ISSN 2304-0122 Ufa Mathematical Journal. Vol. 16. No 4 (2024). P. 1-11.

doi:10.13108/2024-16-4-1

ON ZAREMBA PROBLEM FOR SECOND–ORDER LINEAR ELLIPTIC EQUATION WITH DRIFT IN CASE OF LIMIT EXPONENT

M.D. ALIYEV, Yu.A. ALKHUTOV, G.A. CHECHKIN

Abstract. We establish the unique solvability of the Zaremba problem with the homogeneous Dirichlet and Neumann boundary conditions for an inhomogeneous linear second order second order equation in the divergence form with measurable coefficients and lower order terms. The problem is considered in a bounded strictly Lipschitz domain. We suppose that the domain is contained in an n–dimensional Euclidean space, where $n \geqslant 2$. If $n > 2$, then the lower coefficient belong to the Lebesgue space with the limiting summability exponent from the Sobolev embedding theorem. If $n = 2$, then the lower coefficients are summable at each power exceeding two. Apart of the unique solvability, we establish an energy estimate for the solution.

Keywords: Zaremba problem, solvability, drift, limiting exponent, capacity.

Mathematics Subject Classification: 35A01, 35B45, 35D30, 35J25

1. Introduction

In the paper we study the unique solvability of the Zaremba problem for an elliptic operator with lower terms defined in a bounded strictly Lipschitz domain $D \in \mathbb{R}^n$, where $n > 1$, of the form

$$\mathcal{L}u := \operatorname{div}(a\nabla u) + b \cdot \nabla u. \tag{1.1}$$

Here $a(x) = \{a_{ij}(x)\}$ is a uniformly elliptic real–valued measurable and symmetric matrix, that is, $a_{ij} = a_{ji}$ and

$$\alpha|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqslant \alpha^{-1}|\xi|^2 \tag{1.2}$$

for almost each $x \in D$ and for all $\xi \in \mathbb{R}^n$. The real–valued vector function

$$b(x) = (b_1(x), \dots, b_n(x))$$

in (1.1) obeys the condition

$$b_j \in L_p(D), \qquad p = n \qquad \text{if} \quad n > 2, \quad j = 1, \dots n, \tag{1.3}$$

$$b_j \in L_p(D), \qquad p > 2 \qquad \text{if} \quad n = 2, \quad j = 1, 2. \tag{1.4}$$

M.D. Aliyev, Yu.A. Alkhutov, G.A. Chechkin, On Zaremba problem for second–order linear elliptic equation with drift in case of limit exponent.

The results by Yu.A. Alkhutov in Section 3 were supported in the framework of the state task of Vladimir State University (project FZUN-2023-0004). The results by G.A. Chechkin in Section 2 were supported by the Russian Science Foundation (project no. 20-11-20272). The results by G.A. Chechkin in Section 1 were partially supported by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant no. AP14869553).

Submitted May 1, 2024.

Before we pose the Zaremba problem, we introduce the Sobolev space of the functions $W_2^1(D,F)$, where $F \subset \partial D$ is a closed, as a completion of infinitely differentiable in the closure of D functions, which vanish in the vicinity of F, by the norm

$$\|v\|_{W_2^1(D,F)} = \Big(\int\limits_D v^2\,dx + \int\limits_D |\nabla v|^2\,dx \Big)^{1/2}.$$

For the functions $v \in W_2^1(D,F)$ we apriori require the Friedrichs inequality

$$\int\limits_D v^2\,dx \leqslant C \int\limits_D |\nabla v|^2\,dx. \tag{1.5}$$

To formulate the result, we shall need a more detailed clarification for the notion of strictly Lipshitz domain D. In order to do this, we denote by Q a cube centered at a point $x_0 \in \partial D$. We introduce a Cartesian coordinate system with the origin at x_0, in which the edges of the cube are parallel to the coordinate systems and their length is equal $2R_0$. We say that the domain D is strictly Lipshitz if for each point $x_0 \in \partial D$ the set $Q \cap \partial D$ is a graph of a Lipshitz function $x_n = g(x')$, where $x' = (x_1, \dots, x_{n-1})$, with a Lipshitz constant L. We suppose that the length of edge of cube Q and the Lipshitz constant L are independent of x_0.

Let us provide a necessary and sufficient condition for the set $F \subset \partial D$, which ensures the inequality (1.5). This requires the notion of capacity.

We denote by $\mathcal{Q}_d$ an open cube with the edge of length d and sides parallel to the coordinate axes assuming the Lipshitz domain D has a diameter d and $D \subset \mathcal{Q}_d$. We introduce the notion of capacity $C_2(K, \mathcal{Q}_{2d})$ of a compact set $K \subset \overline{\mathcal{Q}}_d$ with respect to the cube $\mathcal{Q}_{2d}$ by the identity

$$C_2(K,\mathcal{Q}_{2d}) = \inf \Big\{ \int\limits_{\mathcal{Q}_{2d}} |\nabla \varphi|^2\,dx : \ \varphi \in C_0^\infty(\mathcal{Q}_{2d}), \ \varphi \geqslant 1 \text{ on } K \Big\}.$$

It follows from the results by Mazya [1, Sect. 14.1.2] and comments to the results of Chapter 14 of monograph [1] that for the inequality (1.5) holds for the functions $v \in W_2^1(D,F)$ if and only if

$$C_2(F, \mathcal{Q}_{2d}) > 0. \tag{1.6}$$

Letting $G = \partial D \setminus F$, we consider the Zaremba problem

$$\mathcal{L}u = l \quad \text{in} \quad D, \qquad u = 0 \quad \text{on} \quad F, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad G, \tag{1.7}$$

where $\frac{\partial u}{\partial \nu}$ denotes the outward conormal derivative of the function u, and l is a linear functional in the dual space for $W_2^1(D,F)$.

By a solution to the problem (1.7) we mean a function $u \in W_2^1(D,F)$, which satisfies the integral identity

$$\int\limits_D a\nabla u \cdot \nabla \varphi\,dx - \int\limits_D b \cdot \nabla u\, \varphi\,dx = -l(\varphi) \tag{1.8}$$

for all test functions $\varphi \in W_2^1(D,F)$.

By the Friedrichs inequality (1.5) the space $W_2^1(D,F)$ can be equipped with the norm, which involves only the gradient. Then each element in the Sobolev space can be one–to–one isometrically associated with its gradient, which is an element in $(L_2(D))^n$. Using the Hahn — Banach theorem, for instance, as in arguing of Section 1.1.15 in the monograph [1] on the form of the functionals in space dual to the Sobolev space, it is easy to show that the functional l

can be written as

$$l(\varphi) = -\sum_{i=1}^{n} \int_D f_i \varphi_{x_i}\, dx,$$

where $f_i \in L_2(D)$. This is why by (1.8) for a given functional a solution to the problem (1.7) can be treated in the sense of the integral identity

$$\int_D a\nabla u \cdot \nabla\varphi\, dx - \int_D b \cdot \nabla u\, \varphi\, dx = \int_D f \cdot \nabla\varphi\, dx$$

for all test functions $\varphi \in W_2^1(D, F)$, in which the components of the vector functions $f = (f_1, \dots, f_n)$ are functions from $L_2(D)$. We note that for $n > 2$ by the Sobolev embedding theorem the exponent $p = n$ is limiting, see the condition (1.3).

We are in position to formulate the main obtained result.

Theorem 1.1. *If the conditions* (1.2), (1.3) *(or* (1.4)*) and* (1.6) *hold, then the Zaremba problem* (1.7) *is uniquely solvable in* $W_2^1(D, F)$ *and its solution satisfies the estimate*

$$\|\nabla u\|_{L_2(D)} \leqslant C\|f\|_{L_2(D)} \tag{1.9}$$

with a constant C, which depends only on the coefficients of the operator $\mathcal{L}$, the domain D and the dimension of the space.

For $n > 2$ we shall employ the representation of the lower coefficients $b \in (L_n(D))^n$ of the considered equation in the form

$$b = \breve{b} + \widehat{b}, \qquad \breve{b} \in (L_\infty(D))^n, \qquad \widehat{b} \in (L_n(D))^n, \qquad \|\widehat{b}\|_{L_n(D)} \leqslant \theta. \tag{1.10}$$

Here $\theta \in (0, 1)$ is a sufficiently small constant, which is determined during the arguing.

2. Auxiliary statements

As noted above, the Friedrichs inequality (1.5) holds for functions in the space $W_2^1(D, F)$, and this space can be equipped with a norm involving only the gradient. In what follows we use Sobolev embedding theorems for strictly Lipschitz domains having in mind such a norm. In addition, the condition (1.6) is assumed to be satisfied, which implies the Friedrichs inequality (1.5).

We shall need estimates for the bilinear form defined on the functions $u, v \in W_2^1(D, F)$ associated with the operator $\mathcal{L}$, which reads as

$$\ell(u, v) = \int_D a\nabla u \cdot \nabla v\, dx - \int_D (b \cdot \nabla u) v\, dx. \tag{2.1}$$

Lemma 2.1. *If the coefficients of the operator $\mathcal{L}$ in* (1.1) *satisfies the conditions* (1.2), (1.3) *(or* (1.4)*), then*

$$\ell(u, u) \geqslant \frac{\alpha}{2} \int_D |\nabla u|^2\, dx - C(\alpha, b, n, p, D) \int_D u^2\, dx, \tag{2.2}$$

where $C(\alpha, b, n, p, D)$ is a positive constant depending on α, b, n, p and D.

Proof. By the condition (1.2) we have

$$\ell(u, u) \geqslant \alpha \int_D |\nabla u|^2\, dx - \left| \int_D (b \cdot \nabla u) u\, dx \right|. \tag{2.3}$$

We first suppose that $n > 2$ and estimate the second term in the right hand side of the identity (2.1). By (1.10) we have

$$\int\limits_D (b \cdot \nabla u)u\,dx = \int\limits_D (\breve{b} \cdot \nabla u)u\,dx + \int\limits_D (\widehat{b} \cdot \nabla u)u\,dx. \tag{2.4}$$

For the first integral in the right hand side of (2.4) we obtain

$$\Big|\int\limits_D (\breve{b} \cdot \nabla u)u\,dx\Big| \leqslant C(n)\|\breve{b}\|_{L_\infty(D)} \int\limits_D |\nabla u|\,|u|\,dx$$

and by the Cauchy inequality with $\varepsilon > 0$ we find

$$\Big|\int\limits_D (\breve{b} \cdot \nabla u)u\,dx\Big| \leqslant \varepsilon \int\limits_D |\nabla u|^2\,dx + C(\varepsilon, n)\|\breve{b}\|^2_{L_\infty(D)} \int\limits_D u^2\,dx. \tag{2.5}$$

We estimate the second integral in the right hand side of (2.4) by means of the Hölder inequality

$$\Big|\int\limits_D (\widehat{b} \cdot \nabla u)u\,dx\Big| \leqslant \|\widehat{b}\|_{L_n(D)}\Big(\int\limits_D |\nabla u|^2\,dx\Big)^{1/2}\Big(\int\limits_D |u|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{2n}}. \tag{2.6}$$

By the Sobolev inequality we have

$$\Big(\int\limits_D |u|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{2n}} \leqslant C(n, D)\Big(\int\limits_D |\nabla u|^2\,dx\Big)^{1/2}.$$

By (2.6) and (1.10) this implies

$$\Big|\int\limits_D (\widehat{b} \cdot \nabla u)u\,dx\Big| \leqslant C(n, D)\theta \int\limits_D |\nabla u|^2\,dx. \tag{2.7}$$

By (2.5) and (2.7) we find

$$\Big|\int\limits_D (b \cdot \nabla u)u\,dx\Big| \leqslant \varepsilon \int\limits_D |\nabla u|^2\,dx + C(\varepsilon, n)\|\breve{b}\|^2_{L_\infty(D)} \int\limits_D u^2\,dx + C(n, D)\theta \int\limits_D |\nabla u|^2\,dx.$$

Choosing appropriate ε and θ from (2.3), we arrive at the estimate (2.2).

We are going to verify the inequality (2.2) for $n = 2$. In this case we also employ the Hölder inequality with another exponent

$$\Big(\int\limits_D |b|^2 u^2\,dx\Big)^{1/2} \leqslant \Big(\int\limits_D |b|^p\,dx\Big)^{1/p}\Big(\int\limits_D |u|^{\tilde{p}}\,dx\Big)^{1/\tilde{p}}.$$

Here $\tilde{p} = \frac{2p}{p-2}$, $p > 2$, and it is clear that $\tilde{p} > 2$. As a result for the second term in the right hand side (2.3) we have

$$\Big|\int\limits_D (b \cdot \nabla u)u\,dx\Big| \leqslant \Big(\int\limits_D |\nabla u|^2\,dx\Big)^{1/2}\Big(\int\limits_D |b|^p\,dx\Big)^{1/p}\Big(\int\limits_D |u|^{\tilde{p}}\,dx\Big)^{1/\tilde{p}}. \tag{2.8}$$

According to the identity

$$\int\limits_D |u|^{\tilde{p}}\,dx = \int\limits_D |u||u|^{\tilde{p}-1}\,dx,$$

by the Cauchy — Shwartz inequality

$$\int\limits_D |u|^{\tilde p}\,dx \leqslant \Big(\int\limits_D u^2\,dx\Big)^{1/2}\Big(\int\limits_D |u|^{2(\tilde p-1)}\,dx\Big)^{1/2}.$$

Thus, it follows from (2.8) that

$$\Big|\int\limits_D (b\cdot\nabla u)u\,dx\Big| \leqslant \Big(\int\limits_D |b|^p\,dx\Big)^{1/p}\Big(\int\limits_D |\nabla u|^2\,dx\Big)^{1/2}\Big(\int\limits_D |u|^{2(\tilde p-1)}\,dx\Big)^{\frac{1}{2\tilde p}}\Big(\int\limits_D u^2\,dx\Big)^{\frac{1}{2\tilde p}}. \tag{2.9}$$

Then by the Cauchy inequality

$$\begin{aligned}\Big(\int\limits_D |b|^p\,dx\Big)^{1/p}\Big(\int\limits_D |\nabla u|^2\,dx\Big)^{1/2}\Big(\int\limits_D |u|^{2(\tilde p-1)}\,dx\Big)^{\frac{1}{2\tilde p}}\Big(\int\limits_D u^2\,dx\Big)^{\frac{1}{2\tilde p}}&\\ \leqslant \varepsilon\int\limits_D |\nabla u|^2\,dx + \frac{1}{2\varepsilon}\Big(\int\limits_D |b|^p\,dx\Big)^{2/p}\Big(\int\limits_D |u|^{2(\tilde p-1)}\,dx\Big)^{\frac{1}{\tilde p}}\Big(\int\limits_D u^2\,dx\Big)^{\frac{1}{\tilde p}}&\end{aligned} \tag{2.10}$$

and in view of the Young inequality and the identity $\frac{2\tilde p}{p} = \frac{4}{p-2}$ we find

$$\begin{aligned}\Big(\int\limits_D |b|^p\,dx\Big)^{2/p}\Big(\int\limits_D |u|^{2(\tilde p-1)}\,dx\Big)^{\frac{1}{\tilde p}}\Big(\int\limits_D u^2\,dx\Big)^{\frac{1}{\tilde p}}&\\ \leqslant \varepsilon_1\Big(\int\limits_D |u|^{2(\tilde p-1)}\,dx\Big)^{\frac{1}{\tilde p-1}} + C(\varepsilon_1)\Big(\int\limits_D |b|^p\,dx\Big)^{\frac{4}{p-2}}\int\limits_D u^2\,dx.&\end{aligned}$$

By the Sobolev embedding theorem the inequality

$$\Big(\int\limits_D |u|^{2(\tilde p-1)}\,dx\Big)^{\frac{1}{\tilde p-1}} \leqslant C(D,p)\int\limits_D |\nabla u|^2\,dx$$

holds and this is why

$$\begin{aligned}\Big(\int\limits_D |b|^p\,dx\Big)^{2/p}\Big(\int\limits_D |u|^{2(\tilde p-1)}\,dx\Big)^{\frac{1}{\tilde p}}\Big(\int\limits_D u^2\,dx\Big)^{\frac{1}{\tilde p}}&\\ \leqslant \varepsilon_1 C(D,p)\int\limits_D |\nabla u|^2\,dx + C(\varepsilon_1,b,p)\int\limits_D u^2\,dx.&\end{aligned} \tag{2.11}$$

By (2.9)–(2.11) in view of (2.3) under an appropriate choice of ε_1 we again arrive at the inequality (2.2). The proof is complete. □

Lemma 2.2. *If the coefficients of the operator $\mathcal{L}$ in* (1.1) *obey the conditions* (1.2), (1.3) *(or* (1.4)*), then for a fixed $u \in W_2^1(D,F)$ the mapping $v \mapsto \ell(u,v)$, where the form $\ell(u,v)$ is defined in* (2.1)*, is a bounded linear functional on $W_2^1(D,F)$ and the estimate*

$$|\ell(u,v)| \leqslant C(\alpha,b,n,p,D)\|u\|_{W_2^1(D,F)}\|v\|_{W_2^1(D,F)} \tag{2.12}$$

holds.

Proof. Due to the uniform ellipticity condition (1.1)

$$\Big|\int\limits_D a\nabla u\cdot\nabla v\,dx\Big| \leqslant \alpha^{-1}\|u\|_{W_2^1(D,F)}\|v\|_{W_2^1(D,F)}. \tag{2.13}$$

For $n > 2$, in view of the condition (1.3), we estimate the second term in the form (2.1) by the Hölder inequality

$$
\begin{aligned}
\Big|\int\limits_D (b\cdot\nabla u)v\,dx\Big| &\leqslant \|u\|_{W_2^1(D,F)}\Big(\int\limits_D |b|^2v^2\,dx\Big)^{1/2} \\
&\leqslant \|u\|_{W_2^1(D,F)}\Big(\int\limits_D |b|^n\,dx\Big)^{1/n}\Big(\int\limits_D |v|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{2n}}.
\end{aligned}
\tag{2.14}
$$

Using the Sobolev embedding theorem

$$
\Big(\int\limits_D |v|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{2n}} \leqslant C(n,D)\|v\|_{W_2^1(D,F)},
$$

by (2.13), (2.14) we arrive at (2.12).

If $n = 2$, then the second term in the form (2.1) can be also estimated by the Hölder inequality

$$
\begin{aligned}
\Big|\int\limits_D (b\cdot\nabla u)v\,dx\Big| &\leqslant \|u\|_{W_2^1(D,F)}\Big(\int\limits_D |b|^2v^2\,dx\Big)^{1/2} \\
&\leqslant \|u\|_{W_2^1(D,F)}\Big(\int\limits_D |b|^p\,dx\Big)^{1/p}\Big(\int\limits_D |v|^{\frac{2p}{p-2}}\,dx\Big)^{\frac{p-2}{2p}},
\end{aligned}
\tag{2.15}
$$

where $p > 2$. Again by the Sobolev embedding theorem

$$
\Big(\int\limits_D |v|^{\frac{2p}{p-2}}\,dx\Big)^{\frac{p-2}{2p}} \leqslant C(n,D)\|v\|_{W_2^1(D,F)}
$$

and in view of (2.13), (2.15) we arrive at (2.12). The proof is complete. □

Now we consider the Zaremba problem (1.7) for the homogeneous equation. We are going to prove the maximum principle for its solutions. The function $u \in W_2^1(D,F)$ is called a *subsolution* to the Zaremba problem (1.7) for the homogeneous equation in the domain D if

$$
\int\limits_D a\nabla u\cdot\nabla\varphi\,dx - \int\limits_D (b\cdot\nabla u)\varphi\,dx \leqslant 0, \tag{2.16}
$$

where $\varphi \in W_2^1(D,F)$ is an arbitrary non–negative function. In the same way we define a *supersolution* $u \in W_2^1(D,F)$ in the domain D, which obeys the inequality

$$
\int\limits_D a\nabla u\cdot\nabla\varphi\,dx - \int\limits_D (b\cdot\nabla u)\varphi\,dx \geqslant 0
$$

for all non–negative functions $\varphi \in W_2^1(D,F)$.

The following statement for the Dirichlet problem can be found in [2], see also [3, Thm. 3.1].

Lemma 2.3. *If the conditions* (1.2), (1.3) *(or* (1.4)*) are satisfied and the function* $u \in W_2^1(D,F)$ *is a subsolution in the domain* D*, then*

$$
\operatorname*{ess\,sup}_D u \leqslant 0. \tag{2.17}
$$

If $u \in W_2^1(D,F)$ *is a supersolution in the domain* D*, then*

$$
\operatorname*{ess\,inf}_D u \geqslant 0. \tag{2.18}
$$

Proof. First we are going to prove (2.17). We argue by contradiction. Suppose that

$$\operatorname*{ess\,sup}_{D} u > 0.$$

Then there exists a number k such that

$$0 < k < \operatorname*{ess\,sup}_{D} u.$$

We consider a function

$$v = \max(u-k, 0) = (u-k)^+,$$

which belongs to the space $W_2^1(D,F)$ and is non–negative. By (2.16) we have

$$\int\limits_{D} a\nabla v \cdot \nabla v \, dx \leqslant \int\limits_{D} (b \cdot \nabla v) v \, dx.$$

By the choice of the function v this estimate can be rewritten as

$$\int\limits_{D\cap\{u>k\}} a\nabla u \cdot \nabla u \, dx \leqslant \int\limits_{D\cap\{u>k\}} (b \cdot \nabla u) v \, dx. \tag{2.19}$$

We first suppose that $n > 2$. Using the ellipticity condition (1.2) and applying the Hölder inequality in the right hand side, we get

$$\alpha \int\limits_{D\cap\{u>k\}} |\nabla u|^2 \, dx \leqslant \Bigg(\int\limits_{D\cap\{u>k\}} |b|^n \, dx \Bigg)^{1/n} \Bigg(\int\limits_{D\cap\{u>k\}} |\nabla u|^2 \, dx \Bigg)^{1/2} \Bigg(\int\limits_{D} |v|^{\frac{2n}{n-2}} \, dx \Bigg)^{\frac{n-2}{2n}}. \tag{2.20}$$

Since $v \in W_2^1(D,F)$, by the Sobolev embedding theorem

$$\Bigg(\int\limits_{D} |v|^{\frac{2n}{n-2}} \, dx \Bigg)^{\frac{n-2}{2n}} \leqslant C(n,D) \Bigg(\int\limits_{D} |\nabla v|^2 \, dx \Bigg)^{1/2} = C(n,D) \Bigg(\int\limits_{D\cap\{u>k\}} |\nabla u|^2 \, dx \Bigg)^{1/2},$$

and it follows from (2.20) that

$$\alpha \int\limits_{D\cap\{u>k\}} |\nabla u|^2 \, dx \leqslant C(n,D) \Bigg(\int\limits_{D\cap\{u>k\}} |b|^n \, dx \Bigg)^{1/n} \int\limits_{D\cap\{u>k\}} |\nabla u|^2 \, dx. \tag{2.21}$$

If $M = \operatorname*{ess\,sup}_{D} u = \infty$, then the first integral in the right hand side of (2.21) tends to zero as $k \to \infty$, and this leads us to the contradiction.

If $M < \infty$, then $\nabla u = 0$ almost everywhere on the set $D \cap \{u = M\}$ and the estimate (2.21) becomes

$$\alpha \leqslant C(n,D) \Bigg(\int\limits_{M_k} |b|^n \, dx \Bigg)^{1/n},$$

where

$$M_k = \{x \in D : \ k < u(x) < M, \ \nabla u(x) \neq 0\}.$$

It is clear that the n–dimensional Lebesgue measure of the set M_k tends to zero as $k \to M$ and hence

$$\Bigg(\int\limits_{M_k} |b|^n \, dx \Bigg)^{1/n} \longrightarrow 0 \quad \text{as} \quad k \to M,$$

and we again arrive at the contradiction that proves (2.17).

We consider the remaining case for $n = 2$. In view of (2.19), using the ellipticity condition and applying the Hölder inequality with other exponents, we arrive at the estimate

$$\alpha \int\limits_{D\cap\{u>k\}} |\nabla u|^2\,dx \leqslant \Big(\int\limits_{D\cap\{u>k\}} |b|^p\,dx\Big)^{1/p}\Big(\int\limits_{D\cap\{u>k\}} |\nabla u|^2\,dx\Big)^{1/2}\Big(\int\limits_{D} |v|^{\frac{2p}{p-2}}\,dx\Big)^{\frac{p-2}{2p}}, \quad (2.22)$$

where $p > 2$. For $n = 2$ by the Sobolev embedding theorem

$$\Big(\int\limits_{D} |v|^{\frac{2p}{p-2}}\,dx\Big)^{\frac{p-2}{2p}} \leqslant C(p,D)\Big(\int\limits_{D} |\nabla v|^2\,dx\Big)^{1/2} = C(p,D)\Big(\int\limits_{D\cap\{u>k\}} |\nabla u|^2\,dx\Big)^{1/2}$$

and by (2.22) we arrive at the estimate

$$\alpha \int\limits_{D\cap\{u>k\}} |\nabla u|^2\,dx \leqslant C(p,D)\Big(\int\limits_{D\cap\{u>k\}} |b|^p\,dx\Big)^{1/p}\int\limits_{D\cap\{u>k\}} |\nabla u|^2\,dx. \quad (2.23)$$

Further arguing based on (2.23) do not differ from the above given ones in the case $n > 2$ and this again implies (2.17).

The estimate (2.18) can be proved in the same way. We just observe that if a function u is a supersolution to the equation, then the same function with the opposite sign is a subsolution. The proof is complete. □

Corollary 2.1. *Under the conditions* (1.2), (1.3) *(or* (1.4)*) the Zaremba problem* (1.7) *possesses the unique solution.*

3. Proof of Theorem 1.1

For $\sigma > 0$ we define the operator $\mathcal{L}_\sigma$ by the formula $\mathcal{L}_\sigma u = \mathcal{L}u - \sigma u$. The estimate (2.2) of Lemma 2.1 implies that the form associated with the operator $\mathcal{L}_\sigma$

$$\ell_\sigma(u,u) = \int\limits_D a\nabla u\cdot\nabla u\,dx - \int\limits_D (b\cdot\nabla u)u\,dx + \sigma\int\limits_D u^2\,dx$$

is coercive for sufficiently large $\sigma = \sigma_0(\alpha, b, n, p, D)$, that is,

$$\ell_\sigma(u,u) \geqslant \frac{\alpha}{2}\int\limits_D |\nabla u|^2\,dx.$$

We note that under such choice $\sigma = \sigma_0$ the bilinear form

$$\mathcal{L}_{\sigma_0}(u,v) = \int\limits_D a\nabla u\cdot\nabla v\,dx - \int\limits_D (b\cdot\nabla u)v\,dx + \sigma_0\int\limits_D uv\,dx \quad (3.1)$$

is bounded. This is implied by the estimate (2.12) applied to the first two terms in the right hand side of (3.1) and the estimate

$$\int\limits_D uv\,dx \leqslant \|u\|_{L_2(D)}\|v\|_{L_2(D)} \leqslant C(n,D)\|u\|_{W_2^1(D,F)}\|v\|_{W_2^1(D,F)}$$

implied by the Friedrichs inequality (1.5). Thus, the operator $\mathcal{L}_{\sigma_0}$ is bounded and coercive in the Hilbert space $H = W_2^1(D,F)$.

Let H^* be the dual space for H. We define the operator $\mathfrak{I}_u : H \to H^*$ by the identity

$$\mathfrak{I} = \mathfrak{I}_u v = \int\limits_D uv\,dx, \quad v\in H. \quad (3.2)$$

Let us show that the mapping $\mathfrak{I}_u$ is compact. In order to do this we observe that the mapping $\mathfrak{I}_u$ can be represented as the composition

$$\mathfrak{I}_u = \mathfrak{I}_1 \circ \mathfrak{I}_2. \tag{3.3}$$

Here $\mathfrak{I}_2 : H \to L_2(D)$ is the natural embedding. Since the norm in the space H coincides with the norm in the space $W_2^1(D)$, and the domain D is strictly Lipshitz, by the theorem on the compact embedding from [4, Sect. 11.5] the operator $\mathfrak{I}_2$ is compact. The mapping $\mathfrak{I}_1 : L_2(D) \to H^*$ is defined by the formulas (3.2) and (3.3). The continuity of the operator $\mathfrak{I}_1$ and the compactness of the operator $\mathfrak{I}_2$ imply the compactness of the operator $\mathfrak{I}$.

The equation $\mathcal{L}u = l$ for $u \in H$, where l is a functional from the space H^* dual to $H = W_2^1(D, F)$ is equivalent to the equation

$$\mathcal{L}_{\sigma_0} u + \sigma_0 \mathfrak{I}_u u = l.$$

By the Lax — Milgram lemma [5] the inverse operator $\mathcal{L}_{\sigma_0}^{-1}$ defines a continuous one–to–one correspondence of H^* onto H. This is why, applying this operator to the previous equation, we obtain an equivalent equation

$$u + \sigma_0 \mathcal{L}_{\sigma_0}^{-1} \mathfrak{I}_u u = \mathcal{L}_{\sigma_0}^{-1} l. \tag{3.4}$$

By the compactness of $\mathfrak{I}$ the mapping $T = -\sigma_0 \mathcal{L}_{\sigma_0}^{-1} \mathfrak{I}_u$ is also compact. Hence, by the Fredholm alternative, see, for instance, [6, Sect. 5.3, Thm. 5.3], the existence of a function $u \in H$ obeying Equation (3.4) is implied by the uniqueness in H of the trivial solution of the equation $\mathcal{L}u = 0$. Now the unique solvability of the Zaremba problem (1.7) is implied by Corollary of Lemma 2.3.

We proceed to proving the estimate (1.9). In order to do this, we define a formally adjoint for $\mathcal{L}$ operator $\mathcal{L}^\star$ by the formula

$$\mathcal{L}^\star u := \operatorname{div}(a(x)\nabla u) - \operatorname{div}(b(x)u).$$

Since for the corresponding bilinear forms obeys the identity

$$\ell^\star(u, v) = \ell(v, u) \quad \text{for} \quad u, v \in W_2^1(D, F),$$

the operator $\mathcal{L}^\star$ is the adjoint one for the operator $\mathcal{L}$ in the Hilbert space H. Replacing in the above arguing $\mathcal{L}$ by $\mathcal{L}^\star$, we see that the equation $\mathcal{L}_\sigma u = l$ is equivalent to the equation

$$u + (\sigma_0 - \sigma)\mathcal{L}_\sigma^{-1} \mathfrak{I}_u u = \mathcal{L}_{\sigma_0}^{-1} l$$

and the adjoint $T_\sigma^\star$ of the compact mapping $T_\sigma = (\sigma_0 - \sigma)\mathcal{L}_{\sigma_0}^{-1}\mathfrak{I}$ (see (3.2)) is given by the formula

$$T_\sigma^\star = (\sigma_0 - \sigma)(\mathcal{L}_{\sigma_0}^\star)^{-1}\mathfrak{I}.$$

Using the theorem on the contracting mappings in a Banach space, see, for instance, [6, Sect. 5.1, Thm. 5.1], we arrive at the following statement similar to Theorem 8.6 in Section 8.2 of the monograph [6].

Lemma 3.1. *Suppose that the conditions* (1.2), (1.3) *(or* (1.4)*) are satisfied and the Friedrichs inequality* (1.5) *holds. Then there exists an at most countable discrete set* $\Sigma \in (-\infty, 0)$ *such that if* $\sigma \notin \Sigma$, *then the Zaremba problems for the equations*

$$\mathcal{L}_\sigma u = l \quad \textit{and} \quad \mathcal{L}_\sigma^\star u = l$$

are uniquely solvable in $W_2^1(D, F)$ *for an arbitrary linear functional* l *in dual space for* $W_2^1(D, F)$.

To prove the estimate (1.9), we consider the operator $G_\sigma : H^\star \to H$ defined by the identity $G_\sigma = \mathcal{L}_\sigma^{-1}$ for $\sigma \notin \Sigma$. It is natural to call this operator the Green operator for the Zaremba problem (1.7). Using the Fredholm alternative, see, for instance, [6, Sect. 5.3, Thm. 5.3], we conclude that this operator is bounded and hence, the estimate (1.9) holds. The proof of Theorem 1.1 is complete.

Acknowledgments

The authors thank the referees for a careful reading of the work and useful remarks, which allowed to improve the presentation of the results.

The work was motivated by the studied of the Zaremba problem with frequent alternation of boundary conditions, see [7]–[9], as well as by known estimate on higher integrability of solutions to such problems, see [10]–[13].

BIBLIOGRAPHY

1. V. Maz'ya. *Sobolev Spaces. With Applications to Elliptic Partial Differential Equations.* Springer-Verlag, Berlin (2011).
2. G. Stampacchia. *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus* // Ann. Inst. Fourier **15**:1, 189–257 (1965).
3. D.E. Apushkinskaya, A.I. Nazarov. *The normal derivative lemma and surrounding issues* // Usp. Mat. Nauk. **77**:2, 3–68 (2022). [Russ. Math. Surv. **77**:2, 189–249 (2022).]
4. S.L. Sobolev. *Some applications of functional analysis in mathematical physics.* Nauka, Moscow (1988). [American Mathematical Society, Providence, RI (1991).]
5. P.D. Lax, A. Milgram. *Parabolic equations* // Ann. Math. Stud. **33**, 167–190 (1954).
6. D. Gilbarg, N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer–Verlag, Berlin (1983).
7. M.E. Pérez, G.A. Chechkin, E.I. Yablokova. *On eigenvibrations of a body with "light" concentrated masses on a surface* // Usp. Mat. Nauk **57**:6, 195–196 (2002). [Russ. Math. Surv. **57**:6, 1240–1242 (2002).]
8. G.A. Chechkin. *On the vibration of a partially fastened membrane with many "light" concentrated masses on the boundary* // C. R., Méc., Acad. Sci. Paris **332**:12, 949–954 (2004).
9. G.A. Chechkin, Yu.O. Koroleva, L.–E. Persson. *On the precise asymptotics of the constant in Friedrich's inequality for functions vanishing on the part of the boundary with microinhomogeneous structure* // J. Inequal. Appl. **2007**, 34138, (2007).
10. Yu.A. Alkhutov, G.A. Chechkin. *Increased Integrability of the gradient of the solution to the Zaremba problem for the Poisson equation* // Dokl. Ross. Akad. Nauk, Mat. Inform. Protsessy Upr. **497**:1, 3–6 (2021). [Dokl. Math. **103**:2, 69–71 (2021).]
11. Yu.A. Alkhutov, G.A. Chechkin. *The Meyer's estimate of solutions to Zaremba problem for second–order elliptic equations in divergent form* // C. R. Méc. **349**:2, 299–304 (2021).
12. Yu.A. Alkhutov, G.A. Chechkin, V.G. Maz'ya. *Boyarsky – Meyers estimate for solutions to Zaremba problem* // Arch. Ration. Mech. Anal. **245**:2, 1197–1211 (2022).
13. G.A. Chechkin, T.P. Chechkina. *The Boyarsky – Meyers estimate for second order elliptic equations in divergence form. Two spatial examples* // Probl. Mat. Anal. 119, 107–116 (2022). [J. Math. Sci., New York **268**:4, 523–534 (2022).]

Mushfig Jalal ogly Aliyev,
Baku State University,
Academic Zahid Khalilov str. 33,
AZ1148, Baku, Azerbaijan
E-mail: `a.mushfiq@rambler.ru`

Yuri Alexandrovich Alkhutov,
Vladimir State University
named after Alexander and Nikolay Stoletovs,
Stroiteley av. 11,
600000, Vladimir, Russia
E-mail: yurij-alkhutov@yandex.ru

Gregory Aleksandrovich Chechkin
Lomonosov Moscow State University,
Leninskie Gory 1,
119991, Moscow, Russia
Institute of Mathematics,
Ufa Federal Research Center, RAS,
Chernyshevsky str. 112,
450008, Ufa, Russia
Institute of Mathematics and Mathematical Modeling,
Pushkin str. 125,
05010, Almaty, Kazakhstan
E-mail: chechkin@mech.math.msu.su