CONSTRUCTION OF EXACT SOLUTIONS
OF NONLINEAR PDE VIA DRESSING CHAIN IN 3D

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Abstract. The duality between a class of the Davey — Stewartson type coupled systems and a class of two–dimensional Toda type lattices is discussed. A new coupled system related to the recently found lattice is presented. A method for eliminating nonlocalities in coupled systems by virtue of special finite reductions of the lattices is suggested. An original algorithm for constructing explicit solutions of the coupled systems based on the finite reduction of the corresponding lattice is proposed. Some new solutions for coupled systems related to the Volterra lattice are presented as illustrative examples.

Key words: 3D lattices, generalized symmetries, Darboux integrable reductions, Lax pairs, Davey — Stewartson type coupled system.

Mathematics Subject Classification: 35Q51, 39A14

1. INTRODUCTION

In the papers [1], [2] a close connection between integrable two–dimensional lattices and integrable partial differential equations in three independent variables was discovered. More precisely, the class of generalized symmetries of two–dimensional lattices contains a large variety of nonlinear integrable partial differential equations in 3D. In particular, this class contains such an important model as the Davey — Stewartson equation, which is related to the symmetries of the Toda lattice. After such a kind observation it was natural to expect that this duality would lead to the creation of new algorithms for finding particular solutions to coupled systems. However, this did not happen, since, as noted in the mentioned works, the presence of nonlocal variables creates severe problems for efficient use of dressing chains for constructing explicit solutions of coupled systems.

After the papers [3]–[7] it became clear that the integrability of a two–dimensional chain can be fully revealed at the level of its finite–field reductions obtained by imposing a special type of truncating boundary conditions on the chain. The integrability criterion for a three-dimensional lattice is formulated as a requirement of Darboux integrability of the reduced systems. This special type of boundary conditions for the chains has another remarkable property: it is compatible with arbitrary higher symmetry of the lattice under consideration. It immediately follows from this fact that special cut–off constraints preserve the duality between the lattices and the associated coupled systems. In other words, when the lattice is reduced to a finite–field system, its symmetry becomes a symmetry for this reduction. An important step in this scheme consists of finding explicit expressions for the nonlocalities in terms of the local variables (see the formulas (3.6)). Thus, at the reduction level, nonlocal variables can be completely eliminated.

Let us illustrate some of the notions considered below with an example. It was shown in [2] that the two–dimensional Volterra chain

$$u_{n,y} = u_n(v_{n+1} - v_n), \qquad v_{n,x} = v_n(u_n - u_{n-1}) \tag{1.1}$$

has a symmetry (coupled system) of the following form

$$
\begin{aligned}
u_{n,t} &= u_{n,xx} + \left(u_n^2 + 2u_n V_n\right)_x, \\
v_{n,t} &= -v_{n,xx} + \left(V_n^2\right)_y + (2u_n v_n)_x, \quad V_{n,y} = v_{n,x}.
\end{aligned}
\tag{1.2}
$$

In other words, the relations

$$(u_{n,y})_t = (u_{n,t})_y, \qquad (v_{n,x})_t = (v_{n,t})_x$$

are satisfied identically if the derivatives with respect to x, y, and t are replaced taking into account (1.1) and (1.2). For an arbitrary value of n, the relations (1.2) define a system of partial differential equations with independent variables x, y, and t, where n is a hidden parameter. When moving from n to $n+1$, the desired functions are calculated using an invertible Bäcklund transformation generated by the lattice (1.1) (see [2])

$$v_{n+1} = v_n + (\ln u_n)_y, \qquad u_{n+1} = u_n + (\ln v_{n+1})_x, \qquad V_{n+1} = V_n + (\ln u_n)_x. \tag{1.3}$$

In this paper, we consider coupled systems corresponding to integrable lattices as the main object of study. The lattices rewritten as invertible Bäcklund transformations are interpreted as symmetries with discrete time n for coupled systems. The aim of the work is to develop an algorithm for constructing particular solutions of partial differential equations with three independent variables using dressing chains.

2. EXAMPLES OF COUPLED SYSTEMS

The Volterra chain admits a large class of symmetries (see [2]). For instance, one can easily derive from (1.2) another coupled system of the second order

$$
\begin{aligned}
u_{n,\tau} &= u_{n,yy} + \left(U_n^2\right)_x + (2u_n v_n)_y, \\
v_{n,\tau} &= -v_{n,yy} + \left(v_n^2 + 2v_n U_n\right)_y, \quad U_{n,x} = u_{n,y},
\end{aligned}
\tag{2.1}
$$

by using the involutions

$$x \leftrightarrow -y, \quad t \leftrightarrow \tau, \quad u \leftrightarrow v, \quad U \leftrightarrow V, \quad n \leftrightarrow -n. \tag{2.2}$$

Its Bäcklund transformation is given by

$$u_{n-1} = u_n - (\ln v_n)_x, \qquad v_{n-1} = v_n - (\ln u_{n-1})_y, \qquad U_{n-1} = U_n - (\ln v_n)_y.$$

By taking a linear combination of two symmetries given above we find a more complicated symmetry

$$
\begin{aligned}
u_{n,s} &= \lambda u_{n,xx} + \mu u_{n,yy} - \lambda \left(u_n^2 + 2u_n V_n\right)_x - \mu \left(U_n^2\right)_x - \mu(2u_n v_n)_y, \\
V_{n,y} &= v_{n,x}, \quad \lambda \neq 0, \\
v_{n,s} &= -\lambda v_{n,xx} - \mu v_{n,yy} - \lambda \left(V_n^2\right)_y - \lambda(2u_n v_n)_x - \mu \left(v_n^2 + 2v_n U_n\right)_y, \\
U_{n,x} &= u_{n,y}, \quad \mu \neq 0.
\end{aligned}
\tag{2.3}
$$

A large class of the Toda type integrable lattices is presented in [2] where the related coupled systems are given as well. One extra lattice of this type

$$
\begin{aligned}
u_{n,xy} &= \alpha_n(u_{n,x} - u_n^2 - 1)(u_{n,y} - u_n^2 - 1) + 2u_n(u_{n,x} + u_{n,y} - u_n^2 - 1), \\
\alpha_n &= \frac{1}{u_n - u_{n-1}} - \frac{1}{u_{n+1} - u_n}
\end{aligned}
\tag{2.4}
$$

was recently found in [5]. Symmetries of the lattice (2.4) in the directions of x and y are as follows (see [8]):

$$u_{n,t} = u_{n,xx} - 2u_n u_{n,x} + u_n^2 + 1 - 2(u_n^2 - u_{n,x} + 1)H_n,$$

$$H_n = (T-1)^{-1} D_x \log \frac{u_{n,x} - u_n^2 - 1}{u_{n+1} - u_n},$$

$$D_y H_n = -D_x \frac{u_{n,y} - u_n u_{n-1} - 1}{u_n - u_{n-1}}$$

(2.5)

and respectively

$$u_{n,\tau} = u_{n,yy} - 2u_n u_{n,y} + u_n^2 + 1 - 2(u_n^2 - u_{n,y} + 1)Q_n,$$

$$Q_n = (T-1)^{-1} D_y \log \frac{u_{n+1,y} - u_{n+1}^2 - 1}{u_{n+1} - u_n},$$

$$D_x Q_{n-1} = D_y \frac{u_{n-1,x} - u_n u_{n-1} - 1}{u_n - u_{n-1}}.$$

(2.6)

Note that symmetries (2.5) and (2.6) depend significantly on the discrete parameter n since they contain variables with shifted arguments.

The corresponding coupled systems of the lattice (2.4) obtained from the second–order symmetries (2.5) and (2.6) have the form

$$u_t = u_{xx} - 2uu_x + u^2 + 1 - 2(u^2 - u_x + 1)H,$$

$$v_t = -v_{xx} + 2(v_x - v^2 - 1)H - \frac{2v_x^2}{u - v}$$

$$+ \frac{2(v_x - v^2 - 1)u_x}{u - v} + \frac{2(uv + 1)v_x}{u - v} + v^2 + 1,$$

$$D_y H = -D_x \frac{u_y - uv - 1}{u - v}$$

(2.7)

and

$$u_\tau = -u_{yy} + 2(u_y - u^2 - 1)Q + \frac{2u_y^2}{u - v}$$

$$- \frac{2(u_y - u^2 - 1)v_y}{u - v} - \frac{2(uv + 1)u_y}{u - v} + u^2 + 1,$$

$$v_\tau = v_{yy} - 2vv_y + v^2 + 1 + 2(v_y - v^2 - 1)Q,$$

$$D_x Q = D_y \left(\frac{v_x - uv - 1}{u - v} \right),$$

(2.8)

where $u := u_n$ and $v := u_{n-1}$. Obviously systems (2.7) and (2.8) do not contain any variables u and v with shifted values of n.

The lattice (2.4), supplemented by the equation for the nonlocality H_n, defines an invertible Bäcklund transformation

$$v_{n-1} = v_n - \frac{(u_n - v_n)(v_n^2 - v_{n,x} + 1)(v_n^2 - v_{n,y} + 1)}{(u_n - v_n)(v_{n,xy} - 2v_n(v_{n,x} + v_{n,y} - v_n^2 - 1)) + (v_n^2 - v_{n,x} + 1)(v_n^2 - v_{n,y} + 1)},$$

$$u_{n-1} = v_n,$$

$$H_{n-1} = H_n - D_x \log \frac{v_{n,x} - v_n^2 - 1}{u_n - v_n}$$

for the coupled system (2.7). In a similar way one can derive the Bäcklund transformation for coupled system (2.8).

3.   Finite reductions of 3D lattices compatible with symmetries

It is shown in our articles [3]–[6] that each known integrable Toda type lattice admits cut–off boundary conditions allowing to reduce the lattice to a hierarchy of hyperbolic systems integrable in sense of Darboux. Apparently, these boundary conditions are compatible with a large class of higher symmetries of the lattice. We say that a truncation boundary condition for a lattice is compatible with its symmetry if the truncation preserves the commutativity property of the lattice and the symmetry [9]. Below in this section we discuss how to get rid of the nonlocalities that arise within the symmetry approach. To do this, we pass from chains to their finite–dimensional reductions, then use conservation laws to express the nonlocalities in terms of dynamical variables.

Let us concentrate on finite–field reductions of the Volterra chain obtained by imposing the following type truncation conditions $u_{-k} = 0$ and $v_{m+k} = 0$, $k = 1, 2, 3, \ldots$:

$$
\begin{aligned}
u_{0,y} &= u_0(v_1 - v_0), & v_{0,x} &= v_0 u_0, \\
u_{1,y} &= u_1(v_2 - v_1), & v_{1,x} &= v_1(u_1 - u_0), \\
&\cdots & &\cdots \\
u_{m,y} &= -u_m v_m, & v_{m,x} &= v_m(u_m - u_{m-1}).
\end{aligned}
\tag{3.1}
$$

By summing consecutively equations of the obtained system we arrive at the equation

$$
\sum_{i=0}^{m} (u_{i,y} + v_{i,x}) = 0
$$

that can be represented as a conservation law

$$
\frac{\partial}{\partial y} \sum_{i=0}^{m} u_i + \frac{\partial}{\partial x} \sum_{i=0}^{m} v_i = 0.
$$

Replacing $v_{i,x} = V_{n,y}$ due to the definition of the nonlocal variables (see (1.2)) we find that the expression $\sum_{i=0}^{m} (u_i + V_i)$ is independent of y,

$$
\frac{\partial}{\partial y} \sum_{i=0}^{m} (u_i + V_i) = 0.
$$

Therefore, if we require that the relation

$$
\sum_{i=0}^{m} V_i = - \sum_{i=0}^{m} u_i
\tag{3.2}
$$

is satisfied for some fixed value $y = y_0$ then it would hold for all y. Due to the Bäcklund transformation (1.3) we have the first order linear discrete equation

$$
V_{n+1} = V_n + h_n, \qquad h_n = (\ln u_n)_x.
$$

The equation is easily solved

$$
V_j = V_0 + \sum_{i=0}^{j-1} h_i.
\tag{3.3}
$$

By using the explicit solution (3.3) one can derive a useful formula

$$
\sum_{k=1}^{m} V_k = m V_0 + \sum_{j=1}^{m} \sum_{i=0}^{j-1} h_i.
\tag{3.4}
$$

Now we rewrite (3.2) in the following form

$$\sum_{k=1}^{m} V_k = -V_0 - \sum_{i=0}^{m} u_i. \tag{3.5}$$

Comparison of (3.4) and (3.5) leads to an explicit expressions for all of the variables V_j, $j = \overline{0, m}$, in terms of the dynamical variables u_0, ..., u_m and their first order derivatives with respect to x

$$V_0 = -\frac{1}{m+1} \left(\sum_{j=1}^{m} \sum_{i=0}^{j-1} h_i + \sum_{i=0}^{m} u_i \right),$$

$$V_j = -\frac{1}{m+1} \left(\sum_{j=1}^{m} \sum_{i=0}^{j-1} h_i + \sum_{i=0}^{m} u_i \right) + \sum_{i=0}^{j-1} h_i, \quad j = \overline{0, m}. \tag{3.6}$$

As a result we determine a usual second order symmetry (without nonlocalities) of the hyperbolic system (3.1)

$$u_{n,t} = u_{n,xx} + \left(u_n^2 + 2u_n V_n \right)_x \quad \text{for} \quad n = \overline{0, m},$$

$$v_{n,t} = -v_{n,xx} + \left(V_n^2 \right)_y + (2u_n v_n)_x, \tag{3.7}$$

where the functions $V_j = F_j([u])$ are found due to (3.6). In a similar way one can study the nonlocality U_n of the coupled system (2.1) corresponding to the Volterra lattice. The explicit expressions for the variables $U_0, U_1, \ldots, U_m$ are easily derived from (3.6) due to the involutions (2.2).

Example 1. For $m = 0$ formula (3.6) takes the form $V_0 = -u_0$ (see (4.4) below). It is considered in Section 4, where explicit solution for the coupled system is constructed.

Example 2. If $m = 1$ the nonlocalities are given by

$$V_0 = -\frac{1}{2} \left(h_0 + u_0 + u_1 \right), \quad h_0 = (\ln u_0)_x,$$

$$V_1 = -\frac{1}{2} \left(h_0 + u_0 + u_1 \right) + h_0.$$

The corresponding symmetry takes the form

$$u_{0,t} = -u_{0,x} u_1 - u_0 u_{1,x},$$

$$v_{0,t} = -u_0 v_0 u_1,$$

$$u_{1,t} = u_{1,xx} + \frac{u_1}{u_0} u_{0,xx} + \frac{\left(u_{0,x} - u_0^2 \right) u_{1,x}}{u_0} - \frac{\left(u_{0,x} + u_0^2 \right) u_1 u_{0,x}}{u_0^2}, \tag{3.8}$$

$$v_{1,t} = \frac{u_1 v_1 u_{0,x}}{u_0} + u_{1,x} v_1.$$

It is easily verified that (3.8) is really a symmetry to the reduced system:

$$u_{0,y} = u_0(v_1 - v_0), \qquad v_{0,x} = v_0 u_0,$$

$$u_{1,y} = -u_1 v_1, \qquad v_{1,x} = v_1(u_1 - u_0).$$

Recall that the system (3.1) is integrable in sense of Darboux, i.e. it admits a complete set of characteristic integrals, see [11]. It is known that the symmetries of Darboux integrable systems are linearized (see, for instance, [10]). Therefore (3.7) is linearized by an appropriately chosen differential substitution.

4.   CONSTRUCTION OF EXACT SOLUTIONS TO COUPLED SYSTEMS VIA INTEGRABLE REDUCTIONS OF DRESSING CHAINS

Now we discuss how the dressing chain can be used to construct explicit solutions to the coupled systems. As an illustrative example we take the system

$$u_t = u_{xx} + \left(u^2 + 2uV\right)_x,$$
$$v_t = -v_{xx} + \left(V^2\right)_y + (2uv)_x, \quad V_y = v_x, \tag{4.1}$$

corresponding to the Volterra lattice (here $u := u_n$ and $v := v_n$)

$$u_{n,y} = u_n(v_{n+1} - v_n), \qquad v_{n,x} = v_n(u_n - u_{n-1}).$$

Let us consider its reduction

$$u_y = -uv, \qquad v_x = uv \tag{4.2}$$

obtained due to cutting–off constraint $u_{-k} = 0$, $v_k = 0$, $k = 1, 2, 3, \ldots$. Here the sought functions are $u := u_0$ and $v := v_0$. The functions

$$I = \frac{u_x}{u} - u \qquad \text{and} \qquad J = \frac{v_y}{v} + v$$

are characteristic integrals of the system. Indeed, it is checked straightforwardly that the necessary conditions $D_y I = 0$ and $D_x J = 0$ for the integrals definitely hold. Therefore we have a system of differential equations (Bernoulli equations)

$$\frac{u_x}{u} - u = f_1(x), \qquad \frac{v_y}{v} + v = f_2(y)$$

for searching solution to the system (4.2), where f_1 and f_2 are arbitrary functions.

It is easy to verify that general solution of the system can be parametrized in the following form

$$u(x,y) = -\frac{W_x}{W} = \frac{\rho'(x)}{\varphi(y) - \rho(x)}, \qquad v(x,y) = \frac{W_y}{W} = \frac{-\varphi'(y)}{\varphi(y) - \rho(x)}, \tag{4.3}$$

where $W = \varphi(y) - \rho(x)$ and W_x, W_y denote the derivatives of W with respect to x and y. Here the functions $\varphi(y)$, $\rho(x)$ are chosen arbitrarily. Note that $W_{xy} = 0$.

Now we assume that functions $u(x,y)$ and $v(x,y)$ depend on one more independent variable t due to the system (4.1). In other words we have $\varphi = \varphi(y,t)$ and $\rho = \rho(x,t)$. Then by integrating equation $V_y = v_x$ we derive an explicit expression for the nonlocality

$$V = -\int \frac{\varphi_y \rho_x dy}{W^2} = \frac{\rho_x}{W} + R(x,t) = -u + R(x,t).$$

Let us set $R(x,t) = 0$ for simplicity. Then we get (see also Example 1)

$$V = -u. \tag{4.4}$$

Afterward the coupled system (1.2) turns into

$$u_t = u_{xx} - 2uu_x, \qquad v_t = -v_{xx} + 2uu_y + 2(uv)_x. \tag{4.5}$$

Using the substitution (4.3), we reduce the system (4.5) to an overdetermined system of equations with a single sought function W. To apply the substitution (4.3), we have to use the explicit representations of the derivatives of u and v:

$$u_y = -v_x = \frac{W_x W_y}{W^2}, \qquad\qquad u_x = -\frac{W_{xx}}{W} + \frac{W_x W_y}{W^2},$$
$$u_t = -\frac{W_{xt}}{W} + \frac{W_x W_t}{W^2}, \qquad\qquad v_t = \frac{W_{yt}}{W} - \frac{W_y W_t}{W^2},$$
$$v_{xx} = -\frac{W_{xx} W_y}{W^2} + 2\frac{W_x^2 W_y}{W^3}, \qquad u_{xx} = -\frac{W_{xxx}}{W} + 3\frac{W_x W_{xx}}{W^2} - 2\frac{W_x^3}{W^3}.$$

The formulas above allow us to bring the system (4.5) to the following form

$$WW_{ty} = W_yW_t - W_{xx}W_y,$$
$$WW_{tx} = WW_{xxx} + W_xW_t - W_xW_{xx}. \qquad (4.6)$$

The next step is to solve it explicitly. Let us start with the second equation in (4.6). First, we represent the equation as

$$\frac{WW_{xt} - W_xW_t}{W^2} = \frac{WW_{xxx} - W_xW_{xx}}{W^2}.$$

Then integrating the latter we get

$$\frac{W_t}{W} = \frac{W_{xx}}{W} + g(t, y).$$

Now we simplify the first equation in (4.6) due to the relation

$$W_t = W_{xx} + g(t, y)W \qquad (4.7)$$

and obtain an equation of the form

$$W_{yt} = g(t, y)W_y. \qquad (4.8)$$

If we apply the operator D_y of the total differentiation with respect to y to both sides of (4.7) and then simplify it in virtue of the equation $W_{xxy} = 0$, we arrive at the relation

$$W_{yt} = g(t, y)W_y + g_y(t, y)W. \qquad (4.9)$$

Comparing relations (4.8) and (4.9), we arrive at $g_y(t, y) = 0$ or, the same,

$$g(t, y) = g(t).$$

Analyzing the above reasoning, we conclude that the desired function $W = W(x, y, t)$ is a solution to the system

$$W_{xy} = 0,$$
$$W_t = W_{xx} + g(t)W, \qquad (4.10)$$
$$W_{ty} = g(t)W_y.$$

Obviously the third equation of the system is easily integrated, since it is of the form

$$\frac{\partial}{\partial t}\ln W_y = g(t).$$

Hence it implies

$$\ln W_y = \ln G(t) + \ln F_1(y),$$

where

$$\ln G(t) = \int_0^t g(\tau)d\tau$$

and the constant of integration $\ln F_1(y)$ does not depend on x due to the first equation in (4.10). Then we integrate the obtained equation $W_y = G(t)F_1(y)$ with respect to y. It is convenient to present the result in the form

$$W = G(t)(F(y) + S(x, t)), \qquad (4.11)$$

where

$$F(y) = \int_0^y F_1(z)dz.$$

Now we substitute (4.11) into the second equation in (4.10). After a slight simplification we obtain the heat equation for $S(x, t)$

$$S_t = S_{xx}. \qquad (4.12)$$

Therefore general solution to the system (4.10) is given by (4.11) with arbitrary $G(t)$ and $F(y)$ and with an arbitrary solution $S(x,t)$ to (4.12). This leads us to the following theorem.

Theorem 4.1. *Assume that $S(x,t)$ is a solution to Equation (4.12) and $F(y)$ is an arbitrary smooth function, then the functions defined by the rule*

$$u(x,y,t) = -\frac{\partial}{\partial x}\ln(S(x,t) + F(y)),$$

$$v(x,y,t) = \frac{\partial}{\partial y}\ln(S(x,t) + F(y)), \tag{4.13}$$

$$V(x,y,t) = \frac{\partial}{\partial x}\ln(S(x,t) + F(y))$$

give a solution to the coupled system (1.2).

Theorem 4.1 can be easily verified by a simple substitution.

As it is known, the solution of the heat equation (4.12) is given in a closed form by the Poisson formula

$$S(x,t) = \frac{1}{2\sqrt{\pi t}}\int\limits_{R} S_0(\xi)e^{-\frac{(x-\xi)^2}{4t}}\,d\xi,$$

where $S|_{t=0} = S_0(x)$ is a bounded continuous function. Therefore solution (4.13) of coupled system (1.2) depends on two arbitrary functions $S_0(x)$ and $F(y)$.

5.   SECOND EXAMPLE

Taking a linear combination of two symmetries, we find a coupled system which depends symmetrically on x and y (see (2.3) above)

$$
\begin{aligned}
&u_{n,s} = \lambda u_{n,xx} + \mu u_{n,yy} + \lambda\left(u_n^2 + 2u_n V_n\right)_x + \mu\left(U_n^2\right)_x + \mu(2u_n v_n)_y,\\
&V_{n,y} = v_{n,x}, \quad \lambda \neq 0,\\
&v_{n,s} = -\lambda v_{n,xx} - \mu v_{n,yy} + \lambda\left(V_n^2\right)_y + \lambda(2u_n v_n)_x + \mu\left(v_n^2 + 2v_n U_n\right)_y,\\
&U_{n,x} = u_{n,y}, \quad \mu \neq 0.
\end{aligned}
\tag{5.1}
$$

The boundary conditions $u_{-k} = 0$ and $v_k = 0$, $k = 1,2,3,\ldots$ imposed on the Volterra chain are compatible with all symmetries. Therefore, to construct solutions of the system (2.3), one can use the same ansatz

$$u(x,y) = -\frac{W_x}{W}, \qquad v(x,y) = \frac{W_y}{W}$$

as in the previous example. Here $W = \varphi(y) - \rho(x)$ with arbitrary functions $\varphi(y)$, $\rho(x)$ and W_x, W_y denote the derivatives of W with respect to x and y. We choose the nonlocalities as $V = -u$ and $U = -v$.

As a result, we arrive at the following system of equations:

$$W_{sx} - \lambda W_{xxx} - \mu W_{xyy} - \frac{W_x W_s}{W} + \lambda\frac{W_{xx}W_x}{W} + 2\mu\frac{W_{xy}W_y}{W} - \mu\frac{W_x W_{yy}}{W} = 0,$$

$$W_{sy} + \lambda W_{xxy} + \mu W_{yyy} - \frac{W_y W_s}{W} - \mu\frac{W_{yy}W_y}{W} - 2\lambda\frac{W_{xy}W_x}{W} + \lambda\frac{W_y W_{xx}}{W} = 0.$$

We integrate the first equation with respect to x and the second one integrate with respect to y. Finally we get the single equation

$$W_s - \lambda W_{xx} + \mu W_{yy} - g(s)W = 0.$$

It is simplified by the linear transformation $W = h(s)H$, where $h(s)$ is a solution to the equation $h' = gh$. The transformation brings it to the simple equation

$$H_s = \lambda H_{xx} - \mu H_{yy}. \tag{5.2}$$

Theorem 5.1. *Assume that $H(x, y, s)$ is an arbitrary solution to Equation (5.2). Then the functions*

$$u = -\frac{\partial}{\partial x} \ln(H), \qquad v = \frac{\partial}{\partial y} \ln(H),$$

$$U = \frac{\partial}{\partial y} \ln(H), \qquad V = -\frac{\partial}{\partial x} \ln(H) \tag{5.3}$$

define a solution to the coupled system (5.1).

Recall that the solution of the heat equation (5.2) with $\lambda = -\mu = 1$ is given in a closed form by the Poisson formula

$$H(x, y, s) = \frac{1}{4\pi s} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} H_0(\xi, \eta) e^{-\frac{(x-\xi)^2 + (y-\eta)^2}{4s}} \, d\xi d\eta,$$

where $H|_{s=0} = H_0(x, y)$ is an arbitrary continuous and bounded function. Therefore, the formulas (5.3) define a solution to the coupled system (5.1) depending on an arbitrary function of two variables $H_0(x, y)$.

6. CONSTRUCTION OF A PARTICULAR SOLUTION OF LATTICE (1.8)

In this section we construct a particular solution of the lattice (2.4), subject to the following additional constraints

$$u := u_0, \quad \text{for } n \geqslant 1, \qquad u_n = (-1)^{n+1}i, \quad \text{for } n \leqslant -1, \qquad u_n = (-1)^n i. \tag{6.1}$$

Then the lattice (2.4) takes the form

$$u_{xy} = \frac{2uu_xu_y}{u^2 + 1}.$$

Next, we use the integrals of lattice (2.4) found in [11], which for the choice of u_n by virtue of (6.1) read as

$$J = \frac{u_x}{u^2 + 1}, \qquad I = \frac{u_y}{u^2 + 1}.$$

Recall that a function $J(u, u_x, u_{xx}, u_{xxx}, \dots)$ is called a y–integral if the condition

$$D_y J(u, u_x, u_{xx}, u_{xxx}, \dots) = 0$$

is satisfied. The x–integral is defined similarly.

From the condition $D_y J(u, u_x, u_{xx}, u_{xxx}, \dots) = 0$ we find

$$J(u, u_x, u_{xx}, u_{xxx}, \dots) = f(x)$$

or in our case we have

$$\frac{u_x}{u^2 + 1} = f(x).$$

Integrating the last expression, we obtain

$$u = \tan(F(x) + G(y)),$$

where $F'(x) = f(x)$. Thus, $F(x)$ and $G(y)$ are arbitrary functions.

Let us rewrite the found solution in general form

$$u = \tan W(x, y), \tag{6.2}$$

where $W(x, y) = F(x) + G(y)$.

Now we substitute (6.2) into the symmetry (2.7)

$$u_{n,x_2} = u_{n,x_1x_1} - 2u_n u_{n,x_1} + u_n^2 + 1 - 2(u_n^2 - u_{n,x_1} + 1)H_n$$

of Equation (2.4). First, we simplify this symmetry due to the restrictions (6.1), namely, we find out what the value of nonlocality H. We let $x := x_1$, $t := x_2$, $y := y_1$, $H := H_0$:

$$u_t = u_{xx} - 2uu_x + u^2 + 1 - 2(u^2 - u_x + 1)H, \qquad (6.3)$$

$$D_y H = -D_x \left(\frac{u_y + iu - 1}{u + i} \right).$$

We integrate the last equality with respect to y and find

$$H = -\frac{u_x}{u+i} - \varphi(x),$$

where $\varphi(x)$ is an arbitrary function. Here we restrict ourselves to considering the case $\varphi(x) = 0$. Due to this condition the symmetry (6.3) can be written as

$$u_t = u_{xx} - \frac{2u_x^2}{u+i} - 2iu_x + u^2 + 1. \qquad (6.4)$$

Let us substitute $u = \tan W(x, y, t)$ into (6.4) and find

$$W_t = W_{xx} + i \left(2W_x^2 - 2W_x - i \right).$$

We differentiate this equation with respect to x

$$W_{t,x} = W_{xxx} + i \left(4W_x W_{xx} - 2W_{xx} \right)$$

and make the substitution

$$W_x = \frac{1}{2} \left(-i\tilde{W} + 1 \right).$$

Then the equation reduces to the Burgers equation

$$\tilde{W}_t = \tilde{W}_{xx} + 2\tilde{W}\tilde{W}_x.$$

As it is known, the Burgers equation is reduced to the heat equation

$$\bar{W}_t = \bar{W}_{xx}$$

by means of the Cole — Hopf substitution

$$\tilde{W} = -\frac{\bar{W}_x}{\bar{W}}.$$

Therefore, the sought particular solution can be written as

$$u = \tan(W) = \tan \left(D_x^{-1} \left(\frac{1}{2} \left(-i\tilde{W} + 1 \right) \right) \right)$$

$$= \tan \left(D_x^{-1} \left(\frac{1}{2} \left(i\frac{\bar{W}_x}{\bar{W}} + 1 \right) \right) \right)$$

$$= \tan \left(\frac{i}{2} \ln(\bar{W}(x, y, t)) + \frac{x}{2} + C(y, t) \right),$$

where $\bar{W}(x, y, t)$ is a solution of the heat equation.

Let us determine the dependence of function $\bar{W}(x, y, t)$ on the variable y. To do this, we substitute the found solution into the x–integral

$$\frac{u_y}{u^2 + 1} = g(y, t),$$

where $g(y, t)$ is an arbitrary function. After simplification we obtain

$$\bar{W}(x, y, t) = \bar{F}(x, t)\bar{G}(y, t),$$

where $\bar{F}(x,t)$ is arbitrary as well,

$$\bar{G}(y,t) = e^{2iC(y,t) - 2i\int g(y,t)dy}.$$

Finally, the particular solution of the lattice (2.4) becomes

$$u = \tan\left(\frac{i}{2}\ln(\bar{F}(x,t)\bar{G}(y,t)) + \frac{x}{2} + C(y,t)\right).$$

Returning to the corresponding coupled system (2.7), we see that its solution is given by

$$u = \tan\left(\frac{i}{2}\ln(\bar{F}(x,t)\bar{G}(y,t)) + \frac{x}{2} + C(y,t)\right),$$
$$v = -i.$$

CONCLUSION

The problem of constructing explicit solutions for multidimensional integrable models was studied by many authors like Shabat, Zakharov, Novikov, Krichever, Manakov, Grinevich, Santini, Fokas, Taimanov, Konopelcnenko, Bogdanov, Ferapontov, Pavlov, Dryuma and others. A great variety of tools were suggested, see, for instance, [14]–[20].

Here we discuss the dressing chains method that provides an effective tool for constructing explicit solutions for integrable nonlinear PDE in the dimension 1+1 (see, for instance, [12], [13] and references therein). However, in 3D some difficulties arise due to the nonlocal variables (see [1]). Examples considered in the article convince that to overcome these difficulties one can use finite reductions of the dressing chains obtained by imposing cutting off constraints preserving integrability. Besides the degenerate boundary conditions related to reductions integrable in the sense of Darboux, one can use also more general boundary conditions compatible with the integrability property of the lattices (see [9]).

REFERENCES

1. A.N. Leznov, A.B. Shabat, R.I. Yamilov. *Canonical transformations generated by shifts in nonlinear lattices* // Phys. Lett., A **174**:5–6, 397–402 (1993).
2. A.B. Shabat, R.I. Yamilov. *To a transformation theory of two–dimensional integrable systems* // Phys. Lett., A, **227**:1–2, 15–23 (1997).
3. I.T. Habibullin. *Characteristic Lie rings, finitely–generated modules and integrability conditions for (2+1)–dimensional lattices* // Phys. Scr. **87**:6, 065005 (2013).
4. M.N. Poptsova, I.T. Habibullin. *Algebraic properties of quasilinear two–dimensional lattices connected with integrability* // Ufa Math. J. **10**:3, 86–105 (2018).
5. I.T. Habibullin, M.N. Kuznetsova. *A classification algorithm for integrable two–dimensional lattices via Lie − Rinehart algebras* // Theor. Math. Phys. **203**:1, 569–581 (2020).
6. M.N. Kuznetsova. *Classification of a subclass of quasilinear two–dimensional lattices by means of characteristic algebras* // Ufa Math. J. **11**:3, 10–131 (2019).
7. I.T. Habibullin, A.R. Khakimova. *Characteristic Lie algebras of integrable differential–difference equations in 3D* // J. Phys. A, Math. Theor. **54**:29, 295202, 34 p. (2021).
8. I.T. Habibullin, A.R. Khakimova. *Symmetries of Toda type 3D lattices* // Preprint, arXiv:2409.07017 [nlin.SI] (2024).
9. B. Gürel, I. Habibullin. *Boundary conditions for two–dimensional integrable chains* // Phys. Lett., A **233**:1, 68–72 (1997).
10. A.V. Zhiber, V.V. Sokolov. *Exactly integrable hyperbolic equations of Liouville type* // Russ. Math. Surv. **56**:1, 61вЂ“101 (2001).
11. I.T. Habibullin, A.U. Sakieva. *On integrable reductions of two–dimensional Toda–type lattices* // Part. Diff. Eq. Appl. Math. **11** 100854 (2024).
12. A.B. Shabat, R.I. Yamilov. *Symmetries of nonlinear chains* // Leningr. Math. J. **2**:2, 377–400 (1991).

13. V.B. Matveev, A.O. Smirnov. *Dubrovin method and Toda lattice* // St. Petersbg. Math. J. **34**:6, 1019–1037 (2023).

14. P.G. Grinevich, P.M. Santini. *The finite–gap method and the periodic Cauchy problem for (2+1)–dimensional anomalous waves for the focusing Davey − Stewartson 2 equation* // Russ. Math. Surv. **77**:6, 1029–1059 (2022).

15. I.A. Taimanov. *The Moutard transformation for the Davey − Stewartson II equation and its geometrical meaning* // Math. Notes **110**:5, 754–766 (2021).

16. V.E. Zakharov, S.V. Manakov. *Construction of higher–dimensional nonlinear integrable systems and of their solutions* // Funct. Anal. Appl. **19**:2, 89–101 (1985).

17. R.Ch. Kulaev, A.B. Shabat. *Darboux system and separation of variables in the Goursat problem for a third order equation in R^3* // Russ. Math. **64**:4, 35–43 (2020).

18. B.G. Konopelchenko, B.T. Matkarimov. *Inverse spectral transform for the nonlinear evolution equation generating the Davey − Stewartson and Ishimori equations* // Stud. Appl. Math. **82**:4, 319–359 (1990).

19. M.N. Kuznetsova. *Construction of localized particular solutions of chains with three independent variables* // Theor. Math. Phys. **216**:2, 1158–1167 (2023).

20. M.N. Kuznetsova, I.T. Habibullin, A.R. Khakimova. *On the problem of classifying integrable chains with three independent variables* // Theor. Math. Phys. **215**:2, 667–690 (2023).

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