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RECONSTRUCTION OF POTENTIAL OF DISCONTINUOUS STURM — LIOUVILLE OPERATOR FROM SPECTRAL DATA

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Abstract. We deal with the inverse spectral problem of the discontinuous Sturm — Liouville operator. The aim is we to determine the potential $q(x)$ and the boundary constant h by a given spectral data. We provide the algorithm for reconstructing the potential $q(x)$ from the spectral data.

Keywords: discontinuous Sturm — Liouville operator, inverse problem, spectral data.

Mathematics Subject Classification: 34A55, 34A36, 34B24

1. Introduction

This paper is devoted to the inverse spectral problem for the Sturm — Liouville equation with a discontinuous coefficient subject to the discontinuity conditions (or transmission conditions) at an interior point of the finite interval $(0, \pi)$. Unlike other studies, the problem examined in this paper includes both the discontinuous coefficient and the discontinuity condition inside the finite interval. Namely, we consider the discontinuous Sturm — Liouville boundary value problem

$$-y'' + q(x)y = \lambda^2\mu(x)y, \quad 0 < x < \pi, \tag{1.1}$$

$$y(a+0) = \beta y(a-0), \qquad y'(a+0) = \beta^{-1}y'(a-0), \tag{1.2}$$

$$y'(0) = y'(\pi) + hy(\pi) = 0, \tag{1.3}$$

where $q(x) \in L_2(0, \pi)$ is a real–valued function, $\beta > 0$ and h are real constants, $\mu(x)$ is a piecewise–constant function

$$\mu(x) = \begin{cases} 1, & 0 < x < a, \\ \alpha^2, & a < x < \pi, \end{cases}$$

λ is a spectral parameter. We assume that $a > \frac{\alpha\pi}{\alpha+1}$.

The Sturm — Liouville problems containing discontinuity conditions (see [1]–[6]) and Sturm — Liouville problems involving discontinuous coefficients (see, for instance, [7]–[10]) were studied as two separate problems. In this paper, we examine a new generalized problem by combining these two different Sturm — Liouville problems. The direct spectral problem (i.e. the spectral properties and the eigenfunction expansion) of Equation (1.1) with the discontinuity condition (1.2) under the boundary condition $y'(0) - h_1y(0) = y'(\pi) + h_2y(\pi) = 0$ was studied in [11] and the inverse problem was solved by means of the Weyl function [12]. Moreover, the inverse problem of Equation (1.1) with (1.2) under the boundary condition $y(0) = y(\pi) = 0$ according to the spectral data and Weyl function were studied in [13].

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Discontinuous boundary value problems appear in many disciplines from mathematics to engineering. Especially, since such problems are related to discontinuous material properties, it is important and interesting to study the corresponding inverse problems, see [14]–[20] and the reference therein.

In this paper, we pose the inverse problem as follows: to determine the potential function $q(x)$ and the boundary constant h from the spectral data of the problem (1.1)–(1.3). For this purpose, using the Gelfand — Levitan — Marchenko method, we construct the modified main equation which is satisfied by the kernel of new integral representation. We provide this integral representation in Section 2 and the kernel has a discontinuity along the line $t = -\alpha(x-a)+a$ for $a < x < \pi$. We prove the uniqueness theorem for the inverse problem and provide a reconstruction algorithm of the potential function $q(x)$ from the spectral data.

2. Preliminaries

We denote by $e(x,\lambda)$ the solution of Eequation (1.1) with discontinuity conditions (1.2) under the initial conditions

$$e(0,\lambda) = 0, \qquad e'(0,\lambda) = i\lambda.$$

As $q(x) \equiv 0$ in Equation (1.1), the solution $e_0(x,\lambda)$ is

$$e_0(x,\lambda) = \begin{cases} e^{i\lambda x}, & 0 < x < a, \\ \kappa_1 e^{i\lambda \vartheta^+(x)} + \kappa_2 e^{i\lambda\vartheta^-(x)}, & a < x < \pi, \end{cases}$$

with

$$\vartheta^{\pm}(x) = \pm\alpha(x-a)+a, \qquad \kappa_1 = \frac{1}{2}\left(\beta + \frac{1}{\alpha\beta}\right), \qquad \kappa_2 = \frac{1}{2}\left(\beta - \frac{1}{\alpha\beta}\right).$$

Theorem 2.1. [11] *The solution $e(x,\lambda)$ can be expressed by the integral representation:*

$$e(x,\lambda) = e_0(x,\lambda) + \int_{-\sigma(x)}^{\sigma(x)} k(x,t)e^{i\lambda t}dt, \tag{2.1}$$

where

$$\sigma(x) = \begin{cases} x, & 0 < x < a, \\ \vartheta^+(x), & a < x < \pi, \end{cases}$$

the kernel function $k(x,\cdot)$ belongs to $L_1(-\sigma(x),\sigma(x))$ for each fixed $x \in (0,a)\cup(a,\pi)$ and satisfies the inequality

$$\int_{-\sigma(x)}^{\sigma(x)} |k(x,t)|\,dt \leqslant \exp\{cp(x)\} - 1$$

with

$$p(x) = \int_0^x (x-\xi)|q(\xi)|d\xi, \qquad c = (\alpha+4)|\kappa_1| + (\alpha+2)|\kappa_2|.$$

We observe that the kernel $k(x,t)$ possesses the following properties:

$$k(x,-\sigma(x)) = 0,$$

$$k(x,\sigma(x)) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^x q(\xi)\,d\xi, & 0<x<a, \\ \dfrac{\kappa_1}{2}\displaystyle\int_0^x \dfrac{1}{\sqrt{\mu(\xi)}} q(\xi)\,d\xi, & a<x<\pi, \end{cases} \tag{2.2}$$

$$k(x,\vartheta^-(x)+0) - k(x,\vartheta^-(x)-0) = -\frac{\kappa_2}{2}\left(\int_0^a q(\xi)d\xi - \frac{1}{\alpha}\int_a^x q(\xi)d\xi\right), \quad a<x<\pi.$$

We denote by $c(x,\lambda)$ the solution of Equation (1.1) subject to the discontinuity conditions (1.2) and the initial conditions

$$c(0,\lambda)=1, \qquad c'(0,\lambda)=0.$$

The integral representation for the solution $c(x,\lambda)$ implied by formula (2.1) reads as

$$c(x,\lambda) = c_0(x,\lambda) + \int_0^{\sigma(x)} \tilde{k}(x,t)\cos\lambda t\,dt, \tag{2.3}$$

where

$$c_0(x,\lambda) = \begin{cases} \cos\lambda x, & 0<x<a, \\ \kappa_1\cos\lambda\vartheta^+(x) + \kappa_2\cos\lambda\vartheta^-(x), & a<x<\pi, \end{cases}$$

and $\tilde{k}(x,t) = k(x,-t)+k(x,t)$. The latter equality yields

$$\tilde{k}(x,\sigma(x)) = k(x,\sigma(x)). \tag{2.4}$$

Let $\zeta(x,\lambda)$ be the solution of Equation (1.1) subject to (1.2) and the initial conditions

$$\zeta(\pi,\lambda) = -1, \qquad \zeta'(\pi,\lambda) = h.$$

The estimate

$$\zeta(x,\lambda) = O\left(e^{|\operatorname{Im}\lambda|(\vartheta^+(\pi)-\vartheta^+(x))}\right), \quad |\lambda|\to\infty,$$

holds. We define the characteristic function of the boundary value problem (1.1)–(1.3) as

$$\chi(\lambda) = c'(\pi,\lambda) + hc(\pi,\lambda).$$

This function is entire in λ, and hence, it has an at most countable set of zeros $\{\lambda_j\}$, and the numbers $\{\lambda_j^2\}$ are the eigenvalues of the problem (1.1)–(1.3). The functions $c(x,\lambda_j)$ and $\zeta(x,\lambda_j)$ are eigenfunctions and

$$\zeta(x,\lambda_j) = \rho_j c(x,\lambda_j), \quad \rho_j \neq 0. \tag{2.5}$$

We denote the norming constants of the problem (1.1)–(1.3) by

$$\gamma_j := \int_0^\pi c^2(x,\lambda_j)\mu(x)\,dx.$$

The relation and estimate

$$\dot{\chi}(\lambda_j) = 2\lambda_j\gamma_j\rho_j, \quad \dot{\chi}(\lambda) = \frac{d}{d\lambda}\chi(\lambda), \tag{2.6}$$

$$\chi(\lambda) = \lambda\omega(\lambda) + O\left(e^{|\operatorname{Im}\lambda|\vartheta^+(\pi)}\right), \quad |\lambda|\to\infty, \tag{2.7}$$

where

$$\omega(\lambda) = \alpha\left(-\kappa_1\sin\lambda\vartheta^+(\pi) + \kappa_2\sin\lambda\vartheta^-(\pi)\right).$$

The zeros of this function are

$$\tilde{\lambda}_j = \frac{j\pi}{\vartheta^+(\pi)} + d_j, \qquad \sup_j |d_j| = d < \infty.$$

Theorem 2.2. [11] *The boundary value problem (1.1)–(1.3) has a countable set of eigenvalues* $\{\lambda_j^2\}_{j\geqslant 0}$,

$$\lambda_j = \tilde{\lambda}_j + \frac{p_j}{\tilde{\lambda}_j} + \frac{t_j}{j}, \qquad \{p_j\} \in \ell_\infty, \qquad \{t_j\} \in \ell_2.$$

Definition 2.1. *The numbers* $\{\lambda_j^2, \gamma_j\}_{j\geqslant 0}$ *are called the spectral data of the boundary value problem (1.1)–(1.3).*

Theorem 2.3. [11] *The system of eigenfunctions* $\{c(x,\lambda_j)\}_{j\geqslant 0}$ *of boundary value problem (1.1)–(1.3) is complete in* $L_2(0,\pi;\mu)$. *The function* $f(x) \in AC[0,a] \cap AC[a,\pi]$ *satisfying the discontinuity condition (1.2) and the boundary conditions (1.3) can be expanded into a uniformly convergent series over the eigenfunctions of the problem (1.1)—(1.3)*

$$f(x) = \sum_{j=0}^{\infty} s_j c(x,\lambda_j), \qquad s_j = \frac{1}{\gamma_j} \int_0^\pi c(x,\lambda_j) f(x) \mu(x)\, dx. \tag{2.8}$$

For $f(x) \in L_2(0,\pi;\mu)$, *the series (2.8) converges in* $L_2(0,\pi;\mu)$ *and Parseval's identity holds:*

$$\int_0^\pi |f(x)|^2 \mu(x)\, dx = \sum_{j=0}^{\infty} \gamma_j |s_j|^2.$$

3. Main results

Consider the function

$$f(x,t) = \mu(t) \sum_{j=0}^{\infty} \left(\frac{c_0(x,\lambda_j)c_0(t,\lambda_j)}{\gamma_j} - \frac{c_0(x,\tilde{\lambda}_j)c_0(t,\tilde{\lambda}_j)}{\gamma_j^0} \right), \tag{3.1}$$

where the numbers γ_j^0 are the norming constants of the problem (1.1)–(1.3) for $q(x) \equiv 0$.

Remark 3.1. *The integral representation (2.3) can be written as*

$$c(x,\lambda) = c_0(x,\lambda) + \int_0^x h(x,t) c_0(t,\lambda)\, dt, \tag{3.2}$$

where

$$h(x,t) = \begin{cases} \tilde{k}(x,t), & 0 < t < x < a, \quad 0 < t < \vartheta^-(x), \quad a < x < \pi, \\ \tilde{k}(x,t) - \dfrac{\kappa_2}{\kappa_1}\tilde{k}(x,2a-t), & \vartheta^-(x) < t < a < x < \pi, \\ \dfrac{\alpha}{\kappa_1}\tilde{k}(x,\vartheta^+(t)), & a < t < x < \pi. \end{cases} \tag{3.3}$$

To justify this formula, we take into consideration the relation

$$\cos\lambda t = \begin{cases} c_0(x,\lambda), & 0 < x < a, \\ \dfrac{1}{\kappa_1} c_0\left(\dfrac{t-a}{\alpha} + a, \lambda\right) - \dfrac{\kappa_2}{\kappa_1} c_0(2a-t,\lambda), & a < x < \pi. \end{cases}$$

Resolving the Volterra equation (3.2) with respect to $c_0(x,\lambda)$, we have

$$c_0(x,\lambda) = c(x,\lambda) + \int_0^x \tilde{h}(x,t)c(t,\lambda)\,dt. \tag{3.4}$$

Theorem 3.1. *For each fixed $x \in (0,\pi]$, the kernel $\tilde{k}(x,t)$ satisfies the linear integral equation*

$$f(x,t) + h(x,t) + \int_0^x h(x,\xi)f(\xi,t)d\xi = 0, \quad t < x. \tag{3.5}$$

This equation is called the Gelfand – Levitan – Marchenko type equation (or modified main equation) of the boundary value problem (1.1)–(1.3).

Доказательство. Using the formulas (3.2) and (3.4), we write

$$\Phi_n(x,t) = \phi_{n_1}(x,t) + \phi_{n_2}(x,t) + \phi_{n_3}(x,t) + \phi_{n_4}(x,t), \tag{3.6}$$

where

$$\Phi_n(x,t) = \sum_{j=0}^{n}\left(\frac{c(x,\lambda_j)c(t,\lambda_j)}{\gamma_j} - \frac{c_0(x,\tilde{\lambda}_j)c_0(t,\tilde{\lambda}_j)}{\gamma_j^0}\right),$$

$$\phi_{n_1}(x,t) = \sum_{j=0}^{n}\left(\frac{c_0(x,\lambda_j)c_0(t,\lambda_j)}{\gamma_j} - \frac{c_0(x,\tilde{\lambda}_j)c_0(t,\tilde{\lambda}_j)}{\gamma_j^0}\right),$$

$$\phi_{n_2}(x,t) = \int_0^x h(x,\xi)\sum_{j=0}^{n}\left(\frac{c_0(\xi,\lambda_j)c_0(t,\lambda_j)}{\gamma_j} - \frac{c_0(\xi,\tilde{\lambda}_j)c_0(t,\tilde{\lambda}_j)}{\gamma_j^0}\right)d\xi,$$

$$\phi_{n_3}(x,t) = \int_0^x h(x,\xi)\sum_{j=0}^{n}\frac{c_0(\xi,\tilde{\lambda}_j)c_0(t,\tilde{\lambda}_j)}{\gamma_j^0}d\xi,$$

$$\phi_{n_4}(x,t) = -\int_0^t \tilde{h}(t,\xi)\sum_{j=0}^{n}\frac{c(x,\lambda_j)c(\xi,\lambda_j)}{\gamma_j}d\xi.$$

Let $g(x) \in AC[0,a] \cap AC[a,\pi]$. In view of Theorem 2.3 and the formula (3.1) we obtain

$$\begin{aligned}\lim_{n\to\infty}\max_{0\leqslant x\leqslant\pi}\left|\int_0^\pi g(t)\Phi_n(x,t)\mu(t)\,dt\right| &= \lim_{n\to\infty}\max_{0\leqslant x\leqslant\pi}\left|\sum_{j=0}^{n}s_jc(x,\lambda_j) - \sum_{j=0}^{n}s_j^0c_0(x,\tilde{\lambda}_j)\right| \\ &\leqslant \lim_{n\to\infty}\max_{0\leqslant x\leqslant\pi}\left|g(x) - \sum_{j=0}^{n}s_jc(x,\lambda_j)\right| \\ &\quad + \lim_{n\to\infty}\max_{0\leqslant x\leqslant\pi}\left|g(x) - \sum_{j=0}^{n}s_j^0c_0(x,\tilde{\lambda}_j)\right| = 0,\end{aligned} \tag{3.7}$$

$$\lim_{n\to\infty}\int_0^\pi g(t)\phi_{n_1}(x,t)\mu(t)\,dt = \int_0^\pi g(t)f(x,t)\,dt, \tag{3.8}$$

$$\lim_{n\to\infty}\int_0^\pi g(t)\phi_{n_2}(x,t)\mu(t)\,dt = \int_0^\pi g(t)\left(\int_0^x h(x,\xi)f(\xi,t)d\xi\right)dt, \tag{3.9}$$

$$\lim_{n\to\infty}\int_0^\pi g(t)\phi_{n_3}(x,t)\mu(t)\,dt=\int_0^x h(x,\xi)g(\xi)d\xi, \tag{3.10}$$

$$\lim_{n\to\infty}\int_0^\pi g(t)\phi_{n_4}(x,t)\mu(t)\,dt=-\frac{1}{\mu(\xi)}\int_\xi^\pi g(t)\tilde{h}(t,\xi)\mu(t)\,dt. \tag{3.11}$$

Substituting the relations (3.7)–(3.11) into the equality (3.6), we find

$$\int_0^\pi g(t)f(x,t)\,dt+\int_0^\pi g(t)\left(\int_0^x h(x,\xi)f(\xi,t)d\xi\right)dt$$
$$+\int_0^x h(x,\xi)f(\xi)d\xi-\frac{1}{\mu(x)}\int_x^\pi g(t)\tilde{h}(t,x)\mu(t)\,dt=0.$$

Since $h(x,t)=\tilde{h}(x,t)=0$ for $x<t$, for an arbitrarily chosen function $g(x)$ we get

$$f(x,t)+\int_0^x h(x,\xi)f(\xi,t)d\xi+h(x,t)-\frac{\mu(t)}{\mu(x)}\tilde{h}(t,x)=0.$$

Consequently, for $t<x$, we obtain the Gelfand — Levitan — Marchenko type equation (3.5). □

Theorem 3.2. *The Gelfand — Levitan — Marchenko type equation (3.5) has a unique solution $h(x,\cdot)\in L_2(0,x;\mu)$ for each fixed $x\in(0,\pi]$.*

Доказательство. We are going to prove that the homogenous equation

$$u(t)+\int_0^x f(s,t)u(s)ds=0 \tag{3.12}$$

has only trivial solution $u(t)=0$. Let $u(t)$ be a solution of Equation (3.12) and $u(t)=0$ for $t\in(x,\pi)$. Then

$$\int_0^x u^2(t)\mu(t)\,dt+\int_0^x\int_0^x f(s,t)u(s)u(t)\mu(t)dsdt=0$$

and using the relation (3.1), we can write

$$\int_0^x u^2(t)\mu(t)\,dt+\sum_{j=0}^\infty\frac{1}{\gamma_j}\left(\int_0^x c_0(t,\lambda_j)g(t)\mu(t)\,dt\right)^2-\sum_{j=0}^\infty\frac{1}{\gamma_j^0}\left(\int_0^x c_0(t,\tilde{\lambda}_j)g(t)\mu(t)\,dt\right)^2=0.$$

By the Parseval's identity

$$\int_0^x u^2(t)\mu(t)\,dt=\sum_{j=0}^\infty\frac{1}{\gamma_j^0}\left(\int_0^x c_0(t,\tilde{\lambda}_j)u(t)\mu(t)\,dt\right)^2,$$

we obtain

$$\sum_{j=0}^\infty\frac{1}{\gamma_j}\left(\int_0^x c_0(t,\lambda_j)u(t)\mu(t)\,dt\right)^2=0,$$

where $\gamma_j>0$ and the system $\{c_0(t,\lambda_j)\}_{j\geqslant0}$ is complete in $L_2(0,\pi;\mu)$. This yields $u(t)=0$. □

Now we consider a boundary value problem similar to the problem (1.1)–(1.3) but with different coefficients $\hat{q}(x)$ and $\hat{h}$. Note that all expressions containing this notation (such as $\hat{q}(x)$) belong to the new problem.

Theorem 3.3. *The boundary value problem (1.1)–(1.3) is uniquely determined by the spectral data* $\left\{\lambda_j^2, \gamma_j\right\}_{j\geqslant 0}$.

Доказательство. Assume that $\lambda_j = \hat{\lambda}_j$ and $\gamma_j = \hat{\gamma}_j$ for $j \geqslant 0$. We are going to show that $q(x) = \hat{q}(x)$ almost everywhere on $(0, \pi)$ and $h = \hat{h}$. The expression for the function $f(x,t)$ and formula (3.1) imply $f(x,t) = \hat{f}(x,t)$. Then it follows from the main equation (3.5) that $h(x,t) = \hat{h}(x,t)$. Taking into consideration the relation

$$k(x,x) = \frac{1}{2}\int\limits_0^x q(\xi)\, d\xi \tag{3.13}$$

implied by the formulas (2.2), (2.4) and (3.3), we see that $q(x) = \hat{q}(x)$ almost everywhere on $(0, \pi)$. It follows from the relations (2.7) and (2.6) that $\dot{\chi}(\lambda_j) \equiv \dot{\hat{\chi}}(\lambda_j)$ and $\rho_j = \hat{\rho}_j$, respectively. Moreover, using the relation (2.5), we get $h = \hat{h}$. □

Algorithm 3.1. *The potential function $q(x)$ is constructed by the spectral data* $\left\{\lambda_j^2, \gamma_j\right\}_{j\geqslant 0}$ *as follows:*

- *using the spectral data in the formula (3.1), construct the function $f(x,t)$,*
- *solving the main equation (3.5), find $h(x,t)$,*
- *calculate the potential $q(x)$ by the relation (3.13).*

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