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TO QUESTION ON EMBEDDING OF REPRODUCING KERNEL HILBERT SPACES

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Abstract. In this work we obtain necessary and sufficient conditions for embedding of one reproducing kernel Hilbert space into another reproducing kernel Hilbert space. The paper is a continuation of works by the authors, in which the problem on coincidence or equivalence of two reproducing kernel Hilbert spaces was studied. An important role is played by the consistence condition of two complete systems of functions with some linear continuous operator introduced by the authors before. The obtained results are demonstrated by particular examples.

Keywords: reproducing kernel Hilbert spaces, description of dual space, orthosimilar expansion systems, Bergman spaces.

Mathematics Subject Classification: 46E22, 30H05, 32A38, 47B32

1. Introduction

In many problems in the complex analysis a question often arises: whether a given reproducing kernel Hilbert space (RKHS) is contained in another wider RKHS? Many problems in the probability theory, mathematical statistics, numerical methods, partial differential equations, etc are reduced to studying RKHSs (see, for instance, [1], [2]).

We address the following question. Suppose that we are given two RKHSs H_1 and H_2 consisting of functions defined on some set $\Omega_1 \subset \mathbb{C}^n$, $n \in \mathbb{N}$. What conditions guarantee the embedding $H_1 \subset H_2$? We consider this problem in an equivalent formulation [3], [4]. Let H be RKHS consisting of functions defined on a set $\Omega \subset \mathbb{C}^m$, $m \in \mathbb{N}$, that is, for an arbitrary point $\xi \in \Omega$, the functional δ_ξ mapping each function $f \in H$ into the value of the function f at the point $\xi \in \Omega$ is a linear continuous functional on H. Suppose that $\{e_1(\,\cdot\,,\xi)\}_{\xi\in\Omega_1}$, $\{e_2(\,\cdot\,,\xi)\}_{\xi\in\Omega_1}$ are some complete systems of functions H; $\Omega_1 \subset \mathbb{C}^n$, $n \in \mathbb{N}$. We denote

$$
\begin{aligned}
&\widetilde{f}(z) \overset{def}{=} (e_1(\,\cdot\,,z), f)_H \quad \text{for all} \quad z \in \Omega_1, \quad \widetilde{H} = \{\widetilde{f},\, f \in H\},\\
&(\widetilde{f}_1, \widetilde{f}_2)_{\widetilde{H}} \overset{def}{=} (f_2, f_1)_H, \quad \|\widetilde{f}_1\|_{\widetilde{H}} = \|f_1\|_H \quad \text{for all} \quad \widetilde{f}_1, \widetilde{f}_2 \in \widetilde{H},
\end{aligned} \tag{1.1}
$$

$$
\begin{aligned}
&\widehat{f}(z) \overset{def}{=} (e_2(\,\cdot\,,z), f)_H\, \forall z \in \Omega_1, \quad \widehat{H} = \{\widehat{f},\, f \in H\},\\
&(\widehat{f}_1, \widehat{f}_2)_{\widehat{H}} \overset{def}{=} (f_2, f_1)_H, \quad \|\widehat{f}_1\|_{\widehat{H}} = \|f_1\|_H \quad \text{for all} \quad \widehat{f}_1, \widehat{f}_2 \in \widehat{H}.
\end{aligned} \tag{1.2}
$$

We need to find a condition, under which the spaces $\widehat{H}$ and $\widetilde{H}$ possess the property $\widehat{H} \subset \widetilde{H}$, that is, as a set of functions, $\widehat{H}$ is contained in $\widetilde{H}$, and there exists a constant $C > 0$ such that

$$\|h\|_{\widetilde{H}} \leqslant C\|h\|_{\widehat{H}} \quad \text{for all} \quad h \in \widehat{H}.$$

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It should be noted that the question on embedding of one RKHS into another RKHS was earlier considered in relation with applications in the probability theory and mathematical statistics, (see, for instance, [1]). This problem was studied in [5, Thm. 2.4] and [6, Thm. 1]. Theorem 2.2 in our work generalises the results of the cited works. The present work is a continuation of the works [3], [4], [7], in which the problem on coincidence or equivalence of two RKHSs was studied. We also study the consistence condition, when one RKHS is embedded into another RKHS. It turns out that the results on the consistence condition are not simply extended from the case of the equivalence of RKHSs to the case of embedding, see Theorems 2.3, 2.4 and Examples 1, 2 in this work. The obtained results are demonstrated by particular examples of weighted Bergman spaces on the unit disk.

2. Main results

Let H be RKHS, $\{e_1(\cdot,\xi)\}_{\xi\in\Omega_1}$, $\{e_2(\cdot,\xi)\}_{\xi\in\Omega_1}$ be some complete systems of functions in H, $\Omega_1\subset\mathbb{C}^n$, $n\in\mathbb{N}$. The spaces $\widetilde{H}$, $\widehat{H}$ are defined as in (1.1), (1.2).

Theorem 2.1. *The embedding $\widehat{H}\subset\widetilde{H}$ is equivalent to the existence of a linear continuous operator $A\colon H\to H$ such that*

$$A\colon e_1(\cdot,z)\mapsto e_2(\cdot,z)\quad \textit{for all}\quad z\in\Omega_1.$$

Proof. Necessity. Let $\widehat{H}\subset\widetilde{H}$. For $\widehat{f}\in\widehat{H}$ we have

$$\widehat{f}(z)=(e_2(\cdot,z),f)_H\quad\text{for all}\quad z\in\Omega_1. \tag{2.1}$$

On the other hand, since $\widehat{H}\subset\widetilde{H}$, there exists a function $g_f\in H$ such that the identity

$$\widehat{f}(z)=(e_1(\cdot,z),g_f)_H=\widetilde{g}_f(z)\quad\text{for all}\quad z\in\Omega_1 \tag{2.2}$$

is true. If $\widehat{f}$ ranges over the entire space $\widehat{H}$, then f ranges over the entire space H. We define the operator B by the rule

$$B\colon f\mapsto g_f.$$

It is easy to show that B is a linear operator. The expression (2.2) means that

$$\widehat{f}(z)=\widetilde{g}_f(z)=\widetilde{Bf}(z)\quad\text{for all}\quad z\in\Omega_1,\quad\text{for all}\quad f\in H. \tag{2.3}$$

The operator B is bounded. Indeed, by the identities (2.2), (2.3) and the condition $\widehat{H}\subset\widetilde{H}$ we get

$$\|Bf\|_H=\big\|\widetilde{Bf}\big\|_{\widetilde{H}}=\|\widehat{f}\|_{\widetilde{H}}\leqslant C\cdot\|\widehat{f}\|_{\widehat{H}}=C\cdot\|f\|_H.$$

The latter means the boundedness of the operator B.

For all $f\in H$, $z\in\Omega_1$ we have

$$\widehat{f}(z)=(e_1(\cdot,z),Bf)_H=(B^*e_1(\cdot,z),f)_H, \tag{2.4}$$

where B^* is the adjoint operator of B. Deducting the identity (2.4) from (2.1), we obtain

$$0\equiv(e_2(\cdot,z)-B^*e_1(\cdot,z),f)_H\quad\text{for all}\quad f\in H,\quad z\in\Omega_1$$

and

$$B^*e_1(\cdot,z)=e_2(\cdot,z)\quad\text{for all}\quad z\in\Omega_1.$$

We let $A=B^*$. We have constructed the operator

$$A\colon H\to H,\qquad A\colon e_1(\cdot,z)\mapsto e_2(\cdot,z)\quad\text{for all}\quad z\in\Omega_1,$$

and this completes the proof of necessity.

Sufficiency. Suppose that there exists an operator A such that

$$A\colon e_1(\,\cdot\,,z)\mapsto e_2(\,\cdot\,,z)\quad \text{for all}\quad z\in\Omega_1.$$

The relation

$$\widehat{f}(z)=(e_2(\,\cdot\,,z),f)_H=(Ae_1(\,\cdot\,,z),f)_H=(e_1(\,\cdot\,,z),A^*f)_H\quad \text{for all}\quad f\in H$$

implies that if $\widehat{f}\in\widehat{H}$, then $\widehat{f}\in\widetilde{H}$. Using the definitions of the norm in the spaces $\widetilde{H}$, $\widehat{H}$, we also have

$$\|\widehat{f}\|_{\widetilde{H}}=\|A^*f\|_H\leqslant C\|f\|_H=C\|\widehat{f}\|_{\widehat{H}}\quad \text{for all}\quad f\in H,$$

where $C>0$ is some constant. Thus, $\widehat{H}\subset\widetilde{H}$. The proof is complete. □

Let H_1 be an RKHS consisting of the functions defined on the set $\Omega_0^1\subset\mathbb{C}^r$, $r\in\mathbb{N}$, the system of function $\{e_1(\,\cdot\,,\xi)\}_{\xi\in\Omega_1}$ belong to H_1 and be complete in it. Let H_2 be RKHS consisting of the functions defined on the set $\Omega_0^2\subset\mathbb{C}^s$, $s\in\mathbb{N}$, the system of function $\{e_2(\,\cdot\,,\xi)\}_{\xi\in\Omega_1}$ belong to H_2 and be complete in it. We define the spaces $\widetilde{H}_1$ and $\widehat{H}_2$:

$$\begin{aligned}
&\widetilde{f}(z)\stackrel{def}{=}(e_1(\,\cdot\,,z),f)_{H_1}\quad \text{for all}\quad z\in\Omega_1,\quad \widetilde{H}_1=\{\widetilde{f},\,f\in H_1\},\\
&(\widetilde{f}_1,\widetilde{f}_2)_{\widetilde{H}_1}\stackrel{def}{=}(f_2,f_1)_{H_1},\quad \|\widetilde{f}_1\|_{\widetilde{H}_1}=\|f_1\|_{H_1}\quad \text{for all}\quad \widetilde{f}_1,\widetilde{f}_2\in\widetilde{H}_1, &(2.5)\\
&\widehat{g}(z)\stackrel{def}{=}(e_2(\,\cdot\,,z),g)_{H_2}\quad \forall z\in\Omega_1,\quad \widehat{H}_2=\{\widehat{g},\,g\in H_2\},\\
&(\widehat{g}_1,\widehat{g}_2)_{\widehat{H}_2}\stackrel{def}{=}(g_2,g_1)_{H_2},\quad \|\widehat{g}_1\|_{\widehat{H}_2}=\|g_1\|_{H_2}\quad \text{for all}\quad \widehat{g}_1,\widehat{g}_2\in\widehat{H}_2. &(2.6)
\end{aligned}$$

The following theorem generalises Theorem 2.1.

Theorem 2.2. *The embedding $\widehat{H}_2\subset\widetilde{H}_1$ is equivalent to the existence of a linear bounded operator A acting from H_1 into H_2 such that*

$$A\colon e_1(\,\cdot\,,\xi)\mapsto e_2(\,\cdot\,,\xi)\quad \textit{for all}\quad \xi\in\Omega_1. \tag{2.7}$$

Proof. Necessity. Let $\widehat{H}_2\subset\widetilde{H}_1$. For $\widehat{g}\in\widehat{H}_2$ we have

$$\widehat{g}(z)=(e_2(\,\cdot\,,z),g)_{H_2}\quad \text{for all}\quad z\in\Omega_1. \tag{2.8}$$

If $\widehat{g}$ ranges over the entire space $\widehat{H}_2$, then g ranges over the entire space H_2. On the other hand, since $\widehat{H}_2\subset\widetilde{H}_1$, there exists $h_g\in H_1$ such that the identity

$$\widehat{g}(z)=(e_1(\,\cdot\,,z),h_g)_{H_1}=\widetilde{h}_g(z)\quad \text{for all}\quad z\in\Omega_1 \tag{2.9}$$

holds.

We define the operator B as

$$B\colon g\mapsto h_g.$$

It is easy to show that $B\colon H_2\to H_1$ is a linear operator. The expression (2.9) means that

$$\widehat{g}(z)=\widetilde{h}_g(z)=\widetilde{Bg}(z)\quad \text{for all}\quad z\in\Omega_1. \tag{2.10}$$

The operator B is bounded. Indeed, by the identities (2.9), (2.10) and the conditions $\widehat{H}_2\subset\widetilde{H}_1$ we find

$$\|Bg\|_{H_1}=\|\widetilde{Bg}\|_{\widetilde{H}_1}=\|\widehat{g}\|_{\widetilde{H}_1}\leqslant C\cdot\|\widehat{g}\|_{\widehat{H}_2}=C\cdot\|g\|_{H_2}\quad \text{for all}\quad g\in H_2.$$

The latter means the boundedness of the operator B. For each $g\in H_2$, $z\in\Omega_1$, we have

$$\widehat{g}(z)=(e_1(\,\cdot\,,z),Bg)_{H_1}=(B^*e_1(\,\cdot\,,z),g)_{H_2}. \tag{2.11}$$

Here $B^*\colon H_1 \to H_2$ is the adjoint operator for B. Deducting the identity (2.11) from (2.8), we get

$$0 \equiv (e_2(\,\cdot\,, z) - B^* e_1(\,\cdot\,, z), g)_{H_2} \quad \text{for all} \quad g \in H_2,\ \forall z \in \Omega_1$$

and

$$B^* e_1(\,\cdot\,, z) = e_2(\,\cdot\,, z) \quad \text{for all} \quad z \in \Omega_1.$$

We let $A = B^*$. We have constructed the operator $A\colon H_1 \to H_2$ such that

$$A\colon e_1(\,\cdot\,, z) \to e_2(\,\cdot\,, z) \quad \text{for all} \quad z \in \Omega_1.$$

This completes the proof of necessity.

Sufficiency. Suppose that there exists an operator $A\colon H_1 \to H_2$ such that

$$A\colon e_1(\,\cdot\,, z) \mapsto e_2(\,\cdot\,, z) \quad \text{for all} \quad z \in \Omega_1.$$

Then the operator A^* acts from the space H_2 into the space H_1 and

$$\widehat{g}(z) = (e_2(\,\cdot\,, z), g)_{H_2} = (Ae_1(\,\cdot\,, z), g)_{H_2} = (e_1(\,\cdot\,, z), A^*g)_{H_1} = \widetilde{A^*g}(z)$$

for all $z \in \Omega_1$ and $g \in H_2$. This implies that if a function $\widehat{g}$ belongs to $\widehat{H}_2$, then $\widehat{g}$ also belongs the space $\widetilde{H}_1$. Due to the definitions of the norms in the spaces $\widetilde{H}_1$, $\widehat{H}_2$, for the functions $g \in H_2$, $\widehat{g} \in \widehat{H}_2$ the estimate

$$\|\widehat{g}\|_{\widetilde{H}_1} = \big\|\widetilde{A^*g}\big\|_{\widetilde{H}_1} = \|A^*g\|_{H_1} \leqslant C\|g\|_{H_2} = C\|\widehat{g}\|_{\widehat{H}_2} \quad \text{for all} \quad g \in H_2$$

holds, where $C > 0$ is some constant. Thus, we have shown that $\widehat{H}_2 \subset \widetilde{H}_1$. The proof is complete. □

In what follows we study the issue on condition of consistence of complete systems of functions with some linear continuous operator [3] for the case of embedding one RKHS into another RKHS. We recall a definition from [3].

Definition 2.1. *Systems of functions $\{e_j(\,\cdot\,, z)\}_{z\in\Omega_1}$, $j = 1, 2$, belonging to RKHS H, are called consistent with an operator $T\colon H \to H$ if*

$$(e_1(\,\cdot\,, z_1), e_2(\,\cdot\,, z_2))_H = \overline{(e_1(\,\cdot\,, z_2), Te_2(\,\cdot\,, z_1))_H} \quad \textit{for all} \quad z_1, z_2 \in \Omega_1. \tag{2.12}$$

There arises a question: whether an analogue of Statements 3 and 4 from the work [3] (see also [4]) is true? We suppose that the systems of functions $\{e_j(\,\cdot\,, z)\}_{z\in\Omega_1}$, $j = 1, 2$, are orthosimilar expansion systems in RKHS H with the same measure μ [8] (see also [9]). Orthosimilar expansions systems were introduced in works by T.P. Lukashenko. As it turns out, the condition that the systems $\{e_j(\,\cdot\,, z)\}_{z\in\Omega_1}$, $j = 1, 2$, are orthosimilar expansion systems in the space H with the same measure μ is a very strong condition. The space H consists of the functions defined on the set $\Omega \subset \mathbb{C}^m$, $m \in \mathbb{N}$.

Theorem 2.3. *Let $\{e_j(\,\cdot\,, z)\}_{z\in\Omega_1}$, $j = 1, 2$, be two orthosimilar expansion systems in an RKHS H with the same measure μ. Suppose that there exists a linear continuous operator $T\colon H \to H$ such that the systems of functions $\{e_j(\,\cdot\,, z)\}_{z\in\Omega_1}$, $j = 1, 2$, are consistent with the operator T, that is,*

$$(e_1(\,\cdot\,, z_1), e_2(\,\cdot\,, z_2))_H = \overline{(e_1(\,\cdot\,, z_2), Te_2(\,\cdot\,, z_1))_H} \quad \textit{for all} \quad z_1, z_2 \in \Omega_1.$$

Then the space $\widehat{H}$ is equivalent to the space $\widetilde{H}$.

Remark 2.1. *In contrast to* [3, Statems. 3, 4], *the operator $T\colon H \to H$ is not supposed to be invertible; it is only a linear continuous operator.*

Proof. Let $\{e_j(\,\cdot\,,z)\}_{z\in\Omega_1}$, $j=1,2$, be two orthosimilar expansion systems in the space H, which are consistent with the operator T, that is, the relation (2.12) holds, or, equivalently,

$$(e_1(\,\cdot\,,z_1),e_2(\,\cdot\,,z_2))_H=(Te_2(\,\cdot\,,z_1),e_1(\,\cdot\,,z_2))_H \quad \text{for all} \quad z_1,z_2\in\Omega_1.$$

This identity implies

$$\left(\sum_{k=1}^{m} c_k e_1(\,\cdot\,,z_k),e_2(\,\cdot\,,\xi)\right)_H=\left(\sum_{k=1}^{m} c_k Te_2(\,\cdot\,,z_k),e_1(\,\cdot\,,\xi)\right)_H \quad \text{for all} \quad \xi\in\Omega_1. \tag{2.13}$$

Here $\{z_k\}_{k=1}^m$ is an arbitrary set of the points in Ω_1, while $\{c_k\}_{k=1}^m$ is an arbitrary set of complex numbers. Then

$$p(t)\overset{def}{=}\sum_{k=1}^{m} c_k e_1(t,z_k), \quad t\in\Omega,$$

is an arbitrary function in the linear span of the system $\{e_1(\,\cdot\,,z)\}_{z\in\Omega_1}$, and

$$q(t)\overset{def}{=}\sum_{k=1}^{m} c_k Te_2(t,z_k), \quad t\in\Omega,$$

is a function in the linear span of the system $\{Te_2(\,\cdot\,,z)\}_{z\in\Omega_1}$. Then the functions p, q belong to H, while the systems of functions $\{e_j(\,\cdot\,,z)\}_{z\in\Omega_1}$, $j=1,2$, are orthosimilar expansion systems in H with the measure μ. Hence,

$$p(t)=\int\limits_{\Omega_1}(p,e_2(\,\cdot\,,\xi))_H e_2(t,\xi)\,d\mu(\xi) \quad \text{for all} \quad t\in\Omega,$$

$$q(t)=\int\limits_{\Omega_1}(q,e_1(\,\cdot\,,\xi))_H e_1(t,\xi)\,d\mu(\xi) \quad \text{for all} \quad t\in\Omega.$$

By the analogue of the Parseval identity [8, Thm. 1] and relation (2.13) the identity holds

$$\|p\|_H^2=\int\limits_{\Omega_1}|(p,e_2(\,\cdot\,,\xi))|^2\,d\mu(\xi)=\int\limits_{\Omega_1}|(q,e_1(\,\cdot\,,\xi))|^2\,d\mu(\xi)=\|q\|_H^2$$

or $\|p\|_H=\|q\|_H$. On the linear span of the system of functions $\{e_1(\,\cdot\,,z)\}_{z\in\Omega_1}$ we define an operator L_1 by the rule $L_1\colon p\mapsto q$. We denote by H_0^1 the closure of the linear span of the system of functions $\{Te_2(\,\cdot\,,z)\}_{z\in\Omega_1}$ by the norm of the space H. By the Banach theorem [10], the operator L_1 is continued to a linear continuous bijective unitary operator acting from H onto H_0^1 and $L_1\colon e_1(\,\cdot\,,\xi)\mapsto Te_2(\,\cdot\,,\xi)$ for all $\xi\in\Omega_1$. This is why the operator $A\overset{def}{=}L_1^{-1}\circ T$ is a linear continuous operator acting from H into H and $A\colon e_2(\,\cdot\,,\xi)\mapsto e_1(\,\cdot\,,\xi)$ for all $\xi\in\Omega_1$. Applying Theorem 2.1, we obtain $\widetilde{H}\subset\widehat{H}$.

It follows from relation (2.12) that

$$(e_1(\,\cdot\,,z_1),e_2(\,\cdot\,,z_2))_H=\overline{(T^*e_1(\,\cdot\,,z_2),e_2(\,\cdot\,,z_1))_H} \quad \text{for all} \quad z_1,z_2\in\Omega_1, \tag{2.14}$$

T^* is the adjoint operator for T. By (2.14) we get

$$(e_2(\,\cdot\,,z_2),e_1(\,\cdot\,,z_1))_H=(T^*e_1(\,\cdot\,,z_2),e_2(\,\cdot\,,z_1))_H \quad \text{for all} \quad z_1,z_2\in\Omega_1. \tag{2.15}$$

Let $\{z_k\}_{k=1}^m$ be an arbitrary set of points in Ω_1 and $\{c_k\}_{k=1}^m$ be an arbitrary set of complex numbers and

$$p(t)\overset{def}{=}\sum_{k=1}^{m} c_k e_2(t,z_k), \quad t\in\Omega,$$

be a function from the linear span of the system $\{e_2(\,\cdot\,,z)\}_{z\in\Omega_1}$, and

$$q(t) \stackrel{def}{=} \sum_{k=1}^{m} c_k T^* e_1(t,z_k), \quad t\in\Omega,$$

be a function from the linear span of the system $\{T^*e_1(\,\cdot\,,z)\}_{z\in\Omega_1}$. We define an operator L_2 by the rule $L_2\colon p\mapsto q$. We denote by H_0^2 the closure of the linear span of the system of functions $\{T^*e_1(\,\cdot\,,z)\}_{z\in\Omega_1}$ in the norm of the space H. By the Banach theorem [10] the operator L_2 is continued to a linear continuous bijective unitary operator acting from H onto H_0^2, and this operator possesses the property $L_2\colon e_2(\,\cdot\,,\xi)\mapsto T^*e_1(\,\cdot\,,\xi)$ for all $\xi\in\Omega_1$. This is why the operator $A\stackrel{def}{=} L_2^{-1}\circ T^*$ is a linear continuous operator acting from H into H and $A\colon e_1(\,\cdot\,,\xi)\mapsto e_2(\,\cdot\,,\xi)$ for all $\xi\in\Omega_1$. Applying Theorem 2.1, we get $\widehat{H}\subset\widetilde{H}$, that is, as a set of function, $\widehat{H}$ is embedded into $\widetilde{H}$ and for each $h\in\widehat{H}$ the estimate $\|h\|_{\widetilde{H}}\leqslant C_1\|h\|_{\widehat{H}}$ holds with a constant $C_1>0$ independent of h. Earlier we have shown that $\widetilde{H}\subset\widehat{H}$, that is, as a set of functions, $\widetilde{H}$ is embedded into $\widehat{H}$ and for each $h\in\widetilde{H}$ the estimate $\|h\|_{\widehat{H}}\leqslant C_2\|h\|_{\widetilde{H}}$ holds with a constant $C_2>0$ independent of h. Thus, the spaces $\widehat{H}$ and $\widetilde{H}$ are equivalent, that is, $\widehat{H}$, $\widetilde{H}$ consist of the same functions and the relation

$$\frac{1}{C_2}\|h\|_{\widehat{H}}\leqslant\|h\|_{\widetilde{H}}\leqslant C_1\|h\|_{\widehat{H}} \quad \text{for all} \quad h\in\widetilde{H}$$

holds. The proof is complete. □

The proven theorem implies the following corollary.

Corollary 2.1. *Let $\{e_j(\,\cdot\,,z)\}_{z\in\Omega_1}$, $j=1,2$, be two orthosimilar expansions systems in the space H with the same measure μ. Assume that there exists a linear continuous operator $T\colon H\to H$ such that the systems of functions $\{e_j(\,\cdot\,,z)\}_{z\in\Omega_1}$, $j=1,2$, are consistent with the operator T, that is,*

$$(e_1(\,\cdot\,,z_1),e_2(\,\cdot\,,z_2))_H=\overline{(e_1(\,\cdot\,,z_2),Te_2(\,\cdot\,,z_1))_H} \quad \textit{for all} \quad z_1,z_2\in\Omega_1. \tag{2.16}$$

Then the operator T has a continuous inverse operator.

Proof. Under the made assumptions, by Theorem 2.3, the space $\widehat{H}$ is equivalent to the space $\widetilde{H}$. Applying [3, Statem. 2], we conclude on the existence of a linear continuous invertible operator T_1 such that the condition holds

$$(e_1(\,\cdot\,,z_1),e_2(\,\cdot\,,z_2))_H=\overline{(e_1(\,\cdot\,,z_2),T_1e_2(\,\cdot\,,z_1))_H} \quad \text{for all} \quad z_1,z_2\in\Omega_1. \tag{2.17}$$

On the other hand, by the assumptions, the identity (2.16) is true. Comparing (2.16) and (2.17), we find the identity

$$(e_1(\,\cdot\,,z_2),T_1e_2(\,\cdot\,,z_1))_H=(e_1(\,\cdot\,,z_2),Te_2(\,\cdot\,,z_1))_H \quad \text{for all} \quad z_1,z_2\in\Omega_1$$

or

$$(e_1(\,\cdot\,,z_2),T_1e_2(\,\cdot\,,z_1)-Te_2(\,\cdot\,,z_1))_H\equiv 0 \quad \text{for all} \quad z_1,z_2\in\Omega_1. \tag{2.18}$$

Since the system of functions $\{e_1(\,\cdot\,,\xi)\}_{\xi\in\Omega_1}$ is complete in the space H, it follows from (2.18) that

$$T_1e_2(\tau,z_1)-Te_2(\tau,z_1)=0 \quad \text{for all} \quad \tau\in\Omega, \quad z_1\in\Omega_1$$

or

$$T_1e_2(\tau,z_1)=Te_2(\tau,z_1) \quad \text{for all} \quad \tau\in\Omega, \quad z_1\in\Omega_1$$

and

$$T_1r(\tau)=Tr(\tau) \quad \text{for all} \quad \tau\in\Omega,$$

where $r(\tau)$ is an arbitrary function from the linear span of system $\{e_2(\tau,\xi)\}_{\xi\in\Omega_1}$. Since the system of functions $\{e_2(\,\cdot\,,\xi)\}_{\xi\in\Omega_1}$ is complete in the space H, and T_1, T are continuous operator, the latter identity implies that

$$T_1 f(\tau) = Tf(\tau) \quad \text{for all} \quad \tau\in\Omega, \quad f\in H,$$

that is, the invertible operator T_1 coincides with the operator T. The proof is complete. □

The question arises: whether there are analogues of Statements 1 and 2 in [3] formulated for the case of coincidence or equivalence of RKHSs? If $\widehat{H}\subset\widetilde{H}$, whether the consistence condition

$$(e_1(\,\cdot\,,z_1), e_2(\,\cdot\,,z_2))_H = \overline{(e_1(\,\cdot\,,z_2), Te_2(\,\cdot\,,z_1))_H} \quad \text{for all} \quad z_1,z_2\in\Omega_1$$

is true? In the next section we provide Example 1, which shows in the general case the answer to this question is negative. The condition $d\mu_1 \leqslant C\cdot d\mu_2$ means that if the set $P\subset\Omega_1$ is μ_2–measurable, then P is also μ_1–measurable and there exists a constant independent of the choice of the set $P\subset\Omega_1$ such that $\mu_1(P)\leqslant C\mu_2(P)$.

Theorem 2.4. *Let H be an RKHS consisting of the functions defined on the set Ω and $\{e_1(\,\cdot\,,\xi)\}_{\xi\in\Omega_1}$ be an orthosimilar expansion system in H with the measure μ_1, while $\{e_2(\,\cdot\,,\xi)\}_{\xi\in\Omega_1}$ is an orthosimilar expansion system in H with the measure μ_2. Let there exist a constant $C>0$ such that $d\mu_1\leqslant C\cdot d\mu_2$ and a linear continuous operator $T\colon H\to H$ such that*

$$(e_1(\,\cdot\,,z_1), e_2(\,\cdot\,,z_2))_H = \overline{(e_1(\,\cdot\,,z_2), Te_2(\,\cdot\,,z_1))_H} \quad \textit{for all} \quad z_1,\, z_2\in\Omega_1. \tag{2.19}$$

Then the space $\widehat{H}$ is embedded into the space $\widetilde{H}$.

Proof. Let T^* be the adjoint operator for T. The relation (2.19) implies

$$(e_1(\,\cdot\,,z_1), e_2(\,\cdot\,,z_2))_H = \overline{(T^*e_1(\,\cdot\,,z_2), e_2(\,\cdot\,,z_1))_H} \quad \text{for all} \quad z_1,\, z_2\in\Omega_1$$

or

$$(T^*e_1(\,\cdot\,,z_2), e_2(\,\cdot\,,z_1))_H = (e_2(\,\cdot\,,z_2), e_1(\,\cdot\,,z_1))_H \quad \text{for all} \quad z_1,\, z_2\in\Omega_1. \tag{2.20}$$

Let $\{z_k\}_{k=1}^m$ be an arbitrary set of the points in Ω_1 and $\{c_k\}_{k=1}^m$ be an arbitrary set of complex numbers and

$$p(t) \stackrel{def}{=} \sum_{k=1}^m c_k T^* e_1(t,z_k), \quad q(t) \stackrel{def}{=} \sum_{k=1}^m c_k e_2(t,z_k), \quad t\in\Omega.$$

Using the linearity of the scalar product in the first argument, by (2.20) we get the identity

$$(p, e_2(\,\cdot\,,\xi))_H = (q, e_1(\,\cdot\,,\xi))_H \quad \text{for all} \quad \xi\in\Omega_1. \tag{2.21}$$

Applying the analogue of the Parseval identity for orthosimilar expansion systems [8, Thm. 1] and taking into consideration the condition $d\mu_1\leqslant C\cdot d\mu_2$, by (2.21) we get the inequality

$$\begin{aligned}\|q\|_H^2 = \int\limits_{\Omega_1} |(q, e_1(\,\cdot\,,\xi))_H|^2\, d\mu_1(\xi) &= \int\limits_{\Omega_1} |(p, e_2(\,\cdot\,,\xi))_H|^2\, d\mu_1(\xi)\\ &\leqslant C\int\limits_{\Omega_1} |(p, e_2(\,\cdot\,,\xi))_H|^2\, d\mu_2(\xi) = C\|p\|_H^2.\end{aligned} \tag{2.22}$$

Let H_0 be the closure of the linear span of the system of functions $\{T^*e_1(\,\cdot\,,\xi)\}_{\xi\in\Omega_1}$ by the norm of the space H. Then H_0 is a closed subspace of the space H. We consider an operator B acting on the linear span of the system $\{T^*e_1(\,\cdot\,,\xi)\}_{\xi\in\Omega_1}$ by the rule

$$B: p\mapsto q.$$

Estimate (2.22) implies

$$\|Bp\|_H \leqslant C\|p\|_H \quad \text{for all} \quad p \in \operatorname{span}\{T^* e_1(\,\cdot\,,\xi)\}_{\xi\in\Omega_1}. \tag{2.23}$$

By the Banach theorem the operator B is continued to a linear continuous operator acting from H_0 into H. Thus,

$$\|Bh\|_H \leqslant C\|h\|_H \quad \text{for all} \quad h \in H_0. \tag{2.24}$$

The operator B possesses the property

$$B\colon T^* e_1(\,\cdot\,,\xi) \mapsto e_2(\,\cdot\,,\xi) \quad \text{for all} \quad \xi \in \Omega_1.$$

Then the operator $A \overset{def}{=} B \circ T^*$ is a linear continuous operator acting from H into H and possesses the property

$$A\colon e_1(\,\cdot\,,\xi) \mapsto e_2(\,\cdot\,,\xi) \quad \text{for all} \quad \xi \in \Omega_1.$$

By Theorem 2.1 the space $\widehat{H}$ is embedded into the space $\widetilde{H}$. The proof is complete. □

3. Examples

3.1. Example 1. We consider the Bergman space $B_2(D)$, which consists of the functions analytic in the unit disk $D = \{z \in \mathbb{C}\colon |z| < 1\}$. By l_2 we denote the space of sequences

$$l_2 = \{\mathbf{x} = \{x_k\}_{k\in\mathbb{N}_0}\colon \|\mathbf{x}\|_{l_2}^2 = \sum_{k=0}^{\infty} |x_k|^2 < \infty\},$$

where $\mathbb{N}_0 := \mathbb{N}\bigcup\{0\}$. It is known that the operator A defined on the space l_2 by the rule

$$A\colon \{x_0, x_1, x_2, \ldots, x_n, \ldots\} \mapsto \{0, x_0, x_1, x_2, \ldots, x_n, \ldots\},$$

acts in the space l_2 and is bounded (but not invertible). The image of the operator $\operatorname{Im} A$ is a closed subspace of the space l_2. The operator A generates a linear continuous operator $A_1\colon B_2(D) \to B_2(D)$, which acts by the rule: if

$$f(z) = \sum_{k=0}^{\infty} f_k \cdot \sqrt{k+1} \cdot z^k, \quad \{f_k\}_{k\in\mathbb{N}_0} \in l_2,$$

then

$$A_1 f(z) := \sum_{k=0}^{\infty} f_k \cdot \sqrt{k+2} \cdot z^{k+1}, \quad \{f_k\}_{k\in\mathbb{N}_0} \in l_2.$$

Since $\operatorname{Im} A$ is a closed subspace of the space l_2, then $\operatorname{Im} A_1$ is a closed subspace of the space $B_2(D)$. It is easy to see that $\operatorname{Im} A_1$ does not coincide with the space $B_2(D)$. We let

$$e_1(k,z) := \sqrt{k+1} \cdot z^k, \qquad k \in \mathbb{N}_0, \quad z \in D;$$
$$e_2(k,z) := \sqrt{k+2} \cdot z^{k+1}, \quad k \in \mathbb{N}_0, \quad z \in D.$$

Let $H = l_2$, then (see relations (1.1), (1.2)) $\Omega = \mathbb{N}_0$, $\Omega_1 = D$, $\widetilde{H} = B_2(D)$, $\widehat{H} = \operatorname{Im} A_1$. Suppose that there exists an operator $T\colon H \to H$ such that the consistence condition holds

$$(e_1(\,\cdot\,,z_1), e_2(\,\cdot\,,z_2))_H = \overline{(e_1(\,\cdot\,,z_2), Te_2(\,\cdot\,,z_1))_H} \quad \text{for all} \quad z_1, z_2 \in \Omega_1. \tag{3.1}$$

Then by Theorem 2.3 the spaces $B_2(D)$ and $\operatorname{Im} A_1$ are equivalent. However, we have just established that this is not true. Thus, we have provided an example, when the space $\widehat{H}$ is embedded into the space $\widetilde{H}$, but the consistence condition (3.1) for the systems of functions $\{e_j(\,\cdot\,,z)\}_{z\in\Omega_1}$, $j = 1, 2$, fails for any choice of the linear continuous operator T.

3.2. Example 2. We provide an example illustrating Theorem 2.4. Let $\alpha > -1$. We consider the weighted Bergman space

$$B_2(D,\alpha) := \left\{ f \in H(D)\colon \|f\|^2_{B_2(D,\alpha)} = \int\limits_D |f(z)|^2 (1-|z|)^\alpha \, dv(z) \right\},$$

where $dv(z)$ is the planar Lebesgue measure, $B_2(D,\alpha)$ is an RKHS with the scalar product

$$(f,g)_{B_2(D,\alpha)} = \int\limits_D f(z)\cdot \overline{g(z)} (1-|z|)^\alpha \, dv(z) \quad \text{for all} \quad f,g \in B_2(D,\alpha).$$

It is known that $\{z^k\}_{k=0}^\infty$, $z \in D$, forms an orthogonal basis in the space $B_2(D,\alpha)$ (see, for instance, [11]).

Then the system of functions $\{z^k/\|z^k\|_{B_2(D,\alpha)}\}_{k=0}^\infty$, $z \in D$, is an orthonormal basis in the space $B_2(D,\alpha)$. It is easy to see that if $\alpha \geqslant 0$, then $B_2(D) \subset B_2(D,\alpha)$, while if $-1 < \alpha \leqslant 0$, then $B_2(D,\alpha) \subset B_2(D)$. Indeed, it is easy to verify the inequalities

$$\|f\|^2_{B_2(D,\alpha)} = \int\limits_D |f(z)|^2 (1-|z|)^\alpha \, dv(z) \leqslant \int\limits_D |f(z)|^2 \, dv(z) = \|f\|^2_{B_2(D)}, \quad \alpha \geqslant 0 \quad \text{for all} \quad f \in B_2(D);$$

$$\|f\|^2_{B_2(D,\alpha)} = \int\limits_D |f(z)|^2 (1-|z|)^\alpha \, dv(z) \geqslant \int\limits_D |f(z)|^2 \, dv(z) = \|f\|^2_{B_2(D)}, \quad -1 < \alpha \leqslant 0 \quad \text{for all} \quad f \in B_2(D).$$

We also have

$$(1-|z|)^\alpha \, dv(z) \leqslant dv(z), \quad \alpha \geqslant 0;$$
$$(1-|z|)^\alpha \, dv(z) \geqslant dv(z), \quad -1 < \alpha \leqslant 0.$$

In our notation

$$H = l_2, \quad \{e_1(\,\cdot\,,\xi)\}_{\xi \in D} := \left\{ \frac{\xi^k}{\|\xi^k\|^2_{B_2(D,\alpha)}} \right\}_{\xi \in D}, \quad k \in \mathbb{N}_0;$$
$$\{e_2(\,\cdot\,,\xi)\}_{\xi \in D} := \left\{ \frac{\xi^k}{\|\xi^k\|^2_{B_2(D)}} \right\}_{\xi \in D}, \quad k \in \mathbb{N}_0.$$

We also get $\widehat{H} = B_2(D)$, $\widetilde{H} = B_2(D,\alpha)$.

The system of functions $\{e_1(\,\cdot\,,\xi)\}_{\xi \in D}$ is an orthosimilar expansion system with the measure $d\mu_1(z) := (1-|z|)^\alpha \, dv(z)$ in the space l_2. The system of functions $\{e_2(\,\cdot\,,\xi)\}_{\xi \in D}$ is an orthosimilar expansion system with the measure $d\mu_2(z) := dv(z)$ in the space l_2. It is easy to verify that the systems $\{e_j(\,\cdot\,,\xi)\}_{\xi \in D}$, $j = 1,2$, satisfy the consistence condition

$$(e_1(\,\cdot\,,z_1), e_2(\,\cdot\,,z_2))_{l_2} = \overline{(e_1(\,\cdot\,,z_2), id[e_2(\,\cdot\,,z_1)])}_{l_2} \quad \text{for all} \quad z_1,\, z_2 \in D,$$

where id stands for the identity mapping. If $\alpha \geqslant 0$, then $d\mu_1 \leqslant d\mu_2$; all assumptions of Theorem 2.4 are satisfied and $\widehat{H} \subset \widetilde{H}$. If $-1 < \alpha \leqslant 0$, then $d\mu_1 \geqslant d\mu_2$; all assumptions of Theorem 2.4 are satisfied and $\widehat{H} \supset \widetilde{H}$.

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