|\sum_{i,j=1}^{n}\left(A_{ij}\frac{\partial u_0}{\partial x_j}\frac{\partial \chi_1^\varepsilon}{\partial x_j}, v_\varepsilon\right)_{L_2(\Omega^\varepsilon)}\right| \leqslant C\frac{\varepsilon^{\frac{1}{2}}}{\eta^{\frac{n-2}{2}}}\|f\|_{L_2(\Omega)}\|v_\varepsilon\|_{W_2^1(\Omega^\varepsilon)}. \tag{4.8}$$

$$\left|\sum_{i,j=1}^{n}\left(A_{ij}u_0\frac{\partial \chi_1^\varepsilon}{\partial x_i}, \frac{\partial v_\varepsilon}{\partial x_j}\right)_{L_2(\Omega^\varepsilon)}\right| \leqslant C\varepsilon^{\frac{1}{2}}\|f\|_{L_2(\Omega)}\|v_\varepsilon\|_{W_2^1(\Omega^\varepsilon)}, \tag{4.9}$$

$$\begin{aligned}\left|\sum_{j=1}^{n}\left(A_j u_0\frac{\partial \chi_1^\varepsilon}{\partial x_j}, v_\varepsilon\right)_{L_2(\Omega^\varepsilon)} - \sum_{j=1}^{n}\left(u_0\frac{\partial \chi_1^\varepsilon}{\partial x_j}, A_j v_\varepsilon\right)_{L_2(\Omega^\varepsilon)}\right|& \\ \leqslant C\frac{\varepsilon^{\frac{3}{2}}}{\eta^{\frac{n-2}{2}}}\|f\|_{L_2(\Omega)}\|v_\varepsilon\|_{W_2^1(\Omega^\varepsilon)}.&\end{aligned} \tag{4.10}$$

Using inequalities (4.6), (4.7), (4.8), (4.9) and (4.10), we obtain the estimate for v_ε

$$\|v_\varepsilon\|_{W_2^1(\Omega^\varepsilon)} \leqslant C\left(\frac{\varepsilon}{\eta^{n-2}}\right)^{\frac{1}{2}}\|f\|_{L_2(\Omega)}. \tag{4.11}$$

Now we are going to estimate the norm of the difference $u_\varepsilon - u_0$. We represent this difference as

$$u_\varepsilon - u_0 = u_\varepsilon - (1-\chi^\varepsilon)u_0 + u_0\chi^\varepsilon = v_\varepsilon + u_0\chi^\varepsilon.$$

Lemma 3.5 and the inequality (4.1) imply the estimate for $u_0\chi_1^\varepsilon$

$$\|u_0\chi_1^\varepsilon\|_{L_2(\Omega^\varepsilon)} \leqslant C\varepsilon^{\frac{3}{2}}\|f\|_{L_2(\Omega)}. \tag{4.12}$$

In the same way we get the estimate for $\nabla(u_0\chi^\varepsilon)$

$$\|\nabla u_0\chi^\varepsilon\|_{L_2(\Omega^\varepsilon)} \leqslant C(\|\nabla u_0\|_{L_2(\Omega^\varepsilon)} + \varepsilon^{-1}\|u_0\|_{L_2(\Omega^\varepsilon)}) \leqslant C\varepsilon^{\frac{1}{2}}\|f\|_{L_2(\Omega)}. \tag{4.13}$$

Using inequalities (4.11), (4.12) and (4.13), we get the estimate (2.5). The proof of Theorem 2.1 is complete.

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