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RECONSTRUCTION OF POTENTIAL OF DISCONTINUOUS STURM — LIOUVILLE OPERATOR FROM SPECTRAL DATA

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Abstract. We deal with the inverse spectral problem of the discontinuous Sturm — Liouville operator. The aim is we to determine the potential $q(x)$ and the boundary constant h by a given spectral data. We provide the algorithm for reconstructing the potential $q(x)$ from the spectral data.

Keywords: discontinuous Sturm — Liouville operator, inverse problem, spectral data.

Mathematics Subject Classification: 34A55, 34A36, 34B24

1. Introduction

This paper is devoted to the inverse spectral problem for the Sturm — Liouville equation with a discontinuous coefficient subject to the discontinuity conditions (or transmission conditions) at an interior point of the finite interval $(0,\pi)$. Unlike other studies, the problem examined in this paper includes both the discontinuous coefficient and the discontinuity condition inside the finite interval. Namely, we consider the discontinuous Sturm — Liouville boundary value problem

$$-y'' + q(x)y = \lambda^2\mu(x)y, \quad 0 < x < \pi, \tag{1.1}$$

$$y(a+0) = \beta y(a-0), \qquad y'(a+0) = \beta^{-1}y'(a-0), \tag{1.2}$$

$$y'(0) = y'(\pi) + hy(\pi) = 0, \tag{1.3}$$

where $q(x) \in L_2(0,\pi)$ is a real–valued function, $\beta > 0$ and h are real constants, $\mu(x)$ is a piecewise–constant function

$$\mu(x) = \begin{cases} 1, & 0 < x < a, \\ \alpha^2, & a < x < \pi, \end{cases}$$

λ is a spectral parameter. We assume that $a > \frac{\alpha\pi}{\alpha+1}$.

The Sturm — Liouville problems containing discontinuity conditions (see [1]–[6]) and Sturm — Liouville problems involving discontinuous coefficients (see, for instance, [7]–[10]) were studied as two separate problems. In this paper, we examine a new generalized problem by combining these two different Sturm — Liouville problems. The direct spectral problem (i.e. the spectral properties and the eigenfunction expansion) of Equation (1.1) with the discontinuity condition (1.2) under the boundary condition $y'(0)-h_1y(0) = y'(\pi)+h_2y(\pi) = 0$ was studied in [11] and the inverse problem was solved by means of the Weyl function [12]. Moreover, the inverse problem of Equation (1.1) with (1.2) under the boundary condition $y(0) = y(\pi) = 0$ according to the spectral data and Weyl function were studied in [13].

Discontinuous boundary value problems appear in many disciplines from mathematics to engineering. Especially, since such problems are related to discontinuous material properties,

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it is important and interesting to study the corresponding inverse problems, see [14]–[20] and the reference therein.

In this paper, we pose the inverse problem as follows: to determine the potential function $q(x)$ and the boundary constant h from the spectral data of the problem (1.1)–(1.3). For this purpose, using the Gelfand — Levitan — Marchenko method, we construct the modified main equation which is satisfied by the kernel of new integral representation. We provide this integral representation in Section 2 and the kernel has a discontinuity along the line $t = -\alpha(x-a)+a$ for $a < x < \pi$. We prove the uniqueness theorem for the inverse problem and provide a reconstruction algorithm of the potential function $q(x)$ from the spectral data.

2. Preliminaries

We denote by $e(x, \lambda)$ the solution of Eequation (1.1) with discontinuity conditions (1.2) under the initial conditions

$$e(0,\lambda)=0, \qquad e'(0,\lambda)=i\lambda.$$

As $q(x) \equiv 0$ in Equation (1.1), the solution $e_0(x,\lambda)$ is

$$e_0(x,\lambda)=\begin{cases} e^{i\lambda x}, & 0<x<a,\\ \kappa_1 e^{i\lambda\vartheta^+(x)}+\kappa_2 e^{i\lambda\vartheta^-(x)}, & a<x<\pi,\end{cases}$$

with

$$\vartheta^{\pm}(x)=\pm\alpha(x-a)+a, \qquad \kappa_1=\frac{1}{2}\left(\beta+\frac{1}{\alpha\beta}\right), \qquad \kappa_2=\frac{1}{2}\left(\beta-\frac{1}{\alpha\beta}\right).$$

Theorem 2.1. [11] *The solution $e(x,\lambda)$ can be expressed by the integral representation:*

$$e(x,\lambda)=e_0(x,\lambda)+\int\limits_{-\sigma(x)}^{\sigma(x)} k(x,t)e^{i\lambda t}dt, \tag{2.1}$$

where

$$\sigma(x)=\begin{cases} x, & 0<x<a,\\ \vartheta^+(x), & a<x<\pi,\end{cases}$$

the kernel function $k(x,\cdot)$ belongs to $L_1(-\sigma(x),\sigma(x))$ for each fixed $x \in (0,a)\cup(a,\pi)$ and satisfies the inequality

$$\int\limits_{-\sigma(x)}^{\sigma(x)} |k(x,t)|\,dt \leqslant \exp\{cp(x)\}-1$$

with

$$p(x)=\int\limits_0^x (x-\xi)|q(\xi)|d\xi, \qquad c=(\alpha+4)|\kappa_1|+(\alpha+2)|\kappa_2|.$$

We observe that the kernel $k(x,t)$ possesses the following properties:

$$k(x,-\sigma(x))=0,$$

$$k(x,\sigma(x))=\begin{cases} \dfrac{1}{2}\displaystyle\int\limits_0^x q(\xi)\,d\xi, & 0<x<a,\\ \dfrac{\kappa_1}{2}\displaystyle\int\limits_0^x \frac{1}{\sqrt{\mu(\xi)}}q(\xi)\,d\xi, & a<x<\pi,\end{cases} \tag{2.2}$$

$$k(x,\vartheta^-(x)+0)-k(x,\vartheta^-(x)-0)=-\frac{\kappa_2}{2}\left(\int\limits_0^a q(\xi)d\xi-\frac{1}{\alpha}\int\limits_a^x q(\xi)d\xi\right),\quad a<x<\pi.$$

We denote by $c(x,\lambda)$ the solution of Equation (1.1) subject to the discontinuity conditions (1.2) and the initial conditions

$$c(0,\lambda)=1,\qquad c'(0,\lambda)=0.$$

The integral representation for the solution $c(x,\lambda)$ implied by formula (2.1) reads as

$$c(x,\lambda)=c_0(x,\lambda)+\int\limits_0^{\sigma(x)}\tilde{k}(x,t)\cos\lambda t\,dt, \tag{2.3}$$

where

$$c_0(x,\lambda)=\begin{cases}\cos\lambda x, & 0<x<a,\\ \kappa_1\cos\lambda\vartheta^+(x)+\kappa_2\cos\lambda\vartheta^-(x), & a<x<\pi,\end{cases}$$

and $\tilde{k}(x,t)=k(x,-t)+k(x,t)$. The latter equality yields

$$\tilde{k}(x,\sigma(x))=k(x,\sigma(x)). \tag{2.4}$$

Let $\zeta(x,\lambda)$ be the solution of Equation (1.1) subject to (1.2) and the initial conditions

$$\zeta(\pi,\lambda)=-1,\qquad \zeta'(\pi,\lambda)=h.$$

The estimate

$$\zeta(x,\lambda)=O\left(e^{|\operatorname{Im}\lambda|(\vartheta^+(\pi)-\vartheta^+(x))}\right),\quad |\lambda|\to\infty,$$

holds. We define the characteristic function of the boundary value problem (1.1)–(1.3) as

$$\chi(\lambda)=c'(\pi,\lambda)+hc(\pi,\lambda).$$

This function is entire in λ, and hence, it has an at most countable set of zeros $\{\lambda_j\}$, and the numbers $\{\lambda_j^2\}$ are the eigenvalues of the problem (1.1)–(1.3). The functions $c(x,\lambda_j)$ and $\zeta(x,\lambda_j)$ are eigenfunctions and

$$\zeta(x,\lambda_j)=\rho_j c(x,\lambda_j),\quad \rho_j\neq 0. \tag{2.5}$$

We denote the norming constants of the problem (1.1)–(1.3) by

$$\gamma_j:=\int\limits_0^\pi c^2(x,\lambda_j)\mu(x)\,dx.$$

The relation and estimate

$$\dot{\chi}(\lambda_j)=2\lambda_j\gamma_j\rho_j,\quad \dot{\chi}(\lambda)=\frac{d}{d\lambda}\chi(\lambda), \tag{2.6}$$

$$\chi(\lambda)=\lambda\omega(\lambda)+O\left(e^{|\operatorname{Im}\lambda|\vartheta^+(\pi)}\right),\quad |\lambda|\to\infty, \tag{2.7}$$

where

$$\omega(\lambda)=\alpha\left(-\kappa_1\sin\lambda\vartheta^+(\pi)+\kappa_2\sin\lambda\vartheta^-(\pi)\right).$$

The zeros of this function are

$$\tilde{\lambda}_j=\frac{j\pi}{\vartheta^+(\pi)}+d_j,\qquad \sup_j|d_j|=d<\infty.$$

Theorem 2.2. [11] *The boundary value problem (1.1)–(1.3) has a countable set of eigenvalues* $\{\lambda_j^2\}_{j\geqslant 0}$,

$$\lambda_j = \tilde{\lambda}_j + \frac{p_j}{\tilde{\lambda}_j} + \frac{t_j}{j}, \qquad \{p_j\} \in \ell_\infty, \qquad \{t_j\} \in \ell_2.$$

Definition 2.1. *The numbers* $\{\lambda_j^2, \gamma_j\}_{j\geqslant 0}$ *are called the spectral data of the boundary value problem (1.1)–(1.3).*

Theorem 2.3. [11] *The system of eigenfunctions* $\{c(x, \lambda_j)\}_{j\geqslant 0}$ *of boundary value problem (1.1)–(1.3) is complete in* $L_2(0, \pi; \mu)$. *The function* $f(x) \in AC[0, a] \cap AC[a, \pi]$ *satisfying the discontinuity condition (1.2) and the boundary conditions (1.3) can be expanded into a uniformly convergent series over the eigenfunctions of the problem (1.1)—(1.3)*

$$f(x) = \sum_{j=0}^{\infty} s_j c(x, \lambda_j), \qquad s_j = \frac{1}{\gamma_j} \int_0^\pi c(x, \lambda_j) f(x) \mu(x)\, dx. \tag{2.8}$$

For $f(x) \in L_2(0, \pi; \mu)$, *the series (2.8) converges in* $L_2(0, \pi; \mu)$ *and Parseval's identity holds:*

$$\int_0^\pi |f(x)|^2 \mu(x)\, dx = \sum_{j=0}^{\infty} \gamma_j |s_j|^2.$$

3. Main results

Consider the function

$$f(x, t) = \mu(t) \sum_{j=0}^{\infty} \left(\frac{c_0(x, \lambda_j) c_0(t, \lambda_j)}{\gamma_j} - \frac{c_0(x, \tilde{\lambda}_j) c_0(t, \tilde{\lambda}_j)}{\gamma_j^0} \right), \tag{3.1}$$

where the numbers γ_j^0 are the norming constants of the problem (1.1)–(1.3) for $q(x) \equiv 0$.

Remark 3.1. *The integral representation (2.3) can be written as*

$$c(x, \lambda) = c_0(x, \lambda) + \int_0^x h(x, t) c_0(t, \lambda)\, dt, \tag{3.2}$$

where

$$h(x, t) = \begin{cases} \tilde{k}(x, t), & 0 < t < x < a, \quad 0 < t < \vartheta^-(x), \quad a < x < \pi, \\ \tilde{k}(x, t) - \dfrac{\kappa_2}{\kappa_1} \tilde{k}(x, 2a - t), & \vartheta^-(x) < t < a < x < \pi, \\ \dfrac{\alpha}{\kappa_1} \tilde{k}(x, \vartheta^+(t)), & a < t < x < \pi. \end{cases} \tag{3.3}$$

To justify this formula, we take into consideration the relation

$$\cos \lambda t = \begin{cases} c_0(x, \lambda), & 0 < x < a, \\ \dfrac{1}{\kappa_1} c_0 \left(\dfrac{t - a}{\alpha} + a, \lambda \right) - \dfrac{\kappa_2}{\kappa_1} c_0(2a - t, \lambda), & a < x < \pi. \end{cases}$$

Resolving the Volterra equation (3.2) with respect to $c_0(x, \lambda)$, we have

$$c_0(x, \lambda) = c(x, \lambda) + \int_0^x \tilde{h}(x, t) c(t, \lambda)\, dt. \tag{3.4}$$

Theorem 3.1. *For each fixed $x \in (0,\pi]$, the kernel $\tilde{k}(x,t)$ satisfies the linear integral equation*

$$f(x,t) + h(x,t) + \int_0^x h(x,\xi) f(\xi,t)\,d\xi = 0, \quad t < x. \tag{3.5}$$

This equation is called the Gelfand — Levitan — Marchenko type equation (or modified main equation) of the boundary value problem (1.1)–(1.3).

Proof. Using the formulas (3.2) and (3.4), we write

$$\Phi_n(x,t) = \phi_{n_1}(x,t) + \phi_{n_2}(x,t) + \phi_{n_3}(x,t) + \phi_{n_4}(x,t), \tag{3.6}$$

where

$$\Phi_n(x,t) = \sum_{j=0}^{n} \left(\frac{c(x,\lambda_j)c(t,\lambda_j)}{\gamma_j} - \frac{c_0(x,\tilde{\lambda}_j)c_0(t,\tilde{\lambda}_j)}{\gamma_j^0} \right),$$

$$\phi_{n_1}(x,t) = \sum_{j=0}^{n} \left(\frac{c_0(x,\lambda_j)c_0(t,\lambda_j)}{\gamma_j} - \frac{c_0(x,\tilde{\lambda}_j)c_0(t,\tilde{\lambda}_j)}{\gamma_j^0} \right),$$

$$\phi_{n_2}(x,t) = \int_0^x h(x,\xi) \sum_{j=0}^{n} \left(\frac{c_0(\xi,\lambda_j)c_0(t,\lambda_j)}{\gamma_j} - \frac{c_0(\xi,\tilde{\lambda}_j)c_0(t,\tilde{\lambda}_j)}{\gamma_j^0} \right) d\xi,$$

$$\phi_{n_3}(x,t) = \int_0^x h(x,\xi) \sum_{j=0}^{n} \frac{c_0(\xi,\tilde{\lambda}_j)c_0(t,\tilde{\lambda}_j)}{\gamma_j^0} d\xi,$$

$$\phi_{n_4}(x,t) = -\int_0^t \tilde{h}(t,\xi) \sum_{j=0}^{n} \frac{c(x,\lambda_j)c(\xi,\lambda_j)}{\gamma_j} d\xi.$$

Let $g(x) \in AC[0,a] \cap AC[a,\pi]$. In view of Theorem 2.3 and the formula (3.1) we obtain

$$\begin{aligned}
\lim_{n\to\infty} \max_{0\leqslant x\leqslant\pi} \left| \int_0^\pi g(t)\Phi_n(x,t)\mu(t)\,dt \right| &= \lim_{n\to\infty} \max_{0\leqslant x\leqslant\pi} \left| \sum_{j=0}^{n} s_j c(x,\lambda_j) - \sum_{j=0}^{n} s_j^0 c_0(x,\tilde{\lambda}_j) \right| \\
&\leqslant \lim_{n\to\infty} \max_{0\leqslant x\leqslant\pi} \left| g(x) - \sum_{j=0}^{n} s_j c(x,\lambda_j) \right| \\
&\quad + \lim_{n\to\infty} \max_{0\leqslant x\leqslant\pi} \left| g(x) - \sum_{j=0}^{n} s_j^0 c_0(x,\tilde{\lambda}_j) \right| = 0,
\end{aligned} \tag{3.7}$$

$$\lim_{n\to\infty} \int_0^\pi g(t)\phi_{n_1}(x,t)\mu(t)\,dt = \int_0^\pi g(t) f(x,t)\,dt, \tag{3.8}$$

$$\lim_{n\to\infty} \int_0^\pi g(t)\phi_{n_2}(x,t)\mu(t)\,dt = \int_0^\pi g(t) \left(\int_0^x h(x,\xi) f(\xi,t)\,d\xi \right) dt, \tag{3.9}$$

$$\lim_{n\to\infty} \int_0^\pi g(t)\phi_{n_3}(x,t)\mu(t)\,dt = \int_0^x h(x,\xi) g(\xi)\,d\xi, \tag{3.10}$$

$$\lim_{n\to\infty}\int_0^\pi g(t)\phi_{n_4}(x,t)\mu(t)\,dt=-\frac{1}{\mu(\xi)}\int_\xi^\pi g(t)\tilde{h}(t,\xi)\mu(t)\,dt. \tag{3.11}$$

Substituting the relations (3.7)–(3.11) into the equality (3.6), we find

$$\int_0^\pi g(t)f(x,t)\,dt+\int_0^\pi g(t)\left(\int_0^x h(x,\xi)f(\xi,t)d\xi\right)dt$$
$$+\int_0^x h(x,\xi)f(\xi)d\xi-\frac{1}{\mu(x)}\int_x^\pi g(t)\tilde{h}(t,x)\mu(t)\,dt=0.$$

Since $h(x,t)=\tilde{h}(x,t)=0$ for $x<t$, for an arbitrarily chosen function $g(x)$ we get

$$f(x,t)+\int_0^x h(x,\xi)f(\xi,t)d\xi+h(x,t)-\frac{\mu(t)}{\mu(x)}\tilde{h}(t,x)=0.$$

Consequently, for $t<x$, we obtain the Gelfand — Levitan — Marchenko type equation (3.5). □

Theorem 3.2. *The Gelfand — Levitan — Marchenko type equation (3.5) has a unique solution $h(x,\cdot\,)\in L_2(0,x;\mu)$ for each fixed $x\in(0,\pi]$.*

Proof. We are going to prove that the homogenous equation

$$u(t)+\int_0^x f(s,t)u(s)ds=0 \tag{3.12}$$

has only trivial solution $u(t)=0$. Let $u(t)$ be a solution of Equation (3.12) and $u(t)=0$ for $t\in(x,\pi)$. Then

$$\int_0^x u^2(t)\mu(t)\,dt+\int_0^x\int_0^x f(s,t)u(s)u(t)\mu(t)dsdt=0$$

and using the relation (3.1), we can write

$$\int_0^x u^2(t)\mu(t)\,dt+\sum_{j=0}^\infty\frac{1}{\gamma_j}\left(\int_0^x c_0(t,\lambda_j)g(t)\mu(t)\,dt\right)^2-\sum_{j=0}^\infty\frac{1}{\gamma_j^0}\left(\int_0^x c_0(t,\tilde{\lambda}_j)g(t)\mu(t)\,dt\right)^2=0.$$

By the Parseval's identity

$$\int_0^x u^2(t)\mu(t)\,dt=\sum_{j=0}^\infty\frac{1}{\gamma_j^0}\left(\int_0^x c_0(t,\tilde{\lambda}_j)u(t)\mu(t)\,dt\right)^2,$$

we obtain

$$\sum_{j=0}^\infty\frac{1}{\gamma_j}\left(\int_0^x c_0(t,\lambda_j)u(t)\mu(t)\,dt\right)^2=0,$$

where $\gamma_j>0$ and the system $\{c_0(t,\lambda_j)\}_{j\geqslant 0}$ is complete in $L_2(0,\pi;\mu)$. This yields $u(t)=0$. □

Now we consider a boundary value problem similar to the problem (1.1)–(1.3) but with different coefficients $\hat{q}(x)$ and $\hat{h}$. Note that all expressions containing this notation (such as $\hat{q}(x)$) belong to the new problem.

Theorem 3.3. *The boundary value problem (1.1)–(1.3) is uniquely determined by the spectral data* $\left\{\lambda_j^2, \gamma_j\right\}_{j\geqslant 0}$.

Proof. Assume that $\lambda_j = \hat{\lambda}_j$ and $\gamma_j = \hat{\gamma}_j$ for $j \geqslant 0$. We are going to show that $q(x) = \hat{q}(x)$ almost everywhere on $(0, \pi)$ and $h = \hat{h}$. The expression for the function $f(x,t)$ and formula (3.1) imply $f(x,t) = \hat{f}(x,t)$. Then it follows from the main equation (3.5) that $h(x,t) = \hat{h}(x,t)$. Taking into consideration the relation

$$k(x,x) = \frac{1}{2}\int\limits_0^x q(\xi)\, d\xi \tag{3.13}$$

implied by the formulas (2.2), (2.4) and (3.3), we see that $q(x) = \hat{q}(x)$ almost everywhere on $(0, \pi)$. It follows from the relations (2.7) and (2.6) that $\dot{\chi}(\lambda_j) \equiv \dot{\hat{\chi}}(\lambda_j)$ and $\rho_j = \hat{\rho}_j$, respectively. Moreover, using the relation (2.5), we get $h = \hat{h}$. □

Algorithm 3.1. *The potential function $q(x)$ is constructed by the spectral data $\left\{\lambda_j^2, \gamma_j\right\}_{j\geqslant 0}$ as follows:*

- *using the spectral data in the formula (3.1), construct the function $f(x,t)$,*
- *solving the main equation (3.5), find $h(x,t)$,*
- *calculate the potential $q(x)$ by the relation (3.13).*

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