UNKNOWN COEFFICIENT PROBLEM FOR MIXED EQUATION OF PARABOLIC-HYPERBOLIC TYPE WITH NON-LOCAL BOUNDARY CONDITIONS ON CHARACTERISTICS

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Abstract. For an equation of a mixed parabolic-hyperbolic type with a characteristic line of type change, we study the inverse problem associated with the search for an unknown coefficient at the lowest term of the parabolic equation. In the direct problem, we consider an analog of the Tricomi problem for this equation with a nonlocal condition on the characteristics in the hyperbolic part and the Dirichlet condition in the parabolic part of the domain. In order to determine the unknown coefficient by the solution on the parabolic part of the domain, the integral overdetermination condition is proposed. Global results on the unique solvability of the inverse problem in the sense of the classical solution are proved.

Keywords: parabolic-hyperbolic equation, characteristic, Green's function, inverse problem, contraction principle mapping.

Mathematics Subject Classification: 35A01; 35A02; 35L02; 35L03; 35R03.

1. Formulation of problem

Let Ω_{lT} be a domain in the plane of variables x, y, consisting of the union of two subdomains, i.e. $\Omega_{lT} = \Omega_{1lT} \cup \Omega_{2l}$, where

$$\Omega_{1lT} = \Big\{(x,y):\, 0 < x < l,\, 0 < y \leqslant T\Big\},$$

$$\Omega_{2l} = \Big\{(x,y):\, -y < x \leqslant y + l,\, -\frac{l}{2} < y < 0\Big\},$$

and l, T are fixed positive numbers. In this domain we consider the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1-\operatorname{sign} y}{2}\frac{\partial^2 u}{\partial y^2} - \frac{1+\operatorname{sign} y}{2}\frac{\partial u}{\partial y} - \frac{1+\operatorname{sign} y}{2} q(x)u(x,y) = 0. \tag{1.1}$$

Equation (1.1) is of a mixed parabolic–hyperbolic type. For this equation, the line of change of type $y = 0$ is a characteristic (parabolic degeneration of the second kind [1]).

Direct problem. In the domain Ω_{lT} find the solution of equation (1.1) satisfying the following boundary conditions:

$$u(0,y) = \varphi_1(y), \qquad u(l,y) = \varphi_2(y), \quad y \in [0,T], \tag{1.2}$$

$$u\left(\frac{x}{2}, -\frac{x}{2}\right) + u\left(\frac{x+l}{2}, \frac{x-l}{2}\right) = \psi(x), \quad x \in [0,l]\,, \tag{1.3}$$

where $\varphi_1 = \varphi_1(y)$, $\varphi_2 = \varphi_2(y)$, $\psi = \psi(x)$ are given functions.

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A classical solution to direct problem (1.1)–(1.3) is a function $u(x,y)$ in the class $C\left(\overline{\Omega}_{lT}\right) \cap C^1\left(\Omega_{lT}\right) \cap C^{1,2}_{x,y}\left(\Omega_{1lT}\right) \cap C^2\left(\Omega_{2l}\right)$, which solves equation (1.1) and satisfies conditions (1.2), (1.3).

An *inverse problem* is on finding a pair of functions $u = u(x,y)$, $q = q(x)$, in the classes

$$u \in C\left(\overline{\Omega}_{lT}\right) \cap C^1\left(\Omega_{lT}\right) \cap C^{1,2}_{x,y}\left(\Omega_{1lT}\right) \cap C^2\left(\Omega_{2l}\right), \qquad q \in C[0,l],$$

such that these functions solve equation (1.1) and satisfy boundary conditions (1.2), (1.3) and the following overdetermination condition:

$$\int\limits_0^T h(y)u(x,y)\,dy = f(x), \quad x \in [0,\, l]\,, \tag{1.4}$$

where $h = h(y)$, $f = f(x)$ are given sufficiently smooth functions.

Direct and inverse problems for mixed type equations are not studied in so many details as similar problems for classical equations. Nevertheless, such problems are relevant from the point of view of applications. The importance of considering equations of mixed type, where the equation is of parabolic type in one part of the domain and hyperbolic in the other, was first pointed out by Gel'fand in his work [2]. Another example is the following phenomenon in electrodynamics: a mathematical study of the tension of an electromagnetic field in an inhomogeneous medium consisting of a dielectric and a conducting medium leads to a system consisting of a wave equation and a heat equation, see [3]. There are many examples of such kind.

For the first time, an analogue of the Tricomi problem for a hyperbolic–parabolic equation was studied in [4]. Further, such problems with different boundary and non–local conditions for parabolic–hyperbolic equations with both characteristic and non–characteristic type change lines were formulated and studied in [5]–[8].

Methods for solving direct and inverse problems for finding the solution of an initial boundary value problem for equations of the parabolic-hyperbolic type and the unknown right–hand side (linear problem) of the equation in a rectangular domain were proposed in [9]–[11]. In this direction, we also point out work [12], in which such problems were studied for equations of mixed parabolic–hyperbolic type with the time fractional derivative in the parabolic part of the equation.

Various inverse problems for particular second order equations of hyperbolic and parabolic types can be found in monographs [13]–[16], see also the references therein.

For equations of mixed parabolic–hyperbolic type, inverse coefficient problems were not studied before. This article continues the study of the author [17], in which the local unique solvability of the inverse problem on determining the variable coefficient at the lowest term of a hyperbolic equation for a mixed hyperbolic–parabolic equation with a noncharacteristic line of type change was investigated. Note that the problems considered below, in addition to their independent interest, are also of interest from the point of view of studying the solvability of inverse coefficient problems for parabolic equations.

Throughout this paper we shall assume that the following conditions are satisfied:

(B1) $(\varphi_1(y),\, \varphi_2(y)) \in C^1[0,T]$, $\psi(x) \in C^3\,[0,\lambda]\,$;
(B2) $\varphi_1(0) = \varphi_2(0)$, $\varphi_1(0) - \varphi_2(0) = \psi(0) - \psi(l)$;
(B3) $h(y) \in C^1[0,T]$, $h(0) = h(T) = 0$, $f(x) \in C^3[0,l]$, $\int\limits_0^T h(y)\varphi_1(y)dy = f(0)$,

$$\int\limits_0^T h(y)\varphi_2(y)dy = f(l), \qquad f(x) \neq 0 \quad \text{for all} \quad x \in [0,l].$$

2. Direct problem

Assume that the function $q(x)$ is known.

Theorem 2.1. *Let conditions B1, B2, $q(x)\in C^1[0,l]$ be satisfied. Then there exists an unique solution to direct problem (1.1)–(1.3) in the domain Ω_{lT}.*

We denote $\tau(x):=u(x,0)$, $\nu(x)=\frac{\partial}{\partial y}u(x,0)$. Due to the unique solvability of the Cauchy problem for the wave equation, the solution to equation (1.1) in the domain Ω_{2l} can be written using the d'Alembert formula:

$$u(x,y)=\frac{1}{2}[\tau(x+y)+\tau(x-y)]-\frac{1}{2}\int_{x+y}^{x-y}\nu(s)\,ds. \tag{2.1}$$

Taking into account condition (1.3) and the identities $\tau(0)=\varphi_1(0)$, $\tau(l)=\varphi_2(0)$ (a consequence of the definition of the classical solution), we obtain the identity

$$2\tau(x)+\varphi_1(0)+\varphi_2(0)-\int_0^l\nu(s)ds=2\psi(x),\quad x\in[0,l]. \tag{2.2}$$

It follows from (1.3) at $x=l$, $y=-l$ that

$$u\left(\frac{l}{2},-\frac{l}{2}\right)=\psi(l)-\varphi_2(0).$$

Comparing this with (2.1) at $x=\frac{l}{2}$, $y=-\frac{l}{2}$, we have

$$\int_0^l\nu(s)\,ds=\varphi_1(0)+3\varphi_2(0)-2\psi(l).$$

Substituting this into identity (2.2), we find

$$\tau(x)=\psi(l)-\varphi_2(0)+\psi(x). \tag{2.3}$$

Thus, we have found the function $\tau(x)$.

In order to find $\nu(x)$ we use equation (1.1) in domain Ω_{1lT} and calculate $\lim_{y\to+0}$. Then we easily obtain $\nu(x)=\tau''(x)-q(x)\tau(x)$ and the same

$$\nu(x)=\psi''(x)-q(x)\left(\psi(l)-\varphi_2(0)+\psi(x)\right).$$

It is clear that for known $\tau(x)$ and $\nu(x)$ the solution to direct problem (1.1)–(1.3) in Ω_{2l} is given by formula (2.1). Under the assumptions of Theorem 2.1 we have $u\in C^2(\Omega_{2l})$.

It is known [1] that the Green's function of the first initial boundary value problem for the equation

$$u_{xx}-u_y=0,\quad x\in(0,\,l),\quad y>0,$$

is of the form

$$G(x,\xi,y)=\frac{1}{2\sqrt{\pi y}}\sum_{n=-\infty}^{\infty}\left[\exp\left(-\frac{(x-\xi+2n)^2}{4y}\right)-\exp\left(-\frac{(x+\xi+2n)^2}{4y}\right)\right].$$

In view of this, we rewrite equation (1.1) in the domain Ω_{1lT} with conditions (1.2) to an integral equation

$$u(x,y)=\int\limits_0^l G(x,\xi,y)\tau(\xi)\,d\xi+\int\limits_0^y G_\xi(x,0,y-\eta)\varphi_1(\eta)\,d\eta$$
$$-\int\limits_0^y G_\xi(x,l,y-\eta)\varphi_2(\eta)\,d\eta-\int\limits_0^y\int\limits_0^l G(x,\xi,y-\eta)q(\xi)u(\xi,\eta)\,d\xi d\eta. \tag{2.4}$$

We also note that integral equation (2.4) is of Volterra type since $\tau(x)$ is known. In view of the conditions imposed on φ_1, φ_2 in (B1), this equation determines the function $u\in C^{1,2}_{x,y}(\Omega_{1lT})$, that is, the solution to problem (1.1), (1.2) in the domain Ω_{1lT}.

Thus, the constructed functions in Ω_{1lT} and Ω_{2l} is the classical solution to direct problem (1.1)–(1.3) in the domain Ω_{lT}. This completes the proof of Theorem 2.1.

3. Inverse problem

Assume that conditions (B3) are satisfied. Multiplying equation (1.1) in the domain Ω_{1lT} by the function $h(y)$, integrating over the segment $[0,T]$ and taking into consideration (1.4), we find

$$q(x)=\frac{f''(x)}{f(x)}+\frac{1}{f(x)}\int\limits_0^T h'(y)u(x,y)\,dy,\quad x\in[0,l]. \tag{3.1}$$

Using this formula, we eliminate the function $q(x)$ in (2.4) and write the resulting equation in the operator form:

$$u(x,y)=U[u](x,y),\quad (x,y)\in\overline{\Omega}_{1lT}, \tag{3.2}$$

where the operator U is defined by the identity

$$Uu(x,y)=u_0(x,y)+\int\limits_0^y\int\limits_0^l G(x,\xi,y-\eta)\left[\frac{f''(\xi)}{f(\xi)}+\frac{1}{f(\xi)}\int\limits_0^T h'(s)u(\xi,s)\,ds\right]u(\xi,\eta)\,d\xi d\eta, \tag{3.3}$$

and here u_0 denotes the sum of terms of integral equation, which do not involve the unknown function:

$$u_0(x,y):=\int\limits_0^l G(x,\xi,y)\tau(\xi)d\xi+\int\limits_0^y G_\xi(x,0,y-\eta)\varphi_1(\eta)d\eta-\int\limits_0^y G_\xi(x,l,y-\eta)\varphi_2(\eta)d\eta.$$

We recall that the function $\tau(x)$ has been defined by formula (2.3).

The main result of this section is the following theorem.

Theorem 3.1. *Let conditions (B1)–(B3) be satisfied. Then, there exist positives numbers T^* such that equation (3.2) has an unique continuous solution in the domain Ω_{1lT} for $T\in(0,T^*)$.*

Proof. Owing to (3.3) it is clear that under the assumptions of the theorem the operator U maps a function $u\in C\left(\overline{\Omega}_{1lT}\right)$ into a function belonging to the same space. We define the norm in $C\left(\overline{\Omega}_{1lT}\right)$ as follows:

$$\|v\|_{lT}=\max_{(x,y)\in\overline{\Omega}_{1lT}}|u(x,y)|\,.$$

For the sake of brevity, we introduce notations

$$f_0:=\min_{x\in[0,l]}|f(x)|\,,\qquad f_1:=\max_{x\in[0,l]}|f''(x)|\,,\qquad h_0:=\max_{x\in[0,T]}|h'(y)|\,.$$

We are going to show that for sufficiently small T the operator U is a contraction of the ball

$$S(u_0,r):=\left\{u\in C\left(\overline{\Omega}_{1lT}\right):\,\|u-u_0\|_{lT}\leqslant r\right\}\subset C\left(\overline{\Omega}_{1lT}\right)$$

into itself, where a radius r is a known number and a center is an element $u_0 = u_0(x, y)$ of the functional space $C\left(\overline{\Omega}_{1lT}\right)$. This fact will prove that in the domain $\overline{\Omega}_{1lT}$ equation (3.2) has an unique continuous solution satisfying the inequality $\|u - u_0\|_{lT} \leqslant r$.

It is obvious that each element $u \in S\left(u_0, r\right)$ satisfies an estimate

$$\|u\|_{lT} \leqslant \|u_0\|_{lT} + r =: R,$$

where R denotes a known positive number.

Let us estimate $\|u_0\|_{lT}$. In order to do this, we need estimates for integrals involving the functions G, G_ξ in the definitions of the function $u_0(x, y)$. In this case, we use the identity

$$\int\limits_0^l G(x, \xi, y) d\xi = 1,$$

which follows from the definition of the function G. Taking this into account and explicit form (2.3) of the function $\tau(x)$, the first term of $u_0(x, y)$ can be easily estimated:

$$\left|\int\limits_0^l G(x, \xi, y)\tau(\xi) d\xi\right| \leqslant \|\varphi_2\|_{C[0,T]} + 2\|\psi\|_{C[0,l]}. \tag{3.4}$$

We then observe that G has an expression [3]:

$$G(x, \xi, y) = \frac{2}{l}\sum_{n=1}^{\infty} \exp\left[-\left(\frac{n\pi}{l}\right)^2 y\right] \sin\frac{n\pi x}{l} \sin\frac{n\pi\xi}{l}.$$

In view of this expression we have the identities

$$G_\xi(x, 0, y - \eta) = \frac{2}{l}\sum_{n=1}^{\infty} \exp\left[-\left(\frac{n\pi}{l}\right)^2 (y - \eta)\right] \frac{n\pi}{l} \sin\frac{n\pi x}{l} = \frac{1}{l}\int\limits_0^l G_\eta(x, \xi, y - \eta)\,(l - \xi)\, d\xi,$$

which can be confirmed straightforwardly. Using these relations, we transform the following integral:

$$\begin{aligned}
\int\limits_0^y G_\xi(x, 0, y - \eta)\varphi_1(\eta) d\eta =& \frac{1}{l}\int\limits_0^l (l - \xi)\int\limits_0^y G_\eta(x, \xi, y - \eta)\varphi_1(\eta)\, d\eta d\xi \\
=& \frac{1}{l}\int\limits_0^l (l - \xi)\left\{\Big[G(x, \xi, y - \eta)\varphi_1(\eta)\Big]_0^y - \int\limits_0^y G(x, \xi, y - \eta)\varphi_1'(\eta)\, d\eta\right\} d\xi \\
=& \frac{l - x}{l}\varphi_1(y) \\
& - \frac{1}{l}\int\limits_0^l (l - \xi)\left[G(x, \xi, y)\varphi_1(0) + \int\limits_0^y G(x, \xi, y - \eta)\varphi_1'(\eta)\, d\eta\right] d\xi.
\end{aligned}$$

Here in the intermediate calculations we have used the relation $\lim_{\eta\to y} G(x, \xi, y - \eta) = \delta(x - \xi)$, where $\delta(\cdot)$ is the Dirac delta function and

$$\int\limits_0^l a(\xi)\delta(x - \xi)\, d\xi = a(x),$$

which is valid for each continuous function $a(x)$ on the interval $(0, l)$.

These relations for $(x,y) \in C\left(\overline{\Omega}_{1lT}\right)$ yield the estimate

$$\left| \int_0^y G_\xi(x,0,y-\eta)\varphi_1(\eta)\,d\eta \right| \leqslant (2+T)\,\|\varphi_1\|_{C^1[0,T]}\,. \tag{3.5}$$

For $G_\xi(x,l,y-\eta)$ we then have

$$G_\xi(x,l,y-\eta) = \frac{2}{l}\sum_{n=1}^{\infty} \exp\left[-\left(\frac{n\pi}{l}\right)^2 (y-\eta)\right](-1)^n \frac{n\pi}{l}\sin\frac{n\pi x}{l} = -\frac{1}{l}\int_0^l G_\eta(x,\xi,y-\eta)\xi\,d\xi.$$

Using this, we transform the last term of $u_0(x,y)$ as follows:

$$\begin{aligned}
\int_0^y G_\xi(x,l,y-\eta)\varphi_2(\eta)\,d\eta &= -\frac{1}{l}\int_0^l \xi \int_0^y G_\eta(x,\xi,y-\eta)\varphi_2(\eta)\,d\eta d\xi \\
&= -\frac{1}{l}\int_0^l \xi \left\{ \Big[G(x,\xi,y-\eta)\varphi_2(\eta)\Big]_0^y - \int_0^y G(x,\xi,y-\eta)\varphi_2'(\eta)\,d\eta \right\} d\xi \\
&= -\frac{x}{l}\varphi_2(y) + \frac{1}{l}\int_0^l \xi \left[G(x,\xi,y)\varphi_2(0) + \int_0^y G(x,\xi,y-\eta)\varphi_2'(\eta)d\eta \right] d\xi.
\end{aligned}$$

For $(x,y) \in C\left(\overline{\Omega}_{1lT}\right)$ we also have the estimate

$$\left| \int_0^y G_\xi(x,l,y-\eta)\varphi_2(\eta)d\eta \right| \leqslant (2+T)\,\|\varphi_2\|_{C^1[0,T]}\,. \tag{3.6}$$

Then inequalities (3.4)–(3.6) imply the estimate

$$\|u_0\|_{lT} \leqslant (2+T)\,\|\varphi_1\|_{C^1[0,T]} + (3+T)\,\|\varphi_2\|_{C^1[0,T]} + 2\,\|\psi\|_{C[0,l]}\,. \tag{3.7}$$

We observe that the right side of estimate (3.7) is independent of l.

We now proceed to obtaining conditions for T, under which it is possible to apply the fixed point theorem to the operator U. Let $u \in S\left(u_0, r\right)$, then, for all $(x,y) \in \overline{\Omega}_{1lT}$ we have the inequalities

$$\begin{aligned}
|Uu - u_0| &\leqslant \int_0^y \int_0^l G(x,\xi,y-\eta)\left[\frac{|f''(\xi)|}{|f(\xi)|} + \frac{1}{|f(\xi)|}\int_0^T |h'(s)|\,|u(\xi,s)|\,ds\right] |u(\xi,\eta)|\,d\xi d\eta \\
&\leqslant \frac{R^2 h_0}{f_0}T^2 + \frac{f_1 R}{f_0}T =: m_1(T).
\end{aligned}$$

Condition $\|u-u_0\|_{lT} \leqslant r$ (that is, $Uu \in S(u_0,r)$) is satisfied if T is chosen by the condition $m_1(T) < r$. Let T_1 be a positive root of the quadratic equation $m_1(T) - r = 0$, that is

$$T_1 = \frac{1}{2Rh_0}\left[\sqrt{f_1^2 + 4rf_0h_0} - f_1\right].$$

It is easy to see that $m_1(T)$ increases monotonically in $T \in (0,T_1)$, $m_1(T) \to 0$ at $T \to 0$ and $m_1(T) < r$ for $T \in (0,T_1)$. This means that $Uu \in S(u_0,r)$ for $T < T_1$.

Let us now to show that the operator U contracts the distance between elements of the ball $S(u_0,r)$. To prove this, we take any two elements $(u^1,\, u^2) \in S(u_0,r)$ and estimate the norm of the difference between their images Uu^1, Uu^2. For this purpose, using (3.3) we have the inequality

$$\left\|Uu^1 - Uu^2\right\|_{lT} \leqslant \int\limits_0^y \int\limits_0^l G(x,\xi,y-\eta)\left[\left|\frac{f''(\xi)}{f(\xi)}\right| \left|u^1(\xi,\eta) - u^2(\xi,\eta)\right| \right.$$
$$\left. + \frac{1}{|f(\xi)|}\int\limits_0^T |h'(s)| \left|u^1(\xi,s)u^1(\xi,\eta) - u^2(\xi,s)u^2(\xi,\eta)\right| ds\right] d\xi d\eta. \tag{3.8}$$

Here to estimate the expression $|u^1(\xi,s)u^1(\xi,\eta) - u^2(\xi,s)u^2(\xi,\eta)|$, we use inequality

$$\begin{aligned}\left|u^1(\xi,s)u^1(\xi,\eta) - u^2(\xi,s)u^2(\xi,\eta)\right| &\leqslant \left|u^1(\xi,s)\right| \left|u^1(\xi,\eta) - u^2(\xi,\eta)\right| \\ &\quad + \left|u^2(\xi,\eta)\right| \left|u^1(\xi,s) - u^2(\xi,s)\right| \\ &\leqslant 2R\left\|u^1 - u^2\right\|_{lT}, \quad (s,\xi,\eta) \in [0,T]\times[0,l]\times[0,y],\end{aligned}$$

which holds for all $(u^1,\, u^2) \in S\,(u_0, r)$. Continuing estimating in (3.8), we get

$$\left\|Uu^1 - Uu^2\right\|_{lT} \leqslant \left[\frac{2Rh_0}{f_0}T^2 + \frac{f_1}{f_0}T\right] \left\|u^1 - u^2\right\|_{lT} =: m_2(T)\left\|u^1 - u^2\right\|_{lT}.$$

We choose T so that the inequality $m_2(T) < 1$ holds, then the operator U contracts the distance between elements of the ball $S(u_0; r)$. It is clear that this condition is satisfied by $T \in (0, T_2)$, where

$$T_2 = \frac{1}{4Rh_0}\left[\sqrt{f_1^2 + 8Rf_0h_0} - f_1\right]$$

is a positive root of the quadratic equation $m_2(T) - 1 = 0$.

Let $T^* = \min\,(T_1,\, T_2)$. Then the operator U with $T \in (0, T^*)$ is a contraction mapping of the ball $S(u_0, r)$ into itself. Hence, according to the contraction mapping principle, equation (3.2) possesses a unique solution $u(x,y) \in S(u_0, r)$. The proof is complete. □

Once we have found the function $u(x,y)$, the function $q(x)$ is determined by formula (3.1). In this formula $T \in (0, T^*)$ and $l > 0$ is an arbitrary fixed number. We also note that T^* does depend on l.

Thus the following theorem is valid.

Theorem 3.2. *Let conditions (B1)–(B3) be satisfied and $T \in (0, T^*)$. Then formula (3.1) determines $q(x) \in C^1[0,l]$ for each fixed $l > 0$.*

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