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EMBEDDING THEOREMS FOR SUBSPACES IN SPACES OF FAST DECAYING FUNCTIONS

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Abstract. By means of the family $\mathfrak{M} = \{M_\nu\}_{\nu=1}^\infty$ of separately radial convex functions $M_\nu : \mathbb{R}^n \to \mathbb{R}$ we define the space $GS(\mathfrak{M})$ of type W_M, which is a natural generalization of the space W_M introduced in works by B.L. Gurevich, I.M. Gelfand, and G.E. Shilov. By a certain rule, each function M_ν is associated with a non–negative separately radial convex function h_ν in $\mathbb{R}^n$. The properties of the functions h_ν allows one to form, by the family $\mathcal{H} = \{h_\nu\}_{\nu=1}^\infty$, the space $\mathbb{S}_\mathcal{H}$, which is the inner inductive limit of countably–normed spaces $\mathbb{S}(h_\nu)$ of the functions $f \in C^\infty(\mathbb{R}^n)$ with the finite norms

$$\|f\|_{m,\nu} = \sup_{\substack{x\in\mathbb{R}^n,\beta\in\mathbb{Z}_+^n,\\ \alpha\in\mathbb{Z}_+^n:\|\alpha\|\leqslant m}} \frac{\|x^\beta (D^\alpha f)(x)\|}{\beta! e^{-h_\nu(\beta)}}, \qquad m \in \mathbb{Z}_+.$$

We consider the problem on finding conditions on $\mathfrak{M}$, which ensure continuous embedding of the spaces $GS(\mathfrak{M})$ and $\mathbb{S}_\mathcal{H}$ one to the other.

Key words: Gelfand — Shilov space of type W_M, convex functions.

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1. Introduction

1.1. Aim of work. Let $\mathfrak{M} = \{M_\nu\}_{\nu=1}^\infty$ be a family of separately radial convex functions $M_\nu : \mathbb{R}^n \to \mathbb{R}$ such that for each $\nu \in \mathbb{N}$

j_1) $\displaystyle\lim_{x\to\infty} \frac{M_\nu(x)}{\|x\|} = +\infty$;

j_2) $\displaystyle\lim_{x\to\infty} (M_\nu(x) - M_{\nu+1}(x)) = +\infty$.

For each $\nu \in \mathbb{N}$ and $m \in \mathbb{Z}_+$ we define the space

$$GS_m(M_\nu) = \left\{ f \in C^m(\mathbb{R}^n) : q_{m,\nu}(f) = \sup_{\substack{x\in\mathbb{R}^n,\\ \|\alpha\|\leqslant m}} \|(D^\alpha f)(x)\| e^{M_\nu(x)} < \infty \right\}.$$

We let

$$GS(M_\nu) = \bigcap_{m\in\mathbb{Z}_+} GS_m(M_\nu).$$

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We equip $GS(M_\nu)$ with the topology defined by the family of norms $q_{m,\nu}$ $(m \in \mathbb{Z}_+)$ and introduce the space

$$GS(\mathfrak{M}) = \bigcup_{\nu \in \mathbb{N}} GS(M_\nu).$$

Being equipped with usual summation and multiplication by the complex numbers, $GS(\mathfrak{M})$ is a linear space. In $GS(\mathfrak{M})$ we define the topology of inner inductive limit of the spaces $GS(M_\nu)$. We note that the space $GS(\mathfrak{M})$ is constructively more general that the space W_M [1]–[5], and the space of type W_M from [6].

By the family $\mathfrak{M}$ we form one more family of non–negative separately radial convex functions h_ν in $\mathbb{R}^n$. First we recall that the Young — Fenchel transform g^* of a function $g : \mathbb{R}^n \to [-\infty, +\infty]$ is the function $g^* : \mathbb{R}^n \to [-\infty, +\infty]$ defined by the rule [7]

$$g^*(x) = \sup_{y \in \mathbb{R}^n} (\langle x, y \rangle - g(y)).$$

It will be convenient to employ the following notation: if u is a function on a set $X \subset \mathbb{R}^n$ containing $(0, \infty)^n$, then $u[e](x) := u(e^{x_1}, \ldots, e^{x_n})$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Now for each $\nu \in \mathbb{N}$ we define the functions u_ν on $\mathbb{R}^n_+$ and h_ν on $\mathbb{R}^n$ be letting

$$u_\nu(t) = \sup_{y \in \mathbb{R}^n_+} (\langle t, y \rangle - M^*_\nu[e](y)), \qquad t \in \mathbb{R}^n_+,$$
$$h_\nu(t) = u_\nu(\|t_1\|, \ldots, \|t_n\|) - u_\nu(0), \qquad t = (t_1, \ldots, t_n) \in \mathbb{R}^n.$$

Due to Condition j_1) the functions u_ν and h_ν take finite values on $\mathbb{R}^n$ and

$$\lim_{x \to \infty} \frac{h_\nu(x)}{\|x\|} = +\infty,$$

while Conditions j_1) and j_2) yield

$$\lim_{y \to +\infty} (M^*_{\nu+1}[e](y) - M^*_\nu[e](y)) = +\infty,$$

and in it turn, this implies

$$\lim_{x \to \infty} (h_\nu(x) - h_{\nu+1}(x)) = +\infty.$$

It is easy to verify that for each $Q > 0$ there exists a number $C_Q > 0$ such that

$$h_\nu(x) \leqslant \sum_{1 \leqslant j \leqslant n : x_j \neq 0} x_j \ln \frac{x_j}{Q} + C_Q, \qquad x = (x_1, \ldots, x_n) \in [0, \infty)^n.$$

Moreover, since the function u_ν is convex and non–decreasing in each variable in $\mathbb{R}^n_+$, the function h_ν is convex in $\mathbb{R}^n$, see, for instance, [8, Lm. 4]. It is obvious that $h_\nu \in C(\mathbb{R}^n)$. We form the family $\mathcal{H} = \{h_\nu\}_{\nu=1}^\infty$.

By the family $\mathcal{H}$ we define the space $\mathbb{S}_\mathcal{H}$ as the inner inductive limit of countably–normed spaces $\mathbb{S}(h_\nu)$, each being the projective limit of the spaces

$$\mathcal{S}_m(h_\nu) = \left\{ f \in C^m(\mathbb{R}^n) : \|f\|_{m,\nu} = \sup_{\substack{x \in \mathbb{R}^n, \beta \in \mathbb{Z}^n_+, \\ \alpha \in \mathbb{Z}^n_+ : \|\alpha\| \leqslant m}} \frac{\|x^\beta (D^\alpha f)(x)\|}{\beta! e^{-h_\nu(\beta)}} < \infty \right\}, \qquad m \in \mathbb{Z}_+.$$

The spaces of form $\mathbb{S}_\mathcal{H}$ were considered in the work [9].

The aim of this note is to find conditions for $\mathfrak{M}$, which ensure continuous embedding of the spaces $GS(\mathfrak{M})$ and $\mathbb{S}_\mathcal{H}$ one to the other. The study of this problem can be interesting for the embedding theory of spaces of differentiable functions.

1.2. Results. In the second section, by using auxiliary statements from the first section, we prove the following two results.

Theorem 1.1. *The space* $GS(\mathfrak{M})$ *is continuously embedded into* $\mathbb{S}_{\mathcal{H}}$.

Theorem 1.2. *Let the functions in the family* $\mathfrak{M}$ *be such that for each* $\nu \in \mathbb{N}$

1) *for some* $a_\nu > 0$

$$M^*_{\nu+1}(x) - M^*_\nu(x) \geqslant \sum_{j=1}^{n} \ln x_j - a_\nu, \qquad x = (x_1, \dots, x_n) \in [1, \infty)^n;$$

2) *for some* $b_\nu > 0$

$$M_\nu(x) - M_{\nu+1}(x) \geqslant \sum_{j=1}^{n} \|x_j\| - b_\nu, \qquad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Then the space $\mathbb{S}_{\mathcal{H}}$ *is continuously embedded into* $GS(\mathfrak{M})$.

Thus, under the assumptions of Theorem 1.2 the spaces $\mathbb{S}_{\mathcal{H}}$ and $GS(\mathfrak{M})$ coincide.

Remark 1.1. *The most essential part of Theorem 4 in* [8] *corresponds to a particular case of Theorem 1.2 when the functions in the family* $\mathfrak{M} = \{M_\nu\}_{\nu=1}^{\infty}$ *satisfy the condition: for each* $\nu \in \mathbb{N}$ *there exists a number* $C_\nu > 0$ *such that*

$$M_{\nu+1}(2x) \leqslant M_\nu(x) + C_\nu, \quad x \in \mathbb{R}^n, \quad \nu \in \mathbb{N}.$$

A condition of such kind is typical for all earlier studied spaces of type W_M. *It is also easy to show that in this case for some* $K_\nu > 0$

$$h_\nu(x) - h_{\nu+1}(x) \geqslant \ln 2 \sum_{j=1}^{n} x_j - K_\nu, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n_+,$$

$$h_{\nu+1}(x + y) \leqslant h_\nu(x) + h_\nu(y) + K_\nu, \quad x, y \in \mathbb{R}^n_+.$$

1.3. Notation. $\mathbb{R}^n_+ := [0, \infty)^n$. For $t \geqslant 0$ we let $t^+ = \max(t, 1)$.
For $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_+$ we let

$$\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n,$$

$\|x\|$ is the Euclidean norm x,

$$\|\alpha\| = \alpha_1 + \ldots + \alpha_n, \qquad \alpha! = \alpha_1! \cdots \alpha_n!, \qquad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

By $U(\mathbb{R}^n)$ we denote the set of all separately radial convex functions $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{x \to \infty} \frac{u(x)}{\|x\|} = +\infty.$$

2. Auxiliary results

In the proofs of Theorems 1.1 and 1.2 we shall need the following statements.

Proposition 2.1. *Let* $g = (g_1, \dots, g_n)$ *be a vector function in* $\mathbb{R}^n$ *with convex components* $g_j : \mathbb{R}^n \to [0, \infty)$ *and a function* $f : \mathbb{R}^n \to \mathbb{R}$ *be such that* $f_{|[0,\infty)^n}$ *is convex and non–decreasing in each variable. Then* $f \circ g$ *is convex in* $\mathbb{R}^n$.

The proof can be found in [8].

Proposition 2.2. *Let $u \in U(\mathbb{R}^n)$. Then*

$$(u[e])^*(x) + (u^*[e])^*(x) = \sum_{\substack{1\leqslant j\leqslant n:\\ x_j\neq 0}} (x_j \ln x_j - x_j), \quad x = (x_1, \ldots, x_n) \in [0, \infty)^n \setminus \{0\};$$

$$(u[e])^*(0) + (u^*[e])^*(0) = 0.$$

For the functions $u \in U(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ this proposition was proved in [8], while the general case was proved in [10].

Proposition 2.3. *Suppose that for some $a_\nu > 0$ ($v \in \mathbb{N}$)*

$$M^*_{\nu+1}(x) - M^*_\nu(x) \geqslant \sum_{j=1}^{n} \ln x_j - a_\nu, \quad x = (x_1, \ldots, x_n) \in [1, \infty)^n.$$

Then

$$h_{\nu+1}(x + y) \leqslant h_\nu(x) + c_\nu, \quad x \in \mathbb{R}^n_+, y \in [0, 1])^n, \tag{2.1}$$

where $c_\nu = u_\nu(0) - u_{\nu+1}(0) + a_\nu$.

Proof. Let $x \in \mathbb{R}^n_+, y \in [0, 1])^n$. Then

$$\begin{aligned} u_{\nu+1}(x + y) &= \sup_{t\in\mathbb{R}^n_+} (\langle x + y, t\rangle - M^*_{\nu+1}[e](t)) \\ &= \sup_{t\in\mathbb{R}^n_+} (\langle x, t\rangle - (M^*_{\nu+1}[e](t) - M^*_\nu[e](t)) + \langle y, t\rangle - M^*_\nu[e](t)) \\ &\leqslant \sup_{t\in\mathbb{R}^n_+} (\langle x, t\rangle - M^*_\nu[e](t)) + a_\nu = u_\nu(x) + a_\nu. \end{aligned}$$

This implies the inequality (2.1). □

Proposition 2.4. *Suppose that for each $\nu \in \mathbb{N}$ for some $b_\nu > 0$*

$$M_\nu(x) - M_{\nu+1}(x) \geqslant \sum_{j=1}^{n} \|x_j\| - b_\nu, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Then for all $t \in \mathbb{R}^n_+$

$$(M^*_{\nu+1}[e])^*(t) \leqslant h_\nu(t) + d_\nu, \quad t = (t_1, \ldots, t_n) \in \mathbb{R}^n_+, \tag{2.2}$$

where $d_\nu = u_\nu(0) + b_\nu$.

Proof. Using the separate radiality of the functions M_ν and the assumptions, we have

$$M^*_{\nu+1}(x) \geqslant M^*_\nu(x + y) - b_\nu, \qquad x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+, \qquad y \in [0, 1]^n.$$

Then for all $t = (t_1, \ldots, t_n) \in \mathbb{R}^n_+$

$$\begin{aligned} (M^*_{\nu+1}[e])^*(t) &= \max\Big(\sup_{y\in\mathbb{R}^n_+} (\langle t, y\rangle - M^*_{\nu+1}[e])(y)), \sup_{y\in\mathbb{R}^n\setminus\mathbb{R}^n_+} (\langle t, y\rangle - M^*_{\nu+1}[e](y))\Big) \\ &\leqslant \max\Big(\sup_{y\in\mathbb{R}^n_+} (\langle t, y\rangle - M^*_{\nu+1}[e])(y)), \sup_{y\in\mathbb{R}^n_+} (\langle t, y\rangle - M^*_\nu[e])(y) + b_\nu)\Big) \\ &\leqslant \sup_{y\in\mathbb{R}^n_+} (\langle t, y\rangle - M^*_\nu[e])(y)) + b_\nu = u_\nu(t) + b_\nu = h_\nu(t) + d_\nu. \end{aligned}$$

The proof is complete. □

3. Proof of Theorem 1.1

Let $f \in GS(\mathfrak{M})$. Then $f \in GS(M_\nu)$ for some $\nu \in \mathbb{N}$. Let $m \in \mathbb{Z}_+$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ be an arbitrary point with non–zero coordinates. Then for $\alpha \in \mathbb{Z}_+^n$ with $\|\alpha\| \leqslant m$ we have

$$\|(D^\alpha f)(x)\| \leqslant q_{m,\nu}(f) e^{-M_\nu[e](\ln\|x_1\|, \ldots, \ln\|x_n\|)}.$$

Since the function $M_\nu[e]$ with finite values in $\mathbb{R}^n$ is convex on $\mathbb{R}^n$, we have $M_\nu[e] = ((M_\nu[e])^*)^*$. This is why the previous inequality implies

$$\|(D^\alpha f)(x)\| \leqslant q_{m,\nu}(f) e^{-\sum\limits_{j=1}^n t_j \ln\|x_j\| + (M_\nu[e])^*(t)}, \qquad t = (t_1, \ldots, t_n) \in \mathbb{R}_+^n.$$

By Proposition 2.2 this implies that for all $t = (t_1, \ldots, t_n) \in \mathbb{R}_+^n$

$$\|(D^\alpha f)(x)\| \leqslant q_{m,\nu}(f) e^{-\sum\limits_{j=1}^n t_j \ln\|x_j\| + \sum\limits_{1 \leqslant j \leqslant n : t_j \neq 0} (t_j \ln t_j - t_j) - (M_\nu^*[e])^*(t)}. \tag{3.1}$$

Since for $t \in \mathbb{R}_+^n$

$$(M_\nu^*[e])^*(t) \geqslant \sup_{y \in \mathbb{R}_+^n} (\langle t, y\rangle - M_\nu^*[e])(y)) = u_\nu(t) = h_\nu(t) + u_\nu(0),$$

continuing estimating in (3.1), we obtain that for all $t = (t_1, \ldots, t_n) \in \mathbb{R}_+^n$

$$\|(D^\alpha f)(x)\| \leqslant e^{-u_\nu(0)} q_{m,\nu}(f) e^{-\sum\limits_{j=1}^n t_j \ln\|x_j\| + \sum\limits_{1 \leqslant j \leqslant n : t_j \neq 0} (t_j \ln t_j - t_j) - h_\nu(t)}.$$

In particular, for all $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n$

$$\|x^\beta (D^\alpha f)(x)\| \leqslant e^{-u_\nu(0)} q_{m,\nu}(f) e^{-h_\nu(\beta)} \prod_{1 \leqslant j \leqslant n : \beta_j \neq 0} \frac{\beta_j^{\beta_j}}{e^{\beta_j}}.$$

This inequality is obviously true for each $x \in \mathbb{R}^n$. This implies that for all $x \in \mathbb{R}^n$, $\alpha \in \mathbb{Z}_+^n$ with $\|\alpha\| \leqslant m$ and $\beta \in \mathbb{Z}_+^n$

$$\|x^\beta (D^\alpha f)(x)\| \leqslant e^{-u_\nu(0)} q_{m,\nu}(f) \beta! e^{-h_\nu(\beta)}.$$

Therefore,

$$\|f\|_{m,\nu} \leqslant e^{-u_\nu(0)} q_{m,\nu}(f).$$

Since $m \in \mathbb{Z}_+$ was arbitrary, we have $f \in \mathbb{S}(h_\nu)$. Hence, $f \in \mathbb{S}_\mathcal{H}$. The latter inequality also implies the continuity of embedding of the space $GS(\mathfrak{M})$ into the space $\mathbb{S}_\mathcal{H}$.

4. Proof of Theorem 1.2

Let $f \in \mathbb{S}_\mathcal{H}$. Then $f \in \mathbb{S}(h_\nu)$ for some $\nu \in \mathbb{N}$. Let $m \in \mathbb{Z}_+$ be arbitrary. Then for all $\alpha \in \mathbb{Z}_+^n$ with $\|\alpha\| \leqslant m$, $\beta \in \mathbb{Z}_+^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with non–zero coordinates

$$\|(D^\alpha f)(x)\| \leqslant \frac{\|f\|_{m,\nu} \beta! e^{-h_\nu(\beta)}}{\prod\limits_{j=1}^n \|x_j\|^{\beta_j}}.$$

Taking into consideration that

$$j! \leqslant e\sqrt{2\pi(j+1)} \frac{(j^+)^j}{e^j}$$

for each $j \in \mathbb{Z}_+$, we then find

$$\|(D^\alpha f)(x)\| \leqslant (e\sqrt{2\pi})^n \|f\|_{m,\nu} e^{-h_\nu(\beta)} \prod_{j=1}^n \frac{(\beta_j^+)^{\beta_j}}{(e\|x_j\|)^{\beta_j}}. \tag{4.1}$$

Let us estimate from above the quantity

$$e^{-h_\nu(\beta)}\prod_{j=1}^{n}\frac{(\beta_j^+)^{\beta_j}}{(e\|x_j\|)^{\beta_j}}.$$

For $\beta\in\mathbb{Z}_+^n$ we let

$$\Omega_\beta=\{t=(t_1,\dots,t_n)\in\mathbb{R}^n:\beta_j\leqslant t_j<\beta_j+1\ (j=1,\dots,n)\}.$$

Using the non–decreasing of h_ν in each variable in $\mathbb{R}_+^n$ and Proposition 2.3, for $\mu=(\mu_1,\dots,\mu_n)\in(0,\infty)^n$ and $t\in\Omega_\beta$ we have

$$e^{-h_\nu(\beta)}\prod_{j=1}^{n}\frac{(\beta_j^+)^{\beta_j}}{\mu_j^{\beta_j}}\leqslant e^{-h_{\nu+1}(t)+c_\nu}\prod_{j=1}^{n}\frac{\mu_j^+(t_j+1)^{t_j}}{\mu_j^{t_j}}.$$

Therefore,

$$\inf_{\beta\in\mathbb{Z}_+^n}e^{-h_\nu(\beta)}\prod_{j=1}^{n}\frac{(\beta_j^+)^{\beta_j}}{\mu_j^{\beta_j}}\leqslant e^{c_\nu}e^{\inf\limits_{t=(t_1,\dots,t_n)\in\mathbb{R}_+^n}\left(\sum\limits_{j=1}^{n}(\ln\mu_j^++t_j\ln(t_j+1)-t_j\ln\mu_j)-h_{\nu+1}(t)\right)},$$

Using then Proposition 2.4, we get

$$\inf_{\beta\in\mathbb{Z}_+^n}e^{-h_\nu(\beta)}\prod_{j=1}^{n}\frac{(\beta_j^+)^{\beta_j}}{\mu_j^{\beta_j}}\leqslant K_1e^{\inf\limits_{t=(t_1,\dots,t_n)\in\mathbb{R}_+^n}\left(\sum\limits_{j=1}^{n}(\ln\mu_j^++t_j\ln(t_j+1)-t_j\ln\mu_j)-(M_{\nu+2}^*[e])^*(t)\right)},$$

where $K_1=e^{a_\nu+d_{\nu+1}}$. By Proposition 2.2 this yields

$$\inf_{\beta\in\mathbb{Z}_+^n}e^{-h_\nu(\beta)}\prod_{j=1}^{n}\frac{(\beta_j^+)^{\beta_j}}{\mu_j^{\beta_j}}\leqslant K_1e^ne^{\inf\limits_{t=(t_1,\dots,t_n)\in\mathbb{R}_+^n}\left(\sum\limits_{j=1}^{n}\ln\mu_j^+-\sum\limits_{j=1}^{n}t_j\ln\frac{\mu_j}{e}+(M_{\nu+2}[e])^*(t)\right)}.$$

Taking into consideration that the function $(M_{\nu+2}[e])^*$ takes finite values on $[0,\infty)^n$ and $(M_{\nu+2}[e])^*(x)=+\infty$ for $x\notin[0,\infty)^n$, the above inequality can be rewritten as

$$\inf_{\beta\in\mathbb{Z}_+^n}e^{-h_\nu(\beta)}\prod_{j=1}^{n}\frac{(\beta_j^+)^{\beta_j}}{\mu_j^{\beta_j}}\leqslant K_1e^ne^{-\sup\limits_{t\in\mathbb{R}^n}\left(\sum\limits_{j=1}^{n}t_j\ln\frac{\mu_j}{e}-(M_{\nu+2}[e])^*(t)\right)+\sum\limits_{j=1}^{n}\ln\mu_j^+}.$$

We note that by Proposition 2.1 the function $M_{\nu+2}[e]$ takes finite values in $\mathbb{R}^n$ and is convex in $\mathbb{R}^n$. Therefore, $M_{\nu+2}[e]$ is continuous in $\mathbb{R}^n$ [7, Cor. 10.1.1]. Using then the formula for the inverse Young — Fenchel transform [7, Thm. 12.2], we get

$$\sup_{t\in\mathbb{R}^n}\left(\sum_{j=1}^{n}t_j\ln\frac{\mu_j}{e}-(M_{\nu+2}[e])^*(t)\right)=M_{\nu+2}\left(\frac{\mu_1}{e},\dots,\frac{\mu_n}{e}\right).$$

Thus,

$$\inf_{\beta\in\mathbb{Z}_+^n}e^{-h_\nu(\beta)}\prod_{j=1}^{n}\frac{(\beta_j^+)^{\beta_j}}{\mu_j^{\beta_j}}\leqslant K_1e^ne^{-M_{\nu+2}\left(\frac{\mu_1}{e},\dots,\frac{\mu_n}{e}\right)+\sum\limits_{j=1}^{n}\ln\mu_j^+}.$$

By (4.1) this implies

$$\|(D^\alpha f)(x)\|\leqslant K_12^n(e\sqrt{2\pi})^ne^n\|f\|_{m,\nu}e^{-M_{\nu+2}(x)+\sum\limits_{j=1}^{n}\ln(1+\|x_j\|)}.$$

Using the second assumption of Theorem 1.2, we find that for some $K_2>0$, which depends on ν and n, for all $x\in\mathbb{R}^n$ with non–zero coordinates and for all $\alpha\in\mathbb{Z}_+^n$ with $\|\alpha\|\leqslant m$

$$\|(D^\alpha f)(x)\|\leqslant K_2\|f\|_{m,\nu}e^{-M_{\nu+3}(x)}.$$

This inequality is obviously true for all $x \in \mathbb{R}^n$. Thus, $f \in GS(M_{\nu+3})$ and

$$q_{m,\nu+3}(f) \leqslant K_2 \|f\|_{m,\nu}, \quad f \in \mathbb{S}(h_\nu).$$

Hence, $f \in GS(\mathfrak{M})$ and the embedding mapping is continuous.

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