ISSN 2304-0122 Ufa Mathematical Journal. Vol. 16. No 3 (2024). P. 84-91.

doi:10.13108/2024-16-3-84

HYPERCYCLIC AND CHAOTIC OPERATORS IN SPACE OF FUNCTIONS ANALYTIC IN DOMAIN

A.I. RAKHIMOVA

Abstract. We consider the space $H(\Omega)$ of functions analytic in a simply connected domain Ω in the complex plane equipped with the topology of uniform convergence on compact sets. We study issues on hypercyclicity, chaoticity and frequently hypercyclic for some operators in this space. We prove that a linear continuous operator in $H(\Omega)$, which commutes with the differentiation operator, is hypercyclic. We also show that this operator is chaotic and frequently hypercyclic in $H(\Omega)$.

Keywords: space of analytic functions, hypercyclic operator, chaotic operator, frequently hypercyclic operator.

Mathematics Subject Classification: 37C20

1. Introduction

1.1. Aim of work. A linear continuous operator $T : X \to X$ in a topological space X forms a discrete dynamical system $\{T^n\}_{n\in\mathbb{N}\cup\{0\}}$. In order to describe the behavior of this system, many characteristics for operators were introduced, like cyclicity, hypercyclicity, frequently hypercyclicity, chaoticity and many others. The problem on description of hypercyclic operators in the space $H(\mathbb{C})$ of functions analytic in the complex plane was considered by MacLane [1], Birkhoff [2], Godefroy [3], Shapiro [4], Kim [5], [6] and others. Chaotic and frequently hypercyclic operators in $H(\mathbb{C})$ were studied in works by Grosse–Erdmann and Peris Manguillot [7], Bayart and Matheron [8], Devaney [10] and others.

The foundations of the theory of chaotic operators are due to Devaney [10]. In [11], the authors considered the Devaney chaoticity conditions and proved that the conditions of sensitive dependence of operator on initial conditions follows from the topological transitivity and presence of a dense set of periodic points. Godefroy and Shapiro showed [3] that each convolution operator, the characteristic function of which is non–constant, is chaotic in $H(\mathbb{C})$.

The notion of frequently hypercyclic operator was introduced by Bayart and Grivaux in [12] for the space $H(\mathbb{C})$. In [13] Bonilla and Grosse–Erdmann provided examples of such operators and vectors in $H(\mathbb{C})$. In the books [7] and [8] the interested reader can find detailed information on the dynamics of linear operators including chaotic and frequently hypercyclic ones. In [9], the dynamics of linear operators in the Hardy spaces of functions analytic in a circle was studied.

The issues on hypercyclicity, chaoticity and frequently hypercyclicity of linear continuous operators in the space of functions analytic in a simply connected domain in the complex plane were not considered before. The present work is devoted to studying these issues.

Let Ω be an arbitrary simply connected domain in the plane $\mathbb{C}$. We define $H(\Omega)$ as the space of functions analytic in Ω and equip it with the topology of uniform convergence on the compact

A.I. Rakhimova, Hypercyclic and chaotic operators in space of functions analytic in domain.

Submitted August 10, 2023.

sets in Ω defined by the system of norms

$$p_m(f) = \sup_{z \in K_m} |f(z)|, \quad m = 1, 2, \dots,$$

where K_m are compact sets Ω with a non–empty interior such that $K_m \subset \operatorname{int} K_{m+1}$, $m \in \mathbb{N}$, and $\bigcup_{n=1}^{\infty} K_m = \Omega$. By the Riemann theorem, the system of polynomials is complete in $H(\Omega)$, and hence, the space is separable. It is also metrizable. Then $H(\Omega)$ is a Fréchet space. We note that it is invariant with respect to the differentiation and translation if

$$\Omega = \Omega_\sigma = \{z \in \mathbb{C} : \ |\operatorname{Im} z| < \sigma\},$$

which is a horizontal strip in the plane $\mathbb{C}$, where $\sigma \in \mathbb{R}$, $\sigma > 0$.

The results of the paper are formulated in Theorems 2.1, 3.1 and 3.2. We prove that a linear continuous operator T in $H(\Omega)$, which commutes with the differentiation operator, is hypercyclic (Theorem 2.1), chaotic (Theorem 3.1), and frequently hypercyclic (Theorem 3.2).

1.2. Main definitions. Let X be a topological vector space over the field $\mathbb{C}$. An *orbit* of an element x of an operator $T : X \to X$ is the set

$$\operatorname{Orb}(T, x) = \big\{T^n x\big\}_{n=0}^{\infty},$$

see [7, Def. 1.2], [8, Introduction, Def. 0.1]. An element $x \in X$ is called the *periodic point* of the operator T if there exists a number $n \in \mathbb{N}$ such that $T^n x = x$ ([7, Def. 1.23], [8, Def. 6.5]). We denote by $\operatorname{span} E$ the linear span of a set E in a topological vector space.

Linear continuous operator $T : X \to X$ is called *hypercyclic* ([7, Def. 2.15], [8, Introduction, Def. 0.2]) in the space X if there exists an element $x \in X$, the orbit of which is dense in X. The element $x \in X$ is a *hypercyclic vector* of the operator T in X.

A continuous operator $T : X \longrightarrow X$ in a topological vector space X is called *topologically transitive* if for all non–empty open sets A, $B \subset X$ there exists a number $n \in \mathbb{N}$ such that $T^n(A) \cap B \neq \varnothing$ [7, Def. 1.11], [8, Def. 1.2].

An operator $\Phi : Y \to Y$ in a metric space (Y, d) is called *chaotic* if the following Devaney conditions are satisfied [10, Def. 8.5]:

(A) The operator Φ possesses a sensitive dependence on the initial conditions: there exists $\delta > 0$ such that for each element $x \in Y$ and each its neighbourhood U there exist a point $y \in U$ and a number $n \in \mathbb{N}$ such that $d(\Phi^n x, \Phi^n y) > \delta$;

(B) The operator Φ is topologically transitive;

(C) The set of periodic points of the operator Φ is a dense subset in the space Y.

The *lower density* [7, Def. 9.1], [8, Sect. 6.3.1, Def. 0.2] of a set $A \subset \mathbb{N}$ is defined by the formula

$$\underline{\operatorname{dens}} A = \liminf_{N \to \infty} \frac{\#\{n \in A : \ n \leqslant N\}}{N}.$$

Let X be a topological vector space. A linear continuous operator $T : X \to X$ is called a *frequently hypercyclic operator* [7, Def. 9.2], [8, Def. 6.16] if there exists an element $x \in X$ such that for each non–empty open subset $U \subset X$ the condition

$$\underline{\operatorname{dens}} \big\{n \in \mathbb{N} : T^n x \in U\big\} > 0$$

holds. An element $x \in X$ is a *frequently hypercyclic vector* of the operator T in X. We note that the class of frequently hypercyclic operators is contained in the set of hypercyclic operators [7, Def. 9.2], [8, Def. 6.16].

In what follows we shall need the following theorems.

Theorem 1.1 (Godefroy–Shapiro theorem [3, Cor. 1.3]). *Let $T : X \to X$ be a linear continuous operator in a separable Frechét space X, the subspaces*

$$X_0 = \mathrm{span}\{x \in X : Tx = \lambda x, \lambda \in \mathbb{C} : |\lambda| < 1\}$$

and

$$Y_0 = \mathrm{span}\{x \in X : Tx = \lambda x, \lambda \in \mathbb{C} : |\lambda| > 1\}$$

are dense in X. Then T is a hypercyclic operator.

The following facts were proved in [7] and [11].

Theorem 1.2 ([11, Thm. 1]). *If an operator $T : X \to X$ in a metric space X is topologically transitive and the set of its periodic points is everywhere dense in X, then the operator T has sensitive dependence on initial conditions.*

Theorem 1.3 (Birkhoff transitivity theorem [7, Thm. 2.19]). *If X is a Frechét space, then for a linear continuous operator $T : X \to X$ the topological transitivity and hypercyclicity are equivalent.*

By Theorem 1.2, Condition (A) follows from Conditions (B) and (C), this is why we do not need to verify the proof of the chaoticity. It follows from Theorem 1.3 that to ensure the hypercyclicity for an operator the Frechét space, it is sufficient to verify the density of the set of its periodic points. We note the set of periodic points is a linear subspace.

Theorem 1.4 ([7, Thm. 2.33]). *Let T be a linear operator in a complex vector space X. Then the set of periodic points of the operator reads as*

$$\mathrm{Per}(T) = \mathrm{span}\big\{x \in X : \exists\, \alpha \in \mathbb{Q} : Tx = e^{\alpha\pi i}x\big\}.$$

Let T be an operator in a complex topological vector Frechét space X and $\mathbb{T}$ be the unit circumference. A set of functions $E_j : \mathbb{T} \to H(\Omega)$, $j \in J$, is called a spanning eigenvector field associated with the unimodular eigenvalues if $E_j(w) \in \ker(T - wI)$ for all $w \in \mathbb{T}$, $j \in J$ and the set $\mathrm{span}\big\{E_j(w)\big\}_{w\in\mathbb{T}, j\in J}$ is dense X. A vector field is called *continuous* or C^2*–smooth* if each function E_j, $j \in J$, is respectively continuous or twice differentiable in w on $\mathbb{T}$ ([7, Def. 9.21]).

Theorem 1.5 ([7, Thm. 9.22]). *Let T be an operator in a complex separable Frechét space X. Then the following statements are true:*

a) *If the operator T has spanning continuous eigenvector field associated to unimodular eigenvalues, then it is chaotic;*
b) *If the operator T has a spanning C^2–eigenvector field associated to unimodular eigenvalues, then it is frequently hypercyclic.*

We shall also use the following theorem.

Theorem 1.6 ([7, Lm. 2.34]). *Let $\Lambda \subset \mathbb{C}$ a set with an accumulation point. Then the set*

$$\mathrm{span}\big\{e^{\lambda z}, z \in \Omega\big\}_{\lambda\in\Lambda}$$

is dense in $H(\Omega)$.

2. Hypercyclic operators

For operators commuting with the differentiation the following statement is true.

Theorem 2.1. *Let a linear continuous operator T in the space $H(\Omega)$ commutes with the differentiation operator and is not a multiple of the identity mapping. Then T is a hypercyclic operator in $H(\Omega)$.*

Proof. For each $\lambda \in \mathbb{C}$ the Taylor series

$$e^{\lambda z} = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} z^n$$

converges uniformly on compact sets in the plane, and it hence converges in the topology of the space $H(\Omega)$. The function $T(z,\lambda) = T_z(e^{\lambda z})$ can be represented as a pointwise converging in λ series

$$T(z,\lambda) = \sum_{n=0}^{+\infty} \frac{T(z^n)}{n!} \lambda^n, \quad \lambda \in \mathbb{C}.$$

By the Abel theorem, this power series converges uniformly on the compact sets in the plane, and this is why $T_z(e^{\lambda z})$ is an entire function in the variable λ.

Since T commutes with D, the identity

$$T'_z(e^{\lambda z}) = \frac{d}{dz} T_z(e^{\lambda z}) = T_z \frac{d}{dz}(e^{\lambda z}) = \lambda T_z(e^{\lambda z}), \quad z \in \Omega,$$

holds for each $\lambda \in \mathbb{C}$. A solution to the differential equation $T'_z(e^{\lambda z}) = \lambda T_z(e^{\lambda z})$ reads as

$$T_z(e^{\lambda z}) = a_T(\lambda) e^{\lambda z}, \quad z \in \Omega, \quad \lambda \in \mathbb{C}.$$

Since the function $T_z(e^{\lambda z})$ is entire in λ, the same is true for $a_T(\lambda)$. We note that $a_T(\lambda)$ is not identically constant: if $a_T(\lambda) \equiv c$, where $c = const$, then $T_z(e^{\lambda z}) = ce^{\lambda z}$, and since the system of exponentials is complete $H(\Omega)$, we obtain $Tf = cf$, $f \in H(\Omega)$. This contradicts the assumptions of the theorem.

We consider the sets

$$W_1 = \big\{\lambda \in \mathbb{C} : |a_T(\lambda)| < 1\big\}, \qquad W_2 = \big\{\lambda \in \mathbb{C} : |a_T(\lambda)| > 1\big\}.$$

They are open and non–empty: if $W_2 = \emptyset$, then $a_T(\lambda)$ is bounded and hence constant, and if $W_1 = \emptyset$, then $a_T^{-1}(\lambda)$ is bounded. We let

$$X_0 = \operatorname{span}\{e^{\lambda z}\}_{z \in W_1}, \qquad Y_0 = \operatorname{span}\{e^{\lambda z}\}_{z \in W_2}.$$

The sets X_0 and Y_0 are dense in $H(\Omega)$. Thus, by Theorem 1.1, the operator T is hypercyclic in $H(\Omega)$. $\square$

The operators from Corollaries 2.1–2.4 are linear continuous ones and commutes with the differentiation operator. This is why they obey the assumptions of Theorem 2.1.

Corollary 2.1. *Let a polynomial $P(z) = \sum\limits_{n=0}^{m} a_n z^n$, where $m \in \mathbb{N}$ and $a_n \in \mathbb{C}$, $n \in (0;m)$, is not constant. Then the operator $T = \sum\limits_{n=0}^{m} a_n D^n$ is hypercyclic in $H(\Omega)$.*

We let $\Omega_\sigma = \{z \in \mathbb{C} : |\operatorname{Im} z| < \sigma\}$, where $\sigma \in \mathbb{R}$, $\sigma > 0$.

Corollary 2.2. *Given numbers $m \in \mathbb{N}$ and $a_j \in \mathbb{R}$, $c_j \in \mathbb{C}$, $j \in (1;m)$, if the operator*

$$Tf(z) = \sum_{j=1}^{m} c_j f(z + a_j)$$

is not a multiple of the identity, then it is hypercyclic in $H(\Omega_\sigma)$.

Corollary 2.3. *Let $N \in \mathbb{N}$ and $m \in \mathbb{N}$. If for given numbers $c_{jk} \in \mathbb{C}$ and points $a_k \in \mathbb{R}$, $j = 0, 1, \ldots, N$, $k = 1, 2, \ldots, m$, the operator*

$$Tf(z) = \sum_{j=0}^{N} \sum_{k=1}^{m} c_{jk} (D^j f)(z + a_k),$$

acting in $H(\Omega_\sigma)$ and not being a multiple of the identity mapping, is hypercyclic in $H(\Omega_\sigma)$.

Corollary 2.4. *The operator $M_F[f](z) = \langle F_w, f(z+w)\rangle$, where $F \in H^*(\Omega_\sigma)$, the support of which lies in the real axis, is hypercyclic in $H(\Omega_\sigma)$, see* [14, Thm. 17.3].

Let us provide an example of a non–hypercyclic operator.

Example 2.1. *Let $\lambda \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$ be fixed number. Then the operator*

$$Tf(z) = f'(\lambda z + b)$$

is not hypercyclic in $H(\Omega_\sigma)$ once $|\lambda| < 1$.

Proof. The operator T is obviously linear and continuous in $H(\Omega_\sigma)$. We consider an arbitrary function $f \in H(\Omega_\sigma)$. The n–multiple action of the operator T n on the function f reads as

$$T^n f(z) = \lambda^{\frac{n(n-1)}{2}} f^{(n)} \left(\lambda^n z + b \left(\frac{1-\lambda^n}{1-\lambda} \right) \right).$$

We take an arbitrary compact set $K \subset \Omega_\sigma$. It is obvious that

$$\sup_{z \in K} \left| \lambda^n z + b \left(\frac{1-\lambda^n}{1-\lambda} \right) - \frac{b}{1-\lambda} \right| \xrightarrow[n\to\infty]{} 0,$$

and hence, for some $N \in \mathbb{N}$,

$$\left| \lambda^n z + b \left(\frac{1-\lambda^n}{1-\lambda} \right) - \frac{b}{1-\lambda} \right| < \frac{\sigma}{4}, \quad z \in K,$$

for $n \geqslant N$. We let

$$z_n = \lambda^n z + b \left(\frac{1-\lambda^n}{1-\lambda} \right).$$

For $n \geqslant N$ by the Cauchy integral formula we have

$$f^{(n)}(z_n) = \frac{n!}{2\pi i} \int\limits_{|\xi - \frac{b}{1-\lambda}| = \frac{\sigma}{2}} \frac{f(\xi) d\xi}{(\xi - z_n)^{n+1}},$$

and hence,

$$|f^{(n)}(z_n)| \leqslant \frac{2^{2n+1} n!}{\sigma^{n+1}} \max_{|\xi - \frac{b}{1-\lambda}| \leqslant \frac{\sigma}{2}} |f(\xi)| := C_0 \left(\frac{4}{\sigma} \right)^n n!,$$

where

$$C_0 = \frac{2}{\sigma} \max_{|\xi - \frac{b}{1-\lambda}| \leqslant \frac{\sigma}{2}} |f(\xi)|.$$

Then, for $n \geqslant N$, $z \in K$,

$$|T^n f(z)| = |\lambda|^{\frac{n(n-1)}{2}} |f^{(n)}(z_n)| \leqslant C_0 \left(\frac{4}{\sigma} \right)^n n! |\lambda|^{\frac{n(n-1)}{2}},$$

that is,

$$\max_{z \in K} |T^n f(z)| \xrightarrow[n\to\infty]{} 0.$$

Since K is an arbitrary compact set, we have $T^n f(z) \to 0$ as $n \to \infty$ in the topology of the space $H(\Omega_\sigma)$. In particular, the set $\{T^n f\}_{n\in\mathbb{N}}$ is bounded in the space $H(\Omega_\sigma)$ and can not be everywhere dense. The proof is complete. □

3. Chaotic and frequently hypercyclic operators

For the operator commuting with the differentiation the following statements on the chaoticity and frequently hypercyclicity are true.

Theorem 3.1. *Let a linear continuous operator T in the space $H(\Omega)$ commutes with the differentiation operator and is not a multiple of the identity mapping. Then T is a chaotic operator.*

Proof. Let us verify the definition of the chaoticity. It was shown in Theorem 2.1 that T is a hypercyclic operator in the Frechét space $H(\Omega)$. Due to Theorems 1.2 and 1.3 it remains to show that T has a dense set of periodic points in $H(\Omega)$.

In the proof of Theorem 2.1 we have obtained that the action of the operator T on the exponentials is given by the formula

$$T(e^{\lambda z}) = a_T(\lambda)e^{\lambda z},$$

where $a_T(\lambda)$ is a non–constant entire function, $\lambda \in \mathbb{C}$, $z \in \Omega$. We denote $\varphi(\lambda) = a_T(\lambda)$.

Due to Theorem 1.4, the set of periodic points of the operator T is defined as

$$V = \operatorname{span}\Big\{f \in H(\Omega) : \exists \alpha \in \mathbb{Q} : Tf(z) = e^{\alpha\pi i} f(z)\Big\}.$$

Then the set of the eigenvalues associated with the functions in V is

$$W = \Big\{\lambda \in \mathbb{C} : \exists \alpha \in \mathbb{Q} : \varphi(\lambda) = e^{\alpha\pi i}\Big\}.$$

Since $\varphi(\lambda)$ is a non–constant entire function, it takes all values except, possibly, a single one. Hence, its image $\varphi(\mathbb{C})$ intersects the unit circumference in $\mathbb{T}$. Since a non–constant holomorphic function $\varphi(\lambda)$ is an open mapping, then infinitely many points $\lambda = \varphi^{-1}(e^{\alpha\pi i})$, where $\alpha \in \mathbb{Q}$, lie in some compact set in $\mathbb{C}$. This gives that W has a limiting point. By Theorem 1.6, $V = \operatorname{span}\{e^{\lambda z}\}_{\lambda\in W}$ is dense in $H(\Omega)$. Therefore, the operator T is chaotic in $H(\Omega)$. □

Theorem 3.2. *Let a linear continuous operator T in the space $H(\Omega)$ commute with the differentiation operator and is not a multiple of the identity mapping. Then T is a frequently hypercyclic operator.*

Proof. The proof ifs based on Theorem 1.5. It was shown in Theorem 2.1 that T is a hypercyclic operator in the Frechét space $H(\Omega)$. In its proof we have obtained that the action of the operator T on the exponentials is

$$T(e^{\lambda z}) = a_T(\lambda)e^{\lambda z},$$

where $a_T(\lambda)$ is a non–constant entire function, $\lambda \in \mathbb{C}$, $z \in \Omega$. We denote $\varphi(\lambda) = a_T(\lambda)$. The previous equation shows that the numbers $\varphi(\lambda)$ are among the eigenvalues of the operator T, and the exponentials $e^{\lambda z}$ are the eigenfunctions.

Since φ is a non–constant entire function, by the Picard theorem the set $\varphi(\mathbb{C})$ contains the entire unit circumference except for, possibly, a single point. We take a point $w \in \mathbb{T}$ so that $\varphi(z) = w$ and the derivative $\varphi'(z)$ is non–zero, $\varphi'(z) \neq 0$. In this case the function φ maps some open neighbourhood $\widetilde{U}$ of the point z conformally into some open neighbourhood U of the point w.

Let $\psi = \varphi^{-1} : U \to \widetilde{U}$ be an inverse mapping, it is holomorphic in U. We fix some non–trivial closed arc on the unit circumference $\gamma \subset U$, which contains a point w. We choose a

C^2–smooth function $f:\mathbb{T}\to\mathbb{C}$ such that $f(w)\neq 0$ and $f\equiv 0$ outside γ. We define a function $E:\mathbb{T}\to H(\Omega)$ as $E(\lambda)=f(\lambda)e^{\psi(\lambda)z}$. Since the set $V=\{\psi(\lambda),\ \lambda\in\gamma,\ f(\lambda)\neq 0\}$ obviously has limiting points, by Theorem 1.6 the set $\operatorname{span}\{f(\lambda)e^{\psi(\lambda)z},\psi(\lambda)\in V\}$ is dense in $H(\Omega)$. It is obvious that the set of a single function E forms a spanning eigenvector field associated with the unimodular eigenvalues. Now Theorem 3.2 follows from Theorem 1.5. □

Since the operators in Corollaries 3.1–3.4 are linear continuous ones in $H(\Omega)$ and commute with the differentiation operator, they obey Theorems 3.1 and 3.2.

Corollary 3.1. *Let the polynomial*

$$P(z)=\sum_{n=0}^{m}a_nz^n,$$

where $m\in\mathbb{N}$ and $a_n\in\mathbb{C}$, $n\in(0;m)$, be non–constant. Then the operator

$$T=\sum_{n=0}^{m}a_nD^n$$

is chaotic and frequently hypercyclic in $H(\Omega)$.

We let $\Omega_\sigma=\{z\in\mathbb{C}:|\operatorname{Im}z|<\sigma\}$, where $\sigma\in\mathbb{R}$, $\sigma>0$.

Corollary 3.2. *Given numbers $m\in\mathbb{N}$, $a_j\in\mathbb{R}$, $c_j\in\mathbb{C}$, $j\in(1;m)$, if the operator*

$$Tf(z)=\sum_{j=1}^{m}c_jf(z+a_j)$$

is not a multiple of the identity mapping, then it is chaotic and frequently hypercyclic in $H(\Omega_\sigma)$.

Corollary 3.3. *Let $N\in\mathbb{N}$ and $m\in\mathbb{N}$. If for given numbers $c_{jk}\in\mathbb{C}$ and points $a_k\in\mathbb{R}$, $j=0,1,\dots,N$, $k=1,2,\dots,m$, the operator*

$$Tf(z)=\sum_{j=0}^{N}\sum_{k=1}^{m}c_{jk}(D^jf)(z+a_k),$$

acting in $H(\Omega_\sigma)$ and being not a multiple of the identity mapping, is chaotic and frequently hypercyclic in $H(\Omega_\sigma)$.

Corollary 3.4. *The operator $M_F[f](z)=\langle F_w,f(z+w)\rangle$, where $F\in H^*(\Omega_\sigma)$, the support of which is located in the real axis, is chaotic and frequently hypercyclic in $H(\Omega_\sigma)$, see* [14, Thm. 17.3].

BIBLIOGRAPHY

1. G.R. MacLane. *Sequences of derivatives and normal families* // J. Anal. Math. **2**, 72–87 (1952).
2. G.D. Birkhoff. *Démonstration d'un théorème élémentaire sur les fonctions entières* // C. R. **189**, 473–475 (1929).
3. G. Godefroy, J.H. Shapiro. *Operators with dense, invariant, cyclic vector manifolds* // J. Funct. Anal. **98**:2, 229–269 (1991).
4. R.M. Gethner, J. Shapiro. *Universal vectors for operators on spaces of holomorphic functions* // Proc. Amer. Math. Soc. **100**:2, 281–288 (1987).
5. V.E. Kim. *Hypercyclic and chaotic operators on spaces of entire functions* // Trudy Instituta matematiki VC UNC RAN. **1**, 126–130 (2008) (in Russian).

6. V.E. Kim. *Hypercyclicity and chaotic character of generalized convolution operators generated by Gel'fond-Leont'ev operators* // Mat. Zametki **85**:6, 849–856 (2009). [Math. Notes. **85**:6, 807–813 (2009).]
7. K.–G. Grosse–Erdmann, A. Peris Manguillot. *Linear chaos*, Springer, Berlin (2011).
8. F. Bayart, E. Matheron. *Dynamics of Linear Operators*, Cambridge University Press, Cambridge (2009).
9. A.A. Lishanskii. *Existence of hypercyclic subspaces for Toeplitz operators* // Ufim. Mat. Zh. **7**:2, 109–113 (2015). [Ufa Math J. **7**:2, 102–105 (2015).]
10. R.L. Devaney. *An introduction to chaotic dynamical systems*, Addison–Wesley Publishing Company, Inc., Redwood City, CA etc. (1989).
11. J. Banks, J. Brooks, G. Cairns, G. Davis, P. Stacey. *On Devaney's definition of chaos* // Am. Math. Mon. **99**:4, 332–334 (1992).
12. F. Bayart, S. Grivaux. *Frequently hypercyclic operators* // Trans. Am. Math. Soc. **358**:11, 5083–5117 (2006).
13. A. Bonilla, K.–G. Grosse–Erdmann. *Frequently hypercyclic operators and vectors* // Ergodic Theory Dyn. Syst. **27**:2, 383–404 (2007).
14. V.V. Napalkov. *Convolution equations in multidimensional spaces.* Nauka, Moscow (1982) (in Russian).

Alsu Il'darovna Rakhimova,
Institute of Mathematics,
Ufa Federal Research Center, RAS,
Chernyshevsky str., 112,
450008, Ufa, Russia
E-mail: `alsu1405@mail.ru`