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EXTREME POINT OF COMPLETELY CONVEX STATE STRUCTURE

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Abstract. It is well–known that the set of states of a given quantum mechanical system is to be closed from the point of view of the operational approach if we want to make mixed states or convex combinations. That is, s_1 and s_2 are states, then the same is to be true for $\lambda s_1 + (1-\lambda)s_2$, where $0 < \lambda < 1$. We can define a convex combination of elements in a linear space, but unfortunately, in the general case the linear space is artificial for the set of states and has no physical meaning, but the procedure of forming the mixtures of states has a natural meaning. This is why we provide an abstract definition of the mixtures, which is independent of the linearity notion. We call this space a convex structure.

In the work we consider state spaces, generalized state spaces, in which we select pure states, define operations and effects associated with the operations.

We also consider ultraproducts of the sequences of these structures, operations and effects.

Keywords: generalized states, convex states, operation, ultraproducts.

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1. Introduction

Various authors give various definitions of state or state space. For instance, according to Sigal, the state is a real function on the set of bounded observables, which possesses some properties. From the point of view of Mackey [1], this definition admits too many states since not each state in the sense of Sigal associates some probability distribution to each bounded observable. Sigal states are all limits of such states, which indeed associates a probability distribution to each bounded observable. In applications these ideal "limiting states" often turn out to be convenient.

On the other hand, the state is defined as some function acting from the structure of events to the unit segment, see [2]. The value of this function is interpreted as a probability that some event occurs at the current state. Finally, the state space is defined axiomatically and it is a convex structure and even a complete metric space.

In the paper we consider the state spaces and generalized space states. These definitions of the states make the studies more flexible. In the work we also study various approaches to the notion of the operator in the state space, see [2], [3], pure operations and effects associated with the operations.

The work is devoted to defining and studying ultraproducts of abstract state spaces. It is shown that completely convex state structures are stable with respect to the ultraproducts. Ultraproducts of sequences of operations are also considered, and we prove that the ultraproduct of pure states, in the general case, is not a pure operation.

2. Notion of state mixture. Preliminaries

We first recall some definitions.

Definition 2.1 (see, for instance, [2]). *Let $\mathcal{E}$ be a non–empty set, S be the set of functions from $\mathcal{E}$ into the unit interval* $[0,1]$. *A pair $(\mathcal{E}, S)$ is called the event structure if the following two axioms hold:*

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A1. *If* $s(a) = s(b)$ *for each* $s \in S$*, then* $a = b$*;*

A2. *If* $a_1, a_2, \ldots \in \mathcal{E}$ *satisfy the condition* $s(a_i) + s(a_j) \leqslant 1$*,* $i \neq j$*, for each* $s \in S$*, then there exists an element* $b \in \mathcal{E}$*, such that*

$$s(b) + s(a_1) + s(a_2) + \ldots. = 1$$

for each $s \in S$*.*

The study of the properties of event structures and their ultraproducts can be found in work [6].

Definition 2.2 (see, for instance, [2]). *Let* $(\mathcal{E}, S)$ *be an event structure. The elements of the set* $\mathcal{E}$ *are called the events, while the elements of the set* S *are called the states. The state set* S *is called the convex structure if it possesses the following two properties:*

1. *For all positive numbers* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *such that* $\sum_{i=1}^{n} \lambda_i = 1$ *and all states* $s_1, s_2, \ldots, s_n$ *there exists a unique element*
$$\langle \lambda_1, \lambda_2, \ldots, \lambda_n; s_1, s_2, \ldots, s_n \rangle \in S;$$
2. $\langle \lambda_1, \ldots, \lambda_n; s, s, \ldots, s \rangle = s$.

The defined in this way state $\langle \lambda_1, \lambda_2, \ldots, \lambda_n; s_1, s_2, \ldots, s_n \rangle$ is called the mixture of states $s_1, s_2, \ldots, s_n$. For a mixture of two states we employ a simpler notation $\langle \lambda, 1-\lambda; s, t \rangle = \langle \lambda; s, t \rangle$. A state $s \in S$ is called pure if it can not be written as $s = \langle \lambda; t_1, t_2 \rangle$ for some $t_1 \neq t_2$.

We are going to introduce the notion of distance in a convex structure S. The closeness of states s and t can be measured by comparing the mixtures $\langle \lambda; s_1, s \rangle$ and $\langle \lambda; t_1, t \rangle$ with other states.

Definition 2.3. *We define the distance* $\sigma(s, t)$ *for two states* $s, t \in S$ *as follows: if there exist two states* $s_1, t_1 \in S$ *such that the condition* $\langle \lambda; s_1, s \rangle = \langle \lambda; t_1, t \rangle$ *holds, then*

$$\sigma(s, t) = \inf \{0 < \lambda \leqslant 1 : \langle \lambda; s_1, s \rangle = \langle \lambda; t_1, t \rangle\};$$

otherwise, $\sigma(s, t) = 1/2$.

In the general case this function is not metric.

Definition 2.4. *A convex structure* S *is called the* σ*–convex structure if the following conditions are satisfied:*

1. *If*
$$s_n \in S \quad \text{and} \quad \lim_{n,m\to\infty} \sigma(s_n, s_m) = 0,$$
then there exists a unique state $s \in S$ *such that*
$$\lim_{n\to\infty} \sigma(s_n, s) = 0.$$
2. *If*
$$\lambda_i > 0, \qquad \sum_{i=1}^{\infty} \lambda_i = 1, \qquad t_1, t_2, \dots \in S,$$
and
$$s_n = \left\langle \lambda_1, \ldots, \lambda_n, \sum_{i=n+1}^{\infty} \lambda_i; t_1, \ldots, t_n, t_{n+1} \right\rangle,$$
then
$$\lim_{n,m\to\infty} \sigma(s_n, s_m) = 0.$$

Thus, we can consider mixtures of countably many states.

Definition 2.5. *A mapping* $f : S \to \mathbb{R}$ *is called the affine functional if*

$$f(\langle \lambda_1, \lambda_2, \ldots, \lambda_n; s_1, s_2, \ldots, s_n \rangle) = \sum_{i=1}^{n} \lambda_i f(s_i)$$

for each set of states $s_1, \ldots, s_n$ *and positive numbers* $\lambda_1, \ldots, \lambda_n$*, for which* $\sum_{i=1}^{n} \lambda_i = 1$.

We note that the set of affine functionals S^* is a linear space with respect to pointwise operations. We define zero and unit functionals: $\mathbf{0}(s) = 0$, $\mathbf{e}(s) = 1$ for all $s \in S$. On S^* we define a partial order relation: $f \leqslant g \Leftrightarrow f(s) \leqslant g(s)$ for all $s \in S$. A functional $f \in S^*$ is called the effect if $\mathbf{0} \leqslant f \leqslant \mathbf{e}$. We denote the set of effects by $\mathcal{E}(S)$. It forms a convex subset in the linear space S^*.

Definition 2.6. *A completely convex structure is a σ–convex structure possessing the following property: if $f(s) = f(t)$ for each effect f, then $s = t$.*

The meaning of the latter definition is that it fails in some quantum systems, see [7]. It is well–known [2] that if Definition 2.6 is satisfied, then σ is a metric and the space (S, σ) is a complete metric space.

Extreme points of the convex subset $\mathcal{E}(S)$ of the linear space S^* are called the questions. The set of questions $\mathcal{P}(S) \subseteq S^*$ inherits the order S^* and this is why it is a partially ordered set with the smallest element $\mathbf{0}$ and the greatest element $\mathbf{e}$.

Now we equip S^* with a weak $*$–topology. This is a natural topology for S^*, since in this topology a sequence of effects f_n converges to an effect f if and only if $f_n(s) \to f(s)$ for each state s, $n \to \infty$.

Let S be a convex structure. We define the set $S_+ = \{(\alpha, s) : \alpha \geqslant 0, s \in S\}$ and we let $(\alpha, s) = (\beta, t)$ if $\alpha = \beta \neq 0$ and $s = t$, and $(0, s) = (0, t) = 0$ for all $s, t \in S$. If S is a state set, we call S_+ a generalized state set. Then we define a convex structure on S_+ by letting

$$\langle \lambda_1, \lambda_2, \ldots, \lambda_n; (\alpha_1, s_1), \ldots, (\alpha_n, s_n) \rangle = \left(\sum_{i=1}^{n} \lambda_i \alpha_i, \left\langle \frac{\lambda_1 \alpha_1}{\sum_{i=1}^{n} \lambda_i \alpha_i}, \ldots, \frac{\lambda_n \alpha_n}{\sum_{i=1}^{n} \lambda_i \alpha_i}; s_1, \ldots, s_n \right\rangle \right).$$

Here we identify an element of form $(1, s) \in S_+$ as an element $s \in S$. In the same way as in the case of state space S, the generalized state space can be treated as a complete metric space.

We denote by S_+^* the set of affine functionals for S_+. It is known [2] that if $f \in S^*$, then there exists a unique extension $\hat{f} \in S_+^*$, and if $\hat{f} \in S_+^*$, then $\hat{f}((\alpha, s)) = \alpha f(s)$ for all $(\alpha, s) \in S_+$. In particular, there exists a unique extension of the unit functional $\hat{\mathbf{e}} \in S_+^*$ and $\hat{\mathbf{e}}((\alpha, s)) = \alpha$.

Definition 2.7. *An operation is an affine mapping $F : S_+ \to S_+$ obeying the condition*

$$\hat{\mathbf{e}}(F(w)) \leqslant \hat{\mathbf{e}}(w) \tag{2.1}$$

for all $w = (\alpha, s) \in S_+$, $0 \leqslant \alpha \leqslant 1$.

Here we note that the operation describes the change of the state related with some external action, and the operation preserves the state mixtures. If F is an operation such that

$$(\alpha, t) \in S_+, \quad F((\alpha, t)) = (\alpha', t'),$$

then it can be considered as a mapping consisting of two parts: $\alpha \to \alpha'$, $t \to t'$. The part $t \to t'$ is a distortion of a state, while $\alpha \to \alpha'$ is a degree of state weakening due to the action. For $s \in S$ we interpret $\hat{e}(F(s))$ as the probability of transferring the state s due to the operation F.

For the operation F we define a linear mapping $F^* : S_+^* \to S_+^*$ as

$$(F^* \hat{f})(w) = \hat{f}(F(w))$$

for each $\hat{f} \in S_+^*$, $w \in S_+$.

With each operation F we associate its effect defined as

$$f = F^*(\hat{\mathbf{e}})|S.$$

Since $\hat{\mathbf{e}}(F(s)) = (F^* \hat{\mathbf{e}})(s) = f(s)$ for each $s \in S$, the effect f determines only the probability of transferring the state. Thus, the operation itself contains more information than the associated effect.

As in the case of usual states, we introduce the notion of extreme points (pure states) in the generalized state space S_+.

Definition 2.8. *An operation is called pure if it transforms pure states into pure states.*

Example 2.1. *Let H be a complex separable Hilbert space and S be the set of density operators on H. In this case S is a σ–convex structure, see [2]. Let us consider the set of bounded self–adjoint operators A on H obeying the condition $0 \leqslant A \leqslant I$. We let $\mathbf{e} = I$. We define the effect*

$$A(s) = \mathbf{Tr}(As).$$

It is known [3], [4] *that such effects describe all effects on* S, *and the effect* A *is a question if and only if* A *is a projection. A generalized state* $w = (\alpha, s) \in S_+$ *is defined as an operator* αs, $0 < \alpha \leqslant 1$, $s \in S$. *Thus, the class* S_+ *can be regarded as a class of operators with a positive trace.*

In this case

$$\hat{\mathbf{e}}(w) = \mathbf{Tr}(w) = \alpha \mathbf{Tr}(s).$$

Now we approach the conception of the operation from a bit another point of view.

Definition 2.9 ([3]). *Let* S_1 *and* S_2 *be state spaces in Hilbert spaces* H_1 *and* H_2 *(finite– or infinite–dimensional) being* σ*–convex structures, respectively. The operation on* S_1 *is defined as a positive linear mapping* $T : S_1 \to S_2$ *obeying the condition*

$$\mathbf{e}_2(T(s)) \leqslant \mathbf{e}_1(s) \quad \text{or} \quad (\,\mathbf{Tr}(T(s)) \leqslant \mathbf{Tr}(s)\,)$$

for all $s \in S_1$, *where* $\mathbf{e}_1$ *is the unity in* S_1, *and* $\mathbf{e}_2$ *is the unity in* S_2.

If T is a pure operation, the structure of this operation is well–known [3], namely, it is represented in one of the following forms:

$$T(s) = BsB^*,$$

or

$$T(s) = \mathbf{Tr}(sB)|\psi >< \psi|,$$

where $B : H_1 \to H_2$ is a linear bounded operator, $\psi \in H_2$. In the latter case the operation T is degenerate, that is, T maps all states $s \in S_1$ into a single state in S_2 defined by a vector $\psi \in H_2$. In these cases the pure operation possesses a property $\mathbf{e}_2(T(s)) = \mathbf{e}_1(s)$, or, in other words, the pure operation preserves the effect.

3. Ultraproducts of completely convex structures

Definition 3.1. *Let* (S_n, σ_n) *be a sequence of completely convex structures,* $\mathscr{U}$ *be a nontrivial ultrafilter in the set of natural numbers* $\mathbb{N}$. *We consider the Cartesian product* $\prod_{n=1}^{\infty} S_n$ *of a sequence* (S_n) *and introduce in it an equivalence relation by letting*

$$(s_n) \sim (t_n) \Leftrightarrow \lim_{\mathscr{U}} \sigma_n(s_n, t_n) = 0.$$

The set of all equivalence classes $\prod_{n=1}^{\infty} S_n$ *defined by this relation is called the ultraproduct of the sequence* (S_n) *and it is denoted by* $S_{\mathscr{U}} = (S_n)_{\mathscr{U}}$.

In the ultraproduct $(S_n)_{\mathscr{U}}$ *we naturally define the metric* $\sigma_{\mathscr{U}}$ *by letting*

$$\sigma_{\mathscr{U}}(s, t) = \lim_{\mathscr{U}} \sigma_n(s_n, t_n), \qquad t = (t_n)_{\mathscr{U}}, \qquad s = (s_n)_{\mathscr{U}},$$

and effects

$$f_{\mathscr{U}}(s) = \lim_{\mathscr{U}} f_n(s_n), \qquad f_n \in \mathcal{E}(S_n), \qquad s = (s_n)_{\mathscr{U}}.$$

The pair $(S_{\mathscr{U}}, \sigma_{\mathscr{U}})$ *is called the ultraproduct of sequence of completely convex structures.*

Theorem 3.1. *Let* $(S_n, \sigma_n)_{n\geqslant 1}$ *be a sequence of completely convex structures,* $\mathscr{U}$ *be a nontrivial ultrafilter in the set of natural numbers* $\mathbb{N}$. *Then the ultraproduct* $(S_{\mathscr{U}}, \sigma_{\mathscr{U}})$ *is a completely convex structure.*

Proof. We are going to show that the ultraproduct preserves the structure of completely convex space.

The state mixtures are introduced in a natural way: for all positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that $\sum_{i=1}^{m} \lambda_i = 1$ and all states $(s_n^1)_{\mathscr{U}}, (s_n^2)_{\mathscr{U}}, \ldots, (s_n^m)_{\mathscr{U}}$ we let

$$\langle \lambda_1, \lambda_2, \ldots, \lambda_m; (s_n^1)_{\mathscr{U}}, (s_n^2)_{\mathscr{U}}, \ldots, (s_n^m)_{\mathscr{U}} \rangle = \langle \lambda_1, \lambda_2, \ldots, \lambda_m; s_n^1, s_n^2, \ldots, s_n^m \rangle_{\mathscr{U}} \in S_{\mathscr{U}}.$$

It is obvious that

$$\langle \lambda_1, \ldots, \lambda_m; (s_n)_{\mathscr{U}}, (s_n)_{\mathscr{U}}, \ldots, (s_n)_{\mathscr{U}} \rangle = (s_n)_{\mathscr{U}}.$$

We consider a sequence $((s_n^k)_{\mathscr{U}})_{k\geqslant 1}$ such that

$$\lim_{k,m\to\infty} \sigma_{\mathscr{U}}((s_n^k)_{\mathscr{U}}, (s_n^m)_{\mathscr{U}}) = 0.$$

Then for each $\varepsilon > 0$ there exist an element $U \in \mathscr{U}$ and $N \in \mathbb{N}$ such that for all $n \in U$ and all $k > N$, $m > N$ we have $\sigma_n(s_n^k, s_n^m) < \varepsilon$. Then for all $n \in U$ there exists a state s_n such that $\lim_{k\to\infty} \sigma_n(s_n^k, s_n) = 0$. Therefore,

$$\lim_{k\to\infty} \sigma_{\mathscr{U}}((s_n^k)_{\mathscr{U}}, (s_n)_{\mathscr{U}}) = 0.$$

This shows that the ultraproduct of σ–convex structures is a σ–convex structure. The definition of the ultraproduct $(S_n)_{\mathscr{U}}$ and the effect on it implies that $(S_{\mathscr{U}}, \sigma_{\mathscr{U}})$ is a completely convex structure. □

We denote by $\hat{\sigma}$ the metric in S_+. Since $(S_+, \hat{\sigma})$ is a completely convex structure with respect to this metric, the definition of the ultraproduct of sequence of generalized state spaces coincides with the previous definition. We note that in this case

$$S_{\mathscr{U}+} = S_{+\mathscr{U}} = (\tilde{\mathbb{R}}_+)_{\mathscr{U}} \times S_{\mathscr{U}},$$

where

$$(\tilde{\mathbb{R}}_+)_{\mathscr{U}} = \{(\alpha_n)_{\mathscr{U}} : \alpha_n \geqslant 0, \sup_n \alpha_n < \infty\}.$$

Theorem 3.2. *Let $(S_{n+}, \hat{\sigma}_n)_{n\geqslant 1}$ be a sequence of generalized state spaces, $\mathscr{U}$ be a nontrivial filter in the space of natural numbers $\mathbb{N}$. Then the ultraproduct $(S_{\mathscr{U}+}, \hat{\sigma}_{\mathscr{U}})$ is a completely convex structure.*

The proof follows from Theorem 3.1.

Definition 3.2. *Suppose that we are given two sequences of state spaces $(S_n^{(1)})_{n\geqslant 1}$ and $(S_n^{(2)})_{n\geqslant 1}$ being completely convex structures, operations $T_n : S_n^{(1)} \to S_n^{(2)}$, and a nontrivial ultrafilter $\mathscr{U}$ in the set of natural numbers $\mathbb{N}$. We define the ultraproduct of sequence of operations (T_n) as the mapping $T_{\mathscr{U}} : S_{\mathscr{U}}^{(1)} \to S_{\mathscr{U}}^{(2)}$, where*

$$T_{\mathscr{U}}(s_{\mathscr{U}}^{(1)}) = (T_n(s_n^{(1)}))_{\mathscr{U}}, \qquad s_{\mathscr{U}}^{(1)} = (s_n^{(1)})_{\mathscr{U}}.$$

It is easy to see that the ultraproduct of sequence of operations is an operation. The natural question arises: whether the ultraproduct of sequence of pure operations is a pure operation? In the general case the answer turns out to be negative, as the following counterexample shows.

We are going to construct two sequences of state spaces and a sequence of pure operations such that the ultraproduct of the latter is not a pure operation. Let $S_n^{(1)}$ be the set of $H_n^{(1)}$–quasi–invariant probability measures defined on a measurable space $(\Omega_n^{(1)}, \Sigma_n^{(1)}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where

$$H_n^{(1)} = \{x \in \mathbb{R} : |x| < \infty\}, \qquad n \in \mathbb{N}.$$

It is well–known that the Gaussian measures are $H_n^{(1)}$–quasi–invariant and ergodic with respect to the translations by the elements of $H_n^{(1)}$, and hence, (see, for instance, [5]), are extreme points in the set of $H_n^{(1)}$–quasi–invariant measures, that is, in the set $S_n^{(1)}$, $n \in \mathbb{N}$. The pair $(\Sigma_n^{(1)}, S_n^{(1)})$ is an event structure (see, for instance, [6]), and at the same time $S_n^{(1)}$ is a completely convex structure, $n \in \mathbb{N}$.

Let $S_n^{(2)}$ be a set of $H_n^{(2)}$–quasi–invariant probability measures $\mu^n = \prod_{k=1}^n \mu_k$, $\mu_k \in S_n^{(1)}$, defined on a measurable space $(\Omega_n^{(2)}, \Sigma_n^{(2)}) = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, $n \in \mathbb{N}$, where

$$H_n^{(2)} = \{x_n \in \mathbb{R}^n : \|x_n\| < \infty\}, \qquad n \in \mathbb{N}.$$

At the same time the Gaussian measures are extreme points in $S_n^{(2)}$ since they are $H_n^{(2)}$–ergodic, $n \in \mathbb{N}$. The pair $(\Sigma_n^{(2)}, S_n^{(2)})$ is an event structure, $S_n^{(2)}$ is a completely convex structure, $n \in \mathbb{N}$.

We consider a pure operation $T_n : S_n^{(1)} \to S_n^{(2)}$ letting

$$T_n(\mu_n) = \mu^n,$$

where μ_n is the Gaussian measure from $S_n^{(1)}$ with the parameters $N(0,1)$, μ^n is the Gaussian measure from $S_n^{(2)}$ with the parameters $N(0, I_n)$, where I_n is the unit matrix, $n \in \mathbb{N}$.

We consider ultraproducts of sequences of two event structures $(\Sigma_n^{(1)}, S_n^{(1)})_{n\geqslant 1}$ and $(\Sigma_n^{(2)}, S_n^{(2)})_{n\geqslant 1}$ with respect to the nontrivial ultrafilter $\mathscr{U}$ on the set of natural numbers $\mathbb{N}$. In this case the state spaces are represented as

$$S_{\mathscr{U}}^{(1)} = \Big\{\mu_{\mathscr{U}}^{(1)} : \mu_{\mathscr{U}}^{(1)}(\,\cdot\,) = \lim_{\mathscr{U}} \mu_n(\,\cdot\,)\Big\}, \qquad S_{\mathscr{U}}^{(2)} = \Big\{\mu_{\mathscr{U}}^{(2)} : \mu_{\mathscr{U}}^{(2)}(\,\cdot\,) = \lim_{\mathscr{U}} \mu^n(\,\cdot\,)\Big\},$$

for more detail see [6]. Then the measure $\mu^{(1)}_{\mathscr{U}}$ is $(\tilde{H}^{(1)}_n)_{\mathscr{U}}$–quasi–invariant, where

$$\tilde{H}^{(1)}_n = \Big\{x_n \in \mathbb{R} : \sup_n |x_n| < \infty\Big\},$$

the measure $\mu^{(2)}_{\mathscr{U}}$ is $(\tilde{H}^{(2)}_n)_{\mathscr{U}}$–quasi–invariant,

$$\tilde{H}^{(2)}_n = \Big\{x_n \in \mathbb{R}^n : \sup_n \|x_n\| < \infty\Big\}.$$

The measure $\mu^{(1)}_{\mathscr{U}}$ is also $(\tilde{H}^{(1)}_n)_{\mathscr{U}}$–ergodic, therefore, it is an extreme point of the set $S^{(1)}_{\mathscr{U}}$, but the measure $\mu^{(2)}_{\mathscr{U}}$ is not an extreme point of $S^{(2)}_{\mathscr{U}}$ since it is not $(\tilde{H}^{(2)}_n)_{\mathscr{U}}$–ergodic, see [5].

We define the operation $T_{\mathscr{U}} : (S^{(1)}_n)_{\mathscr{U}} \to (S^{(2)}_n)_{\mathscr{U}}$ by letting $T_{\mathscr{U}}(\,\cdot\,) = \lim\limits_{\mathscr{U}} T_n(\,\cdot\,)$. Then the ultraproduct of pure operations

$$T_{\mathscr{U}}(\mu^{(1)}_{\mathscr{U}}) = \lim_{\mathscr{U}} T_n(\mu_n) = \lim_{\mathscr{U}} \mu^n = \mu^{(2)}_{\mathscr{U}}$$

is not a pure operation. Let us formulate the result.

Theorem 3.3. *There exist two sequences of state spaces and a sequence of pure operations such that the ultraproduct of latter is not a pure operation.*

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