ISSN 2304-0122 Ufa Mathematical Journal. Vol. 16. No 3 (2024). P. 113-124.

doi:10.13108/2024-16-3-113

CATEGORICAL CRITERION FOR EXISTENCE OF UNIVERSAL C^*–ALGEBRAS

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Abstract. We deal with categories, which determine universal C^*–algebras. These categories are called the compact C^*–relations. They were introduced by T.A. Loring. Given a set X, a compact C^*–relation on X is a category, the objects of which are functions from X to C^*–algebras, and morphisms are $*$–homomorphisms of C^*–algebras making the appropriate triangle diagrams commute. Moreover, these functions and $*$–homomorphisms satisfy certain axioms. In this article, we prove that every compact C^*–relation is both complete and cocomplete. As an application of the completeness of compact C^*–relations, we obtain the criterion for the existence of universal C^*–algebras.

Keywords: compact C^*–relation, complete category, universal C^*–algebra.

Mathematics Subject Classification: 16B50, 46L05, 46M15

1. Introduction

The motivation for our work comes from the theory of universal C^*–algebras generated by sets of generators subject to relations (see [1]– [6]) and the study of limits for inductive systems consisting of universal C^*–algebras and their $*$–homomorphisms in [7]– [12]. A categorical approach to relations that determine universal C^*–algebras was developed by Loring [5]. In the framework of this approach, one deals with categories called C^*–relations. Given a set X, a C^*–relation $\mathcal{R}$ on X is a category, the objects of which are functions from X to C^*–algebras, and morphisms are $*$–homomorphisms of C^*–algebras making the appropriate triangle diagrams commute. In addition, the objects and the morphisms of $\mathcal{R}$ satisfy certain axioms. The C^*–relations determining universal C^*–algebras are called compact. A necessary and sufficient condition for $\mathcal{R}$ to be compact is the existence of an initial object $C^*(\mathcal{R})$ in the category $\mathcal{R}$ [5]. The universal C^*–algebra for the compact C^*–relation $\mathcal{R}$ is the initial object $C^*(\mathcal{R})$ of this category, that is, an object with precisely one outgoing morphism for each other object of $\mathcal{R}$.

The C^*–relations called the $*$–polynomial relations associated with $*$–polynomial pairs were studied in [13]. A polynomial pair (X, P) consists of a non–empty set X and a non–empty subset P of the free $*$–algebra $F(X)$ generated by X over the field of complex numbers. The objects of the $*$–polynomial relation associated with (X, P) are all functions f from the set X to C^*–algebras satisfying the property: the set P is contained in the kernel of the unique $*$–homomorphism, which is an extension of f to the free $*$–algebra $F(X)$. It was proved in [13] that every C^*–algebra is a universal C^*–algebra determined by a $*$–polynomial relation and every compact C^*–relation is isomorphic to a $*$–polynomial relation.

In this article we continue the study of properties of the compact C^*–relations initiated in [13]. We show that each compact C^*–relation is both complete and cocomplete. To obtain

R.N. Gumerov, E.V. Lipacheva, K.A. Shishkin, A categorical criterion for the existence of universal C^*–algebras.

Submitted November 3, 2023.

this result, we use of the completeness and cocompleteness of the category of C^*–algebras and their $*$–homomorphisms [16]. The completeness of every compact C^*–relation together with the aforementioned equivalence between the compactness of a C^*–relation $\mathcal{R}$ and the existence of an initial object in $\mathcal{R}$ yields the criterion for the existence of the universal C^*–algebra $C^*(\mathcal{R})$. Namely, $C^*(\mathcal{R})$ exists if and only if the category $\mathcal{R}$ is complete.

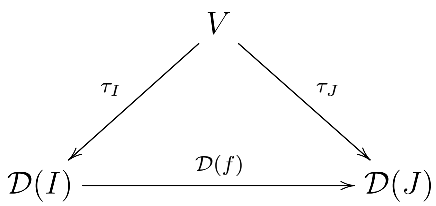
The article is organized as follows. It consists of the Introduction and three sections. Section 2 contains needed notation, definitions and facts from the category theory and the theory of C^*–relations. In Section 3 we prove that every compact C^*–relation is complete. As a consequence of this result, we obtain the criterion for the existence of universal C^*–algebras. Section 4 is devoted to the proof of the cocompleteness of all compact C^*–relations.

2. Preliminaries

In this section, we recall some necessary definitions and facts from the theory of categories and functors. For detail we refer the reader to book [17].

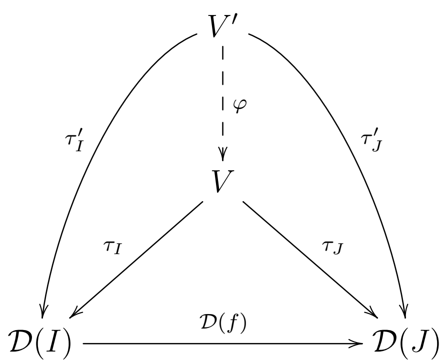
Let $\mathcal{C}$ be a category and $\mathcal{I}$ be a small category. A functor $\mathcal{D}\colon \mathcal{I} \to \mathcal{C}$ is called *a diagram in $\mathcal{C}$ of shape $\mathcal{I}$*.

A cone on the diagram $\mathcal{D}$ is a pair $(\mathcal{V}, \tau)$, where $\mathcal{V}\colon \mathcal{I} \to \mathcal{C}$ is a constant functor and $\tau\colon \mathcal{V} \to \mathcal{D}$ is a natural transformation from $\mathcal{V}$ to $\mathcal{D}$. Thus, the functor $\mathcal{V}$ sends each object I of $\mathcal{I}$ to a fixed object V in $\mathcal{C}$ and $\mathcal{V}(f)$ is the identity $\mathbb{1}_V$ on V for each morphism f of $\mathcal{I}$. Moreover, one has a family of morphisms $\tau_I\colon V \to \mathcal{D}(I)$ indexed by objects I of the category $\mathcal{I}$ such that the diagram



commutes for every morphism $f : I \to J$ in $\mathcal{I}$.

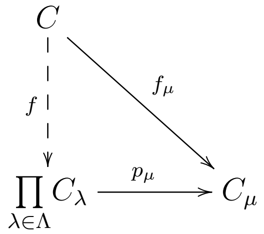
A cone $(\mathcal{V}, \tau)$ on the diagram $\mathcal{D}$ is said to be *universal* if for every cone $(\mathcal{V}', \tau')$ on $\mathcal{D}$ there exists a unique morphism $\varphi\colon V' \to V$ in $\mathcal{C}$ such that $\tau' = \tau \circ \varphi$, that is, the diagram



commutes for every morphism $f : I \to J$ in $\mathcal{I}$. A universal cone on $\mathcal{D}$ is called *a limit of the diagram $\mathcal{D}$*. A category is said to be *complete* if it has a limit for every diagram in this category.

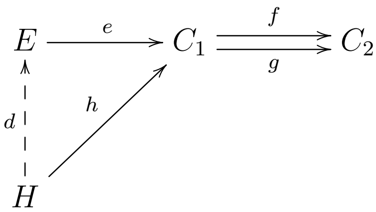
In what follows, two basic types of limits of diagrams are involved in our arguing. These are *products* and *equalizers*; let us recall the definitions.

Let Λ be a set. We denote by $\mathcal{L}$ the discrete category, the objects of which are the elements of Λ and all morphisms are the identities. Let $\{C_\lambda\}_{\lambda\in\Lambda}$ be a family of objects in the category $\mathcal{C}$. Consider the diagram $\mathcal{D}\colon \mathcal{L} \to \mathcal{C}$, which sends an object λ of $\mathcal{L}$ to the object C_λ in $\mathcal{C}$. A limit of the diagram $\mathcal{D}$ is called *the product of the family* $\{C_\lambda\}_{\lambda\in\Lambda}$. It is denoted by $\left(\prod\limits_{\lambda\in\Lambda} C_\lambda, \{p_\lambda\}_{\lambda\in\Lambda}\right)$. The object $\prod\limits_{\lambda\in\Lambda} C_\lambda$ itself is often called the product of the family $\{C_\lambda\}_{\lambda\in\Lambda}$. The morphisms p_λ are called *the projections* of the product. Thus, the product possesses the following *universal property*. For each object C in $\mathcal{C}$ and each Λ–indexed family of morphisms $f_\lambda : C \to C_\lambda$ in $\mathcal{C}$ there exists a unique morphism $f : C \to \prod\limits_{\lambda\in\Lambda} C_\lambda$ such that for each $\mu \in \Lambda$ the diagram



is commutative. We say that a category *has all products* if every family of its objects indexed by a set has a product in this category.

Another basic limit is an equalizer, which is defined as follows. Let $\mathcal{E}$ be a category with two objects, say A and B, with two morphisms $u, v \colon A \to B$, and with no other morphisms except for identities. Let $f, g \colon C_1 \to C_2$ be morphisms of the category $\mathcal{C}$. We refer to pairs of morphisms like f and g as *parallel morphisms*. Consider the diagram $\mathcal{D}$ in $\mathcal{C}$ of shape $\mathcal{E}$ such that $\mathcal{D}(u) = f$ and $\mathcal{D}(v) = g$. A limit of this diagram $\mathcal{D}\colon \mathcal{E} \to \mathcal{C}$ is called *the equalizer* of f and g. Thus, it is a pair (E, e), where E is an object of the category $\mathcal{C}$ and $e\colon E \to C_1$ is a morphism of $\mathcal{C}$ such that $f \circ e = g \circ e$ and the following *universal property* holds:



every morphism $h\colon H \to C_1$ such that $f \circ h = g \circ h$ can be factorized uniquely through e, that is, there exists a unique morphism $d\colon H \to E$ such that $e \circ d = h$. In case each pair of parallel morphisms in a category $\mathcal{C}$ has an equalizer, we say that $\mathcal{C}$ *has all equalizers*.

The next result states that all limits can be built up from products and equalizers [17, Ch. V, Sect. 2, Cor. 2].

Lemma 2.1. *A category is complete if and only if it has all products and equalizers.*

Using the duality principle, one obtains the dual notions, namely, a cocone, a universal cocone, a colimit, a coproduct, a coequalizer, a cocomplete category and the dual of Lemma 2.1. For details, we refer the reader to [17, Ch. II, Sect. 1].

We denote by C^*–**alg** the category of all C^*–algebras and $*$–homomorphisms between them. The trivial C^*–algebra consisting of single zero element is denoted by 0.

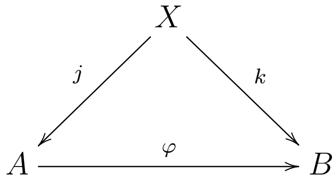
For a family $\{A_\lambda \mid \lambda \in \Lambda\}$ of objects in C^*–**alg** indexed by a set Λ, we consider the direct product

$$\prod_{\lambda\in\Lambda} A_\lambda := \Big\{(a_\lambda) \;\Big|\; \|(a_\lambda)\| = \sup_\lambda \|a_\lambda\| < +\infty\Big\},$$

which is a C^*–algebra with respect to the coordinatewise algebraic operations and the supremum norm.

Further, we give the definitions of categories from Loring's paper [5]. These categories are the main objects of investigation in the present article.

Given a set X, *the null C^*–relation on X* is the category $\mathcal{F}_X$, the objects of which are all functions of the form $j : X \to A$, where A is a C^*–algebra. For two objects $j : X \to A$ and $k : X \to B$ in $\mathcal{F}_X$, a morphism from j to k is each $*$–homomorphism of C^*–algebras $\varphi : A \to B$ making the diagram



commute, i.e., $k = \varphi \circ j$.

A C^*–relation on X is a full subcategory $\mathcal{R}$ of $\mathcal{F}_X$ satisfying the following axioms:

C1 the function $X \to 0$ is an object of $\mathcal{R}$;

C2 if $\varphi : A \to B$ is an injective $*$–homomorphism of C^*–algebras, $f : X \to A$ is a function and $\varphi \circ f$ is an object of $\mathcal{R}$, then f is an object of $\mathcal{R}$;

C3 if $\varphi : A \to B$ is a $*$–homomorphism of C^*–algebras and $f : X \to A$ is an object of $\mathcal{R}$, then $\varphi \circ f$ is an object of $\mathcal{R}$;

C4f if $f_i : X \to A_i$ is an object of $\mathcal{R}$ for every $i = 1, \dots, n$, $n \in \mathbb{N}$, then the function

$$\prod_{i=1}^{n} f_i : X \to \prod_{i=1}^{n} A_i$$

is an object of $\mathcal{R}$.

Objects of C^*–relations are also called the *representations*.

A C^*–relation $\mathcal{R}$ on a set X is said to be *compact* if, in addition, the following condition is fulfilled:

C4 for each non–empty set Λ, if $f_\lambda : X \to A_\lambda$ is an object of $\mathcal{R}$ for every $\lambda \in \Lambda$, then the function

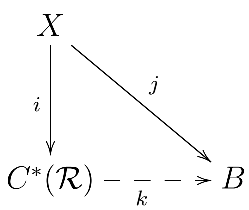
$$\prod_{\lambda\in\Lambda} f_\lambda : X \to \prod_{\lambda\in\Lambda} A_\lambda$$

is also an object of $\mathcal{R}$.

The following statement is a reformulation of Theorem 2.10 from [5] (see also [2, Prop. 1.3.6], [3, Sect. 3.1] and [4, Sect. 1.4]).

Lemma 2.2. *Let $\mathcal{R}$ be a C^*–relation on a set X. Then $\mathcal{R}$ is compact if and only if there exists an initial object in $\mathcal{R}$.*

In what follows, for a compact C^*–relation $\mathcal{R}$ on a set X, we consider an initial object $i : X \to A$ of $\mathcal{R}$. The C^*–algebra A is denoted by $C^*(\mathcal{R})$. Thus, for every representation $j : X \to B$ of $\mathcal{R}$ there exists a unique $*$–homomorphism of C^*–algebras $k : C^*(\mathcal{R}) \to B$ such that the diagram



is commutative, i.e., $j = k \circ i$.

The object $i : X \to C^*(\mathcal{R})$ is called *the universal representation*, and the C^*–algebra $C^*(\mathcal{R})$ is called *the universal C^*–algebra* for the compact C^*–relation $\mathcal{R}$.

Finally, we give three examples of C^*–relations, which are denoted by $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$. Since every C^*–relation must be a full subcategory in the null C^*–relation $\mathcal{F}_X$, we specify only objects for these categories. One can easily verify that Axioms **C1**, **C2**, **C3** and **C4f** hold in $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$, that is, these categories are C^*–relations.

Example 2.1. *Let $X = \{x\}$ be an one–element set. We consider the category $\mathcal{R}_1$, the objects of which are all functions $f : X \to A$, where A is a C^*–algebra, and $f(x)$ is a normal element of A.*

We claim that $\mathcal{R}_1$ is not a compact C^*–relation. Indeed, to see this, we fix a C^*–algebra A and a non–zero normal element $a \in A$. For each $n \in \mathbb{N}$, we consider the object f_n of the category $\mathcal{R}_1$ defined as

$$f_n : X \to A : x \mapsto na.$$

*Since $\sup_{n\in\mathbb{N}} \|f_n(x)\| = +\infty$, Axiom **C4** is not valid for $\mathcal{R}_1$. That is, the C^*–relation $\mathcal{R}_1$ is not compact, as claimed.*

By Lemma 2.2, there is no initial object in the category $\mathcal{R}_1$, and the universal C^*–algebra for $\mathcal{R}_1$ is not defined.

We note that the category $\mathcal{R}_1$ is a $*$–polynomial relation associated with the $*$–polynomial pair $(X, \{x^*x - xx^*\})$. This fact also guarantees that $\mathcal{R}_1$ is a C^*–relation [13, Prop. 2].

Example 2.2. *Let $X = \{x\}$. As objects of the category $\mathcal{R}_2$, we take all functions of the form $f : X \to A$, where A is a unital C^*–algebra and $f(x)$ is a unitary element in A. It is straightforward to verify that Axiom **C4** is satisfied in the C^*–relation $\mathcal{R}_2$, hence, it is compact.*

By Lemma 2.2, there exist the universal representation in $\mathcal{R}_2$ and the universal C^*–algebra $C^*(\mathcal{R}_2)$.

Using the continuous functional calculus, one can see that $C^*(\mathcal{R}_2)$ is isomorphic to the commutative C^*–algebra $C(S^1)$ consisting of all continuous complex–valued functions on the unit circle S^1 in the complex plane.

Example 2.3. *Let $n \geqslant 2$ be an integer and $X = \{x_1, \dots, x_n\}$ be a set consisting of n elements. We define $\mathcal{R}_3$ as the category, the objects of which are all functions of the form $f : X \to A$, where A is a unital C^*–algebra and $f(x_1), \dots, f(x_n)$ are isometries with pairwise orthogonal ranges. It is easy to see that Axiom **C4** holds for the C^*–relation $\mathcal{R}_3$, that is, $\mathcal{R}_3$ is compact.*

Consequently, by Lemma 2.2, there is the universal representation $i : X \to C^*(\mathcal{R}_3)$ in the category $\mathcal{R}_3$.

The universal C^*–algebra $C^*(\mathcal{R}_3)$ is called the Toeplitz — Cuntz algebra for n generators. This algebra was defined and studied by Cuntz [14], [15]. In particular, it was shown that the Toeplitz — Cuntz algebra contains a closed two–sided ideal, which is isomorphic to the compact operators on an infinite–dimensional separable Hilbert space, and the quotient of $C^*(\mathcal{R}_3)$ by this ideal is the Cuntz algebra [14]. In [11], [12], the universal property of $C^*(\mathcal{R}_3)$ is used for constructing the direct sequences of the Toeplitz — Cuntz algebras and studying properties of reduced semigroup C^*–algebras.

3. Completeness of compact C^*–relations

In this section we show that all compact C^*–relations are complete. Our proof is based on the fact that the category C^*–**alg** is complete [16]. More precisely, we explore explicit limit constructions in the category C^*–**alg** from [16]. Using completeness of compact C^*–relations and Lemma 2.2, we obtain the criterion for the existence of universal C^*–algebras for C^*–relations.

Lemma 3.1. *Every compact C^*–relation $\mathcal{R}$ on a set X has all products.*

Proof. Let $\{f_\lambda\colon X \to A_\lambda\}_{\lambda\in\Lambda}$ be a family of objects of $\mathcal{R}$ indexed by elements of a set Λ. Consider the function

$$\prod_{\lambda\in\Lambda} f_\lambda\colon X \to \prod_{\lambda\in\Lambda} A_\lambda\colon x \mapsto (f_\lambda(x))_{\lambda\in\Lambda}, \qquad x \in X.$$

By Axiom **C4**, it is an object of the category $\mathcal{R}$. For each $\lambda \in \Lambda$, we denote by p_λ the natural projection of the direct product of the C^*–algebras $\prod\limits_{\mu\in\Lambda} A_\mu$ onto the C^*–algebra A_λ. Obviously, the $*$–homomorphism p_λ is a morphism of $\mathcal{R}$.

We claim that the pair

$$\left(\prod_{\lambda\in\Lambda} f_\lambda, \{p_\lambda\colon \prod_{\mu\in\Lambda} A_\mu \to A_\lambda\}_{\lambda\in\Lambda}\right)$$

is a product of this family in $\mathcal{R}$. Indeed, to show that this pair satisfies the universal property, we take an object $f\colon X \to A$ and a family of morphisms $\{g_\lambda\colon A \to A_\lambda\}_{\lambda\in\Lambda}$ in the category $\mathcal{R}$ such that

$$g_\lambda \circ f = f_\lambda \quad \text{whenever} \quad \lambda \in \Lambda. \tag{3.1}$$

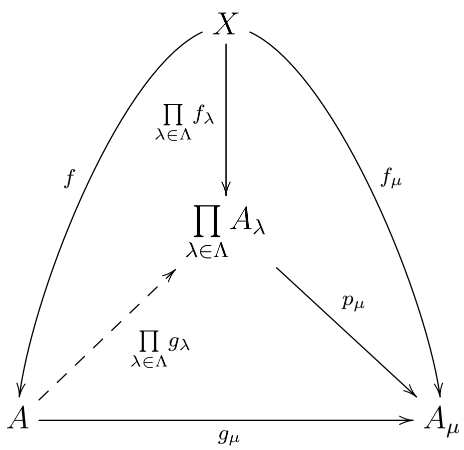
Since the pair $\left(\prod\limits_{\lambda\in\Lambda} A_\lambda, \{p_\lambda\}_{\lambda\in\Lambda}\right)$ is a product [16, Thm. 2.9] of the family $\{A_\lambda\}_{\lambda\in\Lambda}$ in the category of C^*–algebras and their $*$–homomorphisms, there is a unique $*$–homomorphism

$$\prod_{\lambda\in\Lambda} g_\lambda\colon A \to \prod_{\lambda\in\Lambda} A_\lambda\colon a \mapsto (g_\lambda(a))_{\lambda\in\Lambda}$$

such that

$$p_\mu \circ \prod_{\lambda\in\Lambda} g_\lambda = g_\mu \tag{3.2}$$

for each index $\mu \in \Lambda$, that is, in the next diagram the bottom triangle is commutative:



Moreover, using (3.2), (3.1) and the commutativity of the triangle on the right–hand side of the diagram, we have the equalities

$$\left(p_\mu \circ \left(\prod_{\lambda\in\Lambda} g_\lambda\right) \circ f\right)(x) = (g_\mu \circ f)(x) = f_\mu(x) = \left(p_\mu \circ \prod_{\lambda\in\Lambda} f_\lambda\right)(x)$$

for every index $\mu \in \Lambda$ and for every $x \in X$. Consequently, by the definition of an element of a product in category of C^*–algebras, the triangle on the left–hand side of the diagram is commutative:

$$\left(\prod_{\lambda\in\Lambda} g_\lambda\right) \circ f = \prod_{\lambda\in\Lambda} f_\lambda,$$

that is, the $*$–homomorphism $\prod_{\lambda\in\Lambda} g_\lambda$ is a morphism of the C^*–relation $\mathcal{R}$.

Thus, the required universal property is satisfied and the pair $\left(\prod_{\lambda\in\Lambda} f_\lambda, \{p_\lambda\}_{\lambda\in\Lambda}\right)$ is a product in the category $\mathcal{R}$, as claimed. The proof is complete. □

To prove the following statement we use the fact that the category C^*–**alg** has all equalizers [16, Lm. 2.5].

Lemma 3.2. *Every compact C^*–relation $\mathcal{R}$ on a set X has all equalizers.*

Proof. We take two objects $f\colon X \to A$ and $g\colon X \to B$ and two parallel morphisms $\varphi\colon A \to B$ and $\psi\colon A \to B$ from f to g in the category $\mathcal{R}$.

Let us consider the C^*–algebra E and the $*$–homomorphism ε of C^*–algebras defined as

$$E = \{a \in A \mid \varphi(a) = \psi(a)\}, \qquad \varepsilon\colon E \to A\colon a \mapsto a, \quad a \in E.$$

It is clear that

$$E \xrightarrow{\ \varepsilon\ } A \underset{\psi}{\overset{\varphi}{\rightrightarrows}} B$$

is an equalizer diagram in the category of C^*–**alg**.

Further, we define a function $e\colon X \to E$ such that the pair $(e\colon X \to E, \varepsilon)$ is an equalizer of morphisms φ and ψ in the category $\mathcal{R}$. We show that this function is determined by the condition

$$\varepsilon \circ e = f. \tag{3.3}$$

Namely, we let

$$e(x) := f(x), \quad x \in X. \tag{3.4}$$

First of all, we need to verify that the function $e\colon X \to E$ given by the rule (3.4) is well–defined, that is,

$$f(x) \in E \quad \text{whenever} \quad x \in X. \tag{3.5}$$

Since φ and ψ are parallel morphisms from f to g in $\mathcal{R}$, we have

$$\varphi(f(x)) = g(x) = \psi(f(x)).$$

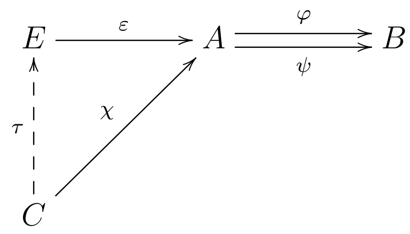
Hence, condition (3.5) holds, as required.

Since $\varepsilon\colon E \to A$ is an injective $*$–homomorphism and $f\colon X \to A$ is an object of the category $\mathcal{R}$, by Axiom **C2**, it follows from the equality (3.3) that the function e is an object of $\mathcal{R}$. Moreover, the equality (3.3) implies that the $*$–homomorphism ε is a morphism of $\mathcal{R}$.

We claim that the pair $(e\colon X \to E, \varepsilon\colon E \to A)$ is an equalizer of the morphisms $\varphi\colon A \to B$ and $\psi\colon A \to B$ in $\mathcal{R}$. Indeed, firstly, we have the equality

$$\varphi \circ \varepsilon = \psi \circ \varepsilon.$$

Secondly, we need to show that the pair (e, ε) possesses the universal property in the category $\mathcal{R}$. To this end, we take a pair $(h\colon X \to C, \chi\colon C \to A)$ consisting of an object h in $\mathcal{R}$ and a morphism χ in $\mathcal{R}$ from h to f such that $\varphi \circ \chi = \psi \circ \chi$. By the universal property of the equalizer (E, ε) in the category C^*–**alg**, there exists a unique $*$–homomorphism $\tau\colon C \to E$ of C^*–algebras making the triangle



commute, that is,

$$\chi = \varepsilon \circ \tau. \tag{3.6}$$

Since the $*$–homomorphism of C^*–algebras χ is a morphism of the category $\mathcal{R}$, we have the equality

$$f = \chi \circ h. \tag{3.7}$$

Using the equalities (3.3), (3.7) and (3.6), we obtain

$$\varepsilon \circ e = f = \chi \circ h = \varepsilon \circ \tau \circ h. \tag{3.8}$$

Since the function ε is a monomorphism in the category of sets and functions, the equality (3.8) implies the equality $e = \tau \circ h$. The latter means that the $*$–homomorphism τ is a morphism in $\mathcal{R}$ from h to e. Thus, the pair (e, ε) is an equalizer of parallel morphisms φ and ψ in $\mathcal{R}$, as claimed. The proof is complete. □

Using Lemma 3.1, Lemma 3.2 and Lemma 2.1, we have

Theorem 3.1. *Every compact C^*–relation is a complete category.*

As an application of Theorem 3.1, we obtain the criterion for the existence of universal C^*–algebra.

Theorem 3.2. *Let $\mathcal{R}$ be a C^*–relation. Then the universal C^*–algebra $C^*(\mathcal{R})$ exists if and only if the category $\mathcal{R}$ complete.*

Proof. By Lemma 2.2, the category $\mathcal{R}$ has a universal representation $i : X \to C^*(\mathcal{R})$ if and only if the C^*–relation $\mathcal{R}$ is compact. By Theorem 3.1, every compact C^*–relation is complete. Conversely, if the C^*–relation $\mathcal{R}$ is complete, then $\mathcal{R}$ has all products and satisfies Axiom **C4**, as required. This completes the proof. □

4. Cocompleteness of compact C^*–relations

In this section we show that every compact C^*–relation is cocomplete. In our proof we employ colimit constructions in the category C^*–**alg** (see [16]).

Lemma 4.1. *Each compact C^*–relation $\mathcal{R}$ on a set X has all coproducts.*

Proof. Let $\{f_\lambda\colon X \to A_\lambda\}_{\lambda\in\Lambda}$ be a family of objects in the category $\mathcal{R}$ and the pair

$$\left(\coprod_{\lambda\in\Lambda} A_\lambda, \{i_\lambda\colon A_\lambda \to \coprod_{\mu\in\Lambda} A_\mu\}_{\lambda\in\Lambda}\right)$$

be a coproduct of the family $\{A_\lambda\}_{\lambda\in\Lambda}$ of C^*–algebras in C^*–**alg** (see [16, Lm. 2.3]).

In the C^*–algebra $\coprod_{\lambda\in\Lambda} A_\lambda$, we consider the closed two-sided ideal I generated by the differences $i_\lambda(f_\lambda(x)) - i_\mu(f_\mu(x))$, where x runs over X and $\lambda, \mu \in \Lambda$:

$$I = \overline{\big\langle \big\{ i_\lambda(f_\lambda(x)) - i_\mu(f_\mu(x)) \;\big|\; x \in X,\ \lambda, \mu \in \Lambda \big\} \big\rangle}.$$

We denote by

$$p\colon \coprod_{\lambda\in\Lambda} A_\lambda \to \coprod_{\lambda\in\Lambda} A_\lambda \Big/ I$$

the canonical $*$–homomorphism between the C^*–algebras.

By the construction of the ideal I, we have

$$p \circ i_\lambda \circ f_\lambda = p \circ i_\mu \circ f_\mu$$

whenever $\lambda, \mu \in \Lambda$. We let $f = p \circ i_\lambda \circ f_\lambda$ for $\lambda \in \Lambda$. By Axiom **C3**, the function f is an object of the category $\mathcal{R}$. Hence, the $*$–homomorphism $p \circ i_\lambda$ is a morphism of $\mathcal{R}$ for every $\lambda \in \Lambda$.

We claim that the pair

$$\left(f\colon X \to \coprod_{\lambda\in\Lambda} A_\lambda \Big/ I, \{ p \circ i_\lambda \colon A_\lambda \to \coprod_{\mu\in\Lambda} A_\mu \Big/ I \}_{\lambda\in\Lambda} \right) \tag{4.1}$$

is a coproduct of the family $\{f_\lambda \colon X \to A_\lambda\}_{\lambda\in\Lambda}$ in the category $\mathcal{R}$. Indeed, we need to verify that (4.1) satisfies the universal property.

To this end, we take a pair

$$(h\colon X \to C, \{g_\lambda \colon A_\lambda \to C\}_{\lambda\in\Lambda})$$

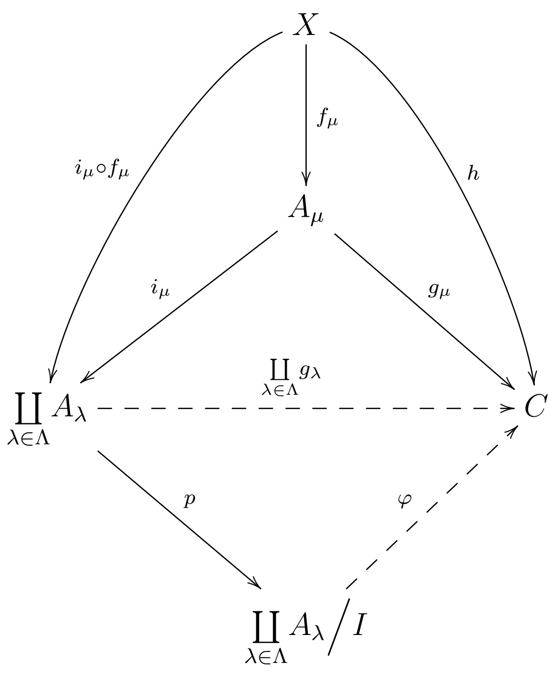
where h is an object of $\mathcal{R}$ and g_λ is a morphism in $\mathcal{R}$ from f_λ to h for every $\lambda \in \Lambda$.

Let us show that there is a unique $*$–homomorphism

$$\varphi\colon \coprod_{\lambda\in\Lambda} A_\lambda \Big/ I \to C$$

such that $\varphi \circ f = h$, that is, φ is a morphism of $\mathcal{R}$ from f to h, and $g_\lambda = \varphi \circ (p \circ i_\lambda)$ for every $\lambda \in \Lambda$.

To do this, for arbitrary index $\mu \in \Lambda$, we consider the diagram



Since $\coprod_{\lambda\in\Lambda} A_\lambda$ is a coproduct in the category C^*–**alg**, there is a unique $*$–homomorphism $\coprod_{\lambda\in\Lambda} g_\lambda$ making the central triangle in the above diagram commute.

For all $\mu,\nu\in\Lambda$, we have

$$\Big(\coprod_{\lambda\in\Lambda} g_\lambda\Big)\circ(i_\mu\circ f_\mu - i_\nu\circ f_\nu) = \Big(\big(\coprod_{\lambda\in\Lambda} g_\lambda\big)\circ i_\mu\circ f_\mu\Big) - \Big(\big(\coprod_{\lambda\in\Lambda} g_\lambda\big)\circ i_\nu\circ f_\nu\Big)$$
$$= \big(g_\mu\circ f_\mu\big) - \big(g_\nu\circ f_\nu\big) = h - h = 0.$$

It follows that the kernel of $\coprod_{\lambda\in\Lambda} g_\lambda$ contains the ideal I, and there is a unique $*$–homomorphism

$$\varphi\colon \coprod_{\lambda\in\Lambda} A_\lambda\Big/ I \to C$$

such that the bottom triangle in the above diagram is commutative, that is,

$$\varphi\circ p = \coprod_{\lambda\in\Lambda} g_\lambda.$$

It is easy to see that $\varphi\circ f = h$. Therefore, φ is a morphism of $\mathcal{R}$. Moreover, we have

$$g_\lambda = \varphi\circ(p\circ i_\lambda) \quad \text{for each} \quad \lambda\in\Lambda.$$

Thus, the required universal property is satisfied, and the pair (4.1) is a coproduct in the category $\mathcal{R}$, as claimed. The proof is complete. □

In the proof of the following statement we use the explicit construction of a coequalizer in the category C^*–**alg** (see [16, Lm. 2.5]).

Lemma 4.2. *Every compact C^*–relation $\mathcal{R}$ on a set X has all coequalizers.*

Proof. We take two objects $f\colon X\to A$ and $g\colon X\to B$ and two parallel morphisms $\varphi\colon A\to B$ and $\psi\colon A\to B$ from f to g in the category $\mathcal{R}$.

In the C^*–algebra B, we construct the closed two–sided ideal I generated by the differences $\varphi(a)-\psi(a)$, where a runs over A:

$$I = \overline{\langle\{\varphi(a)-\psi(a) \mid a\in A\}\rangle}.$$

Let $C = B\big/I$ and $\pi\colon B\to C$ be the canonical surjection. It was shown in the proof of Lemma 2.5 in [16] that

$$A \underset{\psi}{\overset{\varphi}{\rightrightarrows}} B \xrightarrow{\ \pi\ } C$$

is a coequalizer diagram in the category C^*–**alg**.

To construct a coequalizer of the morphisms φ and ψ in the category $\mathcal{R}$, we use Axiom **C3** and define the object $c\colon X\to C$ of $\mathcal{R}$ by

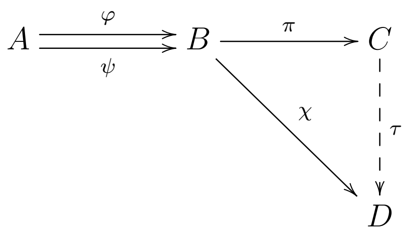
$$c := \pi\circ g, \tag{4.2}$$

which guarantees that the $*$–homomorphism π is a morphism of the category $\mathcal{R}$ from g to c.

We claim that the pair $(c\colon X\to C, \pi\colon B\to C)$ is a coequalizer of the morphisms $\varphi\colon A\to B$ and $\psi\colon A\to B$ in $\mathcal{R}$. Indeed, by the construction of the ideal I, we have the equality

$$\pi\circ\varphi = \pi\circ\psi.$$

We need to prove that the pair (c,π) has the universal property in the category $\mathcal{R}$. To this end, we take a pair $(h\colon X\to D, \chi\colon B\to D)$ consisting of an object h in $\mathcal{R}$ and a morphism χ in $\mathcal{R}$ from g to h such that $\chi\circ\varphi = \chi\circ\psi$. By the universal property of the coequalizer (C,π) in the category C^*–**alg**, there exists a unique $*$–homomorphism $\tau\colon C \to D$ of C^*–algebras making the triangle in the diagram



commute, that is,

$$\chi = \tau \circ \pi. \tag{4.3}$$

It remains to show that the $*$–homomorphism of C^*–algebras τ is a morphism from c to h in the category $\mathcal{R}$. Because the $*$–homomorphism of C^*–algebras χ is a morphism of the category $\mathcal{R}$, we have

$$h = \chi \circ g. \tag{4.4}$$

By the equalities (4.4), (4.3) and (4.2), we get

$$h = \chi \circ g = \tau \circ \pi \circ g = \tau \circ c,$$

which means that τ is a morphism from c to h in the category $\mathcal{R}$, as required. It follows that the pair (c, π) is a coequalizer of parallel morphisms φ and ψ in $\mathcal{R}$, as claimed. This completes the proof. □

As an immediate consequence of Lemma 4.1, Lemma 4.2, Lemma 2.1 and the categorical duality principle [17, Ch. II, Sect. 1], we obtain the following theorem.

Theorem 4.1. *Every compact C^*–relation is a cocomplete category.*

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