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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO OUTER ZAREMBA PROBLEM FOR ELLIPTIC EQUATIONS WITH MEASURE–VALUED POTENTIAL

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Abstract. In the exterior of a ball in the space $\mathbb{R}^n$ we consider the Zaremba and Neumann problems for quasilinear second order elliptic problems with a measure–valued potential. We proved the existence and uniqueness of entropy solution to the Zaremba and Neumann problems.

Key words: nonlinear elliptic equation, entropy solution, Radon measure, Zaremba problem.

Mathematics Subject Classification: 35J62, 35J25, 35A01, 35A02, 35D99

1. Introduction

Let $\Omega = \{x \in \mathbb{R}^n : \ |x| > r_0\}$ be the exterior of a ball, $n \geqslant 2$, $\Gamma \subset \partial\Omega$ be a closed subspace of the boundary, which can also be empty. In the present paper we study the existence of the entropy solution to the outer Zaremba problem for the equation

$$-\operatorname{div}(a(x,u,\nabla u)) + b_0(x,u,\nabla u) + b_1(x,u)\mu = f, \quad f \in L_1(\Omega),$$

where μ is a non–negative Radon measure. On Γ we impose the Dirichlet condition: $u(x) = 0$ for $x \in \Gamma$. On the remaining part of the boundary $\partial\Omega \setminus \Gamma$ we impose the Neumann condition: $a(x,u,\nabla u) \cdot x = 0$ for $x \in \partial\Omega \setminus \Gamma$. For empty Γ we have the Neumann problem. The uniqueness of the entropy solution is proved under additional assumptions.

The notion of the entropy solution of Dirichlet problem was proposed in [1]. In this work, in a domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$ (not necessarily bounded), the elliptic equation with L_1–date

$$-\operatorname{div}(a(x,\nabla u)) = f(x,u), \qquad \sup_{|u|<c} |f(x,u)| \in L_{1,\mathrm{loc}}(\Omega), \quad c > 0,$$

is considered. On the function a certain conditions of boundedness, monotonicity and coercivity are imposed. There were proved the existence and uniqueness of the entropy solution to the Dirichlet problem.

After this work, the study of entropic solutions became the research aim of many foreign and Russian mathematicians since the end of the last century.

Our study was motivated by the recent work [2]. In this work, the problem in the bounded domain

$$-\Delta u + \mu g(u) = \sigma, \qquad u|_{\partial\Omega} = 0$$

was considered. Under certain restrictions for the function g, the Radon measure σ and a non–negative measure μ in the Morrey class, the existence and uniqueness of a very weak solution to the problem were established.

Note that few works were devoted to entropy solutions to the Dirichlet problem in an unbounded domain. Entropy solutions to the Zaremba or Neumann problem in an unbounded domain have not been considered yet.

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In [3] for the equation

$$-\operatorname{div}(a(x,\nabla u)) + a_0(x,u) = \sigma$$

with the Radon measure σ the existence and uniqueness of the renormalized solution to the Dirichlet problem for an arbitrary domain Ω. From our point of view, one of the conditions of this work

$$a_0(x,s)s \geqslant c|s|^p, \qquad s \in \mathbb{R}, \tag{1.1}$$

can be weakened.

In [4] there was established the equivalence of entropy and renormalized solutions to nonlinear elliptic problem in Musielak — Orlicz spaces. In [5] the Dirichlet problem

$$-\operatorname{div}\, a(x,u,\nabla u) + M(x,u)/u + b(x,u,\nabla u) = \sigma, \quad u\Big|_{\partial\Omega} = 0,$$

was considered in an unbounded domain, where the functions a, b had a growth determined by the generalized N–function $M(x,u)$, while the bounded Radon measure σ has a special form. The inequality $b(x,u,\nabla u)u \geqslant 0$ was supposed. The existence of the entropy problem was problem. It is important that the result was established without Δ_2–conditions on M, $\overline{M}$.

In the work [6], the problem with the Fourier boundary condition

$$b(u) - \operatorname{div}(a(x,u,\nabla u)) = f, \quad x \in Q; \quad (a(x,u,\nabla u),\mathbf{n}) + \lambda u = g, \quad x \in \partial Q,$$

was considered in an unbounded domain Q. The function $a(x,u,y)$ was supposed to be Lipshitz in u. The operator $\operatorname{div}(a(x,u,\nabla u))$, in particular, can be $p(u)$–Laplcian. The existence and uniqueness of the entropy solution to the problem were proved. The uniqueness was proved under the apriori assumption that the entropy solution obeys the Lipshitz condition.

In the work [7], in the hyperbolic space, the Dirichlet problem for a nonlinear second order elliptic equation with a singular measure–valued potential was considered. The restrictions for the structure of equation were formulated in terms of a generalized N–function. The existence of the entropy solution to the problem was proved. A more detailed survey of works on entropy and renormalized solutions can be found in [5].

As it is known, the space $C_0^\infty(\mathbb{R}^n)$ can be completed both by the norm

$$\left(\int |\nabla u|^p dx\right)^{\frac{1}{p}}$$

and by the norm $\left(\int(|u|^p + |\nabla u|^p)dx\right)^{\frac{1}{p}}$, and in the second case a narrower space $W_p^1(\mathbb{R}^n) \subset \mathcal{H}_p^1(\mathbb{R}^n)$ arises. Usually, for instance, in the works [3], [5], one goes in the second way. While considering the problems in an unbounded domain, this produces too strict requirements of form (1.1) or similar. In this paper, the space $\mathcal{H}_p^1(\Omega)$ of the first type is used.

The results of the present work are also true for some regions, which are not exterior to a ball. But then we would have to formulate requirements on the set Γ depending on the shape of the domain. And this is a separate problem that we do not consider here.

2. Formulation of problem and main results

It is well–known that the space $L_p(\Omega)$ with $p > 1$ is separable and reflexive. In what follows the number $p \in (1,n)$ is supposed to be fixed.

Let $\mathcal{D}_\Gamma(\Omega)$ be the set of restrictions on Ω of the functions from $\mathcal{D}(\mathbb{R}^n)$ vanishing in the vicinity of Γ.

We define the space $\mathcal{H}_p^1(\Omega)$ as the completion of space $\mathcal{D}_\Gamma(\Omega)$ by the norm

$$\|u\|_{p,1} = \||\nabla u|\|_{L_p(\Omega)} = \|u\|_V.$$

For the brevity, we denote this space by V. The dual space for V with the induced norm is denoted by V^*. The actions of functionals l on elements in V is denoted by angle brackets $\langle l, v\rangle$.

We consider the operator

$$\mathcal{B}u = b_0(x,u,\nabla u) + b_1(x,u)\mu,$$

where μ is a non–negative Radon measure. For $u, v \in \mathcal{D}_\Gamma(\Omega)$ the operator $\mathcal{B}u$ acts by the rule

$$\langle \mathcal{B}u, v\rangle = \int\limits_\Omega b_0(x,u,\nabla u)v\,dx + \int\limits_\Omega b_1(x,u)v\,d\mu.$$

The well–definiteness of this formula under some conditions for the functions b_0, b_1 is established in what follows.

The results are established for the equation

$$-\operatorname{div}(a(x,u,\nabla u)) + \mathcal{B}u = f, \quad f \in L_1(\Omega). \tag{2.1}$$

We prove the existence of an entropy solution to the Zaremba and Neumann problem for this equation. Under additional restrictions we establish the uniqueness of solution.

Let μ be a Radon measure with a finite total variation and a support located in a bounded domain $Q \subset \mathbb{R}^n$. We suppose that the measure is continued by zero outside Q. We recall that μ belongs to the Morrey class $\mathbb{M}_s(Q)$, $s \geqslant 1$ if for each ball centered at x the inequality

$$|B_r(x)|_\mu := \int\limits_{B_r(x)} d|\mu| \leqslant c r^{n(1-1/s)}, \quad r > 0, \quad x \in Q,$$

holds. In other notation, $\mu \in \mathbb{M}_{\frac{n}{n-\theta}}(Q)$ for $\theta \in [0,n]$, $\theta = n(1-1/s)$ if

$$\int\limits_{B_r(x)} d|\mu| \leqslant c r^\theta.$$

It is easy to see that the Dirac delta function δ belongs to the class $\mathbb{M}_1(Q)$. Due to the Hölder inequality, the functions in $L_s(Q)$ define a measure from the class $\mathbb{M}_s(Q)$. If

$$f \in L_q(\Omega \cap \{x^1 = \ldots = x^k = 0\}), \quad x' = (0,\ldots,0,x^{k+1},\ldots,x^n),$$

then for $d\mu = f(x)dx'$ we have

$$\int\limits_{B_r(x_0)} |f(x)|dx' \leqslant \|f\|_q \left(\int\limits_{B_r(x_0)\cap\{x^1=\ldots=x^k=0\}} dx' \right)^{1-1/q} \leqslant c r^{(n-k)(1-1/q)},$$

and this function also defines some measure from the Morrey class with the support on the plane of dimension $n-k$.

We introduce the notation $\mathfrak{B}_r = \{x \in \Omega : \ |x| < r\}$, $r > r_0$.

We suppose that there exists a number $\widehat{s} > \frac{np}{np+p-n}$ such that

$$\mu \in \mathbb{M}_{\widehat{s}}(\mathfrak{B}_r) \quad \text{for all} \quad r > r_0. \tag{2.2}$$

Let $Q \subset \mathbb{R}^n$ be a bounded domain $\widehat{\theta} = n(1-1/\widehat{s})$. Given $q < \frac{\widehat{\theta}p}{n-p}$, for a non–negative measure $\mu \in \mathbb{M}_{\widehat{s}}(Q)$ the compact embedding

$$W_p^1(Q) \hookrightarrow L_{q,\mu}(Q) \tag{2.3}$$

is known. In particular, the elements in the space $W_p^1(Q)$ are μ–measurable functions. This is a particular case of a more general statement [2, Prop. 2.5]. In the case of the Lebesgue measure, the embedding

$$W_p^1(Q) \hookrightarrow L_{q_0}(Q)$$

is compact for $q_0 < \frac{np}{n-p}$.

The vector field $a(x,u,\nabla u)$ in (2.1) satisfies the boundedness condition

$$|a(x,r,y)|^{p'} \leqslant g(|r|)(G(x)+|y|^p), \quad r\in\mathbb{R}, \quad y\in\mathbb{R}^n, x\in\Omega, \quad \frac{1}{p}+\frac{1}{p'}=1, \tag{2.4}$$

for $x\in\Omega$ with an increasing function $g(s)$, $s\geqslant 0$, and a function $G\in L_1(\Omega)$, the coercivity condition

$$a(x,r,y)\cdot y \geqslant c_0|y|^p - G(x), \quad r\in\mathbb{R}, \quad c_0>0, \tag{2.5}$$

and the monotonicity condition

$$(a(x,r,y)-a(x,r,z))(y-z)>0, \quad y\neq z, \quad y,z\in\mathbb{R}^n, \quad r\in\mathbb{R}, \quad x\in\Omega. \tag{2.6}$$

Moreover, let a Caratheodory function b_0 and a μ–Caratheodory function b_1 satisfy the inequalities

$$|b_0(x,s,y)| \leqslant g(r)(\widetilde{G}_0(x)+|y|^p), \quad |s|\leqslant r, \quad |x|\leqslant r; \quad \text{for all} \quad r\geqslant 0, \tag{2.7}$$

$$|b_1(x,s)| \leqslant g(r)\widetilde{G}_1(x), \quad |s|\leqslant r, \quad |x|\leqslant r, \quad \text{for all} \quad r\geqslant 0, \tag{2.8}$$

where $\widetilde{G}_0\in L_{1,\mathrm{loc}}(\mathbb{R}^n)$, $\widetilde{G}_1\in L_{1,\mu,\mathrm{loc}}(\mathbb{R}^n)$;

$$b_0(x,r,y)r\geqslant 0, \quad b_1(x,r)r\geqslant 0, \quad \text{for all} \quad r\in\mathbb{R}. \tag{2.9}$$

We define the function

$$T_k(r)=\begin{cases} k & \text{for} \quad r>k, \\ r & \text{for} \quad |r|\leqslant k, \\ -k & \text{for} \quad r<-k. \end{cases}$$

By $\mathcal{T}_p^1(\Omega)$ we denote the set of measurable functions $u:\Omega\to\mathbb{R}$ such that $T_k(u)\in V$ for each $k>0$.

Definition 2.1. *The entropy solution to the Zaremba problem for Equation* (2.1) *is a function $u\in\mathcal{T}_p^1(\Omega)$ such that for all $k>0$, $\xi\in\mathcal{D}_\Gamma(\Omega)$ the inequality*

$$\int\limits_\Omega (a(x,u,\nabla u)\nabla T_k(u-\xi) - fT_k(u-\xi))dx + \langle \mathcal{B}u, T_k(u-\xi)\rangle \leqslant 0, \tag{2.10}$$

is well–defined and true, that is, all its terms are finite.

One of the main results of work is the following theorem.

Theorem 2.1. *Let the conditions* (2.2)–(2.9) *be satisfied, then there exists an entropy solution to the Zaremba problem for Equation* (2.1).

The uniqueness of the entropy solution is established under additional restrictions. Let the Caratheodory function b_0 and μ–Caratheodory function b_1 satisfies the inequalities

$$|b_0(x,s)| \leqslant \widehat{G}_0(x), \quad |s|\leqslant 1, \quad x\in\Omega; \tag{2.11}$$

$$|b_1(x,s)| \leqslant \widehat{G}_1(x), \quad |s|\leqslant 1, \quad x\in\Omega; \tag{2.12}$$

where $\widehat{G}_0\in L_1(\mathbb{R}^n)$, $\widehat{G}_1\in L_{1,\mu}(\mathbb{R}^n)$. In the next theorem the condition (2.4) is employed with $g(r)=C>0$.

Theorem 2.2. *Let $a=a(x,y)$ and the functions $b_i(x,s)$, $i=0,1$, increase in s and the inequalities* (2.11), (2.12) *hold. Let u_1, u_2 be entropy solutions to the Zaremba problem for Equation* (2.1). *If the conditions* (2.2)–(2.6), (2.9) *are satisfied, then $u_1=u_2$.*

We note that we do not know works, in which the uniqueness of the entropy solution is proved for the Dirichlet or Neumann problem for an elliptic equation in an unbounded domain, in which the flow a explicitly depends on the sought function u.

3. Technical lemmas

Lemma 3.1. *Let $v^j \geqslant 0$, $j \in \mathbb{N}$, be measurable non–negative functions in the domain Q (not necessarily bounded) such that*

$$v^j \to v \quad a.e. \ in \quad Q, \quad j \to \infty,$$

and the integrals

$$\int\limits_Q v^j(x)dx \to \int\limits_Q v(x)dx, \quad j \to \infty,$$

converge. Then

$$v^j \to v \quad strongly\ in \quad L_1(Q), \quad j \to \infty.$$

Proof. The identity

$$\int\limits_Q |v^j(x) - v(x)|dx = \int\limits_Q (v^j(x) - v(x))dx + 2 \int\limits_{x \in Q: v(x) > v^j(x)} (v(x) - v^j(x))dx$$

is obvious. The latter integral tends to the zero by the Lebesgue theorem. The proof is complete. □

Lemma 3.2. *There exists a non–negative increasing function $h(r)$ such that the inequality*

$$\|u\|_{W_p^1(\mathfrak{B}_r)} \leqslant h(r)\|u\|_V, \quad r > r_0, \quad for\ all \quad u \in V, \tag{3.1}$$

holds.

Proof. It is sufficient to establish the inequality for $u \in \mathcal{D}_\Gamma(\Omega)$. The inequality

$$\|\nabla u\|_{p,\mathfrak{B}_r} \leqslant \|u\|_V \tag{3.2}$$

is obvious. This is why the embedding

$$V \hookrightarrow W_p^1(\mathfrak{B}_r)$$

is continuous and the inequality (3.1) holds. Indeed, if the operator of this embedding is not bounded on $\mathcal{D}_\Gamma(\Omega)$, then there existsa sequence of smooth functions v^k such that

$$\|v^k\|_{W_p^1(\mathfrak{B}_R)} \geqslant k\|v^k\|_V.$$

Multiplying this inequality by an appropriate factor, we reduce it to the form

$$1 \geqslant k\|v^k\|_V, \tag{3.3}$$

where $\|v^k\|_{W_p^1(\mathfrak{B}_r)} = 1$. By (3.3) we have

$$\|v^k\|_V \to 0.$$

By the Kondrashov theorem, v^k converges strongly in $L_p(\mathfrak{B}_r)$. In view of (3.2), we establish the convergence $v^k \to C \neq 0$ in the space $W_p^1(\mathfrak{B}_r)$. We can also suppose that $v^k \to C = C(r)$ almost everywhere in $\mathfrak{B}_r$. In order to obtain a contradiction, we consider a sequence $\widehat{v}^k = v^k\zeta(|x| - r_0)$, where $\zeta(t) = \min(1, \max(0,t))$. Since $\operatorname{supp}\widehat{v}^k \subset \overline{\Omega}$, by Nirenberg — Gagliardo — Sobolev inequalities

$$\|\widehat{v}^k\|_{L_{p^*}(\Omega)} \leqslant \alpha(p,n)\|\nabla\widehat{v}^k\|_{L_p(\Omega)}; \quad p^* = \frac{np}{n-p}. \tag{3.4}$$

We have the inequality

$$\|\widehat{v}^k\|_{L_{p^*}(\Omega)} \geqslant \|v^k\|_{L_{p^*}(\mathfrak{B}_r \setminus \mathfrak{B}_{r_0+1})} \to |C| \operatorname{mes}^{1/p^*}(\mathfrak{B}_r \setminus \mathfrak{B}_{r_0+1}).$$

On the other hand, the convergence $\|v^k\|_{L_p(\mathfrak{B}_r)} \to 1$ implies the convergence $|v^k|^p \to |C|^p$ in $L_1(\mathfrak{B}_r)$, see Lemma 3.1, and hence

$$\|\nabla\widehat{v}^k\|_{L_p(\Omega)} \leqslant \|v^k\nabla\zeta(|x| - r_0)\|_{L_p(\mathfrak{B}_{r_0+1})} + \|\nabla v^k\|_{L_p(\Omega)} \to |C|\alpha_1(p,n).$$

Two latter inequalities contradicts (3.4) for large k and r. The proof is complete. $\square$

We note that the functions $u \in C_0^\infty(\overline{\Omega})$ satisfy the Nirenberg — Gagliardo — Sobolev inequality

$$\|u\|_{L_{p^*}(\Omega)} \leqslant \alpha_2(p,n)\|\nabla u\|_{L_p(\Omega)}. \tag{3.5}$$

Indeed, the inequality (3.1) allows to construct the continuation to a function $\widehat{u} \in W^1_{p,\mathrm{loc}}(\mathbb{R}^n)$, which coincides with u in Ω, and to apply the usual Nirenberg — Gagliardo — Sobolev inequality in $\mathbb{R}^n$.

The authors thank V.E. Bobkov, who pointed out the work [8], the results of which imply the statement of Lemma 3.2. But we preferred to give a simple proof for this lemma.

Lemma 3.3. *Let a measurable function $u(x)$ be defined in Ω. The set $\{k : \operatorname{mes}\{x \in \Omega : |u(x)| = k\} > 0\}$ is finite or countable.*

Proof. Let N be an arbitrary natural number. We choose numbers k_i such that

$$\operatorname{mes}\{x \in \mathfrak{B}_r : |u(x)| = k_i\} > \frac{1}{N}.$$

These sets are disjoint and this is why

$$\operatorname{mes}\{x \in \mathfrak{B}_r : \ |u(x)| = k_1\} + \operatorname{mes}\{x \in \mathfrak{B}_r : \ |u(x)| = k_2\} + \ldots \leqslant \operatorname{mes}\mathfrak{B}_r.$$

Therefore, there are at most $N \operatorname{mes}\mathfrak{B}_r$ such sets. Then the set

$$\{k : \operatorname{mes}\{x \in \mathfrak{B}_r : \ |u(x)| = k\} > 0\}$$

is finite or countable. This easily implies the statement of the lemma. The proof is complete. $\square$

We shall the values of k, for which

$$\operatorname{mes}\{x \in \Omega : \ |u(x)| = k\} = 0$$

regular. Let k be a regular value and $u^j(x) \to u(x)$ almost everywhere in Ω. Then

$$\chi(|u^j(x)| < k) \to \chi(|u(x)| < k) \quad \text{a.e. in} \quad \Omega. \tag{3.6}$$

Indeed, if $|u(x)| < k$, then $|u^j(x)| < k$ for large j. This implies the convergence for a chosen x. If $|u(x)| > k$, then $|u^j(x)| > k$ for large j. This implies the convergence also for such x.

Lemma 3.4. *Let a function v be such that $T_k(v) \in V$ for all $k > k_0$ and the inequality*

$$\|T_k(v)\|_V^p \leqslant Ck$$

holds. Then

$$\operatorname{mes}\{x \in \Omega : |v| \geqslant k\} \leqslant \frac{C_1}{k^{p^*(1-p^{-1})}}, \quad k > k_0. \tag{3.7}$$

Proof. Using the inequality (3.5), we establish

$$\|T_k(v)\|_{p^*,\Omega} \leqslant C(p,n)\|T_k(v)\|_V.$$

For $k_1 \in (0,k]$ the inequalities

$$\operatorname{mes}\{x \in \Omega : |v| \geqslant k_1\} \leqslant \frac{\int\limits_{\{x\in\Omega:|v|\geqslant k_1\}} |T_k(v)|^{p^*}\,dx}{k_1^{p^*}} \leqslant \frac{C(p,n)^{p^*}\|T_k(v)\|_V^{p^*}}{k_1^{p^*}} \leqslant C_1\frac{k^{p^*/p}}{k_1^{p^*}}$$

are obvious. Letting $k_1 = k$, we get (3.7). The proof is complete. $\square$

Lemma 3.5. *Let $Q \subset \Omega$, the sequence $\{v^m\}_{m\in\mathbb{N}}$ be bounded in $L_p(Q)$, $v \in L_p(Q)$, and*

$$v^m \to v \quad \textit{a.e. in} \quad Q, \quad m \to \infty.$$

Then

$$v^m \rightharpoonup v \quad \textit{weakly in} \quad L_p(Q), \quad m \to \infty.$$

For a bounded domain $Q \subset \mathbb{R}^n$ the proof was given [10], for an arbitrary domain $Q \subset \Omega$ the proof is similar.

In what follows, to avoid bulky notations, instead of statements like "it is possible to select a subsequence from the sequence u^m, which converges almost everywhere in Ω as $m \to \infty$" we shall simply write "the sequence u^m contains a subsequence converging almost everywhere in Ω as $m \to \infty$". We shall also employ the phrase "weakly converges over some subsequence" omitting the index of the subsequence.

Lemma 3.6. *Let v^j, $j \in \mathbb{N}$, v by functions in $L_p(Q)$ such that*

$$v^j \to v \quad a.e.\ in \quad Q, \quad j \to \infty;$$
$$|v^j|^p \leqslant H \in L_1(Q), \quad j \in \mathbb{N},$$

then

$$v^j \to v \quad strongly\ in \quad L_p(Q), \quad j \to \infty.$$

This lemma is implied by the Lebesgue theorem.

Lemma 3.7. *Let a sequence $\{v^j\}_{j\in\mathbb{N}}$ be bounded in $L_{p'}(Q)$. Then there exists a subsequence such that*

$$v^j \rightharpoonup v \quad weakly\ in \quad L_{p'}(Q), \quad j \to \infty.$$

If h^j, $j \in \mathbb{N}$, h are functions in $L_p(Q)$ such that

$$h^j \to h \quad strongly\ in \quad L_p(Q), \quad j \to \infty,$$

then

$$\int\limits_Q v^j h^j dx \to \int\limits_Q vh dx, \quad j \to \infty.$$

The proof of this lemma is simple and we omit it. In what follows we shall employ the Vitali lemma, see [11, Ch. III, Sect. 6, Thm. 15].

Lemma 3.8. *Let v^j, $j \in \mathbb{N}$, v be measurable functions in a bounded domain Q such that*

$$v^j \to v \quad a.e.\ in \quad Q, \quad j \to \infty,$$

and the integrals

$$\int\limits_Q |v^j(x)| dx, \quad j \in \mathbb{N},$$

be uniformly absolutely continuous. Then

$$v^j \to v \quad strongly\ in \quad L_1(Q), \quad j \to \infty.$$

Lemma 3.9. *Let $H^j \to H$ in $L_1(Q)$ as $j \to \infty$. Let v^j, $j \in \mathbb{N}$, be measurable functions in a bounded domain Q such that*

$$v^j \to v \quad a.e.\ in \quad Q, \quad j \to \infty;$$
$$|v^j| \leqslant H^j, \quad j \in \mathbb{N},$$

then

$$v^j \to v \quad strongly\ in \quad L_1(Q), \quad j \to \infty.$$

This lemma can be easily derived from the Vitali lemma.

The next statement is usually called Levi theorem.

Lemma 3.10. *Let $(S, \sum, \mu)$ be a space with a positive measure, $\{f_n\}$ be a non–decreasing sequence of non–negative measurable not necessarily integrable functions. Then*

$$\lim_{n\to\infty} \int\limits_S f_n(x) d\mu = \int\limits_S \sup_n f_n(x) d\mu.$$

The proof was given in [11, Ch. III, Sect. 6, Cor. 17].

Lemma 3.11. *Let in Ω the conditions (2.4)–(2.6) hold and for $k > 0$ and some sequence $w^j \in V$ the conditions*

$$\begin{aligned} &\nabla w^j \rightharpoonup \nabla w \quad \text{in} \quad L_p(\Omega), \quad j \to \infty, \\ &w^j \to w \quad \text{a.e. in} \quad \Omega, \quad j \to \infty, \\ &\lim_{j\to\infty} \int\limits_{\mathfrak{B}_R} (a(x, T_k(w^j), \nabla w^j) - a(x, T_k(w^j), \nabla w)) \cdot \nabla(w^j - w))dx = 0, \quad \text{for all} \quad R > R_0. \end{aligned}$$

be satisfied. Then on some subsequence

$$\begin{aligned} &\nabla w^j \to \nabla w \quad \text{a.e. in} \quad \Omega, \quad j \to \infty, \\ &\nabla w^j \to \nabla w \quad \text{strongly in} \quad L_{p,\mathrm{loc}}(\overline{\Omega}), \quad j \to \infty, \\ &a(x, T_k(w^j), \nabla w^j) \cdot \nabla w^j \to a(x, T_k(w), \nabla w) \cdot \nabla w \quad \text{in} \quad L_{1,\mathrm{loc}}(\overline{\Omega}), \quad j \to \infty. \end{aligned} \tag{3.8}$$

A similar statement in a more general formulation was proved in [9, Lm. 4.10].

4. Weak solution to approximation problem

By (2.4) the vector field $a^m(x, r, y) = a(x, T_m(r), y)$ defines the operator

$$\widetilde{A} : V \times V \to V^*.$$

It acts by the formula

$$\langle \widetilde{A}(u, v), w\rangle = \int\limits_{\Omega} a^m(x, u, \nabla v) \cdot \nabla w dx, \qquad u, v, w \in V.$$

We let

$$\begin{aligned} &f^m(x) = T_m(f(x))\chi_m(x), \\ &\chi_m(x) = \begin{cases} 1, & \text{if} \quad x \in \mathfrak{B}_m, \\ 0, & \text{if} \quad x \notin \mathfrak{B}_m, \end{cases} \\ &b_0^m(x, r, y) = T_m(b_0(x, r, y))\chi_m(x), \quad b_1^m(x, r) = T_m(b_1(x, r))\chi_m(x). \end{aligned}$$

It is obvious that as $r \in \mathbb{R}$, $y \in \mathbb{R}^n$,

$$|b_0^m(x, r, y)| \leqslant m\chi_m(x), \quad |b_1^m(x, r)| \leqslant m\chi_m(x), \quad x \in \Omega.$$

Moreover, applying (2.9), we find

$$b_0^m(x, r, y)r \geqslant 0, \quad b_1^m(x, r)r \geqslant 0, \quad x \in \Omega, \quad r \in \mathbb{R}. \tag{4.1}$$

Using the inequality (3.1), it is easy to show that $f^m \in V^*$,

$$f^m \to f \quad \text{in} \quad L_1(\Omega), \quad m \to \infty,$$

and at the same time

$$|f^m(x)| \leqslant |f(x)|, \quad |f^m(x)| \leqslant m\chi_m(x), \quad x \in \Omega, \quad m \in \mathbb{N}. \tag{4.2}$$

The operator $\mathcal{B}_m : V \to V^*$ acts by the formula

$$\langle \mathcal{B}_m u, v\rangle = \int\limits_{\Omega} b_0^m(x, u, \nabla u)v dx + \int\limits_{\Omega} b_1^m(x, u)v d\mu = \langle K_0(u), v\rangle + \langle K_1(u), v\rangle.$$

The convergence of the second integrals is ensured by the embedding (2.3) and the inequality (3.1). Using (4.1), it is easy to establish the non–negativity of the operator $\mathcal{B}_m$:

$$\langle \mathcal{B}_m u, u\rangle \geqslant 0, \quad u \in \mathcal{D}(\Omega).$$

We consider the equation

$$-\operatorname{div} a^m(x,u,\nabla u) + \mathcal{B}_m u = f^m(x), \quad x \in \Omega, \quad m \in \mathbb{N}, \tag{4.3}$$

with the function $a^m(x,r,y) = a(x,T_m(r),y)$.

The weak solution to the Zaremba problem for Equation (4.3) is a function $u^m \in V$ obeying the integral identity

$$\int\limits_{\Omega} a(x,T_m(u^m),\nabla u^m)\cdot\nabla v dx + \langle \mathcal{B}_m u^m, v\rangle = \langle f^m, v\rangle \tag{4.4}$$

for each function $v \in \mathcal{D}_\Gamma(\Omega)$. It is easy to prove that the relation (4.4) holds also for all $v \in V$. While proving the existence of a weak solution u^m to the Zaremba problem, we shall omit the superscript m.

We shall seek a weak solution $u^m \in V$ to the Zaremba problem for Equation (4.3) by the Galerkin method.

Let a sequence of functions $\omega_j \in \mathcal{D}_\Gamma(\Omega)$ be orthonormalized and has a dense linear span in $L_2(\Omega)$. We seek approximations for a solution to the problem as $u^N = \sum\limits_{j=1}^{N} h_j^N\omega_j$. We fix N. We let $\mathbf{h} = (h_1,h_2,\ldots,h_N) \in \mathbb{R}^N$ and define the functions $P_k(\mathbf{h})$, $k = 1,2,\ldots,N$, by the formulas

$$P_k(\mathbf{h}) = \int\limits_{\Omega} a^m(x,u^N,\nabla u^N)\cdot\nabla\omega_k dx + \langle \mathcal{B}_m u^N, \omega_k\rangle - \langle f^m, \omega_k\rangle.$$

The vector $\mathbf{h}^N$ is determined by the system of equations $P_k(\mathbf{h}^N) = 0$, $k = 1,2,\ldots,N$.

Let us prove the solvability of equations for the vector $\mathbf{h}^N$. We introduce the notation

$$P(\mathbf{h}^N) = (P_1(\mathbf{h}^N), P_2(\mathbf{h}^N),\ldots,P_N(\mathbf{h}^N)).$$

Using the condition (2.5), the non–negativity of the operator $\mathcal{B}_m$ and the inequalities (3.1),

$$|\langle f^m, u^N\rangle| \leqslant C(m)\|u^N\|_{p,\mathfrak{B}_m},$$

we have

$$\begin{aligned}(P(\mathbf{h}^N),\mathbf{h}^N) &= \int\limits_{\Omega} a^m(x,u^N,\nabla u^N)\cdot\nabla u^N dx + \langle \mathcal{B}_m u^N, u^N\rangle - \langle f^m, u^N\rangle \\ &\geqslant \int\limits_{\Omega} (c_0|\nabla u^N|^p - G(x))dx + \langle \mathcal{B}_m u^N, u^N\rangle - \langle f^m, u^N\rangle \\ &\geqslant \int\limits_{\Omega} c_0|\nabla u^N|^p dx - C(m)\|u^N\|_{p,\mathfrak{B}_m} - C_1.\end{aligned} \tag{4.5}$$

Thus, for $p > 1$ by (4.5) we get the inequality

$$(P(\mathbf{h}),\mathbf{h}) > 0$$

for large $|\mathbf{h}|$. By [12, Ch. 1, Lm. 4.3]), there exists a vector $\mathbf{h}^N$ such that $P_k(\mathbf{h}^N) = 0$, $k = 1,2,\ldots N$. Using (4.5) and the identity $(P(\mathbf{h}^N),\mathbf{h}^N) = 0$, we obtain the inequality

$$\int\limits_{\Omega} |\nabla u^N|^p dx \leqslant C_1 + C_2\|u^N\|_{p,\mathfrak{B}_m}.$$

In view of (3.1) this implies the uniform estimate

$$\|u^N\|_V = \|\nabla u^N\|_{p,\Omega} \leqslant C_3, \quad \forall N = 1,2,\ldots.$$

Similarly, using (4.5) and the non–negativity of $\mathcal{B}_m$ we establish that

$$\langle \mathcal{B}_m u^N, u^N\rangle \leqslant C_3, \quad \text{for all} \quad N = 1,2,\ldots.$$

Then we can choose a subsequence N_k so that

$$u^{N_k} \rightharpoonup u \quad \text{weakly in} \quad V \quad \text{and weakly in} \quad W_p^1(\mathfrak{B}_r), \quad r \geqslant r_0.$$

Using inequalities of form (3.1) and the Rellich — Kondrashov theorem, we obtain

$$u^{N_k} \to u \quad \text{strongly in} \quad L_{p,\mathrm{loc}}(\overline{\Omega}).$$

This is why, choosing an appropriate subsequence, we can suppose that

$$u^{N_k} \to u \quad \text{a.e. in} \quad \Omega.$$

By (2.4) the sequence $a^m(x, u^N, \nabla u^N)$ is bounded in the space $(L_{p'}(\Omega))^n$ and

$$|a^m(x, u^N, \nabla u)|^{p'} \leqslant g(m)(G(x) + |\nabla u|^p) \in L_1(\Omega).$$

Hence, by Lemma 3.6 we have a strong convergence in $L_{p'}(\Omega)$

$$a^m(x, u^N, \nabla u) \to a^m(x, u, \nabla u), \quad N \to \infty. \tag{4.6}$$

Moreover, by Lemma 3.7, in the sequence $a^m(x, u^N, \nabla u^N)$ we can choose a weakly converging subsequence. We shall omit the indices of the subsequence

$$a^m(x, u^N, \nabla u^N) \rightharpoonup \kappa \quad \text{weakly in} \quad (L_{p'}(\Omega))^n. \tag{4.7}$$

Since $|K_0(u^N)| = |b_0^m(x, u^N, \nabla u^N)| \leqslant m$, the sequence $K_0(u^N)$ is bounded in the space $L_{p'}(\mathfrak{B}_m)$. Omitting the indices of subsequence, we can suppose that the sequence $K_0(u^N)$ weakly converges to k_0 in the space $L_{p'}(\mathfrak{B}_m) \subset V^*$. Similarly, the sequence $K_1(u^N)$ weakly converges to k_1 in the space $L_{q',\mu}(\mathfrak{B}_m) \subset V^*$.

Passing to the limit as $N \to \infty$ in the identities $P_k(h^N) = 0$, we arrive at the relation

$$\int\limits_\Omega \kappa \cdot \nabla \omega_k dx + \langle k_0 + k_1, \omega_k \rangle = \langle f^m, \omega_k \rangle. \tag{4.8}$$

Multiplying by h_k^N, we easily get the identity

$$\int\limits_\Omega \kappa \cdot \nabla u^N dx + \langle k_0 + k_1, u^N \rangle = \langle f^m, u^N \rangle.$$

Passing to the limit as $N \to \infty$, we find

$$\int\limits_\Omega \kappa \cdot \nabla u dx + \langle k_0 + k_1, u \rangle = \langle f^m, u \rangle. \tag{4.9}$$

Passing to the limit in the identity $(P(\mathbf{h}^N), \mathbf{h}^N) = 0$ (4.5) and using Lemma 3.7, we get the relation

$$\lim_{N\to\infty} \int\limits_\Omega a^m(x, u^N, \nabla u^N) \cdot \nabla u^N dx + \langle k_0 + k_1, u \rangle = \langle f^m, u \rangle. \tag{4.10}$$

It follows from (4.9) and (4.10) that

$$\lim_{N\to\infty} \int\limits_\Omega a^m(x, u^N, \nabla u^N) \cdot \nabla u^N dx = \int\limits_\Omega \kappa \cdot \nabla u dx. \tag{4.11}$$

Now we are going to prove that $\nabla u^N \to \nabla u$ a.e. The weak convergence of the sequence u^N in the space V and the strong convergence (4.6) imply

$$\lim_{N\to\infty} \int\limits_\Omega a^m(x, u^N, \nabla u) \cdot (\nabla u^N - \nabla u) dx = 0. \tag{4.12}$$

Let

$$\begin{aligned} H_N =&(a^m(x,u^N,\nabla u^N)-a^m(x,u^N,\nabla u))(\nabla u^N-\nabla u) \\ =&a^m(x,u^N,\nabla u^N)\nabla u^N - a^m(x,u^N,\nabla u^N)\nabla u - a^m(x,u^N,\nabla u)(\nabla u^N-\nabla u). \end{aligned} \tag{4.13}$$

It follows from (2.6) that $H_N \geqslant 0$. By (4.7) we ge the identity

$$\lim_{N\to\infty}\int\limits_{\Omega} a^m(x,u^N,\nabla u^N)\nabla u dx = \int\limits_{\Omega}\kappa\nabla u dx. \tag{4.14}$$

Passing to the limit and using (4.11)–(4.14), we arrive at the relation

$$\lim_{N\to\infty}\int\limits_{\Omega} H_N dx = 0,$$

which in other notation

$$\Lambda(x,r,y,z) = (a(x,r,y)-a(x,r,z))\cdot(y-z), \quad y,z\in\mathbb{R}^n, \quad r\in\mathbb{R},$$

is written as

$$\lim_{N\to\infty}\int\limits_{\Omega}\Lambda(x,T_m(u^N),\nabla u^N,\nabla u)dx = 0.$$

Applying Lemma 3.11, we obtain the convergence $\nabla u^N\to\nabla u$ almost everywhere in Ω. Then

$$\kappa = a^m(x,u,\nabla u), \qquad k_0 = b_0^m(x,u,\nabla u), \qquad k_1 = b_1(x,u),$$

by (4.8) we easily find that the function u is a weak solution to the approximated Zaremba problem.

5. Existence of solution

In (4.4) we let $v = T_{k,h}(u^m) = T_k(u^m - T_h(u^m))$. Taking into consideration (4.1), we have

$$\begin{aligned} &\int\limits_{\{\Omega:h\leqslant|u^m|<k+h\}} a^m(x,u^m,\nabla u^m)\nabla u^m dx + k\int\limits_{\{\Omega:|u^m|\geqslant k+h\}}|b_0^m(x,u^m,\nabla u^m)|dx \\ &+k\int\limits_{\{\Omega:|u^m|\geqslant k+h\}}|b_1^m(x,u^m)|d\mu + \int\limits_{\{\Omega:h\leqslant|u^m|<k+h\}} b_0^m(x,u^m,\nabla u^m)u^m(1-h/|u_m|)dx \\ &+\int\limits_{\{\Omega:h\leqslant|u^m|<k+h\}} b_1^m(x,u^m)u^m(1-h/|u_m|)d\mu \leqslant k\int\limits_{\{\Omega:|u^m|\geqslant h\}}|f^m|dx. \end{aligned} \tag{5.1}$$

Applying (4.2), (2.5), by (5.1) we get

$$\begin{aligned} &\int\limits_{\{\Omega:h\leqslant|u^m|<k+h\}}(a^m(x,u^m,\nabla u^m)\cdot\nabla u^m + G(x))dx + k\int\limits_{\{\Omega:|u^m|\geqslant k+h\}}|b_0^m(x,u^m,\nabla u^m)|dx \\ &+k\int\limits_{\{\Omega:|u^m|\geqslant k+h\}}|b_1^m(x,u^m)|d\mu \leqslant r\int\limits_{\{\Omega:|u^m|\geqslant h\}}(k|f|+|G|)dx, \quad m\in\mathbb{N}. \end{aligned} \tag{5.2}$$

Letting $h=0$ in (5.2) and using the inequality (2.5), we obtain

$$\begin{aligned} &\int\limits_{\{\Omega:|u^m|<k\}} c_0|\nabla u^m|^p dx + k\int\limits_{\{\Omega:|u^m|\geqslant k\}}|b_0^m(x,u^m,\nabla u^m)|dx \\ &+k\int\limits_{\{\Omega:|u^m|\geqslant k\}}|b_1^m(x,u^m)|d\mu \leqslant (k+1)C_1, \quad m\in\mathbb{N}. \end{aligned} \tag{5.3}$$

By (5.3) we get the estimate

$$\int\limits_{\{\Omega:|u^m|<k\}} |\nabla u^m|^p dx = \int\limits_{\Omega} |\nabla T_k(u^m)|^p dx \leqslant c_0^{-1} C_1(k+1), \quad m \in \mathbb{N}.$$

Then $T_k(u^m) \in V$ and for each $k > 1$

$$\|T_k(u^m)\|_V^p \leqslant C_2 k, \quad m \in \mathbb{N}. \tag{5.4}$$

The reflexivity of the space V allows us to select a weakly converging in V subsequence

$$T_k(u^m) \rightharpoonup v_k \in V, \qquad m \to \infty. \tag{5.5}$$

The inequality (5.4) allows us to apply Lemma 3.4, which yields the estimate

$$\mathrm{mes}\{x \in \Omega : |u^m(x)| \geqslant k\} \leqslant \frac{C}{k^{p^*(1-p^{-1})}}, \qquad m > k > 1. \tag{5.6}$$

Then, choosing sufficiently large k, we obtain

$$\int\limits_{\{\Omega:|u^m|\geqslant k\}} (|f| + |G|) dx \leqslant \varepsilon(k), \quad m > k, \tag{5.7}$$

where $\varepsilon(k) \to 0$ as $k \to \infty$. We are going to to establish the convergence over a subsequence

$$u^m \to u \quad \text{a.e.} \quad \text{and} \quad \mu-\text{a.e.} \quad \text{in} \quad \Omega, \quad m \to \infty. \tag{5.8}$$

The sequence $T_s(u^m)$ is bounded in the space V and by (3.1) is bounded in the space $W_p^1(\mathfrak{B}_R)$. By the Kondrashov theorem, we can select a converging subsequence $T_s(u^m) \to \widetilde{v}_s$ in $L_p(\mathfrak{B}_R)$ as $m \to \infty$. This implies the convergence $T_s(u^m) \to \widetilde{v}_s$ almost everywhere in $\mathfrak{B}_R$. By (5.5) we have the identity $v_s = \widetilde{v}_s$ almost everywhere in $\mathfrak{B}_R$. Then by diagonal process in $R \in \mathbb{N}$ we establish a convergence over some subsequence $T_s(u^m) \to v_s$ almost everywhere in Ω. By Q we denote the set of points in Ω, at which the sequence $u^m(x)$ has a finite limit. We denote this limit by $u(x)$. For $x \in Q$ the identities

$$v_s(x) = \lim T_s(u^m(x)) = T_s \lim u^m(x) = T_s(u)$$

hold. If for some x we have $\lim |T_s(u^m(x))| < s$, then

$$\lim T_s(u^m(x)) = v_s(x) = \lim u^m(x),$$

that is, $x \in Q$. Then for almost each $x \notin Q$ we have $\lim |T_s(u^m(x))| = s$ for all $s > 0$. In particular, $\lim |T_{s+h}(u^m(x))| = s+h$. Then $|u^m(x)| > s$ for large m, therefore, $\lim |u^m(x)| = \infty$. By (5.6), the measure of the set of such points in the ball $\mathfrak{B}_R$ is equal to zero. We then conclude that the difference $\Omega \backslash Q$ has a zero measure and the convergence (5.8) for the Lebesgue measure is established. Then $v_s(x) = T_s(u)$ for almost each $x \in \Omega$.

We also note the convergence $T_s(u^m) \to v_s$ in $L_{q,\mu}(\mathfrak{B}_R)$ implied by (2.3) and (3.1). Then $T_s(u^m) \to v_s$ μ–almost everywhere in $\mathfrak{B}_R$ (over some subsequence). By the diagonal process in $R \in \mathbb{N}$ we establish the convergence over some subsequence

$$T_s(u^m) \to T_s(u), \quad m \to \infty, \tag{5.9}$$

μ–almost everywhere in Ω, and also (5.8).

The relation (5.5) can be rewritten as

$$\nabla T_k(u^m) \rightharpoonup \nabla T_k(u) \quad \text{in} \quad L_p(\Omega), \quad m \to \infty. \tag{5.10}$$

In what follows we shall establish the strong convergence

$$\nabla T_k(u^m) \to \nabla T_k(u) \quad \text{in} \quad L_{p,\mathrm{loc}}(\overline{\Omega}), \quad m \to \infty. \tag{5.11}$$

By (5.4), (2.4) for each $k > 1$ we have the estimate

$$\|a(x, T_k(u^m), \nabla T_k(u^m))\|_{p',\Omega} \leqslant C_5(k), \quad m \in \mathbb{N}. \tag{5.12}$$

Then we can select a weakly converging subsequence

$$a(x, T_k(u^m), \nabla T_k(u^m)) \rightharpoonup a_k \quad \text{weakly in} \quad L_{p'}(\Omega), \quad m \to \infty. \tag{5.13}$$

Let $k > 0$, $h > k + 1$,

$$z^m = T_k(u^m) - T_k(u), \quad m \in \mathbb{N}.$$

We let $\varphi_k(r) = r\exp(\gamma^2 r^2)$, where $\gamma = \frac{g(k)}{c_0}$. It is obvious that

$$\psi_k(r) = \varphi_k'(r) - \gamma|\varphi_k(r)| \geqslant 7/8, \quad r \in \mathbb{R}.$$

This implies the inequalities

$$7/8 \leqslant \psi_k(z^m) \leqslant \max_{[-2k,2k]} \psi_k(r) = C(k), \quad m \in \mathbb{N}.$$

In view of (5.8), $z^m \to 0$ almost everywhere in Ω and μ–a.e. This is why

$$\varphi_k(z^m) \to 0, \quad \varphi_k'(z^m) \to \varphi_k'(0) = 1, \quad \psi_k(z^m) \to \psi_k(0) = 1, \quad m \to \infty, \tag{5.14}$$

almost everywhere in Ω and μ–a.e. The inequalities

$$|\varphi_k(z^m)| \leqslant \varphi_k(2k), \quad 1 \leqslant \varphi_k'(z^m) \leqslant \varphi_k'(2k), \quad m \in \mathbb{N}, \tag{5.15}$$

are obvious.

We let $\eta_h(r) = \zeta(h - r + 1)$.

For the brevity of writing we shall employ the notation

$$d\nu = \eta_{R-1}(|x|)dx, \qquad \eta^m_{h-1}(x) = \eta_{h-1}(|u^m|), \qquad \widetilde{\eta}_{h-1}(x) = \eta_{h-1}(|u|).$$

The convergences (5.8) imply the convergence

$$\eta^m_{h-1} \to \widetilde{\eta}_{h-1} \quad \text{a.e. in} \quad \Omega, \quad m \to \infty.$$

Choosing $\varphi_k(z^m)\eta_{R-1}(|x|)\eta^m_{h-1}$ as the test function in (4.4), we obtain

$$\begin{aligned}
&\int\limits_{\mathfrak{B}_R} a(x, T_h(u^m), \nabla T_h(u^m))\nabla(\varphi_k(z^m)\eta_{R-1}\eta^m_{h-1})dx \\
&\quad + \int\limits_{\mathfrak{B}_R} b_0^m(x, u^m, \nabla u^m)\eta_{R-1}\varphi_k(z^m)\eta^m_{h-1}dx \\
&\quad + \int\limits_{\mathfrak{B}_R} b_1^m(x, u^m)\varphi_k(z^m)\eta_{R-1}\eta^m_{h-1}d\mu \\
&\quad - \int\limits_{\mathfrak{B}_R} f^m\varphi_k(z^m)\eta^m_{h-1}dx = I_1 + I_2 + I_3 + I_4 = 0, \quad m \geqslant h.
\end{aligned} \tag{5.16}$$

5.1. Estimates for integrals $I_2 - I_4$. In view of the inequalities

$$\eta_{h-1}(|u^m|)|b_1^m(x, u^m)| \leqslant g(h)\widetilde{G}_1(x), \quad x \in \mathfrak{B}_R,$$

implied by (2.8), by the Lebesgue theorem and (5.14), we have

$$|I_3| \leqslant \int\limits_{\mathfrak{B}_R} |\varphi_k(z^m)|g(h)\widetilde{G}_1(x)d\mu = \varepsilon_1(m). \tag{5.17}$$

Hereinafter

$$\lim_{m\to\infty} \varepsilon_i(m) = 0.$$

Similarly, since $f \in L_1(\Omega)$, we obtain

$$|I_4| \leqslant \int\limits_{\mathfrak{B}_R} |f\varphi_k(z^m)|dx = \varepsilon_2(m). \tag{5.18}$$

It is obvious that $z^m u^m \geqslant 0$ as $|u^m| \geqslant k$, $\varphi_k(z^m)u^m \geqslant 0$, and this is why, in view of (4.1), we have

$$b_0^m(x, u^m, \nabla u^m)\varphi_k(z^m) \geqslant 0 \quad \text{as} \quad |u^m| \geqslant k.$$

Using this fact and applying (2.8), we estimate the integrals

$$\begin{aligned}-I_2 &\leqslant \int\limits_{\{\mathfrak{B}_R:|u^m|<k\}} |b_0^m(x, u^m, \nabla u^m)||\varphi_k(z^m)|d\nu \\ &\leqslant g(k)\int\limits_{\mathfrak{B}_R} \left(|\nabla T_k(u^m)|^p + \widetilde{G}_0(x)\right)|\varphi_k(z^m)|d\nu, \quad m \in \mathbb{N}.\end{aligned}$$

Using (2.5), we find

$$\begin{aligned}-I_2 \leqslant& \frac{g(k)}{c_0}\int\limits_{\mathfrak{B}_R} \left(c_0\widetilde{G}_0(x) + G(x)\right)|\varphi_k(z^m)|d\nu \\ &+ \frac{g(k)}{c_0}\int\limits_{\mathfrak{B}_R} a(x, T_k(u^m), \nabla T_k(u^m)\nabla T_k(u^m)|\varphi_k(z^m)|d\nu = I_{21} + I_{22}.\end{aligned} \tag{5.19}$$

In view of (5.14) we get

$$I_{21} = \frac{g(k)}{c_0}\int\limits_{\mathfrak{B}_R} \left(c_0\widetilde{G}_0(x) + G(x)\right)|\varphi_k(z^m)|d\nu = \varepsilon_3(m). \tag{5.20}$$

Since $\varphi_k(z^m)u^m \geqslant 0$ as $|u^m| > h-1 \geqslant k$, we have $\varphi_k(z^m)|u^m| = |\varphi_k(z^m)|u^m$. Using this identity, we estimate the integrals

$$\begin{aligned}I_{12} =& -\int\limits_{\{\mathfrak{B}_R:h-1\leqslant|u^m|<h\}} (a(x, T_h(u^m), \nabla T_h(u^m)) \cdot \nabla|u^m|)\varphi_k(z^m)d\nu \\ =& -\int\limits_{\{\mathfrak{B}_R:h-1\leqslant|u^m|<h\}} (a(x, u^m, \nabla u^m)\cdot\nabla u^m + G(x))|\varphi_k(z^m)|d\nu \\ &+ \int\limits_{\{\mathfrak{B}_R:h-1\leqslant|u^m|<h\}} G(x)|\varphi_k(z^m)|d\nu.\end{aligned}$$

Using (5.2), (5.7), (5.15), we find

$$|I_{12}| \leqslant \varepsilon(h), \quad m \geqslant h, \tag{5.21}$$

where $\varepsilon(h) \to 0$ as $h \to \infty$.

Then, using (5.12) and the inequality $|\nabla\eta_{R-1}(|x|)| \leqslant 1$, we obtain the estimate for the integrals

$$\begin{aligned}&I_{13} = \int\limits_{\{\mathfrak{B}_R:|u^m|<h\}} (a(x, T_h(u^m), \nabla T_h(u^m))\nabla\eta_{R-1}(|x|))\eta_{h-1}(|u^m|)\varphi_k(z^m)dx; \\ &|I_{13}| \leqslant C_7(h)\|\varphi_k(z^m)\|_{p,\mathfrak{B}_R} = \varepsilon_5(m).\end{aligned} \tag{5.22}$$

It is easy to establish the identity $I_1 = I_{11} + I_{12} + I_{13}$, where

$$I_{11} = \int\limits_{\mathfrak{B}_R} a(x, T_h(u^m), \nabla T_h(u^m)) \cdot (\nabla z^m)\eta_{h-1}(|u^m|)\varphi_k'(z^m)d\nu.$$

Now, using the estimates for integrals (5.17)–(5.22), by (5.16) we get the inequalities

$$\begin{aligned} I_5 =& I_{11} - I_{22} = (I_1 + I_2) - I_{12} - I_{13} - I_{22} - I_2 \\ \leqslant& -(I_3 + I_4) + \varepsilon_4(m) + \varepsilon(h) = \varepsilon_5(m) + \varepsilon(h), \quad m \geqslant h. \end{aligned} \tag{5.23}$$

5.2. Representation for I_5. After elementary transformations, we write the identities

$$\begin{aligned} I_5 =& \int\limits_{\mathfrak{B}_R} a(x, T_h(u^m), \nabla T_h(u^m)) \cdot \nabla T_k(u^m) \varphi_k'(z^m) d\nu \\ &- \int\limits_{\mathfrak{B}_R} a(x, T_h(u^m), \nabla T_h(u^m)) \cdot \nabla T_k(u) \varphi_k'(z^m) \eta_{h-1}^m d\nu \\ &- \frac{g(k)}{c_0} \int\limits_{\mathfrak{B}_R} a(x, T_k(u^m), \nabla T_k(u^m)) \cdot \nabla T_k(u^m) |\varphi_k(z^m)| d\nu \\ =& \int\limits_{\mathfrak{B}_R} a(x, T_k(u^m), \nabla T_k(u^m)) \cdot \nabla T_k(u^m) \psi_k(z^m) d\nu \\ &- \int\limits_{\mathfrak{B}_R} a(x, T_h(u^m), \nabla T_h(u^m)) \cdot \nabla T_k(u) \varphi_k'(z^m) \eta_{h-1}^m d\nu \\ =& \int\limits_{\mathfrak{B}_R} a(x, T_k(u^m), \nabla T_k(u^m)) \cdot (\nabla z^m) \psi_k(z^m) d\nu \\ &+ \int\limits_{\mathfrak{B}_R} a(x, T_k(u^m), \nabla T_k(u^m)) \cdot \nabla T_k(u) \psi_k(z^m) d\nu \\ &- \int\limits_{\mathfrak{B}_R} a(x, T_h(u^m), \nabla T_h(u^m)) \cdot \nabla T_k(u) \varphi_k'(z^m) \eta_{h-1}^m d\nu \\ =& I_{51} + I_{52} + I_{53}. \end{aligned}$$

The identity

$$\begin{aligned} I_5 =& I_{51} - \frac{g(k)}{c_0} \int\limits_{\mathfrak{B}_R} a(x, T_k(u^m), \nabla T_k(u^m)) \nabla T_k(u) |\varphi_k(z^m)| d\nu \\ &+ \int\limits_{\mathfrak{B}_R} \left(a(x, T_k(u^m), \nabla T_k(u^m)) - \eta_{h-1}^m a(x, T_h(u^m), \nabla T_h(u^m)) \right) \nabla T_k(u) \varphi_k'(z^m) d\nu \\ =& I_{51} + I_{54} + I_{55}, \quad m \geqslant h \end{aligned}$$

is obvious.

5.3. Estimates for integrals I_{54}, I_{55}. Applying (5.14), (5.15), Lemma 3.6 with

$$H = |\nabla T_k(u) \varphi_k(2k)|^p,$$

we obtain

$$\nabla T_k(u) |\varphi_k(z^m)| \to 0 \quad \text{strongly in} \quad L_p(\Omega), \quad m \to \infty.$$

Hence, in view of the convergence (5.13), we establish

$$I_{54} = \varepsilon(m).$$

In the integral I_{55} the integrand F vanishes for $|u^m| \leqslant k$, and this is why $F = F\chi\{|u^m| > k\}$. Applying (5.14), (5.15), Lemma 3.6, we obtain

$$\nabla T_k(u)\chi\{|u^m| > k\}\varphi_k'(z^m) \to \nabla T_k(u)\chi\{|u| > k\} = 0 \quad \text{strongly in} \quad L_p(\mathfrak{B}_R), \quad m \to \infty.$$

Then in view of the convergence (5.13), we get

$$I_{55} = \varepsilon(m).$$

By (5.23), since I_{51} is independent of h, we find

$$I_{51} \leqslant \varepsilon_6(m). \tag{5.24}$$

We are going to estimate the integral

$$\begin{aligned} I_6 = &\int\limits_\Omega (a(x, T_k(u^m), \nabla T_k(u^m)) - a(x, T_k(u^m), \nabla T_k(u))) \\ &\cdot (\nabla T_k(u^m) - \nabla T_k(u))\psi_k(z^m)d\nu \\ = &\int\limits_\Omega a(x, T_k(u^m), \nabla T_k(u^m))(\nabla T_k(u^m) - \nabla T_k(u))\psi_k(z^m)d\nu \\ &- \int\limits_\Omega a(x, T_k(u^m), \nabla T_k(u))(\nabla T_k(u^m) - \nabla T_k(u))\psi_k(z^m)d\nu = I_{51} - I_{61}. \end{aligned} \tag{5.25}$$

By (2.4) we have the estimates

$$|a(x, T_k(u^m), \nabla T_k(u))|^{p'} \leqslant g(k)(|\nabla T_k(u))|^p + G(x)) \in L_1(\Omega), \quad m \in \mathbb{N}.$$

Then, by the almost everywhere convergence (5.8) and Lemma 3.6 we get the convergence

$$a(x, T_k(u^m), \nabla T_k(u))\psi_k(z^m) \to a(x, T_k(u), \nabla T_k(u)) \quad \text{strongly in} \quad L_{p'}(\Omega), \quad m \to \infty.$$

Applying (5.10) and Lemma 3.7, we find

$$I_{61} = \varepsilon(m), \quad m \in \mathbb{N}.$$

Using (5.24), (5.25), we find

$$I_6 \leqslant \varepsilon_7(m).$$

This is why

$$\begin{aligned} I_7 = &\int\limits_\Omega (a(x, T_k(u^m), \nabla T_k(u^m)) - a(x, T_k(u^m), \nabla T_k(u)))\,(\nabla T_k(u^m) - \nabla T_k(u))d\nu \\ &\leqslant 8/7 I_6 \leqslant \varepsilon(m). \end{aligned}$$

We denote

$$q^j(x) = \Lambda(x, T_k(u^j), \nabla T_k(u^j), \nabla T_k(u)), \quad x \in \Omega, \quad j \in \mathbb{N}. \tag{5.26}$$

Using the notation (5.26), we have

$$0 \leqslant \int\limits_\Omega q^m(x)d\nu = I_7 \leqslant \varepsilon(m).$$

By Lemma 3.11 applied to $w^j = T_k(u^j)$, $w = T_k(u)$, in view of (5.9), we have the convergences (5.11) and

$$a(x, T_k(u^m), \nabla T_k(u^m))\nabla T_k(u^m) \to a(x, T_k(u), \nabla T_k(u))\nabla T_k(u) \quad \text{in} \quad L_{1,\text{loc}}(\overline{\Omega}). \tag{5.27}$$

By (3.8),

$$\nabla T_k(u^m) \to \nabla T_k(u) \quad \text{a.e. in} \quad \Omega, \quad m \to \infty. \tag{5.28}$$

Let us prove that for all $s>0$, $R>0$,

$$b_0^m(x,T_s(u^m),\nabla T_s(u^m))\to b_0(x,T_s(u),\nabla T_s(u)) \quad \text{in} \quad L_1(\mathfrak{B}_R), \quad m\to\infty. \tag{5.29}$$

It follows from (5.8), (5.28) that the convergence

$$b_0(x,T_s(u^m),\nabla T_s(u^m))\to b_0(x,T_s(u),\nabla T_s(u)) \quad \text{a.e. in} \quad \Omega, \quad m\to\infty,$$

holds. By the condition (2.7)

$$|b_0^m(x,T_s(u^m),\nabla T_s(u))|\leqslant g(r)(\widetilde{G}_0(x)+|\nabla T_s(u^m)|^p)\in L_1(\mathfrak{B}_R),$$

where can take $r=s+R$. This is why (5.29) is a corollary of Lemma 3.9 and the convergence (5.11). In the same way we prove the convergence

$$b_1^m(x,T_s(u^m))\to b_1(x,T_s(u)) \quad \text{in} \quad L_{1,\mu}(\mathfrak{B}_R), \quad m\to\infty. \tag{5.30}$$

In order to prove (2.10), we take the test function $v=T_k(u^m-\xi)$, $\xi\in D_\Gamma(\Omega)$, in the identity (4.4) and we obtain

$$\begin{aligned}&\int\limits_\Omega a(x,T_m(u^m),\nabla u^m)\cdot\nabla T_k(u^m-\xi)dx\\&\quad+\int\limits_\Omega (b_0^m(x,u^m,\nabla u^m)-f^m)\,T_k(u^m-\xi)dx+\int\limits_\Omega b_1^m(x,u^m)T_k(u^m-\xi)d\mu=0.\end{aligned}\tag{5.31}$$

We let $\mathbf{M}=k+\|\xi\|_\infty$. If $|u^m|\geqslant\mathbf{M}$, then

$$|u^m-\xi|\geqslant|u^m|-\|\xi\|_\infty\geqslant k,$$

this is why

$$\{\Omega:|u^m-\xi|<k\}\subseteq\{\Omega:|u^m|<\mathbf{M}\},$$

and therefore,

$$\begin{aligned}I^m&=\int\limits_\Omega a(x,T_m(u^m),\nabla u^m)\cdot\nabla T_k(u^m-\xi)dx\\&=\int\limits_\Omega a(x,T_\mathbf{M}(u^m),\nabla T_\mathbf{M}(u^m))\nabla T_k(u^m-\xi)dx\\&=\int\limits_\Omega a(x,T_\mathbf{M}(u^m),\nabla T_\mathbf{M}(u^m))(\nabla T_\mathbf{M}(u^m)-\nabla\xi)\chi_{\{\Omega:|u^m-\xi|<k\}}dx, \quad m\geqslant\mathbf{M}.\end{aligned}$$

We let

$$\begin{aligned}I_1^m:=&\int\limits_{\{\Omega:|u^m-\xi|<k\}}(a(x,T_\mathbf{M}(u^m),\nabla T_\mathbf{M}(u^m))\nabla T_\mathbf{M}(u^m)+G(x))dx\\\geqslant&\int\limits_{\{\Omega:|u^m-\xi|<k,|x|<R\}}(a(x,T_\mathbf{M}(u^m),\nabla T_\mathbf{M}(u^m))\nabla T_\mathbf{M}(u^m)+G(x))dx.\end{aligned}$$

For regular values k the convergence (3.6) of characteristic functions

$$\chi_{\{\Omega:|u^m-\xi|<k\}}\to\chi_{\{\Omega:|u-\xi|<k\}} \quad \text{a.e.} \quad \Omega, \quad m\to\infty.$$

By the convergence (5.27), Lemma 3.9 and Fatou lemma we have

$$\liminf_{m\to\infty} I_1^m\geqslant\int\limits_{\{\Omega:|u-\xi|<k,|x|<R\}}(a(x,T_\mathbf{M}(u),\nabla T_\mathbf{M}(u))\nabla T_\mathbf{M}(u)+G(x))dx, \quad \text{for all} \quad R>0.$$

Then, in view of the non–negativity integrals,

$$\liminf_{m\to\infty} I_1^m \geqslant \int\limits_{\{\Omega:|u-\xi|<k\}} (a(x,T_{\mathbf{M}}(u),\nabla T_{\mathbf{M}}(u))\nabla T_{\mathbf{M}}(u)+G(x))dx.$$

This is why the convergence (5.13) implies the inequality

$$\liminf_{m\to\infty} I^m \geqslant \int\limits_{\{\Omega:|u-\xi|<k\}} a(x,T_{\mathbf{M}}(u),\nabla T_{\mathbf{M}}(u))\cdot(\nabla T_{\mathbf{M}}(u)-\nabla\xi)dx$$
$$=\int\limits_{\Omega} a(x,u,\nabla u)\cdot\nabla T_k(u-\xi)dx = C_I.$$

Using Lemma 3.9 and passing to the limit as $m\to\infty$, we get

$$J_1^m := \int\limits_{\Omega} f^m T_k(u^m-\xi)dx \to \int\limits_{\Omega} fT_k(u-\xi)dx = C_{J_1}. \tag{5.32}$$

We introduce the notation

$$J_2^m := \int\limits_{\Omega} b_0^m(x,u^m,\nabla u^m)T_k(u^m-\xi)dx + \int\limits_{\Omega} b_1^m(x,u^m)T_k(u^m-\xi)d\mu,$$

and by (5.31) we obtain

$$C_I + \liminf_{m\to\infty} J_2^m \leqslant C_{J_1}.$$

Let

$$w^m = u^m-\xi, \qquad w=u-\xi, \qquad \operatorname{supp}\xi\subset\mathfrak{B}_{l_0}, \qquad l\geqslant l_0,$$
$$\mathfrak{B}^m_{l,s}=\{x\in\mathfrak{B}_l:\ |u^m(x)|<s\}, \quad s\geqslant\mathbf{M}, \qquad \mathfrak{B}_{l,s}=\{x\in\mathfrak{B}_l:\ |u(x)|<s\}.$$

We choose the numbers s so that $\operatorname{mes}\{x\in\mathfrak{B}_l:\ |u(x)|=s\}=0$. Then, in view of (4.1) and the inequality $u^m(x)T_k(u^m-\xi)\geqslant 0$ for $|u^m(x)|>\mathbf{M}$ (or for $|x|>l_0$), we have

$$J_2^m = \int\limits_{\Omega\setminus\mathfrak{B}^m_{l,s}} b_0^m(x,u^m,\nabla u^m)T_k(w^m)dx + \int\limits_{\Omega\setminus\mathfrak{B}^m_{l,s}} b_1^m(x,u^m)T_k(w^m)d\mu$$
$$+\int\limits_{\mathfrak{B}^m_{l,s}} b_0^m(x,u^m,\nabla u^m)T_k(w^m)dx + \int\limits_{\mathfrak{B}^m_{l,s}} b_1^m(x,u^m)T_k(w^m)d\mu$$
$$\geqslant \int\limits_{\mathfrak{B}^m_{l,s}} b_0^m(x,T_s(u^m),\nabla u^m)T_k(w^m)dx + \int\limits_{\mathfrak{B}^m_{l,s}} b_1^m(x,T_s(u^m))T_k(w^m)d\mu = J^m_{l,s}.$$

Applying (5.29), (5.30), we pass to the limit as $m\to\infty$ and we obtain

$$\int\limits_{\mathfrak{B}_{l,s}} b_1(x,T_s(u))T_k(u-\xi))d\mu + \int\limits_{\mathfrak{B}_{l,s}} b_0(x,T_s(u),\nabla u)T_k(u-\xi)dx = \lim_{m\to\infty} J^m_{l,s} \leqslant \liminf_{m\to\infty} J_2^m.$$

Since by (2.9)

$$\int\limits_{\mathfrak{B}_{l,s}\setminus\mathfrak{B}_{l_0,s}} b_0(x,T_s(u),\nabla u)T_k(u-\xi)dx = \int\limits_{\mathfrak{B}_{l,s}\setminus\mathfrak{B}_{l_0,s}} |b_0(x,T_s(u),\nabla u)T_k(u)|dx,$$

by Levi theorem we can pass to the limit as $l \to \infty$. Letting $\Omega_s = \{x \in \Omega : \ |u(x)| < s\}$ and passing to the limit as $l \to \infty$, we have

$$\int\limits_{\Omega_s} b_1(x,u)T_k(u-\xi)d\mu + \int\limits_{\Omega_s} b_0(x,u,\nabla u)T_k(u-\xi)dx \leqslant \liminf_{m\to\infty} J_2^m.$$

Since by (2.9),

$$\int\limits_{\Omega_s\setminus\Omega_M} b_1(x,u)T_k(u-\xi)d\mu = \int\limits_{\Omega_s\setminus\Omega_M} |b_1(x,u)T_k(u-\xi)|d\mu,$$

we can pass to the limit as $s \to \infty$. As a result we obtain

$$\int\limits_{\Omega} b_1(x,u)T_k(u-\xi)d\mu + \int\limits_{\Omega} b_0(x,u,\nabla u)T_k(u-\xi)dx \leqslant \liminf_{m\to\infty} J_2^m.$$

Combining (5.31)–(5.32), we obtain (2.10).

6. Uniqueness

Lemma 6.1. *Let u be an entropy solution to the Zaremba problem for Equation* (2.1) *and the assumptions of Theorem 2.2 are satisfied. Then $b_0(x,u) \in L_1(\Omega)$, $b_1(x,u) \in L_{1,\mu}(\Omega)$, and for $k > 1$ the inequalities*

$$\int\limits_{\Omega} |\nabla T_k(u)|^p dx \leqslant Ck \tag{6.1}$$

hold.

Proof. We write the inequality (2.10) for $\xi = 0$

$$\int\limits_{\Omega} (a(x,\nabla u)\cdot \nabla T_k(u) - fT_k(u))dx + \langle \mathcal{B}u, T_k(u)\rangle \leqslant 0.$$

The condition (2.5) implies the inequality

$$\int\limits_{\Omega} a(x,\nabla u)\cdot \nabla T_k(u)dx \geqslant \int\limits_{\Omega} (c_0|\nabla T_k(u)|^p - G(x))dx.$$

Thus,

$$\int\limits_{\Omega} (c_0|\nabla T_k(u)|^p + b_0(x,u)T_k(u))dx + \int\limits_{\Omega} b_1(x,u)T_k(u)d\mu \leqslant \int\limits_{\Omega} (G + fT_k(u))dx < \infty.$$

In view of (2.9), this implies the inequalities (6.1) and

$$\int\limits_{\Omega} |b_0(x,u)|\chi(|u|>1)dx + \int\limits_{\Omega} |b_1(x,u)|\chi(|u|>1)d\mu < \infty. \tag{6.2}$$

The conditions (2.11), (2.12) imply the inequality

$$\int\limits_{\Omega} |b_0(x,u)|\chi(|u|\leqslant 1)dx + \int\limits_{\Omega} |b_1(x,u)|\chi(|u|\leqslant 1)d\mu \leqslant \int\limits_{\Omega} (\widehat{G}_0(x) + \widehat{G}_1(x))dx < \infty,$$

combining which with (6.2), we obtain the first statement of the lemma. The proof is complete. □

Lemma 6.2. *Let u be an entropy solution to the Zaremba problem for Equation* (2.1) *and the assumptions of Theorem 2.2 be satisfied. Then* (2.10) *holds for $\xi \in V \cap L_\infty(\Omega)$.*

Proof. Let $\xi \in V$, $\|\xi\|_\infty \leqslant C_0$. Then there exists a sequence $v_i \in \mathcal{D}_\Gamma(\Omega)$ such that $\|v_i\|_\infty \leqslant C_0$, $\nabla v_i \to \nabla \xi$ in $L_p(\Omega)$. At the same time $v_i \to \xi$ in $L_{p,\mathrm{loc}}(\Omega)$ and a.e. in Ω. By (2.3), we have the convergences $v_i \to \xi$ in $L_{q,\mathrm{loc},\mu}(\Omega)$ and μ–a.e. in Ω. We have

$$T_k(u - v_i) \to T_k(u-\xi) \quad \text{a.e. and } \mu\text{–a.e.}$$

Then,

$$|\nabla T_k(u-v_i)| \leqslant |\nabla T_K(u)| + |\nabla v_i|,$$

where $K = k + C_0$. It is easy to establish that

$$\nabla T_k(u-v_i) \rightharpoonup \nabla T_k(u-\xi) \quad \text{weakly in} \quad L_p(\Omega).$$

Using the definition of the entropy solution, we write the inequality

$$\int\limits_\Omega a(x,\nabla u)\nabla T_k(u-v_i)dx + \int\limits_\Omega b_0(x,u)T_k(u-v_i)dx \\ + \int\limits_\Omega b_1(x,u)T_k(u-v_i)d\mu \leqslant \int\limits_\Omega fT_k(u-v_i)dx.$$

The first of the integrals reads as

$$\int\limits_\Omega a(x,\nabla T_K(u))\nabla T_k(u-v_i)dx,$$

and, in view of (2.4), $a(x,\nabla T_K(u)) \in L_{p'}(\Omega)$. This is why the passage to the limit as $i \to \infty$ is possible in this integral. The passage to the limit in the remaining integrals can be made by the Lebesgue theorem by using Lemma 6.1. The proof is complete. ☐

The proof of Theorem 2.2 is based on an approach from the work [1].

Using the definition of entropy solution, we write the inequality (2.10) for u_1 with $\xi = T_h(u_2)$

$$\int\limits_\Omega a(x,\nabla u_1)\nabla T_k(u_1 - T_h(u_2))dx + \int\limits_\Omega b_0(x,u_1)T_k(u_1-T_h(u_2))dx \\ + \int\limits_\Omega b_1(x,u_1)T_k(u_1-T_h(u_2))d\mu \leqslant \int\limits_\Omega fT_k(u_1-T_h(u_2))dx. \tag{6.3}$$

Applying it in the case $u_1 = u_2 = u$, we find

$$\int\limits_\Omega a(x,\nabla u)\nabla T_k(u - T_h(u))dx + \int\limits_\Omega b_0(x,u)T_k(u-T_h(u))dx \\ + \int\limits_\Omega b_1(x,u)T_k(u-T_h(u))d\mu \leqslant \int\limits_\Omega fT_k(u-T_h(u))dx.$$

Using (2.5) and (2.9), it is easy to obtain the inequality

$$\int\limits_{\{\Omega:\ h\leqslant|u|<h+k\}} |\nabla u|^p dx \leqslant \int\limits_{\{\Omega:\ |u|\geqslant h\}} (G+|f|k)dx = \varepsilon(h), \tag{6.4}$$

where $\varepsilon(h) \to 0$ as $h \to \infty$ (by Lemma 3.4).

Summing the inequality (6.3) with the similar one for u_2, we obtain the relation

$$\begin{aligned}\int\limits_{\{|u_1-T_h(u_2)|<k\}} & a(x,\nabla u_1)\nabla T_k(u_1-T_h(u_2))dx\\ &+\int\limits_{\{|u_2-T_h(u_1)|<k\}} a(x,\nabla u_2)\nabla T_k(u_2-T_h(u_1))dx\\ &+\int\limits_{\Omega}(b_0(x,u_1)T_k(u_1-T_h(u_2))+b_0(x,u_2)T_k(u_2-T_h(u_1)))dx\\ &+\int\limits_{\Omega}(b_1(x,u_1)T_k(u_1-T_h(u_2))+b_1(x,u_2)T_k(u_2-T_h(u_1)))d\mu\\ &\leqslant\int\limits_{\Omega} f(T_k(u_1-T_h(u_2))+T_k(u_2-T_h(u_1)))dx.\end{aligned}\tag{6.5}$$

We denote by I^1 the sum of the two integrals in (6.5), and I^2, I^3, I^4 are other integrals respectively. We have $I^1+I^2+I^3\leqslant I^4$. In order to pass to the limit as $h\to\infty$, we split each of the integrals into several parts. We let

$$A_0=\{x\in\Omega:\ |u_1-u_2|<k,\ |u_1|<h,\ |u_2|<h\}.$$

The sum of the two integrals from (6.5) over this set can be written as

$$I_0=\int\limits_{A_0}(a(x,\nabla u_1)-a(x,\nabla u_2))\nabla(u_1-u_2)dx\geqslant 0.$$

For the integral over the set

$$A_1=\{x\in\Omega:\ |u_1-u_2|<k,\ |u_2|\geqslant h\}$$

we have

$$\int\limits_{A_1}a(x,\nabla u_1)\cdot\nabla(u_1-T_h(u_2))dx=\int\limits_{A_1}a(x,\nabla u_1)\cdot\nabla u_1dx\geqslant-\int\limits_{\{|u_2|\geqslant h\}}G(x)dx=-\varepsilon(h).$$

For the remaining set

$$A_2=\{x\in\Omega:\ |u_1-u_2|<k,\ |u_1|\geqslant h,\ |u_2|<h\}$$

we have the inequality

$$\int\limits_{A_2}a(x,\nabla u_1)\nabla(u_1-T_h(u_2))dx\geqslant-\int\limits_{A_2}(G(x)+a(x,\nabla u_1)\nabla u_2dx.$$

It is clear that

$$\|a(x,\nabla u_1)\nabla u_2\|_{L_1(A_2)}\leqslant\|a(\nabla u_1)\|_{L_{p'}(h\leqslant|u_1|<h+k)}\|\nabla u_2\|_{L_p(h-k\leqslant|u_2|<h)}=\varepsilon_1(h).$$

The latter identity is implied by (6.4) and (2.4).

Making similar calculations for the second integral in (6.5) and summing the obtained results we find that $I^1\geqslant I_0-\varepsilon_2(h)$.

We consider the integral I^3 in the formula (6.5). This integral over the set

$$B_0(h)=\{x\in\Omega:\ |u_1|<h,\ |u_2|<h\}$$

gives the quantity

$$J_0 = \int\limits_{B_0(h)} (b_1(x,u_1) - b_1(x,u_2))T_k(u_1-u_2)d\mu \geqslant 0.$$

The integral I^3 over the set

$$B_1 = \{x \in \Omega : \ |u_1| \geqslant h\},$$

with the vanishing measure, as $h \to \infty$ gives the quantity, which can be estimated as

$$|J_1| \leqslant k \int\limits_{B_1} (|b_1(x,u_1)| + |b_1(x,u_2))|)d\mu \leqslant \varepsilon_3(h).$$

The integral J_2 over the remaining set can be estimated in a similar way $|J_2| \leqslant \varepsilon_4(h)$. As a result we have the inequality

$$I^3 \geqslant \int\limits_{B_0(h)} (b_1(x,u_1) - b_1(x,u_2))T_k(u_1-u_2)d\mu - \varepsilon_5(h).$$

Similarly,

$$I^2 \geqslant \int\limits_{B_0(h)} (b_0(x,u_1) - b_0(x,u_2))T_k(u_1-u_2)dx - \varepsilon_6(h), \qquad I^4 \leqslant \varepsilon_7(h).$$

Summing the above obtained inequalities and omitting some negative terms, we find

$$\int\limits_{B_0(h)} (b_0(x,u_1) - b_0(x,u_2))T_k(u_1-u_2)dx \leqslant \varepsilon_8(h).$$

Using the increasing of the function b_0 in the second variable, Lemma 3.10, and passing to the limit at $h \to \infty$ in this inequality, we obtain

$$\int\limits_{\Omega} (b_0(x,u_1) - b_0(x,u_2))T_k(u_1-u_2)dx \leqslant 0.$$

We then conclude that $u_1 = u_2$ almost everywhere in Ω.

7. Some examples

We provide examples of the functions b_0, b_1, obeying the needed conditions. Let $n = 4$, $p = 3$. The measure μ coincides the Lebesgue measure concentrated on the part of the plane

$$\{x \in \Omega : x_1 = 0, x_2 = 0\}.$$

It is easy to see that this measure belongs to the Morrey class $\mathfrak{M}_2(\Omega)$. Let $g(r)$, $r \geqslant 0$, be an arbitrary increasing function. We let

$$b_0(x,r) = G_0(x)g(|r|)r/|r|, \qquad b_1(x,r) = G_1(x)g(|r|)r/|r|,$$

where $G_0 \in L_1(\Omega)$, $G_1 \in L_{1,\mu}(\Omega)$, and the function G_1 is equal to zero outside the support of the function μ. It is easy to see that the conditions (2.7)–(2.12), except for (2.10), are satisfied. For the existence theorem the function

$$b_0(x,r,y) = G_0(x)g(|r|)\frac{r|y|}{|r|(1+|y|)}$$

is also appropriate.

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