ISSN 2304-0122 Ufa Mathematical Journal. Vol. 16. No 4 (2024). P. 40-52.

doi:10.13108/2024-16-4-40

EFFECTIVE FACTORIZATION OF THIRD ORDER HÖLDER MATRIX FUNCTION IN SOME CLASSES

S.N. KIYASOV

Abstract. We consider a homogeneous linear conjugation problem for a three–dimensional piecewise analytic vector on a simple smooth closed contour partitioning the plane of a complex variable into two regions. To each solution of the problem, we assign a triple of functions which are quotients of limiting values on the contour from the corresponding regions of the components of this solution. We provide identities relating the entries of H–continuous matrix function of the linear conjugation problem ensuring the existence of two of its solutions for which the corresponding components of the triples are proportional and the problem itself admits a solution in closed form.

Key words: matrix function, linear conjugation problem, factorization.

Mathematics Subject Classification: 30-XX

1. Introduction

The qualitative theory of the linear conjugation problem for a piecewise–analytic vector (the vector matrix Riemann boundary value problem, the Hilbert boundary value problem, the Riemann — Hilbert boundary value problem for several unknown functions) under various assumptions on the smoothness of contour had already taken a completely finished form by the end of last century. The theory of problem in classes of Hölder functions for each dimension is presented in the monograph [1]. However, there are comparatively few examples of matrix functions for which the solution of linear conjugation problem can be written in closed form, that is, to express the solution to the problem via Cauchy–type integrals and solution of a certain number of linear algebraic systems.

The homogeneous linear conjugation problem in the class of Hölder functions for a three–dimensional vector is posed as follows. Let Γ be a simple smooth closed contour partitioning the plane of complex variable into two domains D^+ and D^-, $0 \in D^+$, $\infty \in D^-$,

$$G(t) = \begin{pmatrix} g_{11}(t) & g_{12}(t) & g_{13}(t) \\ g_{21}(t) & g_{22}(t) & g_{23}(t) \\ g_{31}(t) & g_{32}(t) & g_{33}(t) \end{pmatrix}, \qquad \Delta(t) = \det G(t) \neq 0, \qquad t \in \Gamma, \tag{1.1}$$

be an H–continuous on Γ third order matrix function.

We need to find a piecewise analytic vector function $\mathbf{w}(z) = (w^1(z), w^2(z), w^3(z))$ of finite order at infinity with H–continuous on Γ limiting values $\mathbf{w}^\pm(t)$ related by the condition

$$\mathbf{w}^+(t) = G(t)\mathbf{w}^-(t), \tag{1.2}$$

S.N. Kiyasov, Efficient factorization of third–order Hölder matrix functions in some classes.

Submitted April 5, 2024.

which can be rewritten in the scalar form as

$$\begin{aligned} w^{1+}(t) &= g_{11}(t)w^{1-}(t) + g_{12}(t)w^{2-}(t) + g_{13}(t)w^{3-}(t), \\ w^{2+}(t) &= g_{21}(t)w^{1-}(t) + g_{22}(t)w^{2-}(t) + g_{23}(t)w^{3-}(t), \\ w^{3+}(t) &= g_{31}(t)w^{1-}(t) + g_{32}(t)w^{2-}(t) + g_{33}(t)w^{3-}(t). \end{aligned} \tag{1.3}$$

We note that even in the case of three zero elements of the matrix function (1.1), unless it is triangular or is reduced to such matrix by a permutation of rows and columns, it is impossible to obtain a solution to the problem in closed form. It was shown in the author's paper [2] that in an arbitrary dimension n with $n-1$ particular solutions to the linear conjugation problem (1.2), the canonical system of solutions to the problem can be constructed in a closed form (the matrix function (1.1) admits effective factorization). We note that a similar result based on the "operator approach" to the study of the problem was obtained in [3]. For a three–dimensional linear conjugation problem, the corresponding statement in [2] can be formulated as follows.

Let

$$\mathbf{w}_i(z) = \left(w_i^1(z), w_i^2(z), w_i^3(z)\right), \quad i = 1, 2 \tag{1.4}$$

be two solutions to the problem (1.3) without finite poles of some orders at the infinity; the positive order stands for the order of the pole. We denote by $\Omega^k(z)$ the second order matrix function obtained from the rectangle matrix function $\Omega(z) = \|w_i^j(z)\|$, $j = 1, 2, 3$, $i = 1, 2$, the columns of which are the components of the vector functions (1.4) obtained by removing the row with the index k. Let for some values of the indices k and s, $k, s = 1, 2, 3$, the determinant of the matrix function $\Omega^k(z)$ does not vanish in $D^+ \cup \Gamma$, while the determinant of the matrix function $\Omega^s(z)$ has no zeroes in $\Gamma \cup D^- \setminus \{\infty\}$. Then the canonical system of solutions of problem (1.2) can be constructed in closed form. For instance, let the determinant $\Delta_3^+(z) = \det \Omega^{3+}(z)$ have no zeroes in $D^+ \cup \Gamma$, and the determinant $\Delta_3^-(z) = \det \Omega^{3-}(z)$ have no zeroes in Γ in a finite part of the domain D^-. We denote the sought systems of solutions by

$$\mathbf{v_i}(z) = \left(v_i^1(z), v_i^2(z), v_i^3(z)\right), \quad i = 1, 2, 3,$$

$\mathbf{v_i}(z)$ has order $(-\varkappa_i)$ at the infinity. Integer numbers $\varkappa_i$, which are partial indices of the matrix function (1.1), are supposed to the taken in ascending order $\varkappa_1 \geqslant \varkappa_2 \geqslant \varkappa_3$, where

$$\varkappa_1 + \varkappa_2 + \varkappa_3 = \varkappa = \operatorname{ind} \det G(t)$$

is the total index of matrix function (1.1). Then the components of the first vector function of canonical systems of solutions (their limiting values on Γ) are determined by the formulas

$$\begin{aligned} v_1^{1+} &= w_1^{1+}\left(P[M] + c^1\right) + w_2^{1+}\left(-P[M_1] + c^2\right), \\ v_1^{2+} &= w_1^{2+}\left(P[M] + c^1\right) + w_2^{2+}\left(-P[M_1] + c^2\right), \\ v_1^{3+} &= \left(p_1\Delta^+ - v_1^{1+}\Delta_1^+ - v_1^{2+}\Delta_2^+\right)/\Delta_3^+; \\ v_1^{1-} &= w_1^{1-}\left(-Q[M] + c^1\right) + w_2^{1-}\left(Q[M_1] + c^2\right), \\ v_1^{2-} &= w_1^{2-}\left(-Q[M] + c^1\right) + w_2^{2-}\left(Q[M_1] + c^2\right), \\ v_1^{3-} &= \left(p_1\Delta^- - v_1^{1-}\Delta_1^- - v_1^{2-}\Delta_2^-\right)/\Delta_3^-, \end{aligned}$$

where

$$P = \frac{1}{2}(I + S), \qquad Q = \frac{1}{2}(I - S),$$

I is the identity operator, S is a singular operator, p_1 is an unknown polynomial, for the degree of which an upper bound is known, c^1 and c^2 are constants,

$$\Delta(t) = \det G(t) = \frac{\Delta^+(t)}{\Delta^-(t)},$$

$\Delta_1^{\pm}(t)$ are the limiting values of the determinant of matrix function $\Omega^1(z)$ on Γ, $\Delta_2^{\pm}(t)$ are the limiting values of the determinant of the matrix function$\Omega^2(z)$ taken with opposite sign on Γ, and

$$M = \frac{p_1\Delta^-}{\Delta_3^+\Delta_3^-}\left(g_{13}w_2^{2+} - g_{23}w_2^{1+}\right), \qquad M_1 = \frac{p_1\Delta^-}{\Delta_3^+\Delta_3^-}\left(g_{13}w_1^{2+} - g_{23}w_1^{1+}\right).$$

The coefficients of polynomial $p_1(z)$ and the constants c^1, c^2 are chosen so that the function $\mathbf{v_1}(z)$ possesses the lowest order $(-\varkappa_1)$ at the infinity. The vector function $\mathbf{v_2}(z)$, of order $-\varkappa_2 \geqslant -\varkappa_1$ at the infinity and not coinciding with the vector function $\mathbf{v_1}(z)$ multiplied by some polynomial, is determined by the same formulas, in which the polynomial $p_1(z)$ should be replaced by the polynomial $p_2(z)$, for the degree of which and the polynomials $c^1(z)$, $c^2(z)$ upper bounds are known. The vector function $\mathbf{v_3}(z)$ is similarly sought as having the order $-\varkappa_3 = \varkappa_1+\varkappa_2-\varkappa$ at infinity and is not relating with $\mathbf{v_1}(z)$ and $\mathbf{v_2}(z)$ by any linear combination with polynomial coefficients.

2. Preliminaries

For the convenience we provide main notion and statements from works [4]–[6], which we employ in the present paper.

Definition 2.1. *Let* $\mathbf{w}(z) = (w^1(z), w^2(z), w^3(z))$ *be a piecewise meromorphic solution to the problem* (1.3). *We call it the solution with triple* $(\lambda_1(t), \lambda_2(t), \lambda_3(t))$ *if*

$$\frac{w^{1+}(t)}{w^{1-}(t)} = \lambda_1(t), \qquad \frac{w^{2+}(t)}{w^{2-}(t)} = \lambda_2(t), \qquad \frac{w^{3+}(t)}{w^{3-}(t)} = \lambda_3(t) \tag{2.1}$$

on Γ.

We suppose that the component of the triple λ_k is equal to zero, unbounded or is undefined that is denoted by 0, ∞, 0/0 if respectively

$$w^{k+}(t) \equiv 0, \quad w^{k-}(t) \equiv 0, \quad w^{k\pm}(t) \equiv 0; \quad k = 1, 2, 3, \quad t \in \Gamma.$$

A triple with components not coinciding with 0, ∞, 0/0 is called non–degenerate.

Under the assumption on existence of two solutions $\mathbf{w}_1(z)$ and $\mathbf{w}_2(z)$ to the problem (1.3) with the same non–degenerate triple (2.1) such that $\mathbf{w}_2(z) \not\equiv r(z)\mathbf{w}_1(z)$, where $r(z)$ is a rational function, in the work [4] there were obtained restrictions of the entries of the matrix function (1.1) ensuring the existence of such solutions and these solutions were written explicitly. According to the above statement, this allows to construct the canonical system of solutions of this problem in a closed form.

It was shown in work [5] that a similar results can be obtained once for solutions of the problem we suppose the coincidence of two components of the triple (2.1), while in [6] there was justified a possibility of constructing in closed form of canonical system of solutions under the assumption on existence of two solutions to the problem (1.3) with the same value of just one component of triple (2.1).

Let $w^{1\pm}(z) \not\equiv 0$ for a solution $\mathbf{w}(z) = (w^1(z), w^2(z), w^3(z))$ of the problem (1.3). We consider well–defined in the corresponding domains relations

$$\Phi(z) = \frac{w^2(z)}{w^1(z)}, \qquad \Psi(z) = \frac{w^3(z)}{w^1(z)}. \tag{2.2}$$

It follows from the boundary conditions (1.3) that the pairs of functions $(\Phi^{\pm}(t), \Psi^{\pm}(t))$ are limiting values on Γ of piecewise meromophic solution $(\Phi(z), \Psi(z))$ to the system of two fractional–linear conjugation problems

$$\Phi^+ = \frac{g_{21} + g_{22}\Phi^- + g_{23}\Psi^-}{g_{11} + g_{12}\Phi^- + g_{13}\Psi^-}, \qquad \Psi^+ = \frac{g_{31} + g_{32}\Phi^- + g_{33}\Psi^-}{g_{11} + g_{12}\Phi^- + g_{13}\Psi^-}. \tag{2.3}$$

And vice versa, if a pair of piecewise meromophic functions $(\Phi(z), \Psi(z))$ is a solution to the system of problem (2.3), then, rewriting these identities in the form

$$g_{11} + g_{12}\Phi^- + g_{13}\Psi^- = \frac{\Phi^-}{\Phi^+}\left(g_{22} + g_{21}\frac{1}{\Phi^-} + g_{23}\frac{\Psi^-}{\Phi^-}\right),$$
$$g_{11} + g_{12}\Phi^- + g_{13}\Psi^- = \frac{\Psi^-}{\Psi^+}\left(g_{33} + g_{31}\frac{1}{\Psi^-} + g_{32}\frac{\Phi^-}{\Psi^-}\right)$$

and taking into consideration (2.2), we obtain that the vector function $\mathbf{w}(z)$, the limiting values of which in Γ are determined by the conditions

$$g_{11} + g_{12}\Phi^- + g_{13}\Psi^- = \frac{w^{1+}}{w^{1-}}, \qquad w^{2\pm} = \Phi^{\pm}w^{1\pm}, \qquad w^{3\pm} = \Psi^{\pm}w^{1\pm}, \tag{2.4}$$

is a solution of the linear conjugation problem (1.3).

Definition 2.2. *Two non–degenerate triples of defined on Γ functions $(\lambda_1(t), \lambda_2(t), \lambda_3(t))$ and $(\tilde\lambda_1(t), \tilde\lambda_2(t), \tilde\lambda_3(t))$ are called similar if the corresponding coefficients are proportional*

$$\frac{\lambda_1(t)}{\tilde\lambda_1(t)} = \frac{\lambda_2(t)}{\tilde\lambda_2(t)} = \frac{\lambda_3(t)}{\tilde\lambda_3(t)}, \quad t \in \Gamma.$$

3. Main results. Auxiliary constructions

In the paper we show the possibility of constructing a canonical system of solutions under the assumption on the existence of two solutions to the linear conjugation problem (1.2) with similar triples. First we provide the necessary conditions for the existence of two such solutions for the problem (1.2), and then we specify additional requirements for the elements of matrix–function (1.1) ensuring the possibility of effectively constructing its canonical system of solutions.

Let $\mathbf{w}_1(z)$ and $\mathbf{w}_2(z)$ be two solutions to the problem (1.2) with similar triples $(\lambda_1(t), \lambda_2(t), \lambda_3(t))$ and $(\tilde\lambda_1(t), \tilde\lambda_2(t), \tilde\lambda_3(t))$. Then the corresponding relations (2.2) determine two piecewise meromorphic solutions $(\Phi_1(z), \Psi_1(z))$ and $(\Phi_2(z), \Psi_2(z))$ of system of fractional–linear conjugation problem (2.3). By the similarity condition of triples in accordance with Definition 2.1 it is easy to conclude that these solutions are related by the identities

$$\Phi_2(z) = r_1(z)\Phi_1(z), \quad \Psi_2(z) = r_2(z)\Psi_1(z), \tag{3.1}$$

in which $r_1(z)$ and $r_2(z)$ are rational functions. Thus, by (2.3) for the limiting values of of the functions (3.1) on Γ we arrive at the system of identities

$$\begin{aligned}
&g_{11}\Phi_1^+ + g_{12}\Phi_1^+\Phi_1^- + g_{13}\Phi_1^+\Psi_1^- = g_{21} + g_{22}\Phi_1^- + g_{23}\Psi_1^-,\\
&g_{11}\Psi_1^+ + g_{12}\Psi_1^+\Phi_1^- + g_{13}\Psi_1^+\Psi_1^- = g_{31} + g_{32}\Phi_1^- + g_{33}\Psi_1^-,\\
&r_1g_{11}\Phi_1^+ + r_1^2g_{12}\Phi_1^+\Phi_1^- + r_1r_2g_{13}\Phi_1^+\Psi_1^- = g_{21} + r_1g_{22}\Phi_1^- + r_2g_{23}\Psi_1^-,\\
&r_2g_{11}\Psi_1^+ + r_1r_2g_{12}\Psi_1^+\Phi_1^- + r_2^2g_{13}\Psi_1^+\Psi_1^- = g_{31} + r_1g_{32}\Phi_1^- + r_2g_{33}\Psi_1^-.
\end{aligned} \tag{3.2}$$

Excluding the products $\Phi_1^+(t)\Phi_1^-(t)$ and $\Psi_1^+(t)\Phi_1^-(t)$ from the identities (3.2) and then the products $\Phi_1^+(t)\Psi_1^-(t)$ and $\Psi_1^+(t)\Psi_1^-(t)$, for $r_1(t) \neq r_2(t)$ we arrive at the identities

$$\begin{aligned}
&r_1\Phi_1^+\left[(r_1-1)g_{11} + (r_1-r_2)g_{13}\Psi_1^-\right] = (r_1^2-1)g_{21} + r_1(r_1-1)g_{22}\Phi_1^- + (r_1^2-r_2)g_{23}\Psi_1^-,\\
&r_2\Psi_1^+\left[(r_1-1)g_{11} + (r_1-r_2)g_{13}\Psi_1^-\right] = (r_1r_2-1)g_{31} + r_1(r_2-1)g_{32}\Phi_1^- + r_2(r_1-1)g_{33}\Psi_1^-,\\
&r_1\Phi_1^+\left[(r_2-1)g_{11} + (r_2-r_1)g_{12}\Phi_1^-\right] = (r_1r_2-1)g_{21} + r_1(r_2-1)g_{22}\Phi_1^- + r_2(r_1-1)g_{23}\Psi_1^-,\\
&r_2\Psi_1^+\left[(r_2-1)g_{11} + (r_2-r_1)g_{12}\Phi_1^-\right] = (r_2^2-1)g_{31} + (r_2^2-r_1)g_{32}\Phi_1^- + r_2(r_2-1)g_{33}\Psi_1^-
\end{aligned} \tag{3.3}$$

on Γ.

We introduce the following notation on Γ :

$$\begin{aligned}
&(r_1-1)g_{11}+(r_1-r_2)g_{13}\Psi_1^- = \phi_1^+/\phi_1^-,\\
&(r_2-1)g_{11}+(r_2-r_1)g_{12}\Phi_1^- = \phi_2^+/\phi_2^-,\\
&\Phi_1^\pm\phi_1^\pm=\phi_3^\pm, \qquad \Psi_1^\pm\phi_1^\pm=\phi_4^\pm,\\
&\Phi_1^\pm\phi_2^\pm=\phi_5^\pm, \qquad \Psi_1^\pm\phi_2^\pm=\phi_6^\pm.
\end{aligned}\tag{3.4}$$

This allows us to write the above obtained nonlinear conditions for the limiting values of the functions $\Phi_1^\pm(t)$, $\Psi_1^\pm(t)$ on Γ as the linear conjugation problem

$$\begin{aligned}
\phi_1^+ &= (r_1-1)g_{11}\phi_1^- + (r_1-r_2)g_{13}\phi_4^-,\\
\phi_2^+ &= (r_2-1)g_{11}\phi_2^- + (r_2-r_1)g_{12}\phi_5^-,\\
\phi_3^+ &= \frac{(r_1^2-1)}{r_1}g_{21}\phi_1^- + (r_1-1)g_{22}\phi_3^- + \frac{(r_1^2-r_2)}{r_1}g_{23}\phi_4^-,\\
\phi_4^+ &= \frac{(r_1r_2-1)}{r_2}g_{31}\phi_1^- + \frac{r_1(r_2-1)}{r_2}g_{32}\phi_3^- + (r_1-1)g_{33}\phi_4^-,\\
\phi_5^+ &= \frac{(r_1r_2-1)}{r_1}g_{21}\phi_2^- + (r_2-1)g_{22}\phi_5^- + \frac{r_2(r_1-1)}{r_1}g_{23}\phi_6^-,\\
\phi_6^+ &= \frac{(r_2^2-1)}{r_2}g_{31}\phi_2^- + \frac{(r_2^2-r_1)}{r_2}g_{32}\phi_5^- + (r_2-1)g_{33}\phi_6^-.
\end{aligned}\tag{3.5}$$

According to (3.4), the solutions of the obtained linear conjugation problem should satisfy the conditions

$$\frac{\phi_3^\pm(t)}{\phi_1^\pm(t)}=\frac{\phi_5^\pm(t)}{\phi_2^\pm(t)}, \qquad \frac{\phi_4^\pm(t)}{\phi_1^\pm(t)}=\frac{\phi_6^\pm(t)}{\phi_2^\pm(t)}, \qquad t\in\Gamma. \tag{3.6}$$

And vice versa, given a solution to the linear conjugation problem (3.5) obeying the conditions (3.6), we rewrite the boundary conditions (3.5) as

$$\begin{aligned}
\frac{\phi_1^+}{\phi_1^-} &= (r_1-1)g_{11} + (r_1-r_2)g_{13}\frac{\phi_4^-}{\phi_1^-},\\
\frac{\phi_2^+}{\phi_2^-} &= (r_2-1)g_{11} + (r_2-r_1)g_{12}\frac{\phi_5^-}{\phi_2^-},\\
\frac{\phi_3^+}{\phi_1^+}\frac{\phi_1^+}{\phi_1^-} &= \frac{(r_1^2-1)}{r_1}g_{21} + (r_1-1)g_{22}\frac{\phi_3^-}{\phi_1^-} + \frac{(r_1^2-r_2)}{r_1}g_{23}\frac{\phi_4^-}{\phi_1^-},\\
\frac{\phi_4^+}{\phi_1^+}\frac{\phi_1^+}{\phi_1^-} &= \frac{(r_1r_2-1)}{r_2}g_{31} + \frac{r_1(r_2-1)}{r_2}g_{32}\frac{\phi_3^-}{\phi_1^-} + (r_1-1)g_{33}\frac{\phi_4^-}{\phi_1^-},\\
\frac{\phi_5^+}{\phi_2^+}\frac{\phi_2^+}{\phi_2^-} &= \frac{(r_1r_2-1)}{r_1}g_{21} + (r_2-1)g_{22}\frac{\phi_5^-}{\phi_2^-} + \frac{r_2(r_1-1)}{r_1}g_{23}\frac{\phi_6^-}{\phi_2^-},\\
\frac{\phi_6^+}{\phi_2^+}\frac{\phi_2^+}{\phi_2^-} &= \frac{(r_2^2-1)}{r_2}g_{31} + \frac{(r_2^2-r_1)}{r_2}g_{32}\frac{\phi_5^-}{\phi_2^-} + (r_2-1)g_{33}\frac{\phi_6^-}{\phi_2^-}
\end{aligned}$$

and letting

$$\Phi_1^\pm(t)=\frac{\phi_3^\pm(t)}{\phi_1^\pm(t)}=\frac{\phi_5^\pm(t)}{\phi_2^\pm(t)}, \qquad \Psi_1^\pm(t)=\frac{\phi_4^\pm(t)}{\phi_1^\pm(t)}=\frac{\phi_6^\pm(t)}{\phi_2^\pm(t)}, \qquad t\in\Gamma, \tag{3.7}$$

we obtain that the functions $(\Phi_1(z),\Psi_1(z))$ and functions (3.1) obey the conditions (3.3) and hence the conditions (3.2).

Let us consider some particular cases of the linear conjugation problem (1.3) allowing to write a solution to linear conjugation problem (3.5) in closed form.

Suppose that the entries of the matrix function (1.1) obey the conditions

$$g_{12}(t) \neq 0, \quad g_{13}(t) \neq 0, \quad g_{21}(t) \neq 0, \quad g_{33}(t) \neq 0, \quad g_{11}(t) \equiv 0, \quad g_{23}(t) \equiv 0, \quad t \in \Gamma. \tag{3.8}$$

In this case the determinant of matrix function of linear conjugation problem (3.5) is equal to

$$\begin{aligned}\Delta_1 =& \frac{(r_1 r_2 - 1)}{r_1 r_2}(r_2 - r_1)^2 (r_1 - 1)(r_2 - 1) g_{12} g_{13} g_{21} g_{33} \\ &\cdot \left[(r_1 + 1)(r_2 - 1) g_{21} g_{32} - (r_1 r_2 - 1) g_{22} g_{31}\right].\end{aligned} \tag{3.9}$$

We suppose that this determinant is not equal to zero and infinity on Γ. Under conditions (3.8) the linear conjugation problem (3.5) is rewritten as

$$\begin{aligned}
\phi_1^+ &= (r_1 - r_2) g_{13} \phi_4^-, \\
\phi_2^+ &= (r_2 - r_1) g_{12} \phi_5^-, \\
\phi_3^+ &= \frac{(r_1^2 - 1)}{r_1} g_{21} \phi_1^- + (r_1 - 1) g_{22} \phi_3^-, \\
\phi_4^+ &= \frac{(r_1 r_2 - 1)}{r_2} g_{31} \phi_1^- + \frac{r_1 (r_2 - 1)}{r_2} g_{32} \phi_3^- + (r_1 - 1) g_{33} \phi_4^-, \\
\phi_5^+ &= \frac{(r_1 r_2 - 1)}{r_1} g_{21} \phi_2^- + (r_2 - 1) g_{22} \phi_5^-, \\
\phi_6^+ &= \frac{(r_2^2 - 1)}{r_2} g_{31} \phi_2^- + \frac{(r_2^2 - r_1)}{r_2} g_{32} \phi_5^- + (r_2 - 1) g_{33} \phi_6^-.
\end{aligned} \tag{3.10}$$

By the first two boundary conditions we find

$$\phi_1^+ = (r_1 - r_2) g_{13}^+, \qquad \phi_4^- = \frac{1}{g_{13}^-}, \qquad \phi_2^+ = (r_2 - r_1) g_{12}^+, \qquad \phi_5^- = \frac{1}{g_{12}^-}, \tag{3.11}$$

where and in what follows for $g_{ij}(t) \neq 0$, $t \in \Gamma$ we let $g_{ij}(t) = g_{ij}^+(t) g_{ij}^-(t)$, $i, j = 1, 2, 3$, which are the factorizations on Γ of the corresponding entries of matrix function (1.1). Then by the last two boundary conditions (3.10) we find

$$\begin{aligned}
\phi_5^+ &= \frac{(r_1 r_2 - 1)}{r_1} g_{21}^+ \left(P[K] + R\right), \qquad & \phi_2^- &= \frac{1}{g_{21}^-} \left(-Q[K] + R\right), \\
\phi_6^+ &= (r_2 - 1) g_{33}^+ \left(P[K_1] + R_1\right), \qquad & \phi_6^- &= \frac{1}{g_{33}^-} \left(-Q[K_1] + R_1\right).
\end{aligned} \tag{3.12}$$

Here R, R_1 are rational functions and the functions K and K_1 are defined by the formulas

$$K = \frac{r_1 (r_2 - 1)}{(r_1 r_2 - 1)} \frac{g_{22}}{g_{21}^+ g_{12}^-}, \qquad K_1 = \frac{(r_2^2 - r_1)}{r_2 (r_2 - 1)} \frac{g_{32}}{g_{33}^+ g_{12}^-} + \frac{(r_2 + 1)}{r_2} \frac{g_{31}}{g_{33}^+ g_{21}^-} \left(-Q[K] + R\right). \tag{3.13}$$

It is impossible to determine the components $\phi_3^\pm(t)$ on Γ as well as the components $\phi_4^+(t)$ and $\phi_1^-(t)$ of solution to the linear conjugation problem (3.10) in the present case without additional restrictions for the entries of matrix function (1.1). However, in order the functions $\Phi_1^\pm(t)$, $\Psi_1^\pm(t)$ to satisfy the identities (3.2) ($g_{11}(t) \equiv 0$, $g_{23}(t) \equiv 0$), they should satisfy the identities (3.7), in accordance with which and the formulas (3.11)–(3.13) we obtain

$$\begin{aligned}
\Phi_1^+ &= \frac{\phi_5^+}{\phi_2^+} = \frac{(r_1r_2-1)}{r_1(r_2-r_1)}\frac{g_{21}^+}{g_{12}^+}\left(P[K]+R\right),\\
\phi_3^+ &= \phi_1^+\Phi_1^+ = -\frac{(r_1r_2-1)}{r_1}\frac{g_{13}^+g_{21}^+}{g_{12}^+}\left(P[K]+R\right),\\
\Phi_1^- &= \frac{\phi_5^-}{\phi_2^-} = \frac{g_{21}^-}{g_{12}^-\left(-Q[K]+R\right)},\\
\Psi_1^+ &= \frac{\phi_6^+}{\phi_2^+} = \frac{(r_2-1)g_{33}^+}{(r_2-r_1)g_{12}^+}\left(P[K_1]+R_1\right),\\
\phi_4^+ &= \phi_1^+\Psi_1^+ = -\frac{(r_2-1)g_{13}^+g_{33}^+}{g_{12}^+}\left(P[K_1]+R_1\right),\\
\Psi_1^- &= \frac{\phi_6^-}{\phi_2^-} = \frac{g_{21}^-\left(-Q[K_1]+R_1\right)}{g_{33}^-\left(-Q[K]+R\right)},\\
\phi_1^- &= \frac{\phi_4^-}{\Psi_1^-} = \frac{g_{33}^-\left(-Q[K]+R\right)}{g_{13}^-g_{21}^-\left(-Q[K_1]+R_1\right)},\\
\phi_3^- &= \phi_1^-\Phi_1^- = \frac{g_{33}^-}{g_{12}^-g_{13}^-\left(-Q[K_1]+R_1\right)}.
\end{aligned} \tag{3.14}$$

Substituting the values $\phi_1^-(t)$, $\phi_3^\pm(t)$, $\phi_4^\pm(t)$ into the third and fourth boundary conditions in (3.10), we see that the following identities should be satisfied on Γ:

$$\begin{aligned}
&(r_1r_2-1)g_{13}g_{21}g_{12}^-\left(P[K]+R\right)\left(-Q[K_1]+R_1\right)\\
&\qquad+(r_1^2-1)g_{12}g_{21}g_{33}^-\left(-Q[K]+R\right)+r_1(r_1-1)g_{22}g_{12}^+g_{21}^-g_{33}^-\equiv 0,
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
&r_2(r_2-1)g_{13}g_{33}^+g_{12}^-g_{21}^-\left(P[K_1]+R_1\right)\left(-Q[K_1]+R_1\right)\\
&\qquad+(r_1r_2-1)g_{12}g_{31}g_{33}^-\left(-Q[K]+R\right)\\
&\qquad+r_2(r_1-1)g_{12}g_{33}g_{21}^-\left(-Q[K_1]+R_1\right)+r_1(r_2-1)g_{32}g_{12}^+g_{21}^-g_{33}^-\equiv 0.
\end{aligned} \tag{3.16}$$

The coefficients of the scalar Riemann problems for determining the first components $w_1^1(z)$ and $w_2^1(z)$ of the solutions (1.4) by the found solutions (3.14) (under the identities (3.15), (3.16)) are determined by the first formula in (2.4) and are respectively equal to

$$\frac{g_{21}^-\left(g_{12}^+g_{33}^-+g_{13}\left(-Q[K_1]+R_1\right)\right)}{g_{33}^-\left(-Q[K]+R\right)}, \tag{3.17}$$

$$\frac{g_{21}^-\left(r_1g_{12}^+g_{33}^-+r_2g_{13}\left(-Q[K_1]+R_1\right)\right)}{g_{33}^-\left(-Q[K]+R\right)}. \tag{3.18}$$

Calculating the determinant of the above introduced matrix function $\Omega^3(z)$ in the corresponding domain, we obtain

$$\det\Omega^3(z) = (r_1(z)-1)\Phi_1(z)w_1^1(z)w_2^1(z). \tag{3.19}$$

4. First main result

Theorem 4.1. *Let Γ be a simple smooth closed contour partitioning the plane of complex variable into two domains D^+ and D^-, $0\in D^+$, $\infty\in D^-$), $G(t)$ be an H–continuous on Γ third order matrix function and* $\det G(t)\neq 0$. *If the entries of matrix function* (1.1) *satisfy the conditions* (3.8) *and are related by the identities* (3.15), (3.16), *in which for the rational functions $r_1\neq 1$ is a constant, r_2, R, R_1, the determinant* (3.9) *and the functions* (3.17), (3.18) *do not equal to zero and infinity on* Γ*, the functions* $\Psi_1^{\pm}(z)$ *defined in* (3.13), (3.14) *have no finite poles, and the function* $\Phi_1^{\pm}(z)$ *and functions* (3.19) *have no zeroes and poles in the domain* D^+*, on* Γ *and the finite part of the domain* D^-*. Then the canonical system of solution of linear conjugation problem* (1.2) *can be effectively constructed.*

Proof. If the assumptions of theorem are satisfied, then the functions $\Phi_1^{\pm}(t)$, $\Psi_1^{\pm}(t)$ defined in (3.13), (3.14) and the functions (3.1) satisfy the corresponding identities (3.2) on Γ. This statement can be verified if we express the products $(P[K]+R)\,(-Q[K_1]+R_1)$ and $(P[K_1]+R_1)\,(-Q[K_1]+R_1)$ from the identities (3.15), (3.16) and take into consideration the properties of the operators P and Q $(P+Q=I)$. We determine the components $w_1^1(z)$ and $w_2^1(z)$ as solutions to the corresponding scalar Riemann problem with the coefficients (3.17), (3.18) in the class of functions without zeroes in the domains D^+ and D^- admitting a polar singularity only at infinity (canonical functions of these problems) and we obtain other two components of the solutions (1.4) by the second and third formula in (2.4). This allows us to obtain two solutions of the linear conjugation problem (1.2), for the which the determinants (3.19) of matrix function $\Omega^3(z)$ have no finite zeroes. This allows to construct effectively the canonical system of solutions for the problem by the proposed in [2] algorithm. The proof is complete. $\square$

We note that owing to the boundedness of the operators $P[K]$ and $Q[K]$ we can take R constant and by choosing it appropriately, we can achieve that the expressions $P[K]+R$ and $-Q[K]+R$ involved in the definitions of functions $\Phi_1^{\pm}(z)$ do not vanish on Γ and the corresponding domains D^+ and D^-.

If we admit the presence of zeroes for the determinants (3.19) at finitely many points in the plane, the proposed in [2] algorithm for constructing solutions to the problem (1.2) can be also realized by using the results in the work [7].

Remark 4.1. *We denote by* $F_{(i,j,k)}$ *the matrix of third order permutations, where the components of index* $i \neq j \neq k$ *take the values* 1, 2, 3, *while its non–zero entries in the first, second and third rows are respectively the elements* $f_{1i}=f_{2j}=f_{3k}=1$*. Under the left multiplication of the matrix function* $G(t)$ *by the matrix* $F_{(i,j,k)}$*, the row of the matrix function with the index* i *becomes the first row, the row with the index* j *becomes the second row, while the row with the index* k *becomes the third row. In the right multiplication by* $F_{(i,j,k)}$*, the first row comes from the column, the index of which is determined by the position of the component of index, which is equal to* 1*, the second row comes from the column, the index of which is determined by position of component of index, which is equal to* 2*, and the third row comes from the column, the index of which is determined by the position of component of index, which is equal to* 3*. The inverse matrix of* $F_{(i,j,k)}$ *coincides with its transposition:* $F^{-1}_{(i,j,k)}=F'_{(i,j,k)}$*. Writing the corresponding assumptions of Theorem 4.1 for the matrix functions* $F_{(i,j,k)}G(t)$, $G(t)F_{(i,j,k)}$, $F_{(i,j,k)}G(t)F_{(i_1,j_1,k_1)}$*, we arrive at other cases of the effective factorization of the matrix function* (1.1).

Example 1. Let the conditions (3.8) and corresponding identities (3.15), (3.16) be satisfied

$$
\begin{aligned}
&g_{13}g_{21}\overset{-}{g}_{12}P\left[\frac{g_{22}}{\overset{+}{g}_{21}\overset{-}{g}_{12}}\right]Q\left[\frac{g_{32}}{\overset{+}{g}_{33}\overset{-}{g}_{12}}\right]+2g_{12}g_{21}\overset{-}{g}_{33}Q\left[\frac{g_{22}}{\overset{+}{g}_{21}\overset{-}{g}_{12}}\right]-g_{22}\overset{+}{g}_{12}\overset{-}{g}_{21}\overset{-}{g}_{33}\equiv 0,\\
&9g_{13}\overset{+}{g}_{33}\overset{-}{g}_{12}\overset{-}{g}_{21}P\left[\frac{g_{32}}{\overset{+}{g}_{21}\overset{-}{g}_{12}}\right]Q\left[\frac{g_{32}}{\overset{+}{g}_{21}\overset{-}{g}_{12}}\right]+8g_{12}g_{31}\overset{-}{g}_{33}Q\left[\frac{g_{22}}{\overset{+}{g}_{21}\overset{-}{g}_{12}}\right]\\
&\qquad\qquad\qquad\qquad+9g_{12}g_{33}\overset{-}{g}_{21}Q\left[\frac{g_{32}}{\overset{+}{g}_{33}\overset{-}{g}_{12}}\right]-8g_{32}\overset{+}{g}_{12}\overset{-}{g}_{21}\overset{-}{g}_{33}\equiv 0,
\end{aligned}
$$

in which

$$r_1=-2,\qquad r_2=-1,\qquad R=R_1=0$$

are constants. If the determinant (3.9), which is equal to

$$\Delta_1 = 3g_{12}g_{13}g_{21}g_{33}(g_{21}g_{32} - g_{22}g_{31}),$$

does not vanish on Γ, then the corresponding functions (3.7), in accordance with (3.14), read as

$$\Phi_1^+ = -\frac{2g_{21}^+}{g_{12}^+}P\left[\frac{g_{22}}{g_{21}^+g_{12}^-}\right], \qquad \Phi_1^- = -g_{21}^-/4g_{12}^-Q\left[\frac{g_{22}}{g_{21}^+g_{12}^-}\right],$$
$$\Psi_1^+ = -\frac{3g_{33}^+}{g_{12}^+}P\left[\frac{g_{32}}{g_{21}^+g_{12}^-}\right], \qquad \Psi_1^- = 3g_{21}^-Q\left[\frac{g_{32}}{g_{33}^+g_{12}^-}\right]/8g_{33}^-Q\left[\frac{g_{22}}{g_{21}^+g_{12}^-}\right].$$

The verification that the functions $\Phi_1^\pm$, $\Psi_1^\pm$ and $\Phi_2^\pm = -2\Phi_1^\pm$, $\Psi_2^\pm = -\Psi_1^\pm$, satisfy the corresponding conditions (2.3) can be made straightforwardly taking into consideration the above given identities.

The coefficients of the scalar homogeneous Riemann problems $w_i^{1+}(t) = G_i(t)w_i^{1-}(t)$, $i = 1, 2$, ($G_i(t)$ are H–continuous on Γ functions) for defining the first components of sought solutions $\mathbf{w}_1(z)$ and $\mathbf{w}_2(z)$ to problem (1.3) are respectively equal to

$$G_1 = -g_{21}^-\left(2g_{12}^+g_{33}^- - 3g_{13}Q\left[\frac{g_{32}}{g_{33}^+g_{12}^-}\right]\right)/8g_{33}^-Q\left[\frac{g_{22}}{g_{21}^+g_{12}^-}\right],$$
$$G_2 = g_{21}^-\left(4g_{12}^+g_{33}^- - 3g_{13}Q\left[\frac{g_{32}}{g_{33}^+g_{12}^-}\right]\right)/8g_{33}^-Q\left[\frac{g_{22}}{g_{21}^+g_{12}^-}\right].$$

If these coefficients are not equal to zero and infinity on Γ, the functions Φ_1^-, Ψ_1^- have no finite poles, and the function Φ_1^+ is is non–zero, then we determine the first components of $w_i^{1\pm}(z)$, $i = 1, 2$ of the solutions (1.4) in accordance with [8], we find that the determinants (3.19) have no zeroes on Γ and in the corresponding domain of the finite and hence, the canonical system of solutions of corresponding problem (1.3) can be effectively constructed.

Now we consider the case of existence of two solutions to the linear conjugation problem (1.3) with similar triples, for which in the identities (3.1) we have

$$r_1(z) = r_2(z) = r(z), \qquad r(t) \neq 1, \qquad t \in \Gamma.$$

In this case the boundary conditions (3.2) are rewritten as

$$\begin{aligned}
&g_{11}\Phi_1^+ + g_{12}\Phi_1^+\Phi_1^- + g_{13}\Phi_1^+\Psi_1^- = g_{21} + g_{22}\Phi_1^- + g_{23}\Psi_1^-,\\
&g_{11}\Psi_1^+ + g_{12}\Psi_1^+\Phi_1^- + g_{13}\Psi_1^+\Psi_1^- = g_{31} + g_{32}\Phi_1^- + g_{33}\Psi_1^-,\\
&rg_{11}\Phi_1^+ + r^2g_{12}\Phi_1^+\Phi_1^- + r^2g_{13}\Phi_1^+\Psi_1^- = g_{21} + rg_{22}\Phi_1^- + rg_{23}\Psi_1^-,\\
&rg_{11}\Psi_1^+ + r^2g_{12}\Psi_1^+\Phi_1^- + r^2g_{13}\Psi_1^+\Psi_1^- = g_{31} + rg_{32}\Phi_1^- + rg_{33}\Psi_1^-,
\end{aligned} \tag{4.1}$$

while the conditions (3.3) are reduced to two conditions

$$\begin{aligned}
&r(r-1)g_{11}\Phi_1^+ = (r^2-1)g_{21} + r(r-1)g_{22}\Phi_1^- + r(r-1)g_{23}\Psi_1^-,\\
&r(r-1)g_{11}\Psi_1^+ = (r^2-1)g_{31} + r(r-1)g_{32}\Phi_1^- + r(r-1)g_{33}\Psi_1^-,
\end{aligned} \tag{4.2}$$

which do not imply the identities (4.1).

Introducing the notation

$$(r-1)g_{11} = \phi_1^+/\phi_1^-, \qquad \Phi_1^\pm\phi_1^\pm = \phi_2^\pm, \qquad \Psi_1^\pm\phi_1^\pm = \phi_3^\pm, \tag{4.3}$$

according to (4.2) and notation (4.3), we arrive at the linear conjugation problem

$$\begin{aligned} \phi_1^+ &= (r-1)g_{11}\phi_1^-, \\ \phi_2^+ &= \frac{(r^2-1)}{r}g_{21}\phi_1^- + (r-1)g_{22}\phi_2^- + (r-1)g_{23}\phi_3^-, \\ \phi_3^+ &= \frac{(r^2-1)}{r}g_{31}\phi_1^- + (r-1)g_{32}\phi_2^- + (r-1)g_{33}\phi_3^-. \end{aligned} \tag{4.4}$$

The determinant of the matrix function of linear conjugation problem (4.4) is equal to

$$\Delta_2(t) = (r(t)-1)^3 g_{11}(t)(g_{22}(t)g_{33}(t) - g_{23}(t)g_{32}(t)).$$

We suppose that this determinant is not equal to zero and infinity on Γ. We suppose that the entries of the matrix function (1.1) obey the conditions

$$g_{11}(t) \neq 0, \quad g_{22}(t) \neq 0, \quad g_{33}(t) \neq 0, \quad g_{23}(t) \equiv 0, \quad t \in \Gamma, \quad (\Delta_2(t) \neq 0). \tag{4.5}$$

Then a partial piecewise meromorphic solution to the corresponding linear conjugation problem (4.4) (its limiting values on Γ) can be written by the formulas

$$\begin{aligned} \phi_1^+ &= (r-1)g_{11}^+, & \phi_1^- &= \frac{1}{g_{11}^-}, \\ \phi_2^+ &= (r-1)g_{22}^+\left(P[K]+R\right), & \phi_2^- &= \frac{1}{g_{22}^-}\left(-Q[K]+R\right), \\ \phi_3^+ &= (r-1)g_{33}^+\left(P[K_1]+R_1\right), & \phi_3^- &= \frac{1}{g_{33}^-}\left(-Q[K_1]+R_1\right), \end{aligned}$$

where R, R_1 are rational functions and K, K_1 are defined by the formulas

$$K = \frac{(r+1)}{r}\frac{g_{21}}{g_{22}^+ g_{11}^-}, \qquad K_1 = \frac{(r+1)}{r}\frac{g_{31}}{g_{33}^+ g_{11}^-} + \frac{g_{32}}{g_{33}^+ g_{22}^-}(-Q[K]+R). \tag{4.6}$$

Then, in view of notation (4.3), the functions

$$\begin{aligned} \Phi_1^+ &= \frac{g_{22}^+}{g_{11}^+}\left(P[K]+R\right), & \Phi_1^- &= \frac{g_{11}^-}{g_{22}^-}\left(-Q[K]+R\right); \\ \Psi_1^+ &= \frac{g_{33}^+}{g_{11}^+}\left(P[K_1]+R_1\right), & \Psi_1^- &= \frac{g_{11}^-}{g_{33}^-}\left(-Q[K_1]+R_1\right) \end{aligned} \tag{4.7}$$

should be the limiting values on Γ of a piecewise meromorphic solution to fractional–linear conjugation problem (2.3), which, in view of (4.5), we rewrite as

$$\begin{aligned} g_{11}\Phi^+ + g_{12}\Phi^+\Phi^- + g_{13}\Phi^+\Psi^- &= g_{21} + g_{22}\Phi^-, \\ g_{11}\Psi^+ + g_{12}\Psi^+\Phi^- + g_{13}\Psi^+\Psi^- &= g_{31} + g_{32}\Phi^- + g_{33}\Psi^-. \end{aligned} \tag{4.8}$$

Substituting the functions (4.7) into the boundary conditions (4.8), we find that on Γ the following system of identities should be satisfied:

$$\begin{aligned} g_{12}\frac{g_{22}^+ g_{11}^-}{g_{11}^+ g_{22}^-}\left(P[K]+R\right)\left(-Q[K]+R\right) + g_{13}\frac{g_{22}^+ g_{11}^-}{g_{11}^+ g_{33}^-}\left(P[K]+R\right)\left(-Q[K_1]+R_1\right) &\equiv -\frac{g_{21}}{r}, \\ g_{12}\frac{g_{33}^+ g_{11}^-}{g_{11}^+ g_{22}^-}\left(P[K_1]+R_1\right)\left(-Q[K]+R\right) + g_{13}\frac{g_{33}^+ g_{11}^-}{g_{11}^+ g_{33}^-}\left(P[K_1]+R_1\right)\left(-Q[K_1]+R_1\right) &\equiv -\frac{g_{31}}{r}. \end{aligned} \tag{4.9}$$

We treat the above identities as an algebraic system for determining the entries $g_{12}(t)$ and $g_{13}(t)$ of the matrix function (1.1). Since the determinant of this system is identically zero, this system is compatible if and only if the rank of its matrix coincides with the rank of its extended matrix, and this leads us to the condition

$$g_{21}g_{33}^+\left(P[K_1]+R_1\right) - g_{31}g_{22}^+\left(P[K]+R\right) \equiv 0. \tag{4.10}$$

It is straightforward to show that the validity of (4.10) and one of the relations (4.9) implies the validity of the other relation. Thus, the functions (4.7) determine a solution to the system of fractional–linear conjugation problem (4.8) if, for instance, the first identity in (4.9) and identity (4.10) are satisfied on Γ.

The obtained conditions can be transformed in another form. Indeed, according to (4.10) and the properties of the operators P and Q, the first identity in (4.9) can be written as

$$\begin{aligned}&(g_{12}g_{21}g_{33}+g_{13}g_{22}g_{31}-g_{13}g_{21}g_{32})\,g_{22}^{+}g_{11}^{-}\,(P[K]+R)^2+\\&\frac{(r+1)}{r}\left(g_{13}g_{21}^{2}g_{32}-g_{12}g_{13}g_{21}g_{31}-g_{12}g_{21}^{2}g_{33}\right)(P[K]+R)+\frac{1}{r}g_{21}^{2}g_{33}g_{11}^{+}g_{22}^{-}=0.\end{aligned}\tag{4.11}$$

This is why, the validity of the first identity in (4.9) and identity (4.10) is reduced to the existence of a root to the square equation (4.11), which can be meromorphically continued in the domain D^{+}.

Now we are going to verify that the functions

$$\Phi_2(z)=r(z)\Phi_1(z),\qquad \Psi_2(z)=r(z)\Psi_1(z),$$

where $(\Phi_1(z),\Psi_1(z))$ is a solution to the system of problems (4.8), also determines the solution to this problem. Indeed, it follows from the formulas (4.6), (4.7) that

$$g_{11}\Phi_1^{+}-g_{22}\Phi_1^{-}=\frac{(r+1)}{r}g_{21}.$$

This is why, according to the first condition in (4.8), we arrive at the identity

$$g_{12}\Phi_1^{+}\Phi_1^{-}+g_{13}\Phi_1^{+}\Psi_1^{-}=-\frac{g_{21}}{r}.$$

Similarly, it follows from the formulas (4.6), (4.7) that

$$g_{11}\Psi_1^{+}-g_{33}\Psi_1^{-}-g_{32}\Phi_1^{-}=\frac{(r+1)}{r}g_{31},$$

while by the second condition in (4.8) we get

$$g_{12}\Psi_1^{+}\Phi_1^{-}+g_{13}\Psi_1^{+}\Psi_1^{-}=-\frac{g_{31}}{r}.$$

Substituting the functions $\Phi_2(z)$, $\Psi_2(z)$ into (4.8) and taking into consideration the obtained relations, we arrive at the corresponding identities.

The coefficients in the scalar Riemann problems for determining the first components $w_1^1(z)$ and $w_2^1(z)$ in the solutions (1.4) by the found solutions to the problem (2.3) are determined by the first formulas (2.4), in which the term $g_{12}\Phi_1^{-}+g_{13}\Psi_1^{-}$ is expressed from the aforementioned identity, are respectively equal to

$$g_{11}-\frac{g_{21}}{r}\frac{g_{11}^{+}}{g_{22}^{+}\,(P[K]+R)}\qquad\text{and}\qquad g_{11}-\frac{g_{21}g_{11}^{+}}{g_{22}^{+}\,(P[K]+R)}.\tag{4.12}$$

If these coefficients are not zero and infinity on Γ, then, determining the components $w_1^1(z)$ and $w_2^1(z)$ as solutions of the corresponding homogeneous scalar Riemann problems in the class of the functions without zeroes in the domains D^{+} and D^{-}, which admit the polar singularity only at the infinity (the canonical functions of these problems), we obtain other two components of solutions (1.4) by the second and third formulas in (2.4).

Calculating the determinants of the matrix functions $\Omega^3(z)$ obtained from the above found rectangle matrix function $\Omega(z)=||w_i^j(z)||$, $j=1,2,3$, $i=1,2$, by removing the third row, we obtain

$$\det\Omega^3(z)=(r(z)-1)\Phi_1(z)w_1^1(z)w_2^1(z).\tag{4.13}$$

5. Second main result

Theorem 5.1. *Let Γ be a simple smooth closed contour, which partitions the plane of complex variable into two domains D^+ and D^-, $(0 \in D^+, \infty \in D^-)$, $G(t)$ be an H–continuous on Γ third order matrix function and $\det G(t) \neq 0$. If the entries of the matrix function (1.1) satisfy the conditions (4.5) and are related by one of the identities in (4.9) and the identity (4.10), in which $r \neq 1$ is a constant and R, R_1 are polynomials, the functions (4.12) do not equal to zero and infinity on Γ, and the functions $\Phi_1^{\pm}(z)$ defined in (4.6), (4.7) and the functions (4.13) have no zeroes in the domain D^+ on Γ and in the finite part of the domain D^-; the aforementioned conditions for the matrix function (1.1) left (right) multiplied by the third order permutation matrix.*

Then the canonical system of solutions to the linear conjugation problem (1.2) can be effectively constructed.

The proof is similar to that of Theorem 4.1.

Example 2. Let the conditions (4.5) be satisfied and the entries of matrix function (1.1) are related by the first identity in (4.9) and the identity (4.10):

$$g_{12}\frac{\overset{+}{g}_{22}\overset{-}{g}_{11}}{\overset{+}{g}_{11}\overset{-}{g}_{22}} - g_{13}\frac{\overset{+}{g}_{22}\overset{-}{g}_{11}}{\overset{+}{g}_{11}\overset{-}{g}_{33}}Q\left[\frac{g_{32}}{\overset{+}{g}_{33}\overset{-}{g}_{22}}\right] \equiv g_{21},$$

$$g_{21}\overset{+}{g}_{33}P\left[\frac{g_{32}}{\overset{+}{g}_{33}\overset{-}{g}_{22}}\right] - g_{31}\overset{+}{g}_{22} \equiv 0,$$

where

$$r \equiv -1, \qquad R \equiv 1, \qquad R_1 \equiv 0, \qquad K \equiv 0, \qquad K_1 = \frac{g_{32}}{\overset{+}{g}_{33}\overset{-}{g}_{22}}.$$

Then the formulas

$$\Phi_1^+ = \frac{\overset{+}{g}_{22}}{\overset{+}{g}_{11}}, \qquad \Phi_1^- = \frac{\overset{-}{g}_{11}}{\overset{-}{g}_{22}}, \qquad \Phi_2^{\pm} = -\Phi_1^{\pm};$$

$$\Psi_1^+ = \frac{\overset{+}{g}_{33}}{\overset{+}{g}_{11}}P\left[\frac{g_{32}}{\overset{+}{g}_{33}\overset{-}{g}_{22}}\right], \qquad \Psi_1^- = -\frac{\overset{-}{g}_{11}}{\overset{-}{g}_{33}}Q\left[\frac{g_{32}}{\overset{+}{g}_{33}\overset{-}{g}_{22}}\right], \qquad \Psi_2^{\pm} = -\Psi_1^{\pm}$$

determine solutions to the system of fractional–linear conjugation problems (4.8) satisfying the assumptions of Theorem 5.1. The verification of this fact can be made straightforwardly by substituting the expressions for $\Phi_1^{\pm}(t)$ and $\Psi_1^{\pm}(t)$ into (4.8), in which the values on Γ of the operators $P[K_1]$ and $Q[K_1]$ are taken from the above identities (under the possibility of analytic continuations of the obtained expressions into the corresponding domains). In the considered case ($r \equiv -1$, $R \equiv 1$) the equation (4.11) can be written as

$$(g_{12}g_{21}g_{33} + g_{13}g_{22}g_{31} - g_{13}g_{21}g_{32})\,\overset{+}{g}_{22}\overset{-}{g}_{11} = g_{21}^2 g_{33}\overset{+}{g}_{11}\overset{-}{g}_{22}$$

and the equation has roots 1 and -1.

On Γ the functions (4.12) become

$$\frac{g_{11}\overset{+}{g}_{22} + g_{21}\overset{+}{g}_{11}}{\overset{+}{g}_{22}}, \qquad \frac{g_{11}\overset{+}{g}_{22} - g_{21}\overset{+}{g}_{11}}{\overset{+}{g}_{22}}.$$

If these functions do not vanish on Γ, then determining the first components $w_1^1(z)$ and $w_2^1(z)$ of the solutions (1.4) as solutions to homogeneous scalar Riemann problems in the class of functions without zeroes in the domain D^+ and a finite part of the domain D^- and taking into consideration that the functions $\Phi_1^{\pm}(z)$ do not vanish in the corresponding domains of the finite plane, we obtain that the determinant for the solutions (2.4) in the formula 4.13 satisfies the assumptions of Theorem 5.1, and the canonical system of solutions to the corresponding linear conjugation problem, can be effectively constructed.

Substituting various admissible values of the above determined rational functions into the assumptions of Theorem 4.1 and Theorem 5.1, we select classes of linear conjugation problems (1.3), which can be resolved in closed form.

BIBLIOGRAPHY

1. N.P. Vekua. *Systems of singular integral equations and some boundary value problems.* Nauka, Moscow (1970). [P. Noordhoff, Ltd., Groningen, The Netherlands (1967).]
2. S.N. Kiyasov. *Contribution to the general linear conjugation problem for a piecewise analytic vector* // Sib. Mat. Zh. **59**:2, 369–377 (2018). [Sib. Math. J. **59**:2, 288–294 (2018).]
3. M.C. Camara, L. Rodman, I.M. Spitkovsky. *One sided invertibility of matrices over commutative rings. corona problems. and Toeplitz operators with matrix symbols* // Linear Algebra Appl. **459**, 58–82 (2014).
4. S.N. Kiyasov. *Some classes of linear conjugation problems for a three-dimensional vector that are solvable in closed form* // Sib. Mat. Zh. **56**:2, 389–408 (2015). [Sib. Math. J. **56**:2, 313–329 (2015).]
5. S.N. Kiyasov. *Closed–form solvability of a linear conjugation problem for three-dimensional piecewise analytic vector* // Izv. Vyssh. Uchebn. Zaved., Mat. 2, 22–28 (2020). [Russ. Math. **64**:10, 59–65 (2020).]
6. S.N. Kiyasov. *Closed form of solutions of certain linear conjugation problems for three-dimensional analytic vector* // Izv. Vyssh. Uchebn. Zaved., Mat. 2, 22–28 (2020). [Russ. Math. **64**:2, 19–25 (2020).]
7. V.M. Adukov. *Wiener – Hopf factorization of meromorphic matrix–valued functions* // Algebra Anal. **4**:1, 54–74 (1992). [St. Petersbg. Math. J. **4**:1, 51–69 (1993).]
8. F.D. Gakhov. *Boundary value problems.* Nauka, Moscow (1977). [Dover Publications, Inc., New York (1990).]

Sergey Nikolaevich,
Kazan (Volga Region) Federal University,
Kremlevskaya str. 35,
420008, Kazan, Russia
E-mail: Sergey.Kijasov@kpfu.ru