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BOREL TRANSFORMS OF FUNCTIONS IN PARAMETRIZED FAMILY OF HILBERT SPACES

K.P. ISAEV, R.S. YULMUKHAMETOV

Abstract. We consider Hilbert spaces of entire functions

$$P_\beta(D)=\left\{F\in H(\mathbb{C}):\ \|F\|^2:=\int\limits_0^{2\pi}\int\limits_0^{\infty}\frac{|F(re^{i\varphi})|^2drd\Delta(\varphi)}{K(re^{i\varphi})r^{2\beta}}<\infty\right\},$$

where D is a bounded convex domain on the complex plane,

$$K(\lambda)=\|e^{\lambda z}\|^2_{L_2(D)}=\int\limits_D|e^{\lambda z}|^2dm(z),\quad \lambda\in\mathbb{C},$$

$$h(\varphi)=\max_{z\in\overline{D}}\operatorname{Re} ze^{i\varphi},\quad \varphi\in[0;2\pi],$$

$$\Delta(\varphi)=h(\varphi)+\int\limits_0^{\varphi}h(\theta)d\theta,\quad \varphi\in[0;2\pi].$$

The interest to these spaces is motivated by the fact that $P_0(D)$ is the space of Laplace transforms of linear continuous functionals on the Bergman space $B_2(D)$, while $P_{\frac{1}{2}}(D)$ is the space of Laplace transforms of linear continuous functionals on the Smirnov space $E_2(D)$. In the paper for the parameters $\beta\in\left(-\frac{1}{2};\frac{3}{2}\right)$ we provide a complete description of the Borel transforms of functions in spaces $P_\beta(D)$. In this way, the Bergman and Smirnov spaces are embedded into a scale of Hilbert spaces and, in the authors' opinion, this could allow to apply the theory of Hilbert scales for studying the problems in these spaces.

Key words: scale of Hilbert space, Borel transform, Bergman space, Smirnov space.

Mathematics Subject Classification: 46E20, 30D15

1. Introduction

Let D be a bounded convex domain in the plane. The Bergman space $B_2(D)$ and Smirnov space $E_2(D)$ are rather well studied because of their importance in problems of complex analysis. We recall that $B_2(D)=H(D)\cap L_2(D)$, where $H(D)$ is the space of functions analytic in D and $L_2(D)$ is the space of square integrable functions. In particular, it was established in works [1], [2] that the Laplace transform of linear continuous functionals $\mathcal{L}: S\to\widehat{S}(\lambda)=S_z(e^{\lambda z})$ makes an isomorphism of the dual to $E_2(D)$ space onto the space

$$\widehat{E}_2(D)=\left\{F\in H(\mathbb{C}):\int\limits_0^{2\pi}\int\limits_0^{\infty}\frac{|F(re^{i\varphi})|^2drd\Delta(\varphi)}{K_1(re^{i\varphi})}<\infty\right\},$$

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where $h(\varphi)$ is the support function of this domain, that is,

$$h(\varphi) = \max_{z\in\overline{D}} \operatorname{Re} ze^{i\varphi}, \quad \varphi \in [0; 2\pi],$$

$$\Delta(\varphi) = h(\varphi) + \int^{\varphi} h(\theta)d\theta, \quad \varphi \in [0; 2\pi],$$

$$K_1(\lambda) = \|e^{\lambda z}\|^2_{L_2(\partial D)} = \int_D |e^{\lambda z}|^2 ds(z), \quad \lambda \in \mathbb{C},$$

and $ds(z)$ is the differential of the arc length of boundary D. It was shown in [3] that the Laplace transform makes an isomorphism of the dual to $B_2(D)$ space onto the space

$$\widehat{B}_2(D) = \left\{ F \in H(\mathbb{C}) : \int_0^{2\pi}\int_0^{\infty} \frac{|F(re^{i\varphi})|^2 dr d\Delta(\varphi)}{K(re^{i\varphi})} < \infty \right\},$$

where

$$K(\lambda) = \|e^{\lambda z}\|^2_{L_2(D)} = \int_D |e^{\lambda z}|^2 dm(z), \quad \lambda \in \mathbb{C}.$$

Taking into consideration that $K_1(\lambda) \cong |\lambda| K(\lambda)$, $|\lambda| \to \infty$, for $\beta \in \mathbb{R}$, it is natural to consider the spaces

$$P_\beta(D) = \left\{ F \in H(\mathbb{C}) : \ \|F\|^2 := \int_0^{2\pi}\int_0^{\infty} \frac{|F(re^{i\varphi})|^2 dr d\Delta(\varphi)}{K(re^{i\varphi}) r^{2\beta}} < \infty \right\}.$$

The spaces $P_\beta(D)$ form a continuous scale of Hilbert spaces and, as it has been said above, $\widehat{P}_0(D)$ is isomorphic to $B_2(D)$ and $\widehat{P}_{\frac{1}{2}}(D)$ is isomorphic to $E_2(D)$. In this way, the Bergman and Smirnov spaces are embedded into a scale of Hilbert spaces and, in the authors' opinion, this could allow to apply the theory of Hilbert scales for studying the problems in these spaces.

A function associated in the Borel sense with an entire function F of exponential type σ is the function

$$\gamma(z) = \sum_{k=0}^{\infty} \frac{F^{(k)}(0)}{z^{k+1}}, \quad |z| > \sigma.$$

Let $H_0(D)$ be the space of functions analytic in $\mathbb{C} \setminus \overline{D}$ and vanishing at infinity. For $\alpha > 0$ we let

$$G^\alpha(D) = \left\{ h(\zeta) \in H_0(\mathbb{C} \setminus \overline{D}) : \ \|h\|^2 := \int_{\mathbb{C}\setminus\overline{D}} |h''(\zeta)|^2 \operatorname{dist}^{2\alpha}(D, \zeta) dm(\zeta) < \infty \right\}.$$

In work we are going to prove the following theorem.

Theorem 1.1. *Let F be an entire function with an indicator diagram D, γ be the function associated in the Borel sense with F and $\beta \in \left(-\frac{1}{2}; \frac{3}{2}\right)$. For some constants $c(\beta, D)$, $C(\beta, D) > 0$ depending on the domain D and parameter β the relation*

$$c(\beta, D)\|\gamma\|^2_{G^{\beta+1}} \leqslant \|F\|_{P_\beta} = \int_0^{\infty}\int_0^{2\pi} \frac{|F(re^{i\varphi})|^2}{K(re^{i\varphi}) r^{2\beta}} d\Delta(\varphi) dr \leqslant C(\beta, D)\|\gamma\|^2_{G^{\beta+1}}$$

holds.

An interest to these spaces is motivated by the fact that the authors has a supported conjecture that the spaces $P_\beta(D)$ admit unconditional bases of reproducing kernels, see [4]–[8]. Hence, this could produce a scale of Hilbert spaces of functions analytic in the convex domain D admitting unconditional bases of exponentials.

In first two sections we prove preliminary theorems. In the first section we estimate the integral

$$\int_0^\infty \frac{|F(re^{i\varphi})|^2 dr}{K(re^{i\varphi})r^{2\beta}}$$

for $\beta > -\frac{1}{2}$, and in the second section we justify the localization of the norm in the spaces G^α.

2. Preliminary statements. Estimates of integral over radius

For a fixed $\varphi \in [0; 2\pi]$ by means of the mapping $z \to w = ze^{i\varphi} - h(\varphi)$ we transform the domain D into the domain D_φ, which is located in the left half–plane and touches the ordinate axis. For $t < 0$ by $s(t, \varphi)$ we denote the area of intersection of the domain D_φ with the strip $\{z = x + iy :\ t < x < 0\}$.

Theorem 2.1. *Let F be an entire function obeying the condition: for some $\beta \in \left(-\frac{1}{2}; +\infty\right)$ and $\varphi \in [0; 2\pi]$*

$$I_\varphi = \int_0^\infty \frac{|F(re^{i\varphi})|^2}{K(re^{i\varphi})r^{2\beta}} dr < \infty,$$

and γ is the Borel transform of the function F. For some constants $a(\beta)$, $A(\beta)$ depending only on the parameter β, see the remark in the end of the section, the estimates

$$a(\beta)I_\varphi \leqslant \int_{-\infty}^{\infty}\int_{-\infty}^{0} \frac{|\gamma''(e^{-i\varphi}(h(\varphi) - (x+iy))|^2 |x|^{2\beta+3}}{s(x,\varphi)} dxdy \leqslant A(\beta)I_\varphi$$

hold.

In the rest of the section we prove this theorem. The proof is mainly based on arguing in [3, Sect. 1]. This is why we keep the corresponding notations. On the half–plane

$$P_\varphi = \{\zeta :\ \operatorname{Re}\zeta e^{i\varphi} > h(\varphi)\}$$

the function γ is recovered by the formula

$$\gamma(\zeta) = \int_0^\infty F(re^{i\varphi}) e^{-\zeta re^{i\varphi}} e^{i\varphi} dr.$$

We represent the points in this half–plane in the form $\zeta = (h(\varphi) - \xi)e^{-i\varphi}$, where ξ ranges in the left half–plane $\operatorname{Re}\xi < 0$. Then

$$\gamma''(e^{-i\varphi}(h(\varphi) - \xi)) = \int_0^\infty \left(F(re^{i\varphi})r^2 e^{3i\varphi} e^{-h(\varphi)r}\right) e^{\xi r} dr. \tag{2.1}$$

We shall make use the following theorem proved in [9].

Theorem 2.2. *Let $v(t)$ be a convex function defined on the interval I, and $\widetilde{v}$ be the Young conjugate function for v*

$$\widetilde{v}(x) = \sup_{t\in I}(xt - v(t)).$$

We let

$$J = \{x \in \mathbb{R} : \ \widetilde{v}(x) < \infty\},$$
$$K_0(x) = \int\limits_I e^{2xt-2v(t)}dt, \quad x \in J.$$

Then for each function g on I with a finite integral

$$\|g\|^2 = \int\limits_I |g(t)|^2 e^{-2v(t)}dt,$$

the function

$$\widehat{g}(z) = \int\limits_I \overline{g}(t)e^{zt-2v(t)}dt$$

satisfies the inequalities

$$a\|g\|^2 \leqslant \int\limits_{-\infty}^{\infty}\int\limits_J \frac{\widehat{g}(x+iy)|^2}{K_0(x)}d\widetilde{v}'(x)dy \leqslant A\|g\|^2,$$

where a, $A > 0$ are absolute constants independent of the function g and the weight v.

Let

$$\eta(r) = \frac{e^{2h(\varphi)r}}{K(re^{i\varphi})}, \qquad u(r) = \frac{1}{2}\ln\frac{\eta(r)}{r^4}, \qquad v(r) = u(r) - \beta\ln r.$$

By means of these functions we write the formula (2.1) as

$$\gamma''(e^{-i\varphi}(h(\varphi)-\xi)) = \int\limits_0^{\infty} \frac{F(re^{i\varphi})e^{3i\varphi}e^{h(\varphi)r}}{r^{2+2\beta}K(re^{i\varphi})}e^{\xi r-2v(r)}dr.$$

We apply Theorem 2.2 letting $I = (0; +\infty)$,

$$g(r) = \frac{\overline{F}(re^{i\varphi})e^{-3i\varphi}e^{h(\varphi)r}}{r^{2+2\beta}K(re^{i\varphi})},$$

and choosing $v(t)$ as the weight function. In order to do this, we need to make sure that function $v(t)$ is convex.

Lemma 2.1. *The following statements are true.*

1) *The function v is convex on $I = (0; +\infty)$ and*

$$\left(\frac{1}{2}+\beta\right)\frac{1}{r^2} \leqslant v''(r) \leqslant (2+\beta)\frac{1}{r^2}, \quad r > 0;$$

2) *If*

$$\widetilde{v}(x) = \sup_{r>0}(xr - v(r)), \quad x \in J = (-\infty; 0),$$

is the function Young conjugate with the function v, then

$$\frac{(1+2\beta)^2}{4(2+\beta)}\frac{1}{x^2} \leqslant \widetilde{v}''(x) \leqslant \frac{2(2+\beta)^2}{1+2\beta}\frac{1}{x^2}, \quad x < 0.$$

Proof. In the first assertion of [3, Lm. 1] the estimates

$$\frac{1}{2}\frac{1}{r^2} \leqslant u''(r) \leqslant \frac{2}{r^2}, \quad r > 0,$$

were proved, and they imply the estimates of Assertion 1). Let us prove the second assertion. It was shown in the proof of [3, Lm. 1, Eq. (5)] that

$$-\frac{2}{r} \leqslant u'(r) \leqslant -\frac{1}{2r}, \quad r > 0.$$

Hence,

$$-(2+\beta)\frac{1}{r} \leqslant v'(r) \leqslant -\left(\frac{1}{2}+\beta\right)\frac{1}{r}. \tag{2.2}$$

In the same proof the relations

$$u(r) \to -\infty, \qquad \left|\frac{u(r)}{\ln r}\right| = O(1) \quad \text{as} \quad r \to +\infty, \qquad \lim_{r\to 0+} u(r) = +\infty,$$

were also established and this is why the function $\widetilde{v}(x)$ is defined on $(-\infty; 0)$ and

$$\lim_{r\to 0+}(xr - v(r)) = \lim_{r\to+\infty}(xr - v(r)) = -\infty.$$

Thus, the supremum in the definition of the function $\widetilde{v}$ is attained at a unique stationary point $r = r(x) > 0$ such that $v'(r) = x$. By the estimates (2.2) we find

$$-\left(\frac{1}{2}+\beta\right)\frac{1}{x} \leqslant r(x) \leqslant -(2+\beta)\frac{1}{x}. \tag{2.3}$$

By the definition of the function $\widetilde{v}(x)$ we have the identity

$$\widetilde{v}(x) \equiv xr(x) - v(r(x)), \quad x < 0,$$

or

$$\widetilde{v}(v'(r)) \equiv v'(r)r - u(r), \quad r > 0.$$

We differentiate twice the latter identity

$$\widetilde{v}''(v'(r))v''(r) \equiv 1, \quad r > 0.$$

In view of (2.3) and Assertion 1) we obtain

$$\frac{(1+2\beta)^2}{4(2+\beta)}\frac{1}{x^2} \leqslant \widetilde{v}''(x) = \frac{1}{v''(r(x))} \leqslant \frac{2(2+\beta)^2}{1+2\beta}\frac{1}{x^2}, \quad x < 0.$$

The proof is complete. □

We apply Theorem 2.2 to the function

$$g(r) = \frac{\overline{F}(re^{i\varphi})e^{-3i\varphi}e^{h(\varphi)r}}{r^{2+2\beta}K(re^{i\varphi})}.$$

This yields that the quantity

$$\|g\|^2 = \int_0^\infty |g(r)|^2 e^{-2v(r)}dr = \int_0^\infty \frac{|F(re^{i\varphi})|^2 dr}{K(re^{i\varphi})r^{2\beta}} = I_\varphi$$

is comparable with the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{0} \frac{|\gamma''(e^{-i\varphi}(h(\varphi)-(x+iy)))|^2}{K_0(x,\varphi)} d\widetilde{v}'(x)dy.$$

Estimating $d\widetilde{v}'(x)$ by Assertion 2) of Lemma 2.1, we obtain

$$\frac{(1+2\beta)^2}{4(2+\beta)}aI_\varphi \leqslant \int_{-\infty}^{\infty}\int_{-\infty}^{0} \frac{|\gamma''(e^{-i\varphi}(h(\varphi)-(x+iy)))|^2}{K_0(x,\varphi)x^2} d\widetilde{v}'(x)dy \leqslant \frac{2(2+\beta)^2}{1+2\beta}AI_\varphi. \tag{2.4}$$

To complete the proof of Theorem 2.1 we shall need one more lemma.

Lemma 2.2. *For all $\varphi \in [0; 2\pi]$, $\beta \in (-\frac{1}{2}; \infty)$ the relation*

$$2^{-(2\beta+5)} a_0(\beta) \frac{s(t,\varphi)}{|t|^{2\beta+5}} \leqslant K_0(t,\varphi) \leqslant a_0(\beta) \left(1 + \frac{a_+(\beta)}{a_-(\beta)}\right) \frac{s(t,\varphi)}{|t|^{2\beta+5}}$$

holds, where

$$a_0(\beta) = \int_0^\infty t^{2\beta+4} e^{-2t} dt, \quad a_-(\beta) = \int_0^1 \frac{tdt}{(1+t)^{2\beta+5}}, \quad a_+(\beta) = \int_1^\infty \frac{tdt}{(1+t)^{2\beta+5}}.$$

Proof. We represent the domain D_φ as

$$D_\varphi = \{z = x + iy : \ f_1(x) < y < f_2(x), \ R_\varphi < x < 0\}.$$

Then $f(x) = f_1(x) - f_2(x)$ is a non–negative concave function on $[-R_\varphi; 0]$ and $t < 0$

$$K_0(t,\varphi) = \int_0^\infty e^{2tr - 2v(r)} dr = \int_0^\infty e^{2rt} \frac{r^{2\beta+4} dr}{\eta(r)} = \int_0^\infty e^{2xr} r^{2\beta+4} K(re^{i\varphi}) e^{-2rh(\varphi)} dr$$

$$= \int_0^\infty e^{2rt} r^{2\beta+4} \left(\int_{D_\varphi} e^{2rx} dxdy \right) dr = \int_{D_\varphi} \left(\int_0^\infty e^{2r(x+t)} r^{2\beta+4} dr \right) dxdy.$$

Hence,

$$K_0(t,\varphi) = a_0(\beta) \int_{D_\varphi} \frac{dxdy}{|x+t|^{2\beta+5}},$$

where

$$a_0(\beta) = \int_0^\infty e^{-2\tau} \tau^{2\beta+4} d\tau.$$

1. Let $t \leqslant -D_\varphi$, then on the integration interval we have $|t| \leqslant |t + x| \leqslant 2|t|$, and this is why

$$\frac{a_0(\beta) 2^{-(2\beta+5)}}{t^{2\beta+5}} |D_\varphi| \leqslant K_0(t,\varphi) \leqslant \frac{a_0(\beta)}{t^{2\beta+5}} |D_\varphi|,$$

where $|D_\varphi|$ is the area of the domain D_φ. The statement of the lemma is true since in this case $s(t,\varphi) = |D_\varphi|$.

2. Let $0 \geqslant t > -D_\varphi$ and $p = f(t)$. The concavity of the function f yields

$$f(x) \leqslant \frac{p}{t} x, \qquad -R_\varphi \leqslant x \leqslant t, \qquad f(x) \geqslant \frac{p}{t} x, \qquad t \leqslant x \leqslant 0,$$

and hence,

$$\int_{-R_\varphi}^t \frac{f(x)dx}{|t+x|^{2\beta+5}} \leqslant \frac{p}{|t|} \int_{|t|}^\infty \frac{rdr}{(r+|t|)^{2\beta+5}} = \frac{a_+(\beta)p}{|t|^{2\beta+4}}, \qquad a_+(\beta) = \int_1^\infty \frac{\tau d\tau}{(1+\tau)^{2\beta+5}},$$

$$\int_t^0 \frac{f(x)dx}{|t+x|^{2\beta+5}} \geqslant \frac{p}{|t|} \int_0^{|t|} \frac{rdr}{(r+|t|)^{2\beta+5}} = \frac{a_-(\beta)p}{|t|^{2\beta+4}}, \qquad a_-(\beta) = \int_0^1 \frac{\tau d\tau}{(1+\tau)^{2\beta+5}},$$

and therefore,

$$K_0(t,\varphi) = a_0(\beta) \int\limits_{-R_\varphi}^{0} \frac{f(x)dx}{|t+x|^{2\beta+5}} \leqslant a_0(\beta)\left(1+\frac{a_+(\beta)}{a_-(\beta)}\right)\int\limits_{t}^{0} \frac{f(x)dx}{|x+t|^{2\beta+5}}$$

$$\leqslant a_0(\beta)\left(1+\frac{a_+(\beta)}{a_-(\beta)}\right)\frac{1}{|t|^{2\beta+5}}\int\limits_{t}^{0} f(x)dx = a_0(\beta)\left(1+\frac{a_+(\beta)}{a_-(\beta)}\right)\frac{s(t,\varphi)}{|t|^{2\beta+5}}.$$

The lower bound

$$K_0(t,\varphi) \geqslant a_0(\beta)\int\limits_{t}^{0} \frac{f(x)dx}{|t+x|^{2\beta+5}} \geqslant \frac{2^{-(2\beta+5)}a_0(\beta)}{|t|^{2\beta+5}}\int\limits_{t}^{0} f(x)dx = \frac{2^{-(2\beta+5)}a_0(\beta)}{|t|^{2\beta+5}}s(t,\varphi)$$

is obvious. The proof is complete. □

Now, to complete the proof of Theorem 2.1, it is sufficient to substitute the estimates of Lemma 2.2 into the relation (2.4). The proof of Theorem 2.1 is complete.

Remark 2.1. *As the constants in Theorem 2.1 we can take*

$$a(\beta) = aa_0(\beta)2^{-(2\beta+5)}\frac{(1+2\beta)^2}{4(2+\beta)}, \qquad A(\beta) = Aa_0(\beta)\frac{2(2+\beta)^2}{(1+2\beta)}\left(1+\frac{a_+(\beta)}{a_-(\beta)}\right),$$

where $a_0(\beta)$, $a_\pm(\beta)$ are defined in Lemma 2.2 and a, A are absolute constants from Theorem 2.2.

3. Preliminary statements. Localization of norm in G^α

The main theorem of this section allows us to localize the integrals over $\mathbb{C}\setminus\overline{D}$ to the integrals over the set $\Omega\setminus\overline{D}$, where Ω is an arbitrary neighbourhood of $\overline{D}$.

By $B(z,\varepsilon)$ we denote the circle centered at the point z with the radius ε; if $z=0$, we do not indicate this. We let $D(\varepsilon) = D + B(\varepsilon)$ and

$$R(D) = \inf\{R>0: \quad D\subset \overline{B(R)}\}.$$

Theorem 3.1. *Let $\gamma\in G^\alpha$ and $\alpha\in[0;\frac{3}{2})$. Then for each $\varepsilon\in(0;R(D))$*

$$\int\limits_{\mathbb{C}\setminus\overline{D}} |\gamma''(\zeta)|^2 \mathrm{dist}^{2\alpha}(\zeta)\, dm(\zeta) \leqslant (1+B_0(\alpha))(1+B(\alpha,D)) \int\limits_{D(\varepsilon)\setminus\overline{D}} |\gamma''(\zeta)|^2 \mathrm{dist}^{2\alpha}(\zeta)\, dm(\zeta),$$

and

$$\int\limits_{\mathbb{C}\setminus\overline{D}} |\gamma''(\zeta)|^2 \mathrm{dist}^{2\alpha+1}(\zeta)\, dm(\zeta)$$

$$\leqslant (1+5R(D)B_1(\alpha))(1+5RR(D)B(\alpha,D)) \int\limits_{D(\varepsilon)\setminus\overline{D}} |\gamma''(\zeta)|^2 \mathrm{dist}^{2\alpha}(\zeta)\, dm(\zeta),$$

where

$$B_0(\alpha) = 4^{2\alpha}(4^{(2-\alpha)}-1)^{-1}, \quad B_1(\alpha) = 4^{2\alpha}(2^{(3-2\alpha)}-1)^{-1},$$

$$B(\alpha,D) = 256\frac{(20R)^{2\alpha}(|\partial D|+\pi\varepsilon)^2}{\pi^2\varepsilon^{2(\alpha+1)}}.$$

If $\alpha\in[\frac{3}{2};\frac{5}{2})$, then the same estimates hold under the additional condition

$$\lim_{|z|\to\infty} |z||\gamma(z)| = 0$$

with the constants B_0, B_1 replaced by

$$B_0'(\alpha) = 4^{2\alpha}(2^{(3-\alpha)} - 1)^{-1}, \qquad B_1'(\alpha) = 4^{2\alpha}(2^{(5-2\alpha)} - 1)^{-1}.$$

In the rest of the section we prove this theorem. The proof consists of two steps. At the first step (Lemma 3.1) we estimate the integral of $|\gamma''(\zeta)|$ over the exterior of the circle $\overline{B(4R(D))}$ by the integral over the set $B(4R(D)) \setminus \overline{D}$. Then (Lemma 3.2) the integral over $B(4R(D)) \setminus \overline{D}$ is estimated from above by the integral over $D(\varepsilon) \setminus \overline{D}$.

The number $R(D)$ is briefly denoted by R.

Lemma 3.1. *Let $\gamma \in G^\alpha$ and $\alpha \in [0; \frac{3}{2})$. The relations*

$$\int\limits_{|\zeta|\geqslant 4R} |\gamma''(\zeta)|^2 \text{dist}^{2\alpha}(\zeta)\, dm(\zeta) \leqslant B_0(\alpha) \int\limits_{B(4R)\setminus\overline{D}} |\gamma''(\zeta)|^2 \text{dist}^{2\alpha}(\zeta)\, dm(\zeta)$$

hold and

$$\int\limits_{\mathbb{C}\setminus\overline{D}} |\gamma''(\zeta)|^2 \text{dist}^{2\alpha+1}(\zeta)\, dm(\zeta) \leqslant 5RB_1(\alpha) \int\limits_{D(\varepsilon)\setminus\overline{D}} |\gamma''(\zeta)|^2 \text{dist}^{2\alpha}(\zeta)\, dm(\zeta).$$

If $\alpha \in [\frac{3}{2}; \frac{5}{2})$, then the same estimates hold under the additional condition

$$\lim_{|z|\to\infty} |z||\gamma(z)| = 0$$

with B_0, B_1 replaced by B_0', B_1'.

Proof. We represent the function $\gamma(\zeta)$ as the Laurent series

$$\gamma(\zeta) = \sum_{k=0}^{\infty} \frac{\gamma_k}{\zeta^{k+1}}, \quad |\zeta| > R.$$

By the assumptions, this series and

$$\gamma''(\zeta) = \sum_{k=0}^{\infty} \frac{(k+2)(k+1)\gamma_k}{\zeta^{k+3}} = \sum_{k=0}^{\infty} \frac{\gamma_k''}{\zeta^{k+3}}, \quad |\zeta| > R,$$

converge uniformly on the set $\mathbb{C} \setminus B(2R)$.

We take a number $t \in [0; 2)$. For ζ, $|\zeta| \geqslant 4R$, we have

$$\text{dist}(\zeta) \leqslant |\zeta| + R < 2|\zeta|,$$

and this is why

$$\int\limits_{|\zeta|\geqslant 4R} |\gamma''(\zeta)|^2 \text{dist}^{2t}(\zeta)\, dm(\zeta) \leqslant 2^{2t} \int\limits_{|\zeta|\geqslant 4R} |\gamma''(\zeta)|^2 |\zeta|^{2t}\, dm(\zeta).$$

Passing to the polar coordinates in the above integral and taking into consideration the orthogonality of the system $e^{ik\varphi}$ with respect to the measure $d\varphi$ over $[0; 2\pi]$, we obtain

$$\begin{aligned}\int\limits_{|\zeta|\geqslant 4R} |\gamma''(\zeta)|^2 \text{dist}^{2t}(\zeta)\, dm(\zeta) &\leqslant 2^{2t+1}\pi \int\limits_{4R}^{\infty} \sum_{k=0}^{\infty} \frac{|\gamma_k''|^2 r^{2t+1}\, dr}{r^{2(k+3)}} \\ &= 2^{2t+1}\pi \sum_{k=0}^{\infty} \frac{|\gamma_k''|^2}{2(k+2-t)(4R)^{2(k+2-t)}}.\end{aligned} \tag{3.1}$$

Again by means of the Laurent series, using the estimates $\operatorname{dist}(\zeta) \geqslant \frac{1}{2}|\zeta|$ for $|\zeta| \geqslant 2R$, we estimate from below the integral over the annulus $B(4R) \setminus B(2R)$

$$\begin{aligned}\int\limits_{2R\leqslant|\zeta|<4R} |\gamma''(\zeta)|^2 \operatorname{dist}^{2t}(\zeta)\, dm(\zeta) &\geqslant 2^{-2t} \int\limits_{2R\leqslant|\zeta|<4R} |\gamma''(\zeta)|^2 |\zeta|^{2t}\, dm(\zeta) \\ &= 2^{-2t+1}\pi \sum_{k=0}^{\infty} \frac{|\gamma''_k|^2}{2(k+2-t)(4R)^{2(k+2-t)}} (2^{2(k+2-t)} - 1),\end{aligned} \tag{3.2}$$

hence,

$$\int\limits_{2R\leqslant|\zeta|<4R} |\gamma''(\zeta)|^2 \operatorname{dist}^{2t}(\zeta)\, dm(\zeta) \geqslant 2^{-2t+1} \left(2^{2(2-t)} - 1\right) \pi \sum_{k=0}^{\infty} \frac{|\gamma''_k|^2}{2(k+2-t)(4R)^{2(k+2-t)}}. \tag{3.3}$$

By (3.1) this implies the estimate

$$\int\limits_{|\zeta|\geqslant 4R} |\gamma''(\zeta)|^2 \operatorname{dist}^{2t}(\zeta)\, dm(\zeta) \leqslant 2^{4t} \left(2^{2(2-t)} - 1\right)^{-1} \int\limits_{2R\leqslant|\zeta|<4R} |\gamma''(\zeta)|^2 \operatorname{dist}^{2t}(\zeta)\, dm(\zeta).$$

Letting $t = \alpha < \frac{3}{2}$, we obtain the first estimate in the first part of the lemma, while letting $t = \alpha + \frac{1}{2} < 2$ and employing the estimate $\operatorname{dist}(\zeta) \leqslant 5R$ for $|\zeta| \leqslant 4R$, we get the second estimate in the first part of the lemma.

If $\lim_{|z|\to\infty} |z||\gamma(z)| = 0$, then $\gamma''_k = 0$ and the summation in the relations (3.1), (3.2) is made over $k \geqslant 1$, respectively, instead of the estimate (3.3) we obtain the relation

$$\int\limits_{2R\leqslant|\zeta|<4R} |\gamma''(\zeta)|^2 \operatorname{dist}^{2t}(\zeta)\, dm(\zeta) \geqslant 2^{-2t+1} \left(2^{2(3-t)} - 1\right) \pi \sum_{k=1}^{\infty} \frac{|\gamma''_k|^2}{2(k+2-t)(4R)^{2(k+2-t)}},$$

which is true for all $t \in [0; 3)$. By the relation (3.1), in which $\gamma''_0 = 0$, we find

$$\int\limits_{|\zeta|\geqslant 4R} |\gamma''(\zeta)|^2 \operatorname{dist}^{2t}(\zeta)\, dm(\zeta) \leqslant 2^{4t} \left(2^{2(3-t)} - 1\right)^{-1} \int\limits_{2R\leqslant|\zeta|<4R} |\gamma''(\zeta)|^2 \operatorname{dist}^{2t}(\zeta)\, dm(\zeta).$$

Letting $t = \alpha$, we obtain the first estimate in the second part of the lemma, while the second estimate can be obtained by letting $t = \alpha + \frac{1}{2}$ and using the estimate $\operatorname{dist}(\zeta) \leqslant 5R$ for $|\zeta| \leqslant 4R$. The proof is complete. □

Lemma 3.2. *If $\gamma \in G^\alpha$, $\alpha > 0$, then*

$$\int\limits_{B(4R)\setminus\overline{D}(\varepsilon)} |\gamma''(\zeta)|^2 \operatorname{dist}^{2\alpha}(\zeta)\, dm(\zeta) \leqslant 256 \frac{(20R)^{2\alpha}(|\partial D| + \pi\varepsilon)^2}{\pi^2 \varepsilon^{2(\alpha+1)}} \int\limits_{D(\varepsilon)\setminus\overline{D}} |\gamma''(\zeta)|^2 \operatorname{dist}^{2\alpha}(\zeta)\, dm(\zeta).$$

Proof. Let $\zeta \notin \overline{D}(\varepsilon)$. Since

$$\operatorname{dist}(\zeta, D) \leqslant \operatorname{dist}(\zeta, D(\varepsilon)) + \varepsilon, \qquad |\partial D(\varepsilon)| = |\partial D| + 2\pi\varepsilon,$$

and by the Cauchy formula we have the upper bound

$$\begin{aligned}|\gamma''(\zeta)| &\leqslant \frac{1}{2\pi} \left| \int\limits_{\partial D(\varepsilon/2)} \frac{\gamma''(z)\, dz}{z - \zeta} \right| \leqslant \frac{|\partial D(\varepsilon/2)|}{2\pi \operatorname{dist}(\zeta, D(\varepsilon/2))} \max_{z\in\partial D(\varepsilon/2)} |\gamma''(z)| \\ &\leqslant \frac{|\partial D| + \pi\varepsilon}{2\pi(\operatorname{dist}(\zeta, D) - \varepsilon/2)} \max_{z\in\partial D(\varepsilon/2)} |\gamma''(z)|.\end{aligned}$$

It is obvious that for $\zeta \notin \overline{D}(\varepsilon)$ the inequality $\operatorname{dist}(\zeta, D) \geqslant \varepsilon$ holds and $\frac{x}{x-\frac{\varepsilon}{2}} \leqslant 2$ for $x \geqslant \varepsilon$. Moreover, $\operatorname{dist}(\zeta) \leqslant 5R$ for $\zeta \in B(4R)$, hence, for $\zeta \in B(4R) \setminus \overline{D}(\varepsilon)$,

$$|\gamma''(\zeta)|^2 \operatorname{dist}^{2\alpha}(\zeta) \leqslant \frac{5^{2\alpha} R^{2(\alpha-1)}(|\partial D| + \pi\varepsilon)^2}{\pi^2} \max_{z \in \partial D(\varepsilon/2)} |\gamma''(z)|^2. \tag{3.4}$$

If $z \in \partial D(\varepsilon/2)$, then the circle $B(z, \varepsilon/4)$ is located in the domain $D(3\varepsilon/4) \setminus \overline{D}$. Moreover, if $w \in \partial B(z, \frac{\varepsilon}{4})$, then $\operatorname{dist}(w) \geqslant \frac{\varepsilon}{4}$. Using the subharmonicity of the function $|\gamma''(z)|^2$, we get the upper bound

$$\begin{aligned}
|\gamma''(z)|^2 &\leqslant \frac{16}{\pi\varepsilon^2} \int\limits_{B(z,\varepsilon/4)} |\gamma''(w)|^2 \, dm(w) \\
&\leqslant \frac{16}{\pi\varepsilon^2} \big(\sup_{B(z,\varepsilon/4)} \operatorname{dist}^{-2\alpha}(w)\big) \int\limits_{B(z,\varepsilon/4)} |\gamma''(w)|^2 \operatorname{dist}^{2\alpha}(w) \, dm(w) \\
&\leqslant \frac{4^{2(\alpha+1)}}{\pi\varepsilon^{2(\alpha+1)}} \int\limits_{D(\varepsilon)\setminus\overline{D}} |\gamma''(w)|^2 \operatorname{dist}^{2\alpha}(w) \, dm(w).
\end{aligned}$$

We substitute this estimate into (3.4) and integrate over $B(4R) \setminus \overline{D}(\varepsilon)$ and this completes the proof. □

The estimates in Theorem 3.1 are implied by the relations in Lemmas 3.1, 3.2. The proof of Theorem 3.1 is complete.

4. Proof of main theorem

It is sufficient to prove the main theorem for the domains not containing straight segments and right angles on the boundary. This implies by the fact that the constants $c(\beta, D)$, $C(\beta, D)$ are continuously (rationally) depend on D. Indeed, suppose that the theorem is true with additional mentioned conditions on the domain.

We take an arbitrary $\varepsilon > 0$. In the set $D(\varepsilon)$ we inscribe the convex polygon, which involves D. Then we replace each side of this polygon by an arc of a sufficiently large circumference so that the obtained domain D' remains convex and is contained in the domain $D(2\varepsilon)$. The boundary D' contains no straight segments, but it contains the angles. In order to get rid of them, we pass to the domain $D' + B(0, \varepsilon)$. In this way we obtain a domain without angles and segments on the boundary, which contains D and is contained in $D(3\varepsilon)$.

Since $P_\beta(D) \subset P_\beta(D(\varepsilon))$, then for the functions $F \in P_\beta(D)$ we can apply the main theorem in the spaces $P_\beta(D(\varepsilon))$ and then we pass to the limit as $\varepsilon \to 0$.

For further purposes we need some geometric objects. By $s(x, \varphi)$ we denote the area of the part of the domain

$$D_\varphi = \{w = x + iy : \ f_1(x) < y < f_2(x), \ -R_\varphi < x < 0\}$$

cut out by the straight line $\operatorname{Re} w = x$. The domain D_φ is obtained from the domain D under the mapping $w = ze^{i\varphi} - h(\varphi)$. Thus, R_φ is the distance between the support lines $L(\varphi)$ and $L(\varphi + \pi)$. By $l(x, \varphi)$ we denote the length of the part of the boundary D_φ cut out by the straight line, while by $u(x, \varphi)$ we denote the length of the chord cut out by the domain D_φ on this straight line. We let

$$\sigma(D) = \inf_{\varphi \in [0; 2\pi]} R_\varphi.$$

It is clear that $\sigma(D)$ is the smallest width of the domain.

By Theorem 2.1 the norm $\|F\|$ of the entire function F defined in the main theorem is equivalent to the triple integral

$$\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{0}\frac{|\gamma''(e^{-i\varphi}(h(\varphi)-(x+iy)))|^2|x|^{2\beta+3}}{s(x,\varphi)}dxdyd\Delta(\varphi).$$

In this integral we make the change of the variables

$$\zeta=e^{-i\varphi}(h(\varphi)-x-iy),\qquad \theta=\varphi.$$

Let us describe the geometric meaning of new and old variables. In what follows by $l(\varphi)$ we denote the directed straight line $\{te^{i\varphi},\ -\infty<t<\infty\}$. On the boundary of domain D we choose the counterclockwise direction. In this way all tangent lines to the boundary get a direction. By $L(\varphi)$ we denote the tangent line parallel and co–directed with the line $l(\frac{\pi}{2}-\varphi)$. If $te^{-i\varphi}$ is the intersection point of the lines $L(\varphi)$ and $l(-\varphi)$, it is easy to verify that $h(\varphi)=t$. If the variables $\varphi\in[0;2\pi)$, $x<0$, y, are given, then ζ is the point in the plane, which in the coordinate system formed by the straight lines $l(-\varphi)$ (abscissa axis) and $l(\frac{\pi}{2}-\varphi)$ (ordinate axis) has the coordinates $(h(\varphi)-x;-y)$. At the same time the condition $x<0$ means that the support line $L(\varphi)$ separates the point ζ from the domain D.

Let us find the range of the variables θ and ζ. The point ζ obviously lies outside $\overline{D}$. For a fixed $\zeta\in\mathbb{C}\setminus\overline{D}$ the angle θ should be so that the support line $L(\theta)$ separates the point ζ from the domain D. We draw two tangent lines at the point ζ to the boundary of domain D. Let them be co–directed with the straight lines $l(\varphi_1)$ and $l(\varphi_2)$, and $0\leqslant\varphi_1\leqslant\varphi_2$. Then the angle θ ranges from $\varphi_-(\zeta)=\frac{\pi}{2}-\varphi_2$ to $\varphi_+(\zeta)=\frac{\pi}{2}-\varphi_1$. The Jacobian of passage from the variables φ, x, y to the variablesζ, θ is identically equal to 1 and $x=h(\theta)-\operatorname{Re}\zeta e^{i\theta}$.

Thus,

$$\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{0}\frac{|\gamma''(e^{-i\varphi}(h(\varphi)-\xi))|^2|x|^{2\beta+3}}{s(x,\varphi)}dxdyd\Delta(\varphi)$$

$$=\int_{\mathbb{C}\setminus\overline{D}}|\gamma''(\zeta)|^2\left(\int_{\varphi_-(\zeta)}^{\varphi_+(\zeta)}\frac{(\operatorname{Re}\zeta e^{i\theta}-h(\theta))^{2\beta+3}}{s(h(\theta)-\operatorname{Re}\zeta e^{i\theta},\theta)}d\Delta(\theta)\right)dm(\zeta).$$

The internal integral in the right hand side is denoted by $p(\zeta)$. Thus, we have proved the following statement.

Theorem 4.1. *Let $F=B(\gamma)$ be an entire function satisfying the condition*

$$\|F\|^2=\int_0^{2\pi}\int_0^{\infty}\frac{|F(re^{i\varphi})|^2}{K(re^{i\varphi})r^{2\beta}}\,drd\Delta(\varphi)<\infty$$

and

$$p(\zeta)=\int_{\varphi_-(\zeta)}^{\varphi_+(\zeta)}\frac{(\operatorname{Re}\zeta e^{i\theta}-h(\theta))^{2\beta+3}}{s(h(\theta)-\operatorname{Re}\zeta e^{i\theta},\theta)}\,d\Delta(\theta).$$

Then

$$a(\beta)\|F\|^2\leqslant\int_{\mathbb{C}\setminus\overline{D}}|\gamma''(\zeta)|^2p(\zeta)\,dm(\zeta)\leqslant A(\beta)\|F\|^2,$$

where the constants $a(\beta)$, $A(\beta)$ depend only on the parameter β, see Remark 2.1.

Lemma 4.1. *Let*

$$p_0(\zeta) = \int\limits_{\varphi_-(\zeta)}^{\varphi_+(\zeta)} \frac{(\mathrm{Re}\,\zeta e^{i\theta} - h(\theta))^{2\beta+2}}{u(h(\theta) - \mathrm{Re}\,\zeta e^{i\theta}, \theta)}\, d\Delta(\theta).$$

Then

1. *For the points ζ such that* $\mathrm{dist}(\zeta) \leqslant \sigma(D)/2$,
$$\frac{2}{3} p_0(\zeta) \leqslant p(\zeta) \leqslant 2p_0(\zeta).$$
2. *If* $\mathrm{dist}(\zeta) > \sigma(D)/2$, *we denote by I_0 the part of the interval $(\varphi_-(\zeta);\ \varphi_+(\zeta))$, on which the condition*
$$\mathrm{Re}\,\zeta e^{i\varphi} - h(\varphi) \geqslant \frac{\sigma(D)}{2}$$
is satisfied, and let I be the remaining part of this interval. Then
$$p(\zeta) \leqslant \frac{4\,\mathrm{diam}^2(D)|\partial D|}{\sigma^2(D)|D|} \mathrm{dist}^{2\beta+3}(\zeta) + 2\int\limits_I \frac{(\mathrm{Re}\,\zeta e^{i\theta} - h(\theta))^{2\beta+2}}{u(h(\theta) - \mathrm{Re}\,\zeta e^{i\theta}, \theta)}\, d\Delta(\theta).$$

Proof. 1. We observe that by the definition of the function $h(\varphi)$

$$\mathrm{Re}\,\zeta e^{i\varphi} - h(\varphi) = \min_{z\in\overline{D}} (\mathrm{Re}\,\zeta e^{i\varphi} - \mathrm{Re}\,z e^{i\varphi}) \leqslant \min_{z\in\overline{D}} |\zeta - z| = \mathrm{dist}(\zeta), \tag{4.1}$$

and this is why for the points ζ with the condition $\mathrm{dist}(\zeta) < \sigma(D)/2$ for all $\varphi \in (\varphi_-(\zeta);\ \varphi_+(\zeta))$ the quantities $s(h(\varphi) - \mathrm{Re}\,\zeta e^{i\varphi},\ \varphi)$ can be estimated by means of Assertion 1) in [3, Stat. 2], which immediately implies the estimates in the first part of the lemma.

2. If the interval I_0 is non–empty, then for $\theta \in I_0$ we apply the latter estimate in [3, Stat. 2]:

$$s(h(\theta) - \mathrm{Re}\,\zeta e^{i\theta}, \theta) \geqslant \frac{\sigma^2(D)|D|}{4\,\mathrm{diam}^2(D)}.$$

In view of (4.1) and the geometric meaning of the function $\Delta(\theta)$ we get Assertion 2 of the lemma. The proof is complete. □

Let us describe the integral in the definition of the function $p_0(\zeta)$ in geometric terms. The integration interval $(\varphi_-(\zeta); \varphi_+(\zeta))$ consists of the angles φ such that $\mathrm{Re}\,\zeta e^{i\varphi} - h(\varphi) \geqslant 0$. In other words, these are the directions φ for which the support line $L(\varphi)$ separates the point ζ from the domain D. In this case, the quantity $\mathrm{Re}\,\zeta e^{i\varphi} - h(\varphi)$ is the distance from the point ζ to the support line $L(\varphi)$. If we translate this support line parallel to itself by a distance $\mathrm{Re}\,\zeta e^{i\varphi} - h(\varphi)$, then on the resulting line the region D will cut off a chord the length of which we have denoted by $u(h(\varphi) - \mathrm{Re}\,\zeta e^{i\varphi}, \varphi)$. Finally, the geometric meaning of the function $\Delta(\varphi)$ is that the difference $\Delta(\varphi_1) - \Delta(\varphi_2)$ for $\varphi_1 \geqslant \varphi_2$ is equal to the length of arc of boundary D from the point of tangency of support line $L(\varphi_1)$ to the point of tangency of support line $L(\varphi_2)$.

The above description is not related with the coordinate system. In what follows we choose the coordinate system with the origin at a fixed point ζ and we partially describe the domain D as the overgraph of some convex function f. We are going to write the integral in the definition of the function $p_0(\zeta)$ in terms of the function f.

As the origin we choose the point ζ. There exists a unique point z_0 on the boundary D such that

$$\mathrm{dist}(\zeta) = \inf_{z\in\partial D} |z - \zeta| = |z_0 - \zeta|.$$

We direct the ordinate system from the point ζ to the point z_0. In this coordinate system the domain D is a part of the overgraph of some convex function $f(x)$ defined on the interval $(X_1; X_2)$, where

$$X_1 = h(\pi), \qquad X_2 = h(0).$$

The angles ranges in the new coordinate system. The slopes to the abscissa axis of two tangent lines to domain D passing through the origin are denoted by φ_1 and φ_2, and $\varphi_1 \leqslant \varphi_2$. Then the integral in the definition of p_0 is calculated from $\varphi_- = \frac{\pi}{2} - \varphi_2$ to $\varphi_+ = \frac{\pi}{2} - \varphi_1$. The distance from the point ζ to the domain D in this coordinate system is expressed as $f(0)$ or $-h(\pi/2)$.

We suppose that the boundary of domain D contains no angles and straight segments, see the remark in the beginning of the section. For the function f this means that the derivative f' is strictly increasing continuous function.

If the variable θ ranges from φ_- to φ_+, then the quantity $\frac{\pi}{2} - \theta$ monotonically varies from φ_2 to φ_1, that is, from the slope of tangent line to the graph of function f at the point X_1 to the slope of tangent line at the point X_2. Therefore, once we define the point $x(\theta)$ by the identity

$$f'(x(\theta)) = \tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta), \tag{4.2}$$

the point $x(\theta)$ varies monotonically from X_1 to X_2 and the point $(x(\theta);\ f(x(\theta)))$ is the support point of the support line $L(\theta)$. By the aforementioned geometric meaning of the function $\Delta(\theta)$ we obtain

$$d\Delta(\theta) = d\left(\int^{x(\theta)} \sqrt{1 + f'(s)^2}\, ds\right) = \sqrt{1 + f'(x(\theta))^2}\, dx(\theta) = \frac{1}{|\sin\theta|}\, dx(\theta).$$

Thus, for the function p_0 we have the following representation

$$p_0(\zeta) = \int_{\varphi_-}^{\varphi_+} \frac{|h(\theta)|^{2\beta+2}}{u(h(\theta), \theta)} \frac{1}{|\sin\theta|}\, dx(\theta), \tag{4.3}$$

where the quantities in the integral in the right hand side are expressed in the coordinate system associated with the point ζ.

In what follows we need to estimate the function $u(h(\theta), \theta)$ in terms of the function f. In order to do this, we introduce the following functions. We take an arbitrary point $x_0 \in [X_1;\ X_2]$ and a positive number δ. We let

$$\rho_+(f, x_0, \delta) = \sup\left\{\rho :\ \rho \leqslant X_2 - x_0,\ \int_0^{\rho} (f'(x_0 + y) - f'(x_0))\, dy \leqslant \delta\right\},$$

$$\rho_-(f, x_0, \delta) = \sup\left\{\rho :\ \rho \leqslant x_0 - X_1,\ \int_0^{\rho} (f'(x_0) - f'(x_0 - y))\, dy \leqslant \delta\right\},$$

$$\widetilde{\rho}(f, x_0, \delta) = \rho_-(f, x_0, \delta) + \rho_+(f, x_0, \delta).$$

Let

$$g(t) = \sup_{x \in [X_1; X_2]} (xt - f(t)$$

be the Young conjugate to the function f. If $T_1 = f'(X_1)$, $T_2 = f'(X_2)$, then the supremum in the definition of g is attained at a unique stationary point $x = x(t)$ determined by the condition $f'(x) = t$, that is,

$$g(t) \equiv x(t)t - f(x(t)), \qquad t \in [T_1; T_2].$$

Differentiating this identity, we get

$$g'(t) \equiv x(t) \quad \text{or} \quad g'(f'(x)) \equiv x.$$

Letting $x = x(\theta)$, by (4.2) we have

$$x(\theta) = g'(\cot(\theta)). \tag{4.4}$$

For each positive number δ we define a quantity $\rho = \rho(g, t_0, \delta)$ by the condition

$$\rho = \sup\left\{ s > 0 : \int_{-s}^{s} |g'(t_0 + t) - g'(t_0)|\, dt \leqslant \delta \right\}.$$

By Lemma 4.1, representations (4.3), (4.4) and [3, Lms. 3, 4] we obtain the following statement.

Lemma 4.2. *For points ζ such that $\operatorname{dist}(\zeta) \leqslant \sigma(D)/2$, the inequalities*

$$p(\zeta) \geqslant \frac{2}{9} \int_{\varphi_-}^{\varphi_+} |h(\theta)|^{2\beta+1} |\sin\theta| \rho\left(g, \cot\theta, \frac{|h(\theta)|}{|\sin\theta|}\right) dg'(\cot\theta),$$

$$p(\zeta) \leqslant \frac{48 \operatorname{diam}^4(D)}{|D|^2} \int_{\varphi_-}^{\varphi_+} |h(\theta)|^{2\beta+1} |\sin\theta| \rho\left(g, \cot\theta, \frac{|h(\theta)|}{|\sin\theta|}\right) dg'(\cot\theta)$$

hold. If

$$\operatorname{dist}(\zeta) > \sigma(D)/2,$$

then by I_0 we denote a part of the interval $(\varphi_-;\ \varphi_+)$, on which the condition

$$-h(\theta) > \frac{\sigma(D)}{2}$$

is satisfied, while I is the remaining part of this interval. Then

$$p(\zeta) \leqslant \frac{4 \operatorname{diam}^2(D) |\partial D|}{\sigma^2(D) |D|} \operatorname{dist}^{2\beta+3}(\zeta) + \frac{48 \operatorname{diam}^4(D)}{|D|^2} \int_{I} |h(\theta)|^{2\beta+1} |\sin\theta| \rho\left(g, \cot\theta, \frac{|h(\theta)|}{|\sin\theta|}\right) dg'(\cot\theta).$$

In the integrals we make the change of variables $\theta = \frac{\pi}{2} - \varphi$ and we let

$$p_1(\zeta) = \int_{\theta_-}^{\theta_+} \left| h\left(\frac{\pi}{2} - \varphi\right) \right|^{2\beta+1} |\cos\varphi| \rho\left(g, \tan\varphi, \frac{|h(\frac{\pi}{2} - \varphi)|}{\cos\varphi}\right) dg'(\tan\varphi), \tag{4.5}$$

where $\theta_\pm = \frac{\pi}{2} - \varphi$ are the slopes to the abscissa axis of tangent line to the graph of the function $f(x)$ passing through the origin. In this coordinate system the distance $\operatorname{dist}(\zeta)$ is equal to $-h(\frac{\pi}{2})$. We denote this distance by d.

We determine the point $x = x(\varphi) \in [X_1;\ X_2]$ by the condition

$$f'(x(\varphi)) = \tan\varphi.$$

Then $x(\varphi)$ is a point, at which the supremum $\sup_x (xt - f(x))$ is attained for $t = \tan\varphi$, and hence,

$$g(\tan\varphi) = \tan\varphi \cdot x(\varphi) - f(x(\varphi)). \tag{4.6}$$

On the other hand, the support line $L(\varphi)$ to the domain D is tangent to the function $f(x)$ at the point $x(\varphi)$, while $-h(\frac{\pi}{2} - \varphi)$ is the distance from this tangent line to the origin. It is easy to see that the number $-h(\frac{\pi}{2} - \varphi)/\cos\varphi$ is equal to the ordinate of the intersection of support line with the ordinate axis. The equation of tangent line at the point $x(\varphi)$ reads as

$$y = (x - x(\varphi))\tan\varphi + f(x(\varphi)),$$

and this is why

$$\frac{-h(\frac{\pi}{2} - \varphi)}{\cos\varphi} = f(x(\varphi)) - x(\varphi)\tan\varphi.$$

In view of (4.6) we obtain

$$\frac{-h(\frac{\pi}{2}-\varphi)}{\cos\varphi} = -g(\tan\varphi).$$

Thus, as φ grows monotonically from 0 to θ_+ or decreases from 0 to θ_-, the value $-g(\tan\varphi)$ monotonically decreases from d to 0. We let $\varphi_0 = 0$ and defined the angles φ_n by the conditions

$$\frac{-h(\frac{\pi}{2}-\varphi_n)}{\cos\varphi_n} = \frac{d}{2^{|n|}}$$

or, what is the same, by the identities $-g(\tan\varphi_n) = 2^{-|n|}d$.

The entire integration interval is partitioned into the intervals $(\varphi_n;\ \varphi_{n+1}]$, $n\in\mathbb{Z}$, and the integral in (4.5) can be represented as a sum of the integrals over these intervals. We note that the quantity $\rho(g,t,\delta)$ is non–decreasing in the variable δ and this is why for $\varphi\in(\varphi_n;\ \varphi_{n+1}]$ we have

$$\rho(g,\tan\varphi,2^{-n-1}d) \leqslant \rho\left(g,\tan\varphi,\frac{-h(\frac{\pi}{2}-\varphi)}{\cos\varphi}\right) \leqslant \rho(g,\tan\varphi,2^{-n}d), \qquad n\geqslant 0,$$
$$\rho(g,\tan\varphi,2^{-|n|}d) \leqslant \rho\left(g,\tan\varphi,\frac{-h(\frac{\pi}{2}-\varphi)}{\cos\varphi}\right) \leqslant \rho(g,\tan\varphi,2^{-|n|+1}d), \qquad n< 0.$$

Thus, after the change of variables $t=\tan\varphi$ we obtain

$$\begin{aligned} p_1(\zeta) \leqslant & \sum_{n=0}^{\infty}\left(\frac{d}{2^n}\right)^{2\beta+1}\int\limits_{\tan\varphi_n}^{\tan\varphi_{n+1}}\rho\left(g,t,\frac{d}{2^n}\right)\,dg'(t) \\ & + \sum_{n=-1}^{-\infty}\left(\frac{d}{2^{|n|-1}}\right)^{2\beta+1}\int\limits_{\tan\varphi_n}^{\tan\varphi_{n+1}}\rho\left(g,t,\frac{d}{2^{|n|-1}}\right)\,dg'(t), \end{aligned} \tag{4.7}$$

$$p_1(\zeta) \geqslant \left(\frac{d}{2}\right)^{2\beta+1}\int\limits_{\tan\varphi_{-1}}^{\tan\varphi_1}\frac{1}{(1+t^2)^{\beta+1}}\rho\left(g,t,\frac{d}{2}\right)\,dg'(t). \tag{4.8}$$

We proceed to upper bounds for $p_1(\zeta)$. We let

$$t_n = \frac{\tan\varphi_n+\tan\varphi_{n+1}}{2}.$$

Then for $n\geqslant 0$ we obviously have

$$g(t_n)\geqslant g(\tan\varphi_n) = -\frac{d}{2^n},$$

while for $n<0$ we have

$$g(t_n)\geqslant g(\tan\varphi_{n+1}) = -\frac{d}{2^{|n|-1}}.$$

This is why

$$g(\tan\varphi_n)+g(\tan\varphi_{n+1})-2g(t_n) \leqslant -\frac{d}{2^{n+1}}-\frac{d}{2^n}+\frac{2d}{2^n} = \frac{d}{2^{n+1}} < \frac{d}{2^n}, \qquad n\geqslant 0,$$
$$g(\tan\varphi_n)+g(\tan\varphi_{n+1})-2g(t_n) \leqslant \frac{d}{2^{|n|}} < \frac{d}{2^{|n|-1}}, \qquad n\leqslant 0.$$

We compare these estimates with the definition of the quantity $\rho(g,t,\delta)$ and we see that

$$\tan\varphi_{n+1}-t_n=t_n-\tan\varphi_n<\rho\left(g,t_n,\frac{d}{2^n}\right),\qquad n\geqslant 0,$$
$$\tan\varphi_{n+1}-t_n=t_n-\tan\varphi_n<\rho\left(g,t_n,\frac{d}{2^{|n|-1}}\right),\qquad n<0.$$

We introduce the notation

$$\rho_n=\rho\left(g,t_n,\frac{d}{2^n}\right),\quad n\geqslant 0,\qquad \rho_n=\rho\left(g,t_n,\frac{d}{2^{|n|-1}}\right),\quad n<0.$$

The relation (4.7) gives the estimate

$$p_1(\zeta)\leqslant\sum_0^\infty\left(\frac{d}{2^n}\right)^{2\beta+1}\int\limits_{t_n-\rho_n}^{t_n+\rho_n}\rho\left(g,t,\frac{d}{2^n}\right)dg'(t)+\sum_{-1}^{-\infty}\left(\frac{d}{2^{|n|-1}}\right)^{2\beta+1}\int\limits_{t_n-\rho_n}^{t_n+\rho_n}\rho\left(g,t,\frac{d}{2^{|n|-1}}\right)dg'(t).$$

By [3, Lm. 5, Assrt. 3] we obtain

$$p_1(\zeta)\leqslant\sum_0^\infty\frac{4d}{2^n}\left(\frac{d}{2^n}\right)^{2\beta+1}+\sum_{-1}^{-\infty}\frac{4d}{2^{|n|-1}}\left(\frac{d}{2^{|n|-1}}\right)^{2\beta+1}=2\frac{4^{\beta+2}}{4^{\beta+1}-1}\mathrm{dist}^{2\beta+2}(\zeta).$$

Let $I'=\{\frac{\pi}{2}-\theta,\ \theta\in I\}$, where the interval $I\subset(\varphi_-;\varphi_+)$ was defined in Lemma 4.2. Then $I'\subset(\theta_-;\theta_+)$ and

$$\int\limits_I|h(\theta)|^{2\beta+1}|\sin\theta|\rho\left(g,\cot\theta,\frac{|h(\theta)|}{|\sin\theta|}\right)dg'(\cot\theta)$$
$$=\int\limits_{I'}\left|h\left(\frac{\pi}{2}-\varphi\right)\right|^{2\beta+1}|\cos\varphi|\rho\left(g,\tan\varphi,\frac{|h(\frac{\pi}{2}-\varphi)|}{|\cos\varphi|}\right)dg'(\tan\varphi)$$
$$\leqslant p_1(\zeta)\leqslant 2\frac{4^{\beta+2}}{4^{\beta+1}-1}\mathrm{dist}^{2\beta+2}(\zeta).$$

We substitute two latter estimates into the upper bounds in Lemma 4.2 and we obtain upper bounds for the function $p(\zeta)$: if $\mathrm{dist}(\zeta)\leqslant\frac{\sigma(D)}{2}$, then

$$p(\zeta)\leqslant\frac{48\,\mathrm{diam}^4(D)}{|D|^2}p_1(\zeta)\leqslant\frac{6\cdot 4^{\beta+4}\,\mathrm{diam}^4(D)}{(4^{\beta+1}-1)|D|^2}\mathrm{dist}^{2\beta+2}(\zeta),\tag{4.9}$$

and if $\mathrm{dist}(\zeta)>\frac{\sigma(D)}{2}$, then

$$p(\zeta)\leqslant\frac{4\,\mathrm{diam}^2(D)|\partial D|}{\sigma^2(D)|D|}\mathrm{dist}^{2\beta+3}(\zeta)+\frac{6\cdot 4^{\beta+4}\,\mathrm{diam}^4(D)}{(4^{\beta+1}-1)|D|^2}\mathrm{dist}^{2\beta+2}(\zeta).\tag{4.10}$$

Now, on the base of the relation (4.8), we proceed to the lower bounds for the function $p_1(\zeta)$ and, respectively, for the function $p(\zeta)$. Without loss of generality we suppose that

$$\min(-\tan\varphi_{-1},\tan\varphi_1)=\tan\varphi_1.$$

To shorten the writing, we introduce the notation $\rho=\rho(g,0,\frac{d}{2})$. By the definition of the quantity $\rho(g,0,\frac{d}{2})$ we have

$$g(\rho)+g(-\rho)-2g(0)=\frac{d}{2},$$

on the other hand,

$$g(0)=\max_{X_1\leqslant x\leqslant X_2}(-f(x))=-\min_{X_1\leqslant x\leqslant X_2}f(x)=-f(0)=-d.$$

This is the identity
$$g(\rho)+g(-\rho)=-\frac{3}{2}d$$
should hold. By the definition of the angles $\varphi_{\pm 1}$ we have
$$g(\tan\varphi_1)+g(-\tan\varphi_1)=-\frac{d}{2}+g(-\tan\varphi_1)\geqslant -\frac{d}{2}+g(0)=-\frac{3}{2}d.$$
Therefore,
$$\tan\varphi_1\geqslant\rho=\rho\left(g,0,\frac{d}{2}\right).$$
By (4.8) this implies
$$\begin{aligned}p_1(\zeta)&\geqslant\left(\frac{d}{2}\right)^{2\beta+1}\int\limits_{-\tan\varphi_1}^{\tan\varphi_1}\frac{1}{(1+t^2)^{\beta+1}}\rho\left(g,t,\frac{d}{2}\right)\,dg'(t)\\&\geqslant\left(\frac{d}{2}\right)^{2\beta+1}\frac{1}{(1+\tan^2\ \varphi_1)^{\beta+1}}\int\limits_{-\rho}^{\rho}\rho\left(g,t,\frac{d}{2}\right)\,dg'(t).\end{aligned}$$
Now we can employ [3, Stat. 3]) to estimate $\tan\varphi_1$ from above and [3, Lm. 5, Assrt. 3] to estimate from below the integral
$$\begin{aligned}p_1(\zeta)&\geqslant\left(\frac{d}{2}\right)^{2\beta+2}\left(1+\frac{25\,\mathrm{diam}^2(D)}{4\sigma^2(D)}\right)^{-(\beta+1)}\\&=4^{-(\beta+1)}\left(1+\frac{25\,\mathrm{diam}^2(D)}{4\sigma^2(D)}\right)^{-(\beta+1)}\mathrm{dist}^{2\beta+2}(\zeta).\end{aligned}$$
This estimate and the relations (4.9), (4.10) allows us to derive the following statement by Lemma 4.2.

Lemma 4.3. *For the points ζ such that* $\mathrm{dist}(\zeta)\leqslant\sigma(D)/2$ *the inequalities*
$$m(\beta,D)\mathrm{dist}^{2\beta+2}(\zeta)\leqslant p(\zeta)\leqslant M(\beta,D)\mathrm{dist}^{2\beta+2}(\zeta)$$
hold and for the points ζ such that
$$\mathrm{dist}(\zeta)>\sigma(D)/2,$$
we have
$$p(\zeta)\leqslant M_0(\beta,D)\mathrm{dist}^{2\beta+3}(\zeta)+M(\beta,D)\mathrm{dist}^{2\beta+2}(\zeta),$$
where
$$\begin{aligned}m(\beta,D)&=\frac{2}{9}\cdot 4^{-(\beta+1)}\left(1+\frac{25\,\mathrm{diam}^2(D)}{4\sigma^2(D)}\right)^{-(\beta+1)},\\M(\beta,D)&=\frac{6\cdot 4^{\beta+4}\,\mathrm{diam}^4\ (D)}{(4^{\beta+1}-1)|D|^2},\\M_0(\beta,D)&=\frac{4\,\mathrm{diam}^2(D)|\partial D|}{\sigma^2(D)|D|},\end{aligned}$$
$\sigma(D)$ is the smallest width of the domain D in the directions, $\mathrm{dist}(\zeta)$ *is the distance from the point ζ to the domain D, $|D|$ is the area of D, $|\partial D|$ is the length of boundary of D and finally* $\mathrm{diam}(D)$ *is the diameter of D.*

To complete the proof of the main theorem it remains to collect all proven estimates.

1. Let $\beta \in (-\frac{1}{2}; \frac{1}{2})$.

1.1. We are going to prove the lower bound in the main theorem. By Theorem 4.1 and Lemma 4.3

$$\|F\|^2 \geqslant \frac{1}{A(\beta)} \int\limits_{\mathbb{C}\setminus\overline{D}} |\gamma''(\zeta)|^2 p(\zeta)\, dm(\zeta) \geqslant \frac{m(\beta, D)}{A(\beta)} \int\limits_{D(\sigma(D)/2)\setminus\overline{D}} |\gamma''(\zeta)|^2 \mathrm{dist}^{2(\beta+1)}\, dm(\zeta),$$

and then we apply Assertion 1 of Theorem 3.1 with

$$\varepsilon = \frac{\sigma(D)}{2}, \qquad \alpha = \beta + 1 \in [0; \frac{3}{2})$$

and this gives

$$\|F\|^2_{P_\beta} \geqslant \frac{m(\beta, D)}{A(\beta)} (1 + B_0(\beta, D))^{-1} (1 + B(\beta, D))^{-1} \int\limits_{\mathbb{C}\setminus\overline{D}} |\gamma''(\zeta)|^2 \mathrm{dist}^{2(\beta+1)}\, dm(\zeta).$$

Thus, the left inequality in the main theorem is proved with the constant

$$c(\beta, D) = \frac{m(\beta, D)}{A(\beta)} (1 + B_0(\beta, D))^{-1} (1 + B(\beta, D))^{-1}.$$

1.2. To prove the upper bound, we apply Theorem 4.1

$$\|F\|^2_{P_\beta} \leqslant \frac{1}{a(\beta)} \int\limits_{\mathbb{C}\setminus\overline{D}} |\gamma''(\zeta)|^2 p(\zeta)\, dm(\zeta).$$

We apply the second estimate from Lemma 4.3

$$\begin{aligned}\|F\|^2_{P_\beta} \leqslant & \frac{M(\beta, D)}{a(\beta)} \int\limits_{\mathbb{C}\setminus\overline{D}} |\gamma''(\zeta)|^2 \mathrm{dist}^{2(\beta+1)}(\zeta)\, dm(\zeta) \\ & + \frac{M_0(\beta, D)}{a(\beta)} \int\limits_{\mathbb{C}\setminus D(\sigma(D)/2)} |\gamma''(\zeta)|^2 \mathrm{dist}^{2\beta+3}(\zeta)\, dm(\zeta).\end{aligned}$$

We estimate the second integral by the second inequality in Assertion 1 of Theorem 3.1 with $\alpha = \beta + 1 \in [0; \frac{3}{2})$. The right inequality in the main theorem in this case is proved with the constant

$$C(\beta, D) = \frac{M(\beta, D)}{a(\beta)} + \frac{M_0(\beta, D)}{a(\beta)} (1 + 5RB_0(\beta + 1, D))(1 + B(\beta + 1, D))^{-1}.$$

2. Let $\beta \in [\frac{1}{2}; \frac{3}{2})$. If $F \in P_\beta(D)$, then it follows from the definition of $\|F\|$ with $\beta \geqslant \frac{1}{2}$ that

$$F(0) = \lim_{|z|\to\infty} |z||\gamma(z)| = 0$$

and this is why we can employ Assertion 2 of Theorem 3.1. As a result, the estimates in the main theorem are proved with the same constants as in the case $\beta < \frac{1}{2}$ with the constant $B_0(\beta + 1, D)$ replaced by $B_0'(\beta + 1, D)$.

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Isaev Konstantin Petrovich,
Ufa University of Science and Technology,
Zaki Validi str. 32,
450000, Ufa, Russia
E-mail: orbit81@list.ru

Yulmukhametov Rinad Salavatovich,
Institute of Mathematics,
Ufa Federal Research Center, RAS,
Chernyshevsky str. 112,
450008, Ufa, Russia
E-mail: yulmukhametov@mail.ru