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SPECIFICATION OF ASYMPTOTIC PÓLYA TYPE ESTIMATE FOR DIRICHLET SERIES CONVERGING IN HALF–PLANE

T.I. BELOUS, A.M. GAISIN, R.A. GAISIN

Abstract. We study the asymptotic behavior of a Dirichlet series with positive exponents, converging in the left half–plane, on an arc of bounded slope ending on the convergence line. In the paper we obtain conditions under which the sum of the Dirichlet series satisfies an asymptotic equality of Pólya type on a set, the upper density of which is equal to one. In 2023 we obtained results related to dual cases. We showed that a Pólya type identity holds on an asymptotic set of positive upper density depending on the slope coefficient (Lipschitz constant) of the arc. In this paper, we prove a common theorem covering both of these cases, and we show that the asymptotic set has an upper density, which is equal to one.

Keywords: Dirichlet series, convergence half–plane, maximal term of series, curve of bounded slope, Pólya type identity.

Mathematics Subject Classification: 30D10

1. Introduction

The paper is devoted to problem on regular growth of the sum of the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \quad s = \sigma + it, \quad 0 < \lambda_n \uparrow \infty, \tag{1.1}$$

which converges only in the left half–plane

$$\Pi_0 = \{s = \sigma + it : \sigma < 0\},$$

on an arc γ of a bounded slope, $\gamma \subset \Pi_0$, which ends at the imaginary axis. The growth regularity of sum of series (1.1) is characterized by the Pólya type identity

$$\ln M_F(\sigma) \sim \ln |F(s)|, \quad s \in \gamma, \quad \sigma \to 0-, \quad \sigma \notin \mathrm{e}, \tag{1.2}$$

where $\mathrm{e} \subset [-1, 0)$ is some exceptional set, for instance, of zero lower density

$$M_F(\sigma) = \sup_{|t|<\infty} |F(\sigma + it)|.$$

To obtain estimates for the relative linear measure

$$\frac{\operatorname{mes}(\mathrm{e} \cap [\sigma, 0))}{|\sigma|}$$

as $\sigma \to 0-$, one usually assumes one of following lower growth estimate for the maximal term

$$\mu(\sigma) = \max_{n\geqslant 1}\{|a_n| e^{\lambda_n \sigma}\}$$

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of the series (1.1)

$$\varlimsup_{\sigma\to 0-}\frac{\ln\mu(\sigma)}{\Phi\left(\frac{1}{|\sigma|}\right)}>0 \qquad \text{or} \qquad \varliminf_{\sigma\to 0-}\frac{\ln\mu(\sigma)}{\Phi\left(\frac{1}{|\sigma|}\right)}>0,$$

where Φ, $\Phi:\mathbb{R}_+\to\mathbb{R}_+$, — is some fixed continuous majorant, $\Phi(x)\uparrow\infty$ as $x\to\infty$. The estimates a) and b) are rather natural and appeared quite often in similar problems before, see, for instance, [1]–[4].

In contrast to the case of arbitrary curve, for arcs of bounded slope (Lipshitz arcs) one can obtain stronger estimates, namely, asymptotic identities of Pólya type (1.2), which are valid everywhere on the half–interval $[-1,0)$ outside an exceptional set of a small relative measure. These problems were considered recently in [3], [4], but it turned out that the results of these works can be strengthened. We discuss this issue in the present note.

We briefly dwell on the history of the issue, introduce needed definitions and formulate previous results. Let $\Lambda=\{\lambda_n\}$, $0<\lambda_n\uparrow\infty$, be a sequence having a bounded upper density

$$D=\varlimsup_{n\to\infty}\frac{n}{\lambda_n}<\infty.$$

We shall employ the following notation for the distribution function of the points $\lambda\in\Lambda$:

$$n(t)=\sum_{\lambda_n\leqslant t}1, \qquad N(t)=\int_0^t\frac{n(x)}{x}dx.$$

By L we denote the class of all positive continuous unboundedly growing on $\mathbb{R}_+$ functions, while W stands for the convergence class, that is, the set of the functions $w\in L$ such that $w(x)(1+x^2)^{-1}$ belongs to $L^1(\mathbb{R}_+)$. Then for each function $\Phi\in L$ by φ we denote the inverse function and consider the following classes of functions:

$$W_\varphi=\left\{w\in W:\ \lim_{t\to\infty}\varphi(t)J(t;w)=0\right\},$$

$$\underline{W}_\varphi=\left\{w\in W:\ \varliminf_{t\to\infty}\varphi(t)J(t;w)=0\right\},$$

where

$$J(t;w)=\int_t^\infty\frac{w(x)}{x^2}dx$$

is the remainder of the converging integral $J(1;w)$.

Let the arc

$$\gamma=\{z=x+iy:\ y=g(x),a\leqslant x\leqslant b\}$$

have a bounded slope, that is,

$$\sup_{x_1\neq x_2}\left|\frac{g(x_1)-g(x_2)}{x_2-x_1}\right|=K<\infty. \tag{1.3}$$

This condition means that the function $g(x)$ satisfies the Lipshitz condition

$$|g(x_2)-g(x_1)|\leqslant K|x_2-x_1|.$$

This is why we call an arc of bounded slope the Lipshitz arc. Geometrical meaning of the condition (1.3) is that the absolute values of the tangents of all angles of chords of arc do not exceed K. This is why γ is called the arc of K–bounded slope.

We provide the results obtained in [3], [4].

Theorem 1.1 ([3]). *Let $\Phi \in L$, $w \in W_\varphi$, where $w(x) = N(ex)$. Suppose that the maximal term of the series (1.1) satisfies the condition*

$$\varlimsup_{\sigma\to 0-} \frac{\ln \mu(\sigma)}{\Phi\left(\frac{1}{|\sigma|}\right)} > 0 \tag{1.4}$$

and for some function $w_0 \in W_\varphi$ the estimates

$$q(\lambda_n) \leqslant w_0(\lambda_n), \quad n \geqslant 1, \tag{1.5}$$

hold, where

$$q(\lambda_n) = -\ln |Q'(\lambda_n)|, \qquad Q(\lambda) = \prod_{n=1}^{\infty}\left(1 - \frac{\lambda^2}{{\lambda_n}^2}\right).$$

Then for each arc γ of K–bounded slope defined by the equation $y = g(x)$, $-1 \leqslant x \leqslant 0$, for $s \in \gamma$, $\operatorname{Re} s = \sigma \to 0-$ over an asymptotic set $A \subset [-1, 0)$, the upper density DA of which satisfies the inequality

$$DA = \varlimsup_{\sigma\to 0-} \frac{\operatorname{mes}(\mathrm{A} \cap [\sigma, 0))}{|\sigma|} \geqslant \frac{1}{\sqrt{1+K^2}},$$

the relation holds

$$\ln M_F(\sigma) = (1 + o(1)) \ln |F(s)|, \quad s \in \gamma, \quad s = \sigma + it. \tag{1.6}$$

We note that under the assumptions of Theorem 1.1 the functions φ and w are consistent:

$$\varphi(x)w(x) = o(x) \quad \text{as} \quad x \to \infty.$$

This is implied by the belonging $w \in W_\varphi$.

In [4], the following result was proved, which dual to Theorem 1.1.

Theorem 1.2 ([4]). *Let $\Phi \in L$, $w \in \underline{W}_\varphi$, where $w(x) = N(ex)$. If φ and w are consistent, the maximal term of the series (1.1) obeys the condition*

$$\varliminf_{\sigma\to 0-} \frac{\ln \mu(\sigma)}{\Phi\left(\frac{1}{|\sigma|}\right)} > 0, \tag{1.7}$$

and for some function $w_0 \in W_\varphi$ the estimates (1.5) hold, then for each arc γ of K–bounded slope defined on the segment $[-1, 0]$, as $s \in \gamma$, $\operatorname{Re} s = \sigma \to 0-$ over an asymptotic set $A \subset [-1, 0)$,

$$DA \geqslant \frac{1}{\sqrt{1+K^2}},$$

Pólya type identity (1.6) holds.

The aim of the present paper is to show that in both theorems the relation (1.6) holds on a set A, the upper density DA of which is equal to one.

2. Lemmas. Main result

The proof of the main theorem is based on the following lemmas, which allows us to use a common approach to the mentioned dual problems.

Lemma 2.1. *Let a function $g(z)$ be analytic in the circle*

$$D(0, R) = \{z : |z| < R\},$$

and $|g(0)| \geqslant 1$. If

$$0 < r < 1 - N_0^{-1}, \qquad N_0 > 1,$$

then there exists at most countably many circles

$$V_n = \{z : |z - z_n| \leqslant \rho_n\}, \qquad \sum_n \rho_n \leqslant R r^{N_0}(1-r),$$

such that in the circle $\{z : |z| \leqslant rR\}$*, but outside the set* $\bigcup_n V_n$*, the estimate*

$$\ln |g(z)| \geqslant \frac{R - |z|}{R + |z|} \ln |g(0)| - 5 N_0 L_0 \tag{2.1}$$

holds, where

$$L_0 = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |g(Re^{i\theta})| d\theta - \ln |g(0)|. \tag{2.2}$$

This lemma was proved in [5].

We shall also need the following Borel — Nevanlinna type lemma, which was proved in [1, Lms. 4, 5]. We formulate it here in an appropriate form.

Lemma 2.2. *Let* $u(t)$ *be a continuous non–decreasing on* $[-1, 0)$ *function,* $u(t) \to \infty$ *as* $t \to 0-$*. By* $v = v(t)$ *we denote the solution to the equation*

$$w(v) = e^{u(\sigma)}, \tag{2.3}$$

where w *is some function from the class* W*.*

If as $t \to 0-$ *outside some set* $\mathrm{e}_0 \subset [-1, 0)$*,* $\mathrm{mes}\,(\mathrm{e}_0 \cap [\tau_j, 0)) = o(|\tau_j|)$ *for some sequence* $\{\tau_j\}$*,* $\tau_j \uparrow 0$*,*

$$\frac{w(v(t))}{|t| v(t)} = o(1),$$

and the condition

$$\lim_{\tau_j \to 0-} \frac{1}{|\tau_j|} J(v_j; w) = 0, \qquad v_j = v(\tau_j),$$

holds, then as $\sigma \to 0-$ *outside the exceptional set* $\mathrm{e} \subset [-1, 0)$*, for which*

$$\mathrm{mes}\,(\mathrm{e} \cap [\tau_j, 0)) = o(|\tau_j|), \quad \tau_j \to 0-,$$

the asymptotic identity

$$u\left(t + \frac{w(v(t))}{v(t)}\right) = u(t) + o(1)$$

holds.

This lemma is more general than the corresponding lemma from [6, Lm. 3.2][1], where $\mathrm{e}_0 = \varnothing$.

Our main result is as follows.

Theorem 2.1. *Let the assumptions of Theorem 1.1 or Theorem 1.2 hold. Then for each arc* γ *of a bounded slope defined on the segment* $[-1, 0]$*, as* $s \in \gamma$*,* $\mathrm{Re}\, s = \sigma \to 0-$ *over the asymptotic set* $A \subset [-1, 0)$*,* $DA = 1$*, Pólya type identity (1.6) holds.*

[1]The proof of Theorem 1.1 in [3] involves a wrong reference: instead of Lemma 2.2, Lemma 3.2 from [6] was mentioned.

3. Proof of Theorem 2.1

Let the assumptions of Theorem 1.1 be satisfied. We let $w_1(x) = w(x) + w_0(x)$, where $w(x) = N(ex)$, $w_0(x)$ is the majorant from the condition (1.5). Since $w \in W_\varphi$, we have $w_1 \in W_\varphi$. Therefore, the functions φ and w_1 are consistent, that is,

$$\lim_{x\to\infty} \frac{\varphi(x)w_1(x)}{x} = 0.$$

Then there exists a function $w^*(x) = \beta(x)w_1(x)$, $\beta \in L$, $1 \leqslant \beta(x)$, which also belongs to W_φ. This is why $\varphi(x)w^*(x) = o(x)$ as $x \to \infty$.

Let $v = v(\sigma)$ be a solution to the equation

$$w^*(v) = 3\ln\mu(\sigma). \tag{3.1}$$

It is obvious that $v(\sigma) \uparrow \infty$ as $\sigma \uparrow 0-$. Equation (3.1) can be written as

$$w^*(v) = e^{u(\sigma)}, \qquad u(\sigma) = \ln 3 + \ln\ln\mu(\sigma). \tag{3.2}$$

Since $w^* \in W_\varphi$, we have

$$\lim_{v\to\infty} \varphi(v)J(v; w^*) = 0, \tag{3.3}$$

where $v = v(\sigma) \to \infty$ as $\sigma \to 0-$. The lower bound (1.4) for $\ln\mu(\sigma)$, in view of the identity (3.1), implies that for some sequence $\{\tau_j\}$, $\tau_j \uparrow 0$,

$$w^*(v(\tau_j)) = 3\ln\mu(\tau_j) > \nu_0\Phi\left(\frac{1}{|\tau_j|}\right), \quad \nu_0 > 0.$$

Since $w^*(x) = o(x)$ as $x \to \infty$, this implies

$$\frac{1}{|\tau_j|} \leqslant \varphi(v_j), \quad v_j = v(\tau_j), \quad j \geqslant j_0.$$ [1]

Therefore, by (3.3) and the consistency condition of the functions φ and w^* we have

a) The condition

$$\lim_{\tau_j\to 0-} \frac{1}{|\tau_j|}J(v_j; w^*) = 0, \quad v_j = v(\tau_j),$$

holds.

b) For all σ in $[-1, 0)$ but outside some set e_0,

$$\mathrm{mes}(\mathrm{e}_0 \cap [\tau_j, 0)) = o(|\tau_j|)$$

the relation

$$\frac{w^*(v(\sigma))}{|\sigma|v(\sigma)} = o(1)$$

holds, see [1, Lm. 4].

If the assumptions of Theorem 1.2 are satisfied, then $w^* \in \underline{W}_\varphi$ (formally, the function w^* is the same as in Theorem 1.1), and $\varphi(x)w^*(x) = o(x)$ as $x \to \infty$ (by the assumptions of Theorem 1.2, the functions φ and w are consistent). Since $w^* \in \underline{W}_\varphi$, there exists a sequence $\{\tau_j\}$, $\tau_j \uparrow 0$, (we keep the same notation for this sequence although it is chosen a bit different) such that

$$\lim_{v_j\to\infty} \varphi(v_j)J(v_j; w^*) = 0, \quad v_j = v(\tau_j). \tag{3.4}$$

[1] In [1] this estimate was obtained under the assumption that there exists a constant $C \in (0, \infty)$ such that $\varphi(2t) \leqslant C\varphi(t)$, $t > 0$. This restriction is unnecessary.

By the condition (1.7) in view of (3.1) we obtain that for $\sigma' \leqslant \sigma < 0$ the inequality

$$w^*(v(\sigma)) > \mu_0 \Phi\left(\frac{1}{|\sigma|}\right), \quad \mu_0 > 0,$$

holds. This implies that for $\sigma' < \sigma'' < \sigma < 0$

$$\frac{1}{|\sigma|} \leqslant \varphi(v), \qquad v = v(\sigma).$$

Therefore, in view of (3.4) and the consistency condition of the functions φ and w^*, we again arrive at the conditions a) and b).

Thus, in both Theorems 1.1 and 1.2, the matter is reduced to Relations a) and b). At the same time, generally speaking, in each theorem the sequences $\{\tau_j\}$, $\tau_j \uparrow 0$, are different; in Theorem 1.1 it is chosen by the condition (1.4), while in Theorem 1.2 by the condition (3.4). Further arguing are same for both theorems and are based on the common Borel — Nevanlinna type Lemma 2.2 and Govorov type Lemma 2.1.

By applying Lemma 2.2, it was shown in [3], [4] that as $\sigma \to 0-$ outside some set e_1,

$$\mathrm{e}_0 \subset \mathrm{e}_1 \subset [-1, 0), \qquad \operatorname{mes} \mathrm{e}_1 \cap [\tau_j, 0) = o(|\tau_j|), \qquad \tau_j \to 0-,$$

the following key estimates hold:

$$\mu(\sigma) \leqslant M_F(\sigma) \leqslant M_F(\sigma + 2h^*) < \mu(\sigma)^{1+o(1)}, \tag{3.5}$$

where

$$h^* = h^*(\sigma) = \frac{w^*(v(\sigma))}{v(\sigma)},$$

and

$$\mu(\sigma)^{1+o(1)} \leqslant \max_{|\xi - \alpha| \leqslant h^{(1)}} |F(\xi)| = |F(\xi^*)|, \tag{3.6}$$

where

$$|\xi^* - \alpha| \leqslant h^{(1)}, \qquad \alpha = \sigma + it \in \gamma, \qquad h^{(1)} = h^{(1)}(\sigma) = \frac{h^*(\sigma)}{\sqrt{\beta(v)}}, \qquad v = v(\sigma).$$

Our specifications concern the following estimates obtained in [3], [4]. We provide the corresponding arguing with all details.

We denote

$$B = [-1, 0) \backslash \mathrm{e}_1, \qquad h = \frac{w_1(v)}{v}, \qquad v = v(\sigma),$$

where $w_1(x) = w(x) + w_0(x)$ is a function from the class W_φ (in Theorem 1.1) or the class $\underline{W}_\varphi$ (in Theorem 1.2). The notation for the functions w and w_0 are the same as in Theorem 1.1.

There exists a sequence

$$\{\sigma_j\}, \qquad \sigma_j \in B, \qquad \sigma_j \uparrow 0, \qquad \sigma_j + h_j \leqslant \sigma_{j+1}, \qquad j \geqslant 1,$$

such that [1]

$$B \subset \bigcup_{j=1}^{\infty} [\sigma_j - h_j, \sigma_j + h_j], \qquad h_j = \frac{w_1(v_j)}{v_j},$$

where $v_j = v(\sigma_j)$, $j = 1, 2, \ldots$.

Let $g(z) = F(z + \xi^*)$. As the estimate (3.6) shows,

$$|g(0)| > 1 \quad \text{for} \quad \sigma' < \sigma'' \leqslant \sigma_0 < \sigma < 0 \quad \text{outside} \quad \mathrm{e}_1.$$

According to Lemma 2.1, under an appropriate choice of R, r, $0 < r < 1$, in the circle $\{z : |z| \leqslant rR\}$, but outside a set of exceptional circles of small measure, the estimate (2.1) holds. Then this estimate is also true on an appropriate subarc $\gamma' \subset \gamma$ except for some portion, which also has a small total length. Our aim is to find the size of this portion.

In order to apply Lemma 2.1 to the function $g(z)$, we let

$$\alpha = \alpha_j = \sigma_j + it_j, \qquad \xi^* = \xi_j^*, \qquad h^{(1)} = h^{(1)}(\sigma_j) = \frac{w_1(v_j)}{v_j}\sqrt{\beta(v_j)},$$

while in the lemma we take

$$N_0 = 4, \qquad r = r(j) = \frac{1}{\sqrt{\beta(v_j)}}, \qquad R = R_j = 2h_j^*, \qquad h_j^* = \frac{w^*(v_j)}{v_j}, \qquad j \geqslant j_1.$$

Here the index j_1 is chosen so that for $j \geqslant j_1$ the condition

$$r = r(j) < 1 - N_0^{-1} = \frac{3}{4}$$

holds. Then, since $rR = 2h_j^{(1)}$, by Lemma 2.1, in the circle $\{z : |z| \leqslant 2h_j^{(1)}\}$ but outside exceptional circles V_{nj} with the total sum of radii obeying the estimate

$$\begin{gathered}\sum_n \rho_{nj} \leqslant R_j r_j^{N_0}(1 - r_j) < \frac{2h_j}{\beta_j}, \qquad j \geqslant j_1, \\ h_j = \frac{w_1(v_j)}{v_j}, \qquad \beta_j = \beta(v_j), \qquad v_j = v(\sigma_j),\end{gathered} \tag{3.7}$$

the function $g(z) = F(z + \xi_j^*)$ satisfies the lower bound (2.1).

Let γ_j be a part of the arc γ connecting the vertical lines passing through the end–points of the segment $\Delta_j = [\sigma_j - h_j, \sigma_j + h_j]$, $j \geqslant j_1$. Since the arc γ has a K–slope, the arc γ_j is contained in the rectangle

$$P_j = \Delta_j \times [c_j, d_j], \qquad d_j - c_j \leqslant 2Kh_j,$$

with the center at the point $\alpha_j = \sigma_j + it_j$ and connects its vertical sides.

Let P_j^* be a translation of P_j by the vector $a_j = -\xi_j^*$. Since $\beta_j = \beta(v_j) \geqslant 1$, the set P_j^*, is obviously contained in the circle $\{z : |z| < 2h_j^{(1)}\}$. This is why the estimate (2.1) holds everywhere in the rectangle P_j^* except for the circles V_{nj} with the total sum of radii obeying the estimate (3.7), that is, for all $z \in P_j^* \setminus \cup_n V_{nj}$ as $j \to \infty$

$$\ln|g(z)| \geqslant \left[1 + o(1) - \frac{20L_0}{\ln|g(0)|}\right] \ln|g(0)|. \tag{3.8}$$

Since

$$|g(0)| = |F(\xi^*)| \leqslant M_F(\sigma + h^{(1)}) \leqslant M_F(\sigma + 2h^*),$$

and $\sigma_j \in B$, $L_0 > 0$, the estimate (3.5) shows that as $j \to \infty$

$$\frac{1}{2\pi}\int_0^{2\pi} \ln^+|g(Re^{i\theta})|d\theta \sim \ln|g(0)|.$$

Hence, by (3.8) we obtain that for all z in the rectangle $P_j^* = P_j + a_j$ (P_j^* is the translation of P_j by the vector $a_j = -\xi_j^*$) but outside the exceptional circles V_{nj} with the total sum of radii not exceeding $2\frac{h_j}{\beta_j}$, as $j \to \infty$ the estimate

$$\ln|g(z)| \geqslant (1 + o(1)) \ln|g(0)| \tag{3.9}$$

holds. Taking into consideration the identity $g(z) = F(z + \xi_j^*)$ and estimates (3.5)–(3.9), we obtain that everywhere in the rectangle P_j centered at the point $\alpha_j = \sigma_j + it_j$ except for the circles $V'_{nj} = V_{nj} - a_j$, $n = 1, 2, \ldots$, with the total sum of radii not exceeding $2\frac{h_j}{\beta_j}$ the estimate

$$\ln|F(s)| > (1 + o(1)) \ln \mu(\sigma_j), \qquad s = z + \xi_j^*, \qquad j \to \infty, \tag{3.10}$$

holds.

Let e_2 be the projection of all exceptional circles of the set $\bigcup\limits_j P_j$ onto B. We are going to confirm that $D\mathrm{e}_2 = 0$. Indeed, let $\sigma_j \leqslant \sigma < \sigma_{j+1}$. Since $\sigma_\mathrm{j} \notin \mathrm{e}_0$, $j \geqslant 1$, according to the above conditions a) and b),

$$h_j \leqslant h_j^{(1)} \leqslant h_j^* = o(\sigma_j) \quad \text{as} \quad j \to \infty.$$

Therefore, in view of the inequality $\sum\limits_{k \geqslant j+1} h_k \leqslant |\sigma|$ we obtain

$$\begin{aligned}\frac{\operatorname{mes}(\mathrm{e}_2 \cap [\sigma, 0))}{|\sigma|} &\leqslant \frac{\operatorname{mes}(e_2 \cap \Delta_j)}{|\sigma|} + \frac{\operatorname{mes}(e_2 \cap [\sigma_{j+1} - h_{j+1}, 0))}{|\sigma|} \\ &\leqslant \frac{2h_j}{|\sigma_j + h_j|\beta_j} + \frac{2}{|\sigma|} \sum_{k \geqslant j+1} \frac{h_k}{\beta_k} \leqslant \frac{2h_j}{|\sigma_j|\beta_j(1 + o(1))} + \frac{2}{\beta_j} = o(1), \quad j \to \infty.\end{aligned}$$

This means $D\mathrm{e}_2 = 0$. Therefore, letting $\mathrm{e} = \mathrm{e}_1 \cup \mathrm{e}_2$, we get $d\mathrm{e} = 0$ since $D\mathrm{e}_2 = 0$, $d\mathrm{e}_1 = 0$.

The projection p_j of the arc γ_j onto $[-1, 0)$ is the segment Δ_j. Let $A = P \backslash \mathrm{e}$, where $P = \bigcup\limits_{j=1}^{\infty} p_j$. On this set the asymptotic estimates (3.6), (3.10) hold; the set A is called the asymptotic set. This implies that as $s \in \gamma$, $\operatorname{Re} s = \sigma \to 0-$ over the set A, the Pólya type relation

$$\ln|F(s)| = (1 + o(1)) \ln \mu(\sigma) = (1 + o(1)) \ln M_F(\sigma)$$

holds. It remains to get the estimate for the upper density DA. Since $DP = 1$, we have

$$DA = \overline{\lim_{\sigma \to 0-}} \frac{\operatorname{mes}(A \cap [\sigma, 0))}{|\sigma|} \geqslant \overline{\lim_{j \to \infty}} \frac{\operatorname{mes}(P \cap [\tau_j, 0))}{|\tau_j|} - \overline{\lim_{j \to \infty}} \frac{\operatorname{mes}(\mathrm{e} \cap [\tau_j, 0))}{|\tau_j|} = 1.$$

Here $\{\tau_j\}$ is the above introduced sequence, for which

$$\operatorname{mes}(\mathrm{e} \cap [\tau_j, 0)) = o(|\tau_j|) \quad \text{as} \quad j \to \infty.$$

Therefore, $DA = 1$. The proof of Theorem 2.1 is complete.

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Tatiana Ivanovna Belous,
Ufa University of Science and Technology,
Zaki Validi str. 32,
450076, Ufa, Russia
E-mail: `belousti@yandex.ru`

Akhtyar Magazovich Gaisin,
Institute of Mathematics,
Ufa Federal Research Center, RAS,
Chernyshevsky str. 112,
450008, Ufa, Russia
E-mail: `gaisinam@mail.ru`

Rashit Akhtyarovich Gaisin,
Institute of Mathematics,
Ufa Federal Research Center, RAS,
Chernyshevsky str. 112,
450008, Ufa, Russia
E-mail: `rashit.gajsin@mail.ru`