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DUAL CONSTRUCTION AND EXISTENCE OF (PLURI)SUBHARMONIC MINORANT

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Abstract. We study the existence and construction of subharmonic or plurisubharmonic function enveloping from below a function on a subset in finite–dimensional real or complex space. These problems naturally arise in theories of uniform algebras, potential and complex potential, which was reflected in works by D.A. Edwards, T.V. Gamelin, E.A. Poletsky, S. Bu and W. Schachermayer, B.J. Cole and T.J. Ransford, F. Lárusson and S. Sigurdsson and many others. In works in 1990s and recently we showed that these problems play a key role in studying nontriviality of weighted spaces of holomorphic functions, in description of zero sets and subsets of functions from such spaces, in representations of meromorphic functions as a quotient of holomorphic functions with growth restrictions, in studying the approximation by exponential systems in functional spaces, etc. The main results of the paper on existence of subharmonic or plurisubharmonic function–minorant are derived from our general theoretical functional scheme, which allows us to provide a dual definition of the lower envelope with respect to a convex cone in the projective limit of vector lattices. We develop this scheme during last years and it is based on an abstract form of balayage. The ideology of the abstract balayage goes back to H. Poincaré and M.V. Keldysh in the framework of balayage of measures and subharmonic functions in the potential theory. It is widely used in the probability theory, for instance, in the known monograph by P. Meyer, and it is also reflected, often implicitly, in monographs by G.P. Akilov, S.S. Kutateladze, A.M. Rubinov and others related with the theory of ordered vector spaces and lattices. Our paper is adapted for convex subcones of the cone of all subharmonic or plurisubharmonic functions. This allows us to obtain new criterion for existence of a subharmonic or plurisubharmonic minorant for functions on a domain.

Keywords: subharmonic function, plurisubharmonic function, lower envelope, vector lattice, projective limit, balayage.

Mathematics Subject Classification: 31B05, 31C10, 46A40, 31B15

1. Introduction. Formulation and discussion of results

1.1. Origination and formulation of problem. Let H be some class consisting of subharmonic or plurisubharmonic functions on a domain D of a finite–dimensional real or complex Euclidean space. The main considered problem is under which relations between H and an extended real function f on D with values in the extension $\mathbb{R}\cup\{\pm\infty\}$ of the field of real numbers $\mathbb{R}$ there exists a function $h\in H$ such that $-\infty\neq h\leqslant f$ on D? A natural condition for H is its convexity. Here we consider the case, when H is a convex cone. The dual solution of the problem when H are cones of *all* (pluri)subharmonic functions on the domain D naturally follows

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from the dual description of the lower (pluri)subharmonic envelope for extended real functions on D provided under certain restrictions for the function f in works by Bu and Schachermayer [1], by Poletsky [2], Cole and Ransford [3], as well as our works [4]–[13] and many others published in 1990–2020. The interest to such problems is motivated by their numerous applications in the theories of uniform algebras, (pluri)potential [2], in issues on nontriviality of weighted spaces of holomorphic functions, description of distribution of zero sets and uniqueness sets for such spaces, representation of meromorphic functions as a quotient of the functions from these spaces [5]–[7], [9]–[12]. In the paper we present a further development of our approach to problems of such kind. This approach is based on the general dual description of the envelopes for the vectors in the projective limits of vector lattices and the notion of balayage.

In [10], [11] a rather detailed formulation on the considered problems was provided as well as their history with a wide bibliography up to recent years including the applications to the theory of functions and approximations. We note once again that other authors working in this field considered only envelopes of particular cones of *all* (pluri)subharmonic functions. In our paper the convex subcones can be rather general and they are essentially narrower than the cones of all (pluri)subharmonic functions. Moreover, the function f was supposed to be locally bounded from above, for instance, a semi–continuous from above on D. This did not allow to cover all cases of the functions f from the difference $H-H$, while exactly this case is of the highest interest due to the applications proposed in [6]–[11]. These restrictions were recently omitted by the third author in the paper [13], but only for the case of the convex cone H of *all* subharmonic functions on the domain. Our paper can be regarded as an essential development of this result [13, Thm. 1], which extends it to wide classes of convex subcones in the cone of all subharmonic functions on the domain and extended real functions f on D. Thus, the main advances in the present paper in comparison with previous ones are, first, the dual construction of the minorant or lower envelope from the wide class of various convex subcones of the cone of subharmonic or plurisubharmonic functions on the domain D, and second, the omitting of the condition of local boundedness from above for the function f, for which the minorant or lower envelope from the convex cone H is constructed. We proceed to rigorous definitions and formulations.

1.2. Definitions, notation, conventions. The sets $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{R}$, $\mathbb{C}$ are respectively ones of *natural, real and complex numbers*, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is the *extended real line*, where $-\infty := \inf \mathbb{R} = \sup \emptyset$, $+\infty := \sup \mathbb{R} = \inf \emptyset$ for the *empty set* $\emptyset$. These sets are considered with their natural algebraic, geometric and topological structures. The Euclidean space $\mathbb{R}^d$ of a dimension $d \in \mathbb{N}$ is considered with the *Euclidean norm* $|x| := \sqrt{x_1^2 + \cdots + x_d^2}$ for $(x_1, \dots, x_d) \in \mathbb{R}^d$ and with the *d–dimensional Lebesgue measure* $\mathfrak{m}_d$. As usually, $\mathbb{C}$ is the field of all *complex* number (complex plane). In the present paper it is convenient to identify the d–dimensional complex space $\mathbb{C}^d = \mathbb{R}^d + i\mathbb{R}^d$ with the $2d$–dimensional space $\mathbb{R}^d \times \mathbb{R}^d = \mathbb{R}^{2d}$ with the Lebesgue measure $\mathfrak{m}_{2d}$.

For a pair of *extended real functions* $f \colon X \to \overline{\mathbb{R}}$ and $g \colon X \to \overline{\mathbb{R}}$ we write $f \leqslant g$ *on* D if $f(x) \leqslant g(x)$ for each point $x \in X$.

By $C(X)$ we denote the *vector space over* $\mathbb{R}$ *of continuous functions* on a topological space X with values in $\mathbb{R}$.

Hereafter by the symbol $D \subset \mathbb{R}^d$ we denote a *domain*, that is, a connected open subset in $\mathbb{R}^d$, and $\overline{B}_o(r) := \bigl\{x \in \mathbb{R}^d \bigm| |x - o| \leqslant r\bigr\}$ is the *ball of radius* $r > 0$ *centered at* $o \in \mathbb{R}^d$.

A subset H of a vector space over the field $\mathbb{R}$ is called *cone* if

$$tH := \bigl\{th \bigm| h \in H\bigr\} \subset H \quad \text{for all} \quad 0 < t \in \mathbb{R}.$$

If the cone H contains a zero vector, that is, $tH \subset H$ for all $0 \leqslant t \in \mathbb{R}$, then H is a *cone with the vertex at zero.* The cone H is *convex* if H is a convex set. Thus, H is a *convex cone with the vertex at zero* if and only if

$$tH + tH := \bigl\{th_1 + th_2 \bigm| h_1 \in H, h_2 \in H\bigr\} \subset H \quad \text{for all} \quad 0 \leqslant t \in \mathbb{R}.$$

By $\mathsf{Meas}_0^+(D)$ we denote a *convex cone with the vertex at zero of all positive Borel measures with compact supports in* D, $\mathsf{sbh}(D)$ is a *convex cone with the vertex at zero of all subharmonic on* D *functions,* which includes the function equalling identically to $-\infty$ on D. While considering a domain $D \subset \mathbb{C}^d$, by $\mathsf{psbh}(D) \subset \mathsf{sbh}(D)$ we denote a *convex cone with the vertex at zero of all (pluri)subharmonic functions on* D*.* All facts about subharmonic and (pluri)subharmonic functions can be found in [14], [15].

As in monograph [16], if the integral of the function with respect to the measure μ exists and takes value in $\overline{\mathbb{R}}$, this function is called integrable with respect to the measure μ, or μ–*integrable,* and if this integral is finite, that is, takes a value in $\mathbb{R}$, then this function is called integrable with respect to the measure μ, or μ–*measurable.* Thus, an extended real function $f\colon D \to \overline{\mathbb{R}}$ is *locally integrable over* D *with respect to the measure* μ, or *locally* μ–*integrable on* D if the integral

$$\int\limits_K f \,\mathrm{d}\mu \in \overline{\mathbb{R}} \quad \text{for each compact set} \quad K \subset D, \tag{1.1}$$

is well–defined. If all integrals in (1.1) are finite, that is, they take values in $\mathbb{R}$, then the function f is *locally summable over* D *with respect to the measure* μ, or *locally* μ–*summable over* D.

Below, the notion of summability or integrability of integrals, as well as of the identities $\overset{\text{a.e.}}{=}$ and inequalities $\leqslant^{\text{a.e.}}$ almost everywhere (a.e.) *without indicating the measure* concerns exactly the measure $\mathfrak{m}_d$.

Each *constant* $c \in \overline{\mathbb{R}}$ is often treated as a *function identically equal to* the quantity c. Thus, for a function $u\colon D \to \overline{\mathbb{R}}$ the writing $u \neq -\infty$ means that the function u is not identically $-\infty$ on D. By

$$\begin{aligned} \mathsf{sbh}_*(D) &:= \bigl\{u \in \mathsf{sbh}(D) \bigm| u \neq -\infty\bigr\}, \\ \mathsf{psbh}_*(D) &:= \bigl\{u \in \mathsf{psbh}(D) \bigm| u \neq -\infty\bigr\} \subset \mathsf{sbh}_*(D) \end{aligned} \tag{1.2}$$

we denote convex cones with the vertex at zero of respectively subharmonic $D \subset \mathbb{R}^d$ and (pluri)subharmonic on $D \subset \mathbb{C}^d$ functions not identically equalling to $-\infty$. Each function $u \in \mathsf{sbh}_*(D)$ is locally summable on D.

For an extended real function $f\colon D \to \overline{\mathbb{R}}$ its *upper–semi–continuous regularization* $f^*\colon D \to \overline{\mathbb{R}}$ is defined as $f^*(x) := \limsup\limits_{x' \to x} f(x')$ at each point $x \in D$. The function $f\colon D \to \overline{\mathbb{R}}$ is *locally bounded from above on* D if $\sup\limits_{x \in K} f(x) < +\infty$ for each compact set $K \subset D$.

Our work is based on functional analytic results from [10] and [11], where also the history of the issue with a wide bibliography were provided; we do not dwell on this here. Then these results are applied for the dual description of the *lower envelope* $x \underset{x \in D}{\longmapsto} \sup\bigl\{h(x) \bigm| H \ni h \leqslant f\bigr\}$ of the functions $f\colon D \to \overline{\mathbb{R}}$ with respect to convex subcones $H \subset \mathsf{sbh}(D)$ as well as for the dual description of the conditions, under which for a pair of functions $v \in H_* \overset{(1.2)}{:=} H \setminus \{-\infty\}$ and $M \in H_*$ and for a continuous function $m \in C(D)$ there exists a function $h \in H_*$ such that $v + h \leqslant M + m$ on D. Our main results can be treated as solutions in partial cases of formulated in [10, Sect. 2.3, Prbs. 1–3], [11, Sect. 1.2, Subsect. 1.2.3, Prbs. 1–3] general problems on the existence of the envelope from the convex cones or sets.

Theorem 1.1. *Let a convex cone $H \subset \mathsf{sbh}(D)$ with a vertex at zero contains a constant -1 and for each locally bounded from above on D sequence $(h_k)_{k\in\mathbb{N}}$ of functions $h_k \in H$ the upper–semi–continuous regularization h^* of the pointwise upper limit*

$$h\colon x \underset{x\in D}{\longmapsto} \limsup_{k\to\infty} h_k(x) := \inf_{n\in\mathbb{N}} \sup_{k\geq n} h_k(x) \tag{1.3}$$

of this sequence $(h_k)_{k\in\mathbb{N}}$ belong to this cone H.

Then for function f defined a.e. on D and equalling a.e. to some function $C(D)+H-H$ also defined a.e. the identity

$$\sup\left\{ \int\limits_{\overline{B}_o(r)} h \,\mathrm{d}\mathfrak{m}_d \;\middle|\; -\infty \neq h \in H, h \leqslant^{a.e.} f \text{ on } D \right\} = \inf_{\mu\in \mathsf{J}_o^r(D;H)} \int\limits_D f\,\mathrm{d}\mu, \tag{1.4}$$

where

$$\mathsf{J}_o^r(D;H) := \left\{ \mu \in \mathsf{Meas}_0^+(D) \;\middle|\; \int\limits_{\overline{B}_o(r)} h\,\mathrm{d}\mathfrak{m}_d \leqslant \int\limits_D h\,\mathrm{d}\mu \text{ for all } h\in H \right\} \tag{1.5}$$

is the class of all linear balayages [8]–[12] of the restriction of $\mathfrak{m}_d$ to $\overline{B_o}(r)$ with respect to H, holds for each choice of the closed ball $\overline{B}_o(r) \subset D$.

Corollary 1.1. *Under the assumptions of Theorem 1.1 the following three statements are equivalent:*

1) *there exists a function $h \in H$, for which $-\infty \neq h \leqslant^{a.e.} f$ on D;*
2) *for each closed ball $\overline{B}_o(r) \subset D$*
   (i) *the infimum* inf *in* (1.4) *over all measures $\mu \in \mathsf{J}_o^r(D;H)$ is not equal to $-\infty$;*
3) *there exists a closed ball $\overline{B}_o(r) \subset D$, for which Statement (i) is true.*

The examples of cones H obeying the assumptions of Theorem 1.1 are convex cones $\mathsf{sbh}(D)$ for $D \subset \mathbb{R}^d$ and $\mathsf{psbh}(D) \subset \mathsf{sbh}(D)$ for $D \subset \mathbb{C}^d$. The most important in Theorem 1.1 is a wide possibility to choose f in $H-H+C(D)$. Earlier the case $f \in C(D)$ was completely studied in [10, Cor. 8.1], [11, Cor. 3.2.1], [9, Thm. 7.2]. For $H := \mathsf{psbh}(D)$ in *identities of the form* (1.4) in [1]–[3] the function f was always supposed to be locally bounded from above on D. But for a very important for further applications option $f \in H-H$ the local boundedness from above is likely not obeying since the functions from $-H$ can be unbounded from above even on each non–empty open subset in D.

Let us provide one more corollary for the cone $\mathsf{psbh}(D)$, which, generally speaking, can not be obtained from the main results of [1]–[3] and others. In the case of the convex cone $H = \mathsf{sbh}(D)$ of *all* subharmonic functions on the domain $D \subset \mathbb{R}^d$, a similar statement is the main result of our recent paper [13, Thm. 1]. In particular, for the dimension $d = 2$ while identifying the complex plane $\mathbb{C}$ with $\mathbb{R}^2$ below given Corollary 1.2 and [13, Thm. 1] almost coincide.

Corollary 1.2. *For plurisubharmonic functions $v \neq -\infty$ and $M \neq -\infty$ on the domain $D \subset \mathbb{C}^d$ and a continuous function $m\colon D \to \mathbb{R}$ the following three statements are equivalent:*

1) *there exists a plurisubharmonic on D function $h \neq -\infty$, for which*

$$v(z) + h(z) \leqslant M(z) + m(z) \quad \text{at each point} \quad z \in D; \tag{1.6}$$

2) *for each closed ball $\overline{B}_o(r) \subset D$ there exists a number $C \in \mathbb{R}$ such that*

$$\int\limits_D v\,\mathrm{d}\mu \leqslant \int\limits_D (M+m)\,\mathrm{d}\mu + C \quad \text{for all} \quad \mu \in \mathsf{J}_o^r\bigl(D; \mathsf{psbh}(D)\bigr); \tag{1.7}$$

3) *there exists a closed ball $\overline{B}_o(r) \subset D$ and $C \in \mathbb{R}$, for which* (1.7) *holds.*

2. Envelopes in projective limits of vector lattices

The arguing in this section will serve as a base for the proof of Theorem 1.1.

An ordered vector space $(X, \leqslant)$ over $\mathbb{R}$ with an order relation is called *vector lattice*, if for each *finite* $F \subset X$ there exists a *supremum* in X denoted in what follows as $X\text{-}\sup F \in X$ [18], [19].

The set of all functions $f\colon X \to Y$ defined on entire X with values in Y is denoted by Y^X. For a pair of vector lattices X and Y by $\mathsf{lin}^+ Y^X$ we denote a convex cone with the vertex at zero of *linear positive functions* $l\colon X \to Y$.

Let $(X_n)_{n\in\mathbb{N}_0}$ be a sequence of *vector lattices* X_n with order relations $\leqslant_n$, respectively, that is, the sequence of all pairs $(X_n, \leqslant_n)$, $n \in \mathbb{N}_0$. By this sequence $(X_n, \leqslant_n)$ we can construct the product

$$\prod X_n := \prod_{n=0}^{\infty} X_n, \tag{2.1}$$

in which for $x = (x_n)_{n\in\mathbb{N}_0} \in \prod X_n$ we let $\mathsf{pr}_n x = x_n \in X_n$ to be the *projection* of the vector $x \in \prod X_n$ onto the space X_n. On the product (2.1) we can introduce the order relation $\leqslant$, for which by definition $x \leqslant x'$ in $\prod X_n$ if $\mathsf{pr}_n x \leqslant_n \mathsf{pr}_n x'$ for each $n \in \mathbb{N}_0$.

Let $(p_n)_{n\in\mathbb{N}_0}$ be a sequence of linear positive functions $p_n \in \mathsf{lin}^+ X_n^{X_{n+1}}$ from X_{n+1} into X_n, $n \in \mathbb{N}_0$, for which we suppose the *preserving of the supremum for finite subsets*, namely,

$$X_n\text{-}\sup p_n(F_{n+1}) = p_n\bigl(X_{n+1}\text{-}\sup F_{n+1}\bigr) \quad \text{for each finite} \quad F_{n+1} \subset X_{n+1}.$$

Then the following subspace in the product (2.1) denoted by

$$X := \operatorname{proj}\lim X_n p_n := \Bigl\{ x \in \prod X_n \Bigm| \mathsf{pr}_n x = p_n(\mathsf{pr}_{n+1} x) \quad \text{for all} \quad n \in \mathbb{N}_0 \Bigr\},$$

with the same order relation $\leqslant$ as on $\prod X_n$ is the vector lattice called *projective limit of the sequence* $(X_n)_{n\in\mathbb{N}_0}$ *of vector lattices over* $(p_n)_{n\in\mathbb{N}_0}$. Without loss of generality we can suppose [10, Prop. 3.1], [11, Prop. 2.1.1] that

$$\mathsf{pr}_n X := \bigl\{ \mathsf{pr}_n x \bigm| x \in X \bigr\} = X_n \quad \text{for each} \quad n \in \mathbb{N}_0,$$

that is, the projections pr_n in the projective limit $X = \operatorname{proj}\lim X_n p_n$ onto X_n are *surjective*.

A subset $B \subset X$ *is bounded from below (above) in* X if there exists a vector x in X, for which $x \leqslant b$ (respectively, $b \leqslant x$) for all $b \in B$. A subset $B \subset X$ *is bounded in* X if B is bounded from above and below in X.

Theorem 2.1 ([10, Thm. 2, Cors. 6.1, 3.1], [11, Thm. 2.4.1, Cors. 2.4.1, 2.1.1])**.** *Let $H_* \subset X := \operatorname{proj}\lim X_n p_n$ be a convex cone with the vertex at zero, and for each bounded in X sequence $\bigl(h^{(k)}\bigr)_{k\in\mathbb{N}}$ of vectors $h^{(k)} \in H_*$ there exists the upper limit*

$$\limsup_{k\to\infty} h^{(k)} := \inf_{n\in\mathbb{N}} \sup_{k\geqslant n} h^{(k)} \in H_*. \tag{2.2}$$

Let $S \subset X$ be a vector subspace containing a cone H_* and for each $n \in \mathbb{N}_0$ and each $s_n \in \mathsf{pr}_n S$ there exists a vector $h_n \in \mathsf{pr}_n H_*$ such that $h_n \leqslant_n s_n$.

Let $q_0 \in \mathsf{lin}^+ \mathbb{R}^{X_0}$ be a linear positive function on X_0. Suppose that for the superposition

$$q := q_0 \circ \mathsf{pr}_0 \in \mathsf{lin}^+ \mathbb{R}^X \tag{2.3}$$

for each decaying in X sequence $(h^{(k)})_{k\in\mathbb{N}}$ of vectors $h^{(k)} \in H_*$, under the finitness of the infimum

$$\inf_{k\in\mathbb{N}} q(h^{(k)}) \overset{(2.3)}{=} \inf_{k\in\mathbb{N}} q_0\bigl(\mathsf{pr}_0 h^{(k)}\bigr) \in \mathbb{R}, \tag{2.4}$$

this sequence $(h^{(k)})_{k\in\mathbb{N}}$ is bounded from below in X and

$$q\bigl(\inf_{k\in\mathbb{N}} h^{(k)}\bigr) \geqslant \inf_{k\in\mathbb{N}} q(h^{(k)}). \tag{2.5}$$

Then for each vector $s\in S$ the quantity

$$\sup\bigl\{q(h) \bigm| H_* \ni h \leqslant s\bigr\} \in \overline{\mathbb{R}} \tag{2.6}$$

is equal to

$$\inf\Bigl\{(l_n\circ \mathsf{pr}_n)(s) \Bigm| n\in\mathbb{N}_0, l_n\in \mathsf{lin}^+\mathbb{R}^{\mathsf{pr}_n S}, q(h)\leqslant (l_n\circ\mathsf{pr}_n)(h) \quad \textit{for all} \quad h\in H_*\Bigr\}\in\overline{\mathbb{R}}. \tag{2.7}$$

3. Proofs of main results

Proof of Theorem 1.1. For a domain $D\subset\mathbb{R}^d$ we choose the exhaustion by a sequence $(D_n)_{n\in\mathbb{N}_0}$ of domains $D_n\subset\mathbb{R}^d$, for which $\overline{B}_o(r)\subset D_0$, the *closure* $\mathsf{clos}\,D_n$ of the domain D_n is contained in the domain D_{n+1} for each $n\in\mathbb{N}_0$ and $D=\bigcup_{n\in\mathbb{N}_0} D_n$. For $n\in\mathbb{N}_0$ we consider the space $X_n:=L^1(\mathsf{clos}\,D_n)$ of summable on $\mathsf{clos}\,D_n$ functions with the pointwise preorder relation $\leqslant_n^{\text{a.e.}}$, the factorization of which with respect to $\overset{\text{a.e.}}{=}$ is denoted by X_n, where $\leqslant_n^{\text{a.e.}}$ is the order relation. As linear positive functions $p_n\in\mathsf{lin}^+X_n^{X_{n+1}}$ we choose *restrictions of functions from* X_{n+1} *onto* $\mathsf{clos}\,D_{n+1}$, which become vectors in X_n. In this case the projective limit $\operatorname{proj\,lim} X_n p_n$ is the space of locally summable on D functions factorized with respect to the relation $\overset{\text{a.e.}}{=}$ with the order relation $\leqslant^{\text{a.e.}}$; this space is denoted by $L^1_{\text{loc}}(D)$. Removing the function equalling identically $-\infty$ from the convex cone $H\subset\mathsf{sbh}(D)$, we let

$$H_*:=H\setminus\{-\infty\}\overset{(1.2)}{\subset}\mathsf{sbh}_*(D)\overset{(1.2)}{:=}\mathsf{sbh}(D)\setminus\{-\infty\}\subset L^1_{\text{loc}}(D).$$

The upper–semi–continuous regularization of the upper limit of a sequence of plurisubharmonic functions on the domain, if this upper limit is not equal to $-\infty$, gives, on one hand, a subharmonic function, and on the other hand differs from the upper limit at most on a set of zero $\mathfrak{m}_d$–measure, and even on a smaller polar one. This is why the cone H_* obeys the condition completed by the relation (2.2).

We let $S:=C(D)+H_*-H_*\subset L^1_{\text{loc}}(D)$. It is obvious that $H_*\subset S$. Let $s_n\in S$, that is, $s_n=g_n+h_n-h_n'$, where $g_n\in C(\mathsf{clos}\,D_n)$, and $h_n\in\mathsf{pr}_n H_*$ and $h_n'\in\mathsf{pr}_n H_*$ are the restrictions to $\mathsf{clos}\,D_n$ of functions from H_*. Then there exist *positive numbers* c and c', for which $g_n\geqslant -c$ on $\mathsf{clos}\,D_n$ and $h_n'\leqslant_n c'$ on $\mathsf{clos}\,D_n$. Therefore, $h_n-c-c'\leqslant_n s_n$, where $h_n\in\mathsf{pr}_n H_*$, and the negative constant $-c-c'=(c+c')(-1)$ belongs to $\mathsf{pr}_n H_*$ since by the condition we have $-1\in H_*$. Thus, the assumptions of the theorem concerning the subspace S are satisfied.

As q_0 in Theorem 2.1 we consider a restriction of the measure $\mathfrak{m}_d$ to $\overline{B}_o(r)$ in the sense that

$$q_0(f_0):=\int\limits_{\overline{B}_o(r)} f_0\,\mathrm{d}\mathfrak{m}_d\in\mathbb{R}\quad\text{for all}\quad f_0\in X_0=\mathsf{pr}_0 L^1_{\text{loc}}(D). \tag{3.1}$$

The function $u\colon D\to\overline{\mathbb{R}}$ is *almost subharmonic* on D if it coincides a.e. with a subharmonic function [20]. For an *arbitrary a.e. decaying sequence* $(h^{(k)})_{k\in\mathbb{N}}$ of almost subharmonic on D functions $h^{(k)}$ the condition $\inf\limits_{k\in\mathbb{N}} q(h^{(k)})\in\mathbb{R}$ means that

$$\inf_{k\in\mathbb{N}}\int\limits_{\overline{B}_o(r)} h^{(k)}\,\mathrm{d}\mathfrak{m}_d>-\infty. \tag{3.2}$$

Hence, the limit of this sequence is an almost subharmonic function on D. Then this is true for the decaying sequence $(h^{(k)})_{k\in\mathbb{N}}$ from the cone H_* contained in $\subset\mathsf{sbh}(D)\setminus\{-\infty\}$. The upper limit (2.2) of the decaying sequence is the infimum of this sequence. This is why by the assumptions of Theorem 1.1 for the sequence $(h^{(k)})_{k\in\mathbb{N}}$ under the conditon (3.2) we obtain $-\infty \neq \inf\limits_{k\in\mathbb{N}} h^{(k)} \in H$. In particular, under (3.2) the decaying sequence $(h^{(k)})_{k\in\mathbb{N}}$ in H_*, which is obviously bounded from above by the function $h^{(1)}$, is also bounded from below by the function $\inf\limits_{k\in\mathbb{N}} h^{(k)} \in H_*$. At the same time we can suppose that all the functions $h^{(k)}$ are upper–semi–continuous. For a decaying sequence of such functions, $\inf\limits_{k\in\mathbb{N}}$ can be moved under the integral

$$\inf_{k\in\mathbb{N}} \int\limits_{\overline{B}_o(r)} h^{(k)}\,\mathrm{d}\mathfrak{m}_d = \int\limits_{\overline{B}_o(r)} \inf_{k\in\mathbb{N}} h^{(k)}\,\mathrm{d}\mathfrak{m}_d.$$

This means that the needed in (2.4), (2.5) inequality

$$q\bigl(\inf_{k\in\mathbb{N}} h^{(k)}\bigr) \geqslant \inf_{k\in\mathbb{N}} q(h^{(k)}) \quad \text{for} \quad q := q_0 \circ \mathsf{pr}_0$$

holds.

Thus, the assumptions of Theorem 1.1 imply ones of Theorem 2.1. It is easy to see that

$$\sup\bigl\{q(h) \bigm| H_* \ni h \leqslant s\bigr\} \in \overline{\mathbb{R}}$$

is exactly the left hand side of the identity (1.4).

Now we are going to verify that in the considered situation (2.7) is the right hand side in (1.4).

As in (2.7), we let $l_n \in \mathsf{lin}^+\mathbb{R}^{\mathsf{pr}_n S}$, where $S = C(D) + H_* - H_*$. Then

$$l_n \in \mathsf{lin}^+\mathbb{R}^{\mathsf{pr}_n C(D)} = \mathsf{lin}^+\mathbb{R}^{C(\mathsf{clos}\, D_n)},$$

which by the Riesz theorem implies that a linear positive function l_n on $C(\mathsf{clos}\, D_n)$ is realized as some positive finite Borel measure μ on D with a compact support in $\mathsf{clos}\, D_n$, and it is uniquely continued to all upper–semi–continuous functions on $\mathsf{clos}\, D_n$ with possible values on $\overline{\mathbb{R}}$. In particular, the measure μ is uniquely continued also to subharmonic functions in H_* due to the upper–semi–continuity. Thus, the condition $q(h) \leqslant (l_n \circ \mathsf{pr}_n)(h)$ for all $h \in H_*$ in (2.7) for $q = q_0 \circ \mathsf{pr}_0$ in accordance with (3.1) can be written in terms of the measure μ as

$$\int\limits_D h\,\mathrm{d}\mathfrak{m}_d \leqslant \int\limits_D h\,\mathrm{d}\mu \quad \text{for all} \quad h \in H_*.$$

The latter implies the finiteness of the integrals

$$\int\limits_D h\,\mathrm{d}\mu \in \mathbb{R} \quad \text{for all} \quad h \in H_*.$$

Therefore, the obtained in this way measures $\mu \in \mathsf{Meas}_0^+(D)$ are well–defined on $S = C(D) + H_* - H_*$ and run exactly over $J_o^r(D; H_*) = J_o^r(D; H_*)$ in (1.5) since the removing of the constant $-\infty$ from H in (1.4)–(1.5) changes nothing. Thus, the identity (1.4) is established and this completes the proof of Theorem 1.1. □

Proof of Corollary 1.1. Statement 1) is equivalent to the fact that the left hand side of (1.4) is not equal to $-\infty$ for each ball $\overline{B}_o(r) \subset D$. By the identity (1.4) in Theorem 1.1 this equivalent to the fact the right hand side of the identity (1.4) is not equal to $-\infty$ for such choices of the ball $\overline{B}_o(r) \subset D$. This yields the equivalence of Statement 1)–3) of Corollary 1.1 and completes the proof. □

Proof of Corollary 1.2. As it has already been mentioned, the convex cone $H := \mathsf{psbh}(D)$ with the vertex at zero obeys the assumptions of Theorem 1.1, and hence, of Corollary 1.1. Under such choice

$$f \overset{(1.2)}{:=} m + M - v \in C(D) + \mathsf{psbh}_*(D) - \mathsf{psbh}_*(D) \tag{3.3}$$

by Corollary 1.1 we obtain the equivalence of Statements 1)–3) of Corollary 1.1. According to the choice (3.3) of the function f this can be written as the equivalence of the following three statements

a) there exists a function $h \in H$, for which $-\infty \neq h \overset{\text{a.e.}}{\leqslant} m + M - v$ on D;

b) for each closed ball $\overline{B}_o(r) \subset D$ the relation

$$\inf_{\mu \in J_o^r(D;\mathsf{psbh}(D))} \int\limits_D (m + M - v)\, \mathrm{d}\mu > -\infty \tag{3.4}$$

holds;

c) there exists a closed ball $\overline{B}_o(r) \subset D$, for which (3.4) holds.

Here (3.4) is exactly the same as in (1.7). This is why to complete the proof of the corollary, it is sufficient to establish that the inequality $v + h \overset{\text{a.e.}}{\leqslant} m + M$ on D in Statement a) yields a stronger inequality (1.6) in Statement 1) of Corollary 1.2. In order to do this, we denote by

$$v^{\bullet r}(z) := \frac{1}{\mathfrak{m}_{2d}(\overline{B}_z(r))} \int\limits_{\overline{B}_z(r)} v \, \mathrm{d}\mathfrak{m}_{2d}$$

the integral means of the function v over the balls $\overline{B}_z(r) \subset D$. By the subharmonicity of plurisubharmonic functions in the inequality $v + h \overset{\text{a.e.}}{\leqslant} m + M$ on D we obtain

$$v(z) + h(z) \leqslant v^{\bullet r}(z) + h^{\bullet r}(z) \leqslant m^{\bullet r}(z) + M^{\bullet r}(z) \quad \text{for each} \quad z \in D$$

for all sufficiently small $r > 0$. Each point in the domain D is a Lebesgue point for the functions m and M, and hence, letting r to tend to zero in the right hand side, we obtain (1.6) everywhere on the domain D. This completes the proof. □

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