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ON MEAN–SQUARE APPROXIMATION OF FUNCTIONS IN BERGMAN SPACE B_2 AND VALUE OF WIDTHS OF SOME CLASSES OF FUNCTIONS

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Abstract. Let $A(U)$ be a set of functions analytic in the circle $U := \{z \in \mathbb{C}, |z| < 1\}$ and $B_2 := B_2(U)$ be the space of the functions $f \in A(U)$ with a finite norm

$$\|f\|_2 = \left(\frac{1}{\pi}\iint_{(U)} |f(z)|^2\, d\sigma\right)^{\frac{1}{2}} < \infty,$$

where $d\sigma$ is the area differential and the integral is treated in the Lebesgue sense.

In the work we study extremal problems related with the best polynomial approximation of the functions $f \in A(U)$. We obtain a series of sharp theorems and calculate the values of various n–widths of some classes of functions defined by the continuity moduluses of mth order for the rth derivative $f^{(r)}$ in the space B_2.

Keywords: Bergman space, extremal problems, polynomial approximation, n–widths.

Mathematics Subject Classification: 41A17, 41A25

1. Introduction

The issues of best polynomial approximation of analytic in a circle functions belonging to the Bergman space B_p, $p \geqslant 1$, were studied, for instance, in works [1]–[15]. Here we consider some questions on square–mean approximation of complex functions in the space B_2 and for some classes of functions we calculate the values of various n–widths in B_2.

We use the notation from [16].

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}_+$, $\mathbb{R}$ be respectively the sets of natural, non–negative integer, positive and real numbers. Let $\mathbb{C}$ be a complex plane, $U := \{z \in \mathbb{C} : |z| < 1\}$ be the unit circle in $\mathbb{C}$, $A(U)$ be the set of functions analytic in circle U.

Definition 1.1 ([2]). *A function $f \in A(U)$ is said to belong to the space B_2 if*

$$\|f\|_2 := \|f\|_{B_2} = \left(\frac{1}{\pi}\iint_{(U)} |f(z)|^2\, d\sigma\right)^{\frac{1}{2}} < \infty.$$

The derivative of rth order of the function

$$f(z) = \sum_{k=0}^{\infty} c_k(f) z^k \in A(U) \tag{1.1}$$

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is defined in a standard way:

$$f^{(r)}(z) := \frac{d^r}{dz^r} f(z) = \sum_{k=r}^{\infty} \alpha_{k,r} c_k(f) z^{k-r}, \quad r \in \mathbb{N}, \tag{1.2}$$

where

$$\alpha_{k,r} = \frac{k!}{(k-r)!}, \quad k, r \in \mathbb{N}, \quad k > r; \qquad \alpha_{k,0} \equiv 1, \quad \alpha_{k,1} \equiv k.$$

By the symbol $B_2^{(r)}$, $r \in \mathbb{Z}_{+}$,, $B_2^{(0)} = B_2$, we denote the set of functions $f \in B_2$, for which the derivative of rth order $f^{(r)}(z)$ is also an element of B_2:

$$B_2^{(r)} := \left\{ f \in B_2 : \|f^{(r)}\|_2 < \infty \right\}.$$

Let $\mathcal{P}_n$ be a subspace of complex algebraic polynomials of degree n of form

$$p_n(z) = \sum_{k=0}^{n} a_k z^k, \quad a_k \in \mathbb{C}.$$

The quantity

$$E_n(f)_2 := E(f, \mathcal{P}_n)_{B_2} = \inf \left\{ \|f - p_n\|_2 : p_n \in \mathcal{P}_n \right\} \tag{1.3}$$

is called the best polynomial mean–square approximation of a function $f \in B_2$ by the subspace $\mathcal{P}_n$. It is well–known [17] that for an arbitrary function $f \in B_2$ the relation

$$E_{n-1}(f)_2 = \|f - T_{n-1}(f)\|_2 = \left\{ \sum_{k=n}^{\infty} \frac{|c_k(f)|^2}{k+1} \right\}^{\frac{1}{2}} \tag{1.4}$$

holds, where T_{n-1} is a partial sum of order $n-1$ of series (1.1).

Representing the norm of a function $f \in B_2$ in a more convenient form

$$\|f\|_2 := \left(\frac{1}{\pi} \int_0^1 \int_0^{2\pi} |f(\rho e^{it})|^2 \rho \, d\rho dt \right)^{\frac{1}{2}},$$

by the symbol

$$\Delta_h^m(\rho e^{it}) := \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(\rho e^{i(t+kh)})$$

we denote a finite difference of mth order of a function $f \in B_2$ in the variable t with a step h. The identity

$$\|\Delta_h^m(f)\|_2 := \left(\frac{1}{\pi} \int_0^1 \int_0^{2\pi} \left| \Delta_h^m f(\rho e^{it}) \right|^2 \rho \, d\rho dt \right)^{\frac{1}{2}}$$

defines a norm of the finite difference of mth order for a function $f \in B_2$.

A modulus of continuity of mth order of a function $f \in B_2$ is defined in a standard way by the identity

$$\omega_m(f, \tau)_2 := \sup \left\{ \|\Delta_h^m(f)\|_2 : |h| \leqslant \tau \right\}. \tag{1.5}$$

Applying formula (1.5) to function (1.2), after simple calculations we obtain

$$\omega_m^2(f^{(r)}, \tau)_2 = 2^m \sup_{|h| \leqslant \tau} \sum_{k=r+1}^{\infty} \alpha_{k,r}^2 \frac{|c_k(f)|^2}{k-r+1} (1 - \cos(k-r)h)^m. \tag{1.6}$$

In what follows we shall make use of the following lemma.

Lemma 1.1 ([13]). *For all $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n > r$, the identity*

$$\sup_{f \in B_2^{(r)}} \frac{E_{n-1}(f)_2}{E_{n-r-1}(f^{(r)})_2} = \sqrt{\frac{n-r+1}{n+1}} \frac{1}{\alpha_{n,r}} \tag{1.7}$$

holds true. The supremum in (1.7) is achieved on the function $f_0(z) = z^n \in L_2^{(r)}$.

The main result of this paper are the following theorems.

Theorem 1.1. *For all $n, m \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n > r$, $0 < (n-r)h \leqslant \pi/2$ the identity*

$$\sup_{f \in B_2^{(r)}} \frac{\alpha_{n,r} \cdot E_{n-1}(f)_2}{\left\{ \int\limits_0^\tau \omega_m^{\frac{2}{m}}(f^{(r)}, t)_2 \, dt \right\}^{\frac{m}{2}}} = \sqrt{\frac{n-r+1}{n+1}} \left\{ \frac{n-r}{2[(n-r)\tau - \sin(n-r)\tau]} \right\}^{\frac{m}{2}} \tag{1.8}$$

holds true.

Proof. Without loss of generality we shall consider functions $f \in B_2$, the Taylor coefficients of which satisfy the conditions $c_k(f) = 0$, $k = 0, 1, \ldots, n-1$, that is, we consider the functions of form

$$f(z) = \sum_{k=n}^\infty c_k(f) z^k \in B_2.$$

For such functions we have

$$\|\Delta_t^m(f)\|_2^2 = 2^m \sum_{k=n}^\infty \frac{|c_k(f)|^2}{k+1} (1 - \cos kt)^m. \tag{1.9}$$

Taking into consideration (1.4) and (1.9), we consider the difference

$$E_{n-1}^2(f)_2 - \sum_{k=n}^\infty \frac{|c_k(f)|^2}{k+1} \cos kt = \sum_{k=n}^\infty \frac{|c_k(f)|^2}{k+1} (1 - \cos kt)$$

$$= \sum_{k=n}^\infty \left(\frac{|c_k(f)|^2}{k+1} \right)^{1-\frac{1}{m}} \left(\frac{|c_k(f)|^2}{k+1} \right)^{\frac{1}{m}} (1 - \cos kt).$$

Applying the Hölder inequality for sums to the right hand side of the obtained relation with $p := m/(m-1)$, $q := 1/m$ and taking into consideration identity (1.9), we find:

$$E_{n-1}^2(f)_2 - \sum_{k=n}^\infty \frac{|c_k(f)|^2}{k+1} \cos kt \leqslant \left(\sum_{k=n}^\infty \frac{|c_k(f)|^2}{k+1} \right)^{1-\frac{1}{m}} \left(\sum_{k=n}^\infty \frac{|c_k(f)|^2}{k+1} (1 - \cos kt)^m \right)^{\frac{1}{m}}$$

$$= \left(E_{n-1}^2(f)_2 \right)^{1-\frac{1}{m}} \frac{1}{2} \left(2^m \sum_{k=n}^\infty \frac{|c_k(f)|^2}{k+1} (1 - \cos kt)^m \right)^{\frac{1}{m}}$$

$$= \left(E_{n-1}^2(f)_2 \right)^{1-\frac{1}{m}} \frac{1}{2} \|\Delta_t^m(f)\|_2^{\frac{2}{m}} \leqslant E_{n-1}^{2-\frac{2}{m}}(f)_2 \frac{1}{2} \omega_m^{\frac{2}{m}}(f, t)_2.$$

Thus, for each function $f \in B_2$ the inequality

$$E_{n-1}^2(f)_2 \leqslant \sum_{k=n}^\infty \frac{|c_k(f)|^2}{k+1} \cos kt + \frac{1}{2} E_{n-1}^{2-\frac{2}{m}}(f)_2 \, \omega_m^{\frac{2}{m}}(f, t)_2 \tag{1.10}$$

holds true. Integrating both sides of inequality (1.10) in the variable t from 0 to τ and dividing the result by τ, we get

$$E_{n-1}^2(f)_2 \leqslant \sum_{k=n}^\infty \frac{|c_k(f)|^2}{k+1} \frac{\sin k\tau}{k\tau} + E_{n-1}^{2-\frac{2}{m}}(f)_2 \frac{1}{2\tau} \int\limits_0^\tau \omega_m^{\frac{2}{m}}(f, t)_2 \, dt. \tag{1.11}$$

Using the identity [19]

$$\max_{u\geqslant nt}\left|\frac{\sin u}{u}\right| = \frac{\sin nt}{nt}, \qquad 0\leqslant nt\leqslant \frac{\pi}{2},$$

by inequality (1.11) we obtain

$$\left(1-\frac{\sin n\tau}{n\tau}\right)E_{n-1}^2(f)_2 \leqslant E_{n-1}^{2-\frac{2}{m}}(f)_2\frac{1}{2\tau}\int\limits_0^\tau \omega_m^{\frac{2}{m}}(f,t)_2\,dt,$$

and hence,

$$E_n(f)_2 \leqslant \left\{\frac{n}{2(n\tau-\sin n\tau)}\right\}^{\frac{m}{2}}\left\{\int\limits_0^\tau \omega_m^{\frac{2}{m}}(f,t)_2\,dt\right\}^{\frac{m}{2}}. \tag{1.12}$$

If $f\in B_2^{(r)}$, then it follows from (1.12) that

$$E_{n-r-1}(f^{(r)})_2 \leqslant \left\{\frac{n-r}{2[(n-r)\tau-\sin(n-r)\tau]}\right\}^{\frac{m}{2}}\left\{\int\limits_0^\tau \omega_m^{\frac{2}{m}}(f^{(r)},t)_2\,dt\right\}^{\frac{m}{2}}$$

and applying the formula

$$E_{n-1}(f)_2 \leqslant \frac{1}{\alpha_{n,r}}\sqrt{\frac{n-r+1}{n+1}}E_{n-r-1}(f^{(r)})_2$$

implied by identity (1.7), we obtain

$$E_{n-1}(f)_2 \leqslant \frac{1}{\alpha_{n,r}}\sqrt{\frac{n-r+1}{n+1}}\left\{\frac{n-r}{2[(n-r)\tau-\sin(n-r)\tau]}\right\}^{\frac{m}{2}}\left\{\int\limits_0^\tau \omega_m^{\frac{2}{m}}(f^{(r)},t)_2\,dt\right\}^{\frac{m}{2}}, \tag{1.13}$$

where $0<(n-r)\tau\leqslant \pi/2$. This inequality implies an estimate for the quantity in the left hand side of identity (1.8):

$$\sup_{f\in B_2^{(r)}}\frac{\alpha_{n,r}E_{n-1}(f)_2}{\left\{\int\limits_0^\tau \omega_m^{\frac{2}{m}}(f^{(r)},t)_2\,dt\right\}^{\frac{m}{2}}} \leqslant \sqrt{\frac{n-r+1}{n+1}}\left\{\frac{n-r}{2[(n-r)\tau-\sin(n-r)\tau]}\right\}^{\frac{m}{2}}. \tag{1.14}$$

In order to obtain a similar estimate from below for the mentioned quantity, we introduce the function $f_0(z)=z^n\in B_2^{(r)}$, which by (1.3) and (1.6) satisfies

$$E_{n-1}(f_0)_2 = \frac{1}{\sqrt{n+1}}, \tag{1.15}$$

$$\omega_m^2(f_0^{(r)},t)_2 = 2^m\frac{\alpha_{n,r}}{n-r+1}(1-\cos(n-r)t)^m, \tag{1.16}$$

and since

$$\left\{\int\limits_0^\tau \omega_m^{\frac{2}{m}}(f_0^{(r)},t)_2\ dt\right\}^{\frac{m}{2}} = \frac{\alpha_{n,r}}{\sqrt{n-r+1}}\left\{\frac{2[(n-r)\tau-\sin(n-r)\tau]}{n-r}\right\}^{\frac{m}{2}}, \tag{1.17}$$

taking into consideration identities (1.15) and (1.17), we write a lower bound

$$\sup_{f\in B_2^{(r)}} \frac{\alpha_{n,r}\cdot E_{n-1}(f)_2}{\left\{\int\limits_0^\tau \omega_m^{\frac{2}{m}}(f^{(r)},t)_2\,dt\right\}^{\frac{m}{2}}} \geqslant \frac{\alpha_{n,r}\cdot E_{n-1}(f_0)_2}{\left\{\int\limits_0^\tau \omega_m^{\frac{2}{m}}(f_0^{(r)},t)_2\,dt\right\}^{\frac{m}{2}}}$$

$$= \frac{\alpha_{n,r}/\sqrt{n+1}}{\alpha_{n,r}/\sqrt{n-r+1}\,\{2[(n-r)\tau-\sin(n-r)\tau]/(n-r)\}^{\frac{m}{2}}} \tag{1.18}$$

$$= \sqrt{\frac{n-r+1}{n+1}}\left\{\frac{n-r}{2[(n-r)\tau-\sin(n-r)\tau]}\right\}^{\frac{m}{2}}, \qquad 0<(n-r)\tau\leqslant\frac{\pi}{2}.$$

Required identity (1.8) is implied by comparing inequalities (1.14) and (1.18) and this completes the proof. □

Theorem 1.2. *For all $n,m\in\mathbb{N}$, $r\in\mathbb{Z}_+$, $n>r$, and $0<h\leqslant\pi/(n-r)$ the identity holds:*

$$\sup_{f\in B_2^{(r)}} \frac{\alpha_{n,r}E_{n-1}(f)_2}{\left\{\omega_m^{\frac{2}{m}}(f^{(r)},h)_2+(n-r)^2\int\limits_0^h(h-\tau)\omega_m^{\frac{2}{m}}(f^{(r)},\tau)_2\,d\tau\right\}^{\frac{m}{2}}} = \sqrt{\frac{n-r+1}{n+1}}\frac{1}{[(n-r)h]^m}. \tag{1.19}$$

Proof. We multiply both sides of (1.10) by 2, integrate then in the variable τ from 0 to h and we obtain the inequality

$$h^2E_{n-1}^2(f)_2\leqslant 2\sum_{k=n}^{\infty}\frac{|c_k(f)|^2}{k+1}\frac{1-\cos kh}{k^2}+E_{n-1}^{2-\frac{2}{m}}(f)_2\int\limits_0^h\left(\int\limits_0^\tau\omega_m^{\frac{2}{m}}(f,t)_2\,dt\right)d\tau.$$

Replacing then $1/k^2$ by $1/n^2$ under the sum and integrating by parts in the double integral, we arrive at the inequality

$$h^2E_{n-1}^2(f)_2\leqslant\frac{2}{n^2}\sum_{k=n}^{\infty}\frac{|c_k(f)|^2}{k+1}(1-\cos kh)+E_{n-1}^{2-\frac{2}{m}}(f)_2\int\limits_0^h(h-\tau)\omega_m^{\frac{2}{m}}(f,\tau)_2\,d\tau. \tag{1.20}$$

By (1.10) the above inequality becomes

$$(nh)^2E_{n-1}^{\frac{2}{m}}(f)_2\leqslant\omega_m^{\frac{2}{m}}(f,h)_2+n^2\int\limits_0^h(h-\tau)\omega_m^{\frac{2}{m}}(f,\tau)_2\,d\tau.$$

This yields

$$E_{n-1}(f)_2\leqslant(nh)^{-m}\left\{\omega_m^{\frac{2}{m}}(f,h)_2+n^2\int\limits_0^h(h-\tau)\omega_m^{\frac{2}{m}}(f,\tau)_2\,d\tau\right\}^{\frac{m}{2}}.$$

We write the latter inequality for the quantity $E_{n-r-1}(f^{(r)})_2$ as follows:

$$E_{n-r-1}(f^{(r)})_2\leqslant((n-r)h)^{-m}\left\{\omega_m^{\frac{2}{m}}(f^{(r)},h)_2+(n-r)^2\int\limits_0^h(h-\tau)\omega_m^{\frac{2}{m}}(f^{(r)},\tau)_2\,d\tau\right\}^{\frac{m}{2}}. \tag{1.21}$$

Using Lemma 1.1 and taking into consideration inequality (1.21), we get

$$E_{n-1}(f)_2\leqslant\sqrt{\frac{n-r+1}{n+1}}\frac{1}{\alpha_{n,r}}\frac{1}{((n-r)h)^m}\cdot\left\{\omega_m^{\frac{2}{m}}(f^{(r)},h)_2+(n-r)^2\int\limits_0^h(h-\tau)\omega_m^{\frac{2}{m}}(f^{(r)},\tau)_2\,d\tau\right\}^{\frac{m}{2}},$$

which implies an upper bound for the quantity in the left hand side of identity (1.19):

$$\sup_{f\in B_2^{(r)}} \frac{\alpha_{n,r}E_{n-1}(f)_2}{\left\{\omega_m^{\frac{2}{m}}(f^{(r)},h)_2+(n-r)^2\int\limits_0^h(h-\tau)\cdot\omega_m^{\frac{2}{m}}(f^{(r)},\tau)_2\,d\tau\right\}^{\frac{m}{2}}} \leqslant \sqrt{\frac{n-r+1}{n+1}}\frac{1}{[(n-r)h]^m}. \tag{1.22}$$

In order to obtain a similar lower bound, we observe that the above considered function $f_0(z) = z^n \in B_2^{(r)}$, apart of identities (1.15)–(1.17), satisfies also the identity

$$\left\{\omega_m^{\frac{2}{m}}(f_0^{(r)},h)_2+(n-r)^2\int\limits_0^h(h-\tau)\omega_m^{\frac{2}{m}}(f_0^{(r)},\tau)_2\,d\tau\right\}^{\frac{m}{2}} = \frac{\alpha_{n,r}}{\sqrt{n-r+1}}[(n-r)h]^m. \tag{1.23}$$

Taking into consideration identities (1.15) and (1.23), we write the lower bound

$$\begin{aligned}\sup_{f\in B_2^{(r)}} &\frac{\alpha_{n,r}E_{n-1}(f)_2}{\left\{\omega_m^{\frac{2}{m}}(f^{(r)},h)_2+(n-r)^2\int\limits_0^h(h-\tau)\omega_m^{\frac{2}{m}}(f^{(r)},\tau)_2\,d\tau\right\}^{\frac{m}{2}}}\\ &\geqslant \frac{\alpha_{n,r}E_{n-1}(f_0)_2}{\left\{\omega_m^{\frac{2}{m}}(f_0^{(r)},h)_2+(n-r)^2\int\limits_0^h(h-\tau)\omega_m^{\frac{2}{m}}(f_0^{(r)},\tau)_2\,d\tau\right\}^{\frac{m}{2}}}\\ &= \frac{\alpha_{n,r}/\sqrt{n+1}}{\alpha_{n,r}/\left(\sqrt{n-r+1}[(n-r)h]^m\right)} = \sqrt{\frac{n-r+1}{n+1}}[(n-r)h]^m.\end{aligned} \tag{1.24}$$

We obtain required identity (1.19) by comparing inequalities (1.22) and (1.24). The proof is complete. □

2. Values of n–widths for some classes of functions

Before exposing further results, we recall needed notions and definitions. Let S be the unit ball in B_2, $\mathfrak{M}$ be a convex central–symmetric subset in B_2, $\mathfrak{L}_n \subset B_2$ be an n–dimensional subspace, $\mathfrak{L}^n \subset B_2$ be a subspace of codimension n; $\Lambda : B_2 \to \mathfrak{L}_n$ be a continuous linear operator mapping the space B_2 into $\mathfrak{L}_n$, $\Lambda^\perp : B_2 \to \mathfrak{L}_n$ be a continuous operator of linear projecting of the space B_2. The quantities

$$\begin{aligned}&b_n(\mathfrak{M};B_2) = \sup\left\{\sup\left\{\varepsilon>0 : \varepsilon S\cap\mathfrak{L}_{n+1}\subset\mathfrak{M}\right\} : \mathfrak{L}_{n+1}\subset B_2\right\},\\ &d_n(\mathfrak{M};B_2) = \inf\left\{\sup\left\{\inf\left\{\|f-\varphi\|_2 : \varphi\in\mathfrak{L}_n\right\} : f\in\mathfrak{M}\right\} : \mathfrak{L}_n\subset B_2\right\},\\ &\delta_n(\mathfrak{M};B_2) = \inf\left\{\inf\left\{\sup\left\{\|f-\Lambda f\|_2 : f\in\mathfrak{M}\right\} : \Lambda B_2\subset\mathfrak{L}_n\right\} : \mathfrak{L}_n\subset B_2\right\},\\ &d^n(\mathfrak{M};B_2) = \inf\left\{\sup\left\{\|f\|_2 : f\in\mathfrak{M}\cap\mathfrak{L}^n\right\} : \mathfrak{L}^n\subset B_2\right\},\\ &\Pi_n(\mathfrak{M};B_2) = \inf\left\{\inf\left\{\sup\left\{\|f-\Lambda^\perp f\|_2 : f\in\mathfrak{M}\right\} : \Lambda^\perp B_2\subset\mathfrak{L}_n\right\} : \mathfrak{L}_n\subset B_2\right\}\end{aligned}$$

are respectively called *Bernstein*, *Kolmogorov*, *linear*, *Gelfand*, *projection n–widths* of a subset $\mathfrak{M} \in B_2$.

Since the Bergman space B_2 is Hilbert, by the general theory of n–widths, see, for instance, [18] ,[3], the introduced n–widths satisfy the relations

$$b_n(\mathfrak{M};B_2) \leqslant d^n(\mathfrak{M};B_2) \leqslant d_n(\mathfrak{M};B_2) = \delta_n(\mathfrak{M};B_2) = \Pi_n(\mathfrak{M};B_2). \tag{2.1}$$

We are going to introduce classes of functions, for which we shall find exact values of the aforementioned n–widths.

Let $\Phi(u)$ be an arbitrary continuous increasing as $u \geqslant 0$ function such that $\Phi(u) = 0$. In view of the result of Theorem 1.1, by $W_m^{(r)}(\Phi)$ we denote the class of functions $f \in B_2^{(r)}$, which satisfy the condition

$$\int_0^t \omega_m^{\frac{2}{m}}(f^{(r)},\tau)_2\,d\tau \leqslant \Phi(t)$$

for all $n, m \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n > r$ and $t > 0$. For $m \in \mathbb{N}$, $r \in \mathbb{Z}_+$ and $h > 0$ we also let

$$W_m^{(r)}(h) := \left\{ f \in B_2^{(r)} : \left[\omega_m^{\frac{2}{m}}(f^{(r)},h) + (n-r)^2 \int_0^h \omega_m^{\frac{2}{m}}(f^{(r)},\tau)\,d\tau \right] \leqslant 1 \right\}.$$

For a subset $\mathfrak{M} \subset B_2$ we let

$$E_{n-1}(\mathfrak{M})_2 := \sup\left\{E_{n-1}(f) : f \in \mathfrak{M}\right\}.$$

Theorem 2.1. *If the majorant $\Phi(t)$ satisfies the conditions*

$$\frac{\Phi(t)}{\Phi(\pi/2(n-r))} \geqslant \frac{2}{\pi-2} \begin{cases} (n-r)t - \sin(n-r)t & \text{if} \quad 0 < t \leqslant \dfrac{\pi}{n-r}, \\ 2(n-r)t - \pi & \text{if} \quad t \geqslant \dfrac{\pi}{n-r}, \end{cases} \tag{2.2}$$

for all $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n > r$ and $t > 0$, then the identities

$$\lambda_n\left(W_m^{(r)}(\Phi), B_2\right) = E_{n-1}\left(W_m^{(r)}(\Phi)\right) = \frac{1}{\alpha_{n,r}}\sqrt{\frac{n-r+1}{n+1}}\left\{\frac{n-r}{\pi-2}\Phi\left(\frac{\pi}{2(n-r)}\right)\right\}^{\frac{m}{2}} \tag{2.3}$$

hold true for al $n \in \mathbb{N}$. The set of majorants Φ obeying (2.2) is non–empty.

Proof. Letting $\tau = \pi/2(n-r)$ in inequality (1.13), we obtain

$$E_{n-1}(f)_2 \leqslant \frac{1}{\alpha_{n,r}}\sqrt{\frac{n-r+1}{n+1}}\left\{\frac{n-r}{\pi-2}\int_0^{\pi/2(n-r)} \omega_m^{\frac{2}{m}}(f^{(r)},t)_2 dt\right\}^{\frac{m}{2}}.$$

For an arbitrary function $f \in W_m^{(r)}(\Phi)$ this implies

$$E_{n-1}(f)_2 \leqslant \frac{1}{\alpha_{n,r}}\sqrt{\frac{n-r+1}{n+1}}\left\{\frac{n-r}{\pi-2}\Phi\left(\frac{\pi}{2(n-r)}\right)\right\}^{\frac{m}{2}}. \tag{2.4}$$

Then by inequalities (2.1) we obtain an upper bound for all aforementioned n–widths

$$\lambda_n\left(W_m^{(r)}(\Phi), B_2\right) \leqslant E_{n-1}\left(W_m^{(r)}(\Phi)\right)_2 \leqslant \frac{1}{\alpha_{n,r}}\sqrt{\frac{n-r+1}{n+1}}\left\{\frac{n-r}{\pi-2}\Phi\left(\frac{\pi}{2(n-r)}\right)\right\}^{\frac{m}{2}}. \tag{2.5}$$

In order to obtain lower bound for the mentioned n–widths, in the set $\mathcal{P}_n \cap B_2$ we consider the ball

$$S_{n+1} = \left\{ p_n(z) \in \mathcal{P}_n : \|p_n\|_2 \leqslant \frac{1}{\alpha_{n,r}}\sqrt{\frac{n-r+1}{n+1}}\left\{\frac{n-r}{\pi-2}\Phi\left(\frac{\pi}{2(n-r)}\right)\right\}^{\frac{m}{2}} \right\}$$

and we are going to show that it belongs to the class $W_m^{(r)}(\Phi)$. In order to do this, we shall need the following inequality:

$$\omega_m^2(p_n^{(r)},\tau)_2 \leqslant 2^m \alpha_{n,r}\frac{n+1}{n-r+1}(1-\cos(n-r)\tau)_*^m \|p_n\|_2^2, \tag{2.6}$$

where

$$(1-\cos u)_*^m := \begin{cases} (1-\cos u)^m & \text{if} \quad 0 < u \leqslant \pi, \\ 2^m & \text{if} \quad u \geqslant \pi. \end{cases} \tag{2.7}$$

In order to prove (2.6), we employ identity (1.6). We have:

$$\begin{aligned}\omega_m^2(p_n^{(r)},\tau)_2 :=& 2^m \sup_{|h|\leqslant\tau}\sum_{k=r+1}^{n}\alpha_{k,r}^2\frac{|c_k(p_n)|^2}{k-r+1}(1-\cos(k-r)h)^m\\ =& 2^m \sup_{|h|\leqslant\tau}\sum_{k=r+1}^{n}\alpha_{k,r}^2\frac{k+1}{k-r+1}\frac{|c_k(p_n)|^2}{k+1}(1-\cos(k-r)h)^m\\ \leqslant& 2^m \max_{r\leqslant k\leqslant n}\alpha_{k,r}^2\frac{k+1}{k-r+1}(1-\cos(n-r)h)_*^m\sum_{k=0}^{n}\frac{|c_k(p_n)|^2}{k+1}\\ =& 2^m \max_{r\leqslant k\leqslant n}\alpha_{k,r}^2\frac{k+1}{k-r+1}(1-\cos(n-r)h)_*^m\,\|p_n\|_2^2.\end{aligned} \tag{2.8}$$

Let us show that

$$\max_{r\leqslant k\leqslant n}\alpha_{k,r}^2\frac{k+1}{k-r+1}=\alpha_{n,r}^2\frac{n+1}{n-r+1}. \tag{2.9}$$

Since a function of a natural variable

$$\begin{aligned}y(k) :=&\alpha_{k,r}^2\frac{k+1}{k-r+1}=[k(k-1)\cdots(k-r+2)(k-r+1)]^2\,\frac{k+1}{k-r+1}\\ =&[k(k-1)\cdots(k-r+2)]^2\,(k-r+1)(k+1)\end{aligned}$$

is increasing for all $k\in[r,n]$, then $\max_{r\leqslant k\leqslant n}y(k)=y(n)$ and this proves identity (2.9). Employing (2.9), by (2.8) we obtain inequality (2.6).

Let $0<t\leqslant\pi/(n-r)$. By the definition of the class $W^{(r)}(\Phi)$, the first of conditions (2.2) and relations (2.7), for each $p_n\in S_{n+1}$ we obtain

$$\begin{aligned}\int_0^t\omega_m^{\frac{2}{m}}(p_n^{(r)},\tau)_2\,d\tau \leqslant& 2\left(\alpha_{n,r}^2\frac{n+1}{n-r+1}\right)^{\frac{1}{m}}\|p_n\|_2^{\frac{2}{m}}\int_0^t(1-\cos(n-r)\tau)\,d\tau\\ \leqslant& \frac{2}{\pi-2}((n-r)t-\sin(n-r)t)\Phi\left(\frac{\pi}{2(n-r)}\right)\leqslant\Phi(t).\end{aligned} \tag{2.10}$$

Let $t\geqslant\pi/(n-r)$. In this case similar arguing in view of (2.6), (2.7) and the second inequality from (2.2) show that for each $p_n\in S_{n+1}$ we have

$$\begin{aligned}\int_0^t\omega_m^{\frac{2}{m}}(p_n^{(r)},\tau)\,d\tau =& \left(\int_0^{\pi/(n-r)}+\int_{\pi/(n-r)}^{t}\right)\omega_m^{\frac{2}{m}}(p_n^{(r)},\tau)\,d\tau\\ \leqslant& \frac{2}{\pi-2}[2(n-r)t-\pi]\Phi\left(\frac{\pi}{2(n-r)}\right)\leqslant\Phi(t).\end{aligned} \tag{2.11}$$

Inequalities (2.10) and (2.11) yield $S_{n+1}\subset W_m^{(r)}(\Phi)$. Using relations (2.1) for the aforementioned n–widths and the definition of *Bernstein n–width*, we write lower bounds:

$$\begin{aligned}\lambda_n\left(W_m^{(r)}(\Phi),B_2\right)\geqslant& b_n\left(W_m^{(r)}(\Phi),B_2\right)\geqslant b_n(S_{n+1},B_2)\\ \geqslant& \frac{1}{\alpha_{n,r}}\sqrt{\frac{n-r+1}{n+1}}\left[\frac{n-r}{\pi-2}\Phi\left(\frac{\pi}{2(n-r)}\right)\right]^{\frac{m}{2}}.\end{aligned} \tag{2.12}$$

Comparing upper bounds (2.5) and lower bounds (2.12), we obtain required identity (2.3). We note that condition (2.2) for $m=1$ first appeared in calculation of the exact value of *Kolmogorov n–width* for the classes $W_m^{(r)}(\Phi)$ of periodic function in the space $L_2:=L_2[0,2\pi]$ in work [19] by L.V. Taikov. It was proven in the same work that the function $\Phi(t)=t^{\frac{\pi}{\pi-2}}$

satisfies condition (2.2) for $m = 1$. This implies immediately that in our case $\Phi(t) = t^{\frac{\pi}{\pi-2}}$ obeys condition (2.2) for each $m \in \mathbb{N}$ and this completes the proof. $\square$

Theorem 2.2. *Let $m, n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n > r$. Then for each $0 < h \leqslant \pi/(n-r)$ the identities*

$$\lambda_n\left(W_m^{(r)}(h), B_2\right) = E_{n-1}\left(W_m^{(r)}(h)\right) = \sqrt{\frac{n-r+1}{n+1}}\frac{1}{\alpha_{n,r}}\frac{1}{[(n-r)h]^m} \tag{2.13}$$

hold true.

Proof. An upper bound for the class $W_m^{(r)}(h)$ is implied by inequality (1.23):

$$\lambda_n\left(W_m^{(r)}(h), B_2\right) \leqslant E_{n-1}\left(W_m^{(r)}(h)\right) \leqslant \sqrt{\frac{n-r+1}{n+1}}\frac{1}{\alpha_{n,r}}\frac{1}{[(n-r)h]^m}. \tag{2.14}$$

In order to obtain a similar lower bound for the aforementioned n–widths, in the set of the complex polynomials $\mathcal{P}_n$ we introduce $(n+1)$–dimensional ball

$$\sigma_{n+1} := \left\{p_n(z) \in \mathcal{P}_n : \|p_n\| \leqslant \sqrt{\frac{n-r+1}{n+1}}\frac{1}{\alpha_{n,r}}\frac{1}{[(n-r)h]^m}\right\}$$

and we are going to show that it belongs to the class $W_m^{(r)}(h)$. Taking into consideration the definition of the class and using inequality (2.6) for $0 < h \leqslant \pi/(n-r)$, we get

$$\left\{\omega_m^{\frac{2}{m}}(p_n^{(r)}, h) + (n-r)^2\int\limits_0^h (h-t)\omega_m^{\frac{2}{m}}(p_n^{(r)}, t)dt\right\}$$

$$\leqslant \left\{2\left(\alpha_{n,r}^2\frac{n+1}{n-r+1}\|p_n\|^2\right)^{\frac{1}{m}}\right.$$

$$\left.\cdot\left[1-\cos(n-r)t + (n-r)^2\int\limits_0^h (h-t)(1-\cos(n-r)t)dt\right]\right\}^{\frac{m}{2}}$$

$$= \frac{1}{[(n-r)h]^m}\left\{2(1-\cos(n-r)h) + 2(n-r)^2\int\limits_0^h\left(t - \frac{\sin(n-r)t}{n-r}\right)dt\right\}^{\frac{m}{2}}$$

$$= \frac{1}{[(n-r)h]^m}[(n-r)h]^m = 1.$$

This proves that $\sigma_{n+1} \subset W_m^{(r)}(h)$ and according to the definition of *Bernstein n–width* and relation (2.1) we write a lower bound

$$\lambda_n\left(W_m^{(r)}(h), B_2\right) \geqslant b_n\left(W_m^{(r)}(h), B_2\right) \geqslant b_n(\sigma_{n+1}, B_2) \geqslant \sqrt{\frac{n-r+1}{n+1}}\frac{1}{\alpha_{n,r}}\frac{1}{[(n-r)h]^m}. \tag{2.15}$$

Inequalities (2.14) and (2.15) imply identities (2.13) and this completes the proof. $\square$

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