# ON LEVEL SETS OF NORM OF GENERALIZED RESOLVENT OF OPERATORS PENCILS

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**Abstract.** We prove that the generalized resolvent operator defined in a Hilbert space cannot remain constant on any open subset of the resolvent set. Under certain conditions we also prove the same result for a complex uniformly convex Banach space. These results extend the known ones.

**Keywords:** $\varepsilon$–pseudospectrum, $\varepsilon$–pseudospectrum of operators pencils, generalized spectrum approximation, operator pencil.

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## 1. Introduction

We consider a bounded operator $T$ in a Banach space $X$. The symbol $\sigma(T)$ denotes the spectrum of operator $T$. The $\varepsilon$–pseudospectrum of $T$ is defined as

$$\sigma_\varepsilon(T) = \{z \in \mathbb{C} : \|(T - zI)^{-1}\| > \varepsilon^{-1}\} \cup \sigma(T)$$

or as

$$\Sigma_\varepsilon(T) = \{z \in \mathbb{C} : \|(T - zI)^{-1}\| \geqslant \varepsilon^{-1}\} \cup \sigma(T)$$

where $\varepsilon > 0$. For more details on this concept, see [4], [12], [15], [17]. The difference between $\Sigma_\varepsilon(T)$ and $\sigma_\varepsilon(T)$ is characterized by the $\varepsilon$–level set of $T$ given as

$$L_\varepsilon(T) = \{z \in \mathbb{C} : \|(T - zI)^{-1}\| = \varepsilon^{-1}\}. \tag{1.1}$$

A pertinent question is whether the set $L_\varepsilon(T)$ can contain an open subset. If so, $\Sigma_\varepsilon(T)$ would be significantly larger than the closure of $\sigma_\varepsilon(T)$. This issue remained unresolved for some time, see [6], and was resolved in [14], [5], [3].

For $T, S \in \mathcal{B}(X)$, where $\mathcal{B}(X)$ is the space of linear bounded operators in a Banach space $X$, the generalized eigenvalue problem is $Tu = \lambda Su$, where $\lambda \in \mathbb{C}$ and $u \in X \setminus \{0\}$. The generalized resolvent set is defined by

$$\rho(T, S) = \big\{z \in \mathbb{C} : (T - zS)^{-1} \in \mathcal{B}(X)\big\}.$$

The generalized spectrum is defined as $\sigma(T, S) = \mathbb{C} \setminus \rho(T, S)$. The pair $(T, S)$ is generally called regular if $\rho(T, S) \neq \emptyset$, a condition that is always met in this work. For a more detailed explanation of these definitions see [13], [16], [8], [10], [1], [2].

The $\varepsilon$–pseudospectrum of operator pencils of $T, S \in \mathcal{B}(X)$ is defined as

$$\sigma_\varepsilon(T, S) = \{z \in \mathbb{C} : \|(T - zS)^{-1}S\| > \varepsilon^{-1}\} \cup \sigma(T, S) \tag{1.2}$$

or as

$$\Sigma_\varepsilon(T, S) = \{z \in \mathbb{C} : \|(T - zS)^{-1}S\| \geqslant \varepsilon^{-1}\} \cup \sigma(T, S)$$

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where $\varepsilon > 0$. This definition is borrowed from [11], where it was proved that it is a natural generalization of the case $S = I$. It shows that the set defined in (1.2) remains consistent and preserves fundamental properties of the $\varepsilon$–pseudospectrum, see [11, Thms. 2.1, 2.3, 2.4]. For other definitions of the $\varepsilon$–pseudospectrum see [4], [1], [2], [17].

The difference between $\Sigma_\varepsilon(T, S)$ and $\sigma_\varepsilon(T, S)$ is the level set $L_\varepsilon(T, S)$

$$L_\varepsilon(T, S) = \{\lambda \in \mathbb{C} : \|(T - \lambda S)^{-1}S\| = \varepsilon^{-1}\}. \tag{1.3}$$

We address the issue about a condition for the set $L_\varepsilon(T, S)$ ensuring that it contains no open set.

In this paper we prove that the set defined in (1.3) contains no open set when $X = H$ is a Hilbert space, see Theorem 2.1. This result is established under the condition that $S$ is a compact injective operator. Our second main result demonstrate that given a pair $(T, S)$ acting in a complex uniformly convex Banach space, if the generalized resolvent operator defined as $(T - zS)^{-1}S$, $z \in \rho(T, S)$, has a constant norm on an open set, then this constant represents the global minimum, see Theorem 2.2. Theorems 2.3, 2.4 establish the same for a complex uniformly convex Banach space $X$, namely, the set defined in (1.3) contains no open set provided $S$ is either invertible with $S^{-1} \in \mathcal{B}(X)$, or is compact and injective.

It was shown in [5, Thm. 2.2] that if $A$ is an unbounded operator with a compact resolvent defined on a uniformly convex Banach space, then the set (1.1) contains no open set. Let $\alpha \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of $A$. We consider the operators $S = (A - \alpha I)^{-1}$ and $T = (A - \alpha I)^{-1}A$. It was shown in [9, Thms. 2.3, 4.5] that $T, S \in \mathcal{B}(X)$ and

$$\sigma(T, S) = \sigma(A), \quad \sigma_\varepsilon(T, S) = \sigma_\varepsilon(A)$$

for $\varepsilon > 0$. It is important to note that the assumption that $S$ is compact and injective represents a correct generalization and contributes significantly to the existing studies in the literature. This extension promotes further exploration and understanding of the established concepts in the field of operator pencils.

## 2. Main results

Let $T, S \in \mathcal{B}(X)$. In what follows, if we write $X = H$, then $H$ is a Hilbert space. We begin with providing an example, in which the difference between two definitions of $\varepsilon$–pseudospectrum

$$\{z \in \mathbb{C} : \|(T - zS)^{-1}\| > \varepsilon^{-1}\} \cup \sigma(T, S) \tag{2.1}$$

and

$$\{z \in \mathbb{C} : \|(T - zS)^{-1}\| \geqslant \varepsilon^{-1}\} \cup \sigma(T, S) \tag{2.2}$$

contains open subset. For more details on these definitions see [17]. We introduce the generalized spectral problem as

$$Tu = \lambda S u,$$

where, $H = \mathbb{R}^3$ and

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

It is clear that $T$ and $S$ are degenerate. By elementary matrix calculations we get

$$\|(T - zS)^{-1}\| = \max\left\{1, \frac{4}{|7z+1|}, \frac{3}{|7z+1|} + \frac{|2z-1|}{|7z^2+z|}\right\}, \quad z \in \mathbb{C} \setminus \left\{-\frac{1}{7}, 0\right\}$$

and

$$\|(T - zS)^{-1}S\| = \frac{4|2z-1|}{|7z^2+z|} + \frac{15}{|7z+1|}, \quad z \in \mathbb{C} \setminus \left\{-\frac{1}{7}, 0\right\}.$$

It is clear that, for any open set in $\mathbb{C}$ obeyin the properties $\operatorname{Re} z > 1$ and $\operatorname{Im} z = 0$, we have

$$\|(T - zS)^{-1}\| = 1.$$

Hence, the difference between the sets (2.1) and (2.2) contain an open subsets.

Our first result describes that the set defined in (1.3) contains no open set when $X = H$ is a Hilbert space of infinite dimension. This result is established under the condition that $S$ is a compact injective operator. In what follows, we use the notation $\operatorname{Re}(z, T, S) = (T - zS)^{-1}$ for all $z \in \rho(T, S)$.

**Theorem 2.1.** *Let $T, S \in \mathcal{B}(H)$, where $S$ is compact and injective operator. Let $U$ be an open subset of $\rho(T, S)$. If*

$$\|\operatorname{Re}(\lambda, T, S)S\| \leqslant M \quad \lambda \in U,$$

*then*

$$\|\operatorname{Re}(\lambda, T, S)S\| < M \quad \lambda \in U.$$

Let us now study the situation where $X$ is a uniformly convex Banach space, see [7]. In the references [3], [5], [14], [15], this situation was studied in the case $S = I$. Here we generalize these results for the operator pencils. The next theorem states that, for a pair $(T, S)$ acting in a complex uniformly convex Banach space, if the generalized resolvent operator $\operatorname{Re}(z, T, S)S$, $z \in \rho(T, S)$, has a norm that remains constant over an open set, then this constant value represents the global minimum.

**Theorem 2.2.** *Let $T$ and $S$ belong to $\mathcal{B}(X)$, where $X$ is a complex uniformly convex Banach space. Assume that there exist an open subset $U \subset \rho(T, S)$ and constant $M > 0$ such that*

$$\|\operatorname{Re}(\lambda, T, S)\, S\| = M, \quad \lambda \in U.$$

*Then*

$$\|\operatorname{Re}(\lambda, T, S)\, S\| \geqslant M \quad \textit{for all} \quad \lambda \in \rho(T, S).$$

The next theorem establishes that if $X$ is a uniformly convex Banach space, then the set described by (1.3) contains no open subsets under the condition that $S$ is invertible operator and $S^{-1} \in \mathcal{B}(X)$.

**Theorem 2.3.** *Let $T$ and $S$ belong to $\mathcal{B}(X)$, where $X$ is a complex uniformly convex Banach space. If $S$ is an invertible operator such that $S^{-1} \in \mathcal{B}(X)$, then there is no open subset in $\rho(T, S)$ such that the function $\|\operatorname{Re}(\,\cdot\,, T, S)S\|$ is constant on it.*

The next theorem establishes that if $X$ is a uniformly convex Banach space, then the set described by (1.3) contains no open set under the condition that $S$ is a compact and injective operator.

**Theorem 2.4.** *Let $X$ be a complex uniformly convex Banach space and $T, S \in \mathcal{B}(X)$. Assume that the operator $S$ is injective and compact. Then there is no open subset in $\rho(T, S)$ such that the function $\|\operatorname{Re}(\cdot, T, S)S\|$ is constant on it.*

Let us discuss the main results of the work. An example obeying the assumption of the Theorem 2.1 reads as follows: $H = L^2(\Omega),$, where $\Omega \subset \mathbb{R}^n$ and $n \geqslant 1$. The operators $T$ and $S$ are defined as

$$Tu(t) = u(t) + \int_{\Omega} k_1(t, s)u(s)\, ds, \qquad Su(t) = \int_{\Omega} k_2(t, s)u(s)\, ds,$$

where the functions $k_1$ and $k_2$ are the kernels of the integral operators. This case agrees with the results in [8], [11]. Let $A$ be an unbounded operator in $X$ and $\alpha \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of $A$. Consider the operators $S = (A - \alpha I)^{-1}$ and $T = (A - \alpha I)^{-1}A$. It was shown in [9, Thms. 2.3, 4.5] that $T, S \in \mathcal{B}(X)$, and additionally, we have

$$\operatorname{Re}(\lambda, T, S)S = \operatorname{Re}(\lambda, A)$$

for all $\lambda \in \rho(A)$. Consequently, if $X$ is a complex uniformly convex Banach space or a Hilbert space, and $S$ is compact operator, then according to Theorems 2.1 and 2.4 there is no subset of $\rho(A)$, on which $\|\operatorname{Re}(\,\cdot\,, A)\|$ remains constant.

## 3. Case of Hilbert space

In this section we prove Theorem 2.1 and the following lemmas will play a crucial role.

**Lemma 3.1.** *Given $A \in \mathcal{B}(X)$, let $\|A\| < 1$. Then $(I - A)$ possesses a bounded inverse in $X$, which is represented by the Neumann series*

$$(I - A)^{-1} = \sum_{k=0}^{+\infty} A^k.$$

**Lemma 3.2.** *Let $T, S \in \mathcal{B}(X)$ and $\lambda_0 \in \rho(T, S)$. If there exists a $\lambda \in \mathbb{C}$ such that*

$$|\lambda - \lambda_0| < \|\operatorname{Re}(\lambda_0, T, S)S\|^{-1}, \tag{3.1}$$

*then $\lambda \in \rho(T, S)$ and*

$$\operatorname{Re}(\lambda, T, S)S = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k [\operatorname{Re}(\lambda_0, T, S)S]^{k+1}.$$

*Доказательство.* Let $\lambda \in \rho(T, S)$ satisfy the relation (3.1). Then

$$T - \lambda S = (T - \lambda_0 S)\,(I - (\lambda - \lambda_0)\operatorname{Re}(\lambda_0, T, S)S)\,.$$

Using Lemma 3.1, we arrive at the desired result. The proof is complete. □

Here we prove Theorem 2.1. We argue by contradiction. Let $\lambda_0 \in U$ such that $\|\operatorname{Re}(\lambda_0, T, S)S\| = M$. Since the set $U$ is open, we can choose $r > 0$ such that

$$\|(\lambda - \lambda_0)\operatorname{Re}(\lambda_0, T, S)S\| < 1, \quad \lambda \in B(\lambda_0, r),$$

where $B$ is of radius $r$ centered at $\lambda_0$. By Lemma 3.2 we have

$$\operatorname{Re}(\lambda, T, S)S = \sum_{k=0}^{+\infty} (\lambda - \lambda_0)^k (\operatorname{Re}(\lambda_0, T, S)S)^{k+1}, \quad \lambda \in B(\lambda_0, r).$$

For each $f \in H$ we get

$$\|\operatorname{Re}(\lambda, T, S)Sf\|^2 = \sum_{k,m=0}^{+\infty} (\lambda - \lambda_0)^k \overline{(\lambda - \lambda_0)}^m \left\langle (\operatorname{Re}(\lambda_0, T, S)S)^{k+1} f, (\operatorname{Re}(\lambda_0, T, S)S)^{m+1} f \right\rangle. \tag{3.2}$$

Integrating Equation (3.2) along the circle $|\lambda - \lambda_0| = r$, where $\lambda = \lambda_0 + re^{i\theta}$, we find

$$\frac{1}{2\pi}\int_0^{2\pi} \|\operatorname{Re}(\lambda_0 + re^{i\theta}, T, S)Sf\|^2 d\theta$$
$$= \sum_{k,m=0}^{+\infty} \frac{1}{2\pi}\int_0^{2\pi} (re^{i\theta})^k (re^{-i\theta})^m d\theta \left\langle (\operatorname{Re}(\lambda_0, T, S)S)^{k+1} f, (\operatorname{Re}(\lambda_0, T, S)S)^{m+1} f \right\rangle.$$

Since

$$\frac{1}{2\pi}\int_{0}^{2\pi}(re^{i\theta})^k(re^{-i\theta})^m d\theta = \frac{1}{2\pi}\int_{0}^{2\pi} r^{k+m}e^{i\theta(k-m)}\,d\theta = \begin{cases} r^{2k} & \text{if} \quad k=m, \\ 0 & \text{otherwise,}\end{cases}$$

we obtain

$$\frac{1}{2\pi}\int_{0}^{2\pi}\|\operatorname{Re}(\lambda_0+re^{i\theta},T,S)Sf\|^2 d\theta = \sum_{k=0}^{+\infty} r^{2k}\|(\operatorname{Re}(\lambda_0,T,S)S)^{k+1}f\|^2.$$

It is clear that

$$\|\operatorname{Re}(\lambda_0,T,S)Sf\|^2 + r^2\|(\operatorname{Re}(\lambda_0,T,S)S)^2 f\|^2 \leqslant \sum_{k=0}^{+\infty} r^{2k}\|(\operatorname{Re}(\lambda_0,T,S)S)^{k+1}f\|^2$$

and therefore

$$\|\operatorname{Re}(\lambda_0,T,S)Sf\|^2 + r^2\|(\operatorname{Re}(\lambda_0,T,S)S)^2 f\|^2 \leqslant \frac{1}{2\pi}\int_{0}^{2\pi}\|\operatorname{Re}(\lambda_0+re^{i\theta},T,S)Sf\|^2 d\theta.$$

Using

$$\|\operatorname{Re}(\lambda_0+re^{i\theta},T,S)Sf\| \leqslant M\|f\|,$$

we find

$$\|\operatorname{Re}(\lambda_0,T,S)Sf\|^2 + r^2\|(\operatorname{Re}(\lambda_0,T,S)S)^2 f\|^2 \leqslant M^2\|f\|^2. \tag{3.3}$$

We choose an arbitrary $\varepsilon > 0$. Since $\|\operatorname{Re}(\lambda_0,T,S)S\| = M$, there exists $f_\varepsilon \in H$ such that

$$\|f_\varepsilon\| = 1 \qquad \text{and} \qquad \|\operatorname{Re}(\lambda_0,T,S)Sf_\varepsilon\|^2 > M^2 - \varepsilon.$$

Therefore, due to (3.3),

$$M^2 - \varepsilon + r^2\|(\operatorname{Re}(\lambda_0,T,S)S)^2 f_\varepsilon\|^2 < M^2.$$

Then

$$\|(\operatorname{Re}(\lambda_0,T,S)S)^2 f_\varepsilon\|^2 < \frac{\varepsilon}{r^2},$$

which implies

$$\lim_{\varepsilon\to 0}\|(\operatorname{Re}(\lambda_0,T,S)S)^2 f_\varepsilon\|^2 = 0,$$

and hence

$$\lim_{\varepsilon\to 0}(\operatorname{Re}(\lambda_0,T,S)S)^2 f_\varepsilon = 0. \tag{3.4}$$

Since $S$ is compact operator and the sequence $(\operatorname{Re}(\lambda_0,T,S)Sf_\varepsilon)_{\varepsilon>0}$ is bounded, there exists an infinite subset $I \subset \mathbb{R}^+$ and $y_0 \in H$ such that

$$\lim_{\varepsilon\to 0}\operatorname{Re}(\lambda_0,T,S)Sf_\varepsilon = y_0 \tag{3.5}$$

for all $\varepsilon \in I$. We also find

$$\operatorname{Re}(\lambda_0,T,S)S\left(\lim_{\varepsilon\to 0}\operatorname{Re}(\lambda_0,T,S)Sf_\varepsilon\right) = \operatorname{Re}(\lambda_0,T,S)Sy_0.$$

According to the continuity of $\operatorname{Re}(\lambda_0,T,S)S$,

$$\lim_{\varepsilon\to 0}(\operatorname{Re}(\lambda_0,T,S)S)^2 f_\varepsilon = \operatorname{Re}(\lambda_0,T,S)Sy_0$$

for all $\varepsilon \in I$. In view of Equation (3.4) this gives

$$\operatorname{Re}(\lambda_0,T,S)Sy_0 = 0.$$

Since $S$ is injective, we conclude $y_0 = 0$. By (3.5)

$$\lim_{\varepsilon\to 0}\operatorname{Re}(\lambda_0,T,S)Sf_\varepsilon = 0$$

for all $\varepsilon \in I$ and this contradicts to

$$\|\operatorname{Re}(\lambda_0, T, S)Sf_\varepsilon\|^2 > M^2 - \varepsilon.$$

The proof is complete.

## 4. Case of complex uniformly convex Banach space

In this section we prove Theorems 2.2, 2.3 and 2.4.

The next theorem plays a crucial role in proving Theorems 2.2, 2.3 and 2.4.

**Theorem 4.1.** *Let $T$ and $S$ belong to $\mathcal{B}(X)$, were $X$ is a complex uniformly convex Banach space. Assume that there exists an open subset $U \subset \rho(T, S)$ and a constant $M > 0$ such that*

$$\|\operatorname{Re}(\lambda, T, S)\, S\| = M, \quad \forall \lambda \in U.$$

*Then there exists $(e_n)_{n\geqslant 0} \subset X$ such that $\|e_n\| = 1$ for all $n \in \mathbb{N}$, where*

$$\lim_{n\to\infty} \|\operatorname{Re}(\lambda_0, T, S)Se_n\| = M$$

*and*

$$\lim_{n\to\infty} \|(\operatorname{Re}(\lambda_0, T, S)S)^2 e_n\| = 0$$

*for all $\lambda_0 \in U$.*

In the proof of Theorem 4.1 we employ the following lemma.

**Lemma 4.1.** *Let*

$$\lambda \mapsto f(\lambda) = \sum_{k=0}^{+\infty} a_k(\lambda - \lambda_0)^k$$

*be a function with values in a complex Banach space $X$, defined and analytic in a neighborhood of the point $\lambda_0$. If $\|f(\lambda)\| = \|a_0\|$ in a neighborhood of the point $\lambda_0$, then for each $k \in \mathbb{N}^*$ there exists $r_k > 0$ such that*

$$\|a_0 + (\lambda - \lambda_0)a_k\| \leqslant \|a_0\|, \qquad |\lambda - \lambda_0| \leqslant r_k.$$

The lemma is implied by [7, Lm. 1.1].

*Proof of Theorem 4.1.* The proof is partially based on the proof of [3, Thm. 3.2] in the case $S = I$. Let $\lambda_0 \in U$, we choose $r > 0$ such that $\|\operatorname{Re}(\lambda_0, T, S)S\|^{-1} > r$. According to Lemma 3.2, the function $\operatorname{Re}(\cdot, T, S)S$ is analytic in the ball $B(\lambda_0, r)$ and

$$\operatorname{Re}(\lambda, T, S)S = \sum_{k=0}^{+\infty} (\lambda - \lambda_0)^k (\operatorname{Re}(\lambda_0, T, S)S)^{k+1}, \quad \text{for all} \quad \lambda \in B(r, \lambda_0).$$

Since $\|\operatorname{Re}(\lambda, T, S)S\| = M$ for all $\lambda \in U$, we have

$$\|\operatorname{Re}(\lambda, T, S)S\| = \|\operatorname{Re}(\lambda_0, T, S)S\| = M \quad \text{for all} \quad \lambda \in U.$$

By Lemma (4.1), for each $k \in \mathbb{N}^*$ there exists $r_k > 0$ such that

$$\|\operatorname{Re}(\lambda_0, T, S)S + (\lambda - \lambda_0)(\operatorname{Re}(\lambda_0, T, S)S)^{k+1}\| \leqslant M, \quad |\lambda - \lambda_0| \leqslant r_k$$

for all $\lambda \in U$. This implies

$$\|\operatorname{Re}(\lambda_0, T, S)Sx + (\lambda - \lambda_0)\operatorname{Re}(\lambda_0, T, S)S)^{k+1}x\| \leqslant M, \quad \lambda \in B(\lambda_0, r_k)$$

for each $x \in X$ with $\|x\| = 1$. Therefore,

$$\|\frac{1}{M}\operatorname{Re}(\lambda_0, T, S)Sx + \frac{(\lambda - \lambda_0)r_k}{M}(\operatorname{Re}(\lambda_0, T, S)S)^{k+1}x\| \leqslant 1 \quad \lambda \in B(\lambda_0, 1).$$

Since $\| \operatorname{Re}(\lambda_0, T, S)S\| = M$, there exists $(e_n)_{n\geqslant 0} \subset X$ such that $\|e_n\| = 1$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \| \operatorname{Re}(\lambda_0, T, S)Se_n\| = M.$$

We define the sequence

$$x_n = \frac{1}{M} \operatorname{Re}(\lambda_0, T, S)Se_n, \quad n \in \mathbb{N},$$

and then $\|x_n\| \to 1$. We let

$$y_n = \frac{r_1}{M}(\operatorname{Re}(\lambda_0, T, S)S)^2 e_n, \quad n \in \mathbb{N}.$$

We are going to show that $\|y_n\| \to 0$. We suppose the opposite, that is, there exist $\varepsilon > 0$ and an infinite subset $I \subseteq \mathbb{N}$ such that $\|y_n\| > \varepsilon$ for all $n \in I$. Then

$$\|x_n + (\lambda - \lambda_0)y_n\| \leqslant 1, \quad \lambda \in B(\lambda_0, 1).$$

Applying the complex uniform convexity of $X$, we get the existence of some $\delta > 0$ such that $\|x_n\| < 1 - \delta$ for all $n \in \mathbb{N}$. This contradicts to $\|x_n\| \to 1$. Then $\|y_n\| \to 0$ for all $n \in \mathbb{N}$. Hence,

$$\lim_{n\to\infty} \|(\operatorname{Re}(\lambda_0, T, S)S)^2 e_n\| = 0,$$

and this completes the proof. □

*Proof of Theorem 2.2.* Let $\lambda_0 \in U$. For an arbitrary $\lambda \in \rho(T, S)$, we use twice the first resolvent identity to find

$$\begin{aligned}\operatorname{Re}(\lambda, T, S)S - \operatorname{Re}(\lambda_0, T, S)S =& (\lambda - \lambda_0) \operatorname{Re}(\lambda, T, S)S \operatorname{Re}(\lambda_0, T, S)S \\ =& (\lambda - \lambda_0) \left((\lambda - \lambda_0) \operatorname{Re}(\lambda, T, S)S + I\right) \left(\operatorname{Re}(\lambda_0, T, S)S\right)^2.\end{aligned}$$

Hence,

$$\| \operatorname{Re}(\lambda, T, S)S\| \geqslant \left|\| \operatorname{Re}(\lambda_0, T, S)S\| - |\lambda - \lambda_0| \, \|(\lambda - \lambda_0) \operatorname{Re}(\lambda, T, S)S + I\| \, \| \left(\operatorname{Re}(\lambda_0, T, S)S\right)^2 \|\right|.$$

Due to Theorem 4.1, there exists $(e_n)_{n\geqslant 0} \subset X$ such that

$$\lim_{n\to\infty} \| \operatorname{Re}(\lambda_0, T, S)Se_n\| = M,$$

and

$$\lim_{n\to\infty} \|(\operatorname{Re}(\lambda_0, T, S)S)^2 e_n\| = 0,$$

and $\|e_n\| = 1$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}\| \operatorname{Re}(\lambda, T, S)S\| \geqslant& \Big|\| \operatorname{Re}(\lambda_0, T, S)Se_n\| \\ &- |\lambda - \lambda_0| \, \|(\lambda - \lambda_0) \operatorname{Re}(\lambda, T, S)S + I\| \, \| \left(\operatorname{Re}(\lambda_0, T, S)S\right)^2 e_n\| \Big|.\end{aligned}$$

Then as $n \to +\infty$ we have

$$\| \operatorname{Re}(\lambda, T, S)S\| \geqslant M, \quad \lambda \in \rho(T, S).$$

The proof is complete. □

*Proof of Theorem 2.3.* Assume that there exists an open set $U \subset \rho(T, S)$ such that

$$\| \operatorname{Re}(\lambda, T, S)S\| = M, \quad \lambda \in U.$$

By Theorem 4.1 there exists $\lambda_0 \in U$ and $(e_n)_n \subset X$ such that $\|e_n\| = 1$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \| \operatorname{Re}(\lambda_0, T, S)Se_n\| = M,$$

and

$$\lim_{n\to\infty} \|(\operatorname{Re}(\lambda_0, T, S)S)^2 e_n\| = 0.$$

Since $S$ is invertible, we find

$$\begin{aligned}\lim_{n\to\infty}\|e_n\| \leqslant &\lim_{n\to\infty}\|S^{-1}(T-\lambda_0 S)S^{-1}(T-\lambda_0 S)\|\|(\mathrm{Re}(\lambda_0,T,S)S)^2 e_n\|\\ \leqslant&\|S^{-1}(T-\lambda_0 S)S^{-1}(T-\lambda_0 S)\| \lim_{n\to\infty}\|(\mathrm{Re}(\lambda_0,T,S)S)^2 e_n\| = 0.\end{aligned}$$

This contradicts to $\|e_n\| = 1$ for all $n \in \mathbb{N}$. The proof is complete. □

*Proof Theorem 2.4.* Let $T$ and $S$ belong to $\mathcal{B}(X)$, where $X$ is a complex uniformly convex Banach space. Assume that there exists an open set $U \subset \rho(T,S)$ such that

$$\|\mathrm{Re}(\lambda,T,S)S\| = M, \quad \lambda \in U.$$

By Theorem 4.1 there exists $(e_n)_n \subset X$ such that $\|e_n\| = 1$ for all $n \in \mathbb{N}$, where

$$\lim_{n\to\infty}\|\mathrm{Re}(\lambda_0,T,S)Se_n\| = M.$$

and

$$\lim_{n\to\infty}\|(\mathrm{Re}(\lambda_0,T,S)S)^2 e_n\| = 0.$$

Since $S$ is compact operator, there exists infinite subset $I \subseteq \mathbb{N}$ and $y \in X$ such that

$$\lim_{n\to\infty}\mathrm{Re}(\lambda_0,T,S)Se_n = y, \quad n \in I.$$

We have

$$\mathrm{Re}(\lambda_0,T,S)S(\lim_{n\to\infty}\mathrm{Re}(\lambda_0,T,S)Se_n) = \mathrm{Re}(\lambda,T,S)Sy$$

for all $n \in I$. Thus, $\mathrm{Re}(\lambda_0,T,S)Sy = 0$. Since $\mathrm{Re}(\lambda_0,T,S)S$ is injective operator, this implies $y = 0$. The latter contradicts to $M > 0$. The proof is complete. □

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