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ON ZEROS AND TAYLOR COEFFICIENTS OF ENTIRE FUNCTION OF LOGARITHMIC GROWTH

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Abstract. In the paper for an important class of entire functions of zero order we find out straightforward relations between the increasing rate of the sequences of zeroes and the decay rate of the Taylor coefficients. Applying the coefficient characterization of the growth of entire functions and some Tauberian theorems from the convex analysis, we obtain asymptotically sharp estimates relating the zeroes λ_n and Hadamard rectified Taylor coefficients $\hat{f}_n$ for entire functions of the logarithmic growth. In the cases, when the function possesses a regular behavior of some kind, the mentioned estimates become asymptotically sharp formulas. For instance, if an entire function has a Borel regular growth and the point $a = 0$ is not its Borel exceptional value, then as $n \to \infty$ the asymptotic identity $\ln|\lambda_n| \sim \ln(\hat{f}_{n-1}/\hat{f}_n)$ holds true. The result is true for the functions of perfectly regular logarithmic growth and in the latter case we can additionally state that $\ln|\lambda_1\lambda_2 \ldots \lambda_n| \sim \ln \hat{f}_n^{\,-1}$ as $n \to \infty$.

Keywords: entire function, sequence of zeroes, Taylor coefficients, Hadamard rectified Taylor coefficients, logarithmic order, logarithmic type.

Mathematics Subject Classification: 30D15, 30B10

1. Introduction and preliminaries

This paper is a continuation of a series of papers [1], [2], in which we studied a joint behavior of zeros and Taylor coefficients of an entire function. Here we begin with considering entire functions of positive order in order to show similar issues in the class of functions of zero order requires an independent study. As in [1], [2], we are interesting in obtaining two–sided estimates and asymptotics identities, which relate the zeros and Taylor coefficients of the considered entire functions.

Each entire function is represented by its Taylor series

$$f(z) = \sum_{n=0}^{\infty} f_n\, z^n, \qquad f_n = \frac{f^{(n)}(0)}{n!}, \qquad z \in \mathbb{C}, \tag{1.1}$$

which converges everywhere in the complex plane.

An entire function of a finite order ρ (we provide the definition of the order below) with a sequence of zeros $\Lambda = \Lambda_f = \{\lambda_n\}_{n=1}^{\infty}$ can be represented as the Hadamard product

$$f(z) = z^m e^{P(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) e^{\frac{z}{\lambda_n} + \frac{z^2}{2\lambda_n^2} + \ldots + \frac{z^p}{p\lambda_n^p}}, \qquad z \in \mathbb{C}, \tag{1.2}$$

where m is the multiplicity of the zero at the point $z = 0$, $p \leqslant \rho$, and $P(z)$ is a polynomial of degree not exceeding ρ, see [3]. In what follows we suppose that f possesses infinitely many zeros, they are arranged in the ascending order of their absolute values counting the multiplicities. For simplicity we suppose that $f_0 = f(0) = 1$.

By $M_f(r)$ we denote the maximum of the absolute value of a function f in the circle $|z| \leqslant r$, that is, the quantity

$$M_f(r) = \max_{|z|\leqslant r} |f(z)| = \max_{|z|=r} |f(z)|, \qquad r > 0.$$

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As it is known, the function $M_f(r)$ is increasing and convex with respect to $\ln r$. The rate of its growth to infinity is related with the asymptotic behavior of the sequence of the Taylor coefficients $\{f_n\}_{n\in\mathbb{N}_0}$ and the sequence of zeros $\{\lambda_n\}_{n\in\mathbb{N}}$ of the function f.

Borel [4] introduced the notion of the order and lower order of a function and proposed to calculate these characteristics by the formulas

$$\rho = \rho_f = \varlimsup_{r\to+\infty} \frac{\ln\ln M_f(r)}{\ln r}, \qquad \lambda = \lambda_f = \varliminf_{r\to+\infty} \frac{\ln\ln M_f(r)}{\ln r}.$$

If these orders coincide, that is, $\rho_f = \lambda_f$, then one says that f has a Borel regular growth of order ρ. The order of an entire function can be calculated by the coefficients of its Taylor series via the known formula

$$\rho = \varlimsup_{n\to\infty} \frac{n\ln n}{\ln |f_n|^{-1}}.$$

A similar formula for the lower order with replacing the upper limit by the lower fails in the general case since the sequence of the Taylor coefficients f_n can involve an infinite zero subsequence; for instance, this happens in the series for even functions and in lacunary series. In order to overcome this obstacle and to obtain the formulas not only for the lower order but also for other lower characteristics of the growth of entire functions considered below, as in work [2], we introduced Hadamard rectified (straightened, regularized) Taylor coefficients of the power series.

Let us briefly describe the procedure of straightening the coefficients f_n. On the plane we mark the points $(n, -\ln|f_n|)$, where $n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, and by Γ we denote a polygonal line, which is the boundary of the convex hull of the set of these points. This polygonal line is called Newton–Hadamard polygonal line of the entire function $f(z)$. If $y = G(x)$, $x \geqslant 0$, is the equation for Γ, then the Hadamard rectified coefficients are defined by the identities

$$\hat{f}_n = e^{-G(n)}, \quad n \in \mathbb{N}_0.$$

By the definition, the points $(n, -\ln \hat{f}_n)$, $n \in \mathbb{N}_0$, are located on the polygonal line Γ. This is why the sequence $\{1/\hat{f}_n\}_{n\in\mathbb{N}_0}$ is logarithmically convex. If the sequence $\{1/|f_n|\}_{n\in\mathbb{N}_0}$ is logarithmically convex, then the identities $\hat{f}_n = |f_n|$, $n \in \mathbb{N}_0$, obviously hold true. For instance, see [5, Th. 2.11.8*], this happens if an entire function has an order not exceeding one and only negative zeros. In the general case for all $n \in \mathbb{N}_0$ the inequality $|f_n| \leqslant \hat{f}_n$ holds and it becomes the identity at the abscissas of the vertices of the Newton–Hadamard polygonal line.

Entire functions of finite order

$$f(z) = \sum_{n=0}^{\infty} f_n\, z^n, \qquad \hat{f}(z) = \sum_{n=0}^{\infty} \hat{f}_n\, z^n$$

satisfy the relation

$$\ln M_f(r) \sim \ln M_{\hat{f}}(r), \qquad r \to +\infty.$$

This is why classical growth characteristics of such pair of entire functions coincide. In particular, f and $\hat{f}$ has the same orders and same lower orders. Thus, we can write

$$\rho = \varlimsup_{n\to\infty} \frac{n\ln n}{\ln|f_n^{-1}|} = \varlimsup_{n\to\infty} \frac{n\ln n}{\ln \hat{f}_n^{-1}}, \qquad \lambda = \varliminf_{n\to\infty} \frac{n\ln n}{\ln \hat{f}_n^{-1}}. \tag{1.3}$$

The origination of the first part of the above formula for the order of the function is clear from the above discussion. The second part of the formula for the lower order can be proved by usual methods taking into consideration the logarithmic convexity of the sequence $\{1/\hat{f}_n\}_{n\in\mathbb{N}_0}$, see [6].

We shall make use of less known formulas. We denote

$$R_n = \hat{f}_{n-1}/\hat{f}_n, \qquad n \in \mathbb{N}, \qquad R_0 = 1.$$

As we make sure later, exactly the sequence R_n is most of all related with the sequence of zeros (more precisely, of the absolute values of zeros) defined by series (1.1). The logarithmic convexity of the sequence $\{1/\hat{f}_n\}$ implies the growth of the sequence R_n. We apply a specification of Stolz theorem [7, Thm. 2.7] to the convex sequences $x_n = n\ln n$ and $y_n = -\ln \hat{f}_n$. According to this theorem we have

$$\rho = \varlimsup_{n\to\infty} \frac{n\ln n}{\ln \hat{f}_n^{-1}} = \varlimsup_{n\to\infty} \frac{(n+1)\ln(n+1) - n\ln n}{\ln \hat{f}_{n-1} - \ln \hat{f}_n} = \varlimsup_{n\to\infty} \frac{\ln n}{\ln R_n}.$$

Similar identities also hold for the lower limit. Thus, the following formulas hold[1]

$$\rho = \varlimsup_{n\to\infty} \frac{\ln n}{\ln R_n}, \qquad \lambda = \varliminf_{n\to\infty} \frac{\ln n}{\ln R_n}. \tag{1.4}$$

We proceed to the growth characteristics for the sequence of zeros of an entire function. Let

$$n(r) = \sum_{|\lambda_n|\leqslant r} 1$$

be a counting function of the sequence $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}$, and

$$N(r) = \int_0^r \frac{n(t)}{t}\,dt$$

be its averaged counting function under the earlier assumption $f(0) = 1$. The convergence exponent of the sequence of the zeros of an entire function f is defined by the identity

$$\tau = \inf\left\{\alpha > 0 : \sum_{n=1}^{\infty} \frac{1}{|\lambda_n|^\alpha} < +\infty\right\}.$$

This exponent can be found by the formulas

$$\tau = \varlimsup_{r\to+\infty} \frac{\ln n(r)}{\ln r} = \varlimsup_{n\to\infty} \frac{\ln n}{\ln |\lambda_n|}. \tag{1.5}$$

We also define a lower characteristics of the growth of the sequence of zeros

$$\mu = \varliminf_{r\to+\infty} \frac{\ln n(r)}{\ln r} = \varliminf_{n\to\infty} \frac{\ln n}{\ln |\lambda_n|}. \tag{1.6}$$

The quantities τ and μ are called upper and lower logarithmic densities of the sequence $\Lambda = \{\lambda_n\}$. We shall say that Λ is logarithmically measurable if $\tau = \mu$. We note that the introduced densities of the sequence remain the same if in their definitions, identities (1.5) and (1.6), we replace the counting function $n(r)$ by the averaged counting function $N(r)$. Such replacement is based on the estimates

$$N(r) = \int_c^r \frac{n(t)}{t}\,dt \leqslant n(r)\ln\frac{r}{c}, \qquad 0 < c < |\lambda_1|, \quad r > c, \tag{1.7}$$

$$N(r) \geqslant \int_{r^\alpha}^r \frac{n(t)}{t}\,dt \geqslant n(r^\alpha)\ln r^{1-\alpha}, \qquad 0 < \alpha < 1, \quad r > 1. \tag{1.8}$$

Applying now the Stolz theorem, we obtain useful formulas

$$\tau = \varlimsup_{r\to+\infty} \frac{\ln N(r)}{\ln r} = \varlimsup_{n\to\infty} \frac{n\ln n}{\ln|\lambda_1\lambda_2\cdot\ldots\cdot\lambda_n|}, \tag{1.9}$$

$$\mu = \varliminf_{r\to+\infty} \frac{\ln N(r)}{\ln r} = \varliminf_{n\to\infty} \frac{n\ln n}{\ln|\lambda_1\lambda_2\cdot\ldots\cdot\lambda_n|}. \tag{1.10}$$

Let $a \in \mathbb{C}$ be a given number. As usually, a–points of an entire function f is the roots of the equation $f(z) = a$ and by τ_a we denote the convergence exponent of its sequence of a–points. For each $a \in \mathbb{C}$ the inequality $\tau_a \leqslant \rho$ holds, where ρ is the order of f. Borel showed that except for possibly a single value

[1]A possibility of using the first formula in (1.4) for calculating the order ρ was pointed out in Problem 52 in classical book [8].

a the identity $\tau_a = \rho$ holds. The value, for which this identity fails, is called Borel exceptional[1]. We note that entire functions of non–integer order has no Borel exceptional values. We also note that the Borel regular growth does not imply the logarithmic measurability of its zeros and vice versa. In order to make sure, it is sufficient to consider an entire function given by infinite product (1.2), the external exponential factor of which contains a polynomial P of degree exceeding the convergence exponent of the sequence of zeros.

Now we are in position to prove the main result of this section.

Theorem 1.1. *Let f be an entire function of order $\rho > 0$ and lower order λ, while τ and μ be the upper and lower logarithmic densities of the sequence of its zeros. Then the Hadamard rectified Taylor coefficients and the zeros of the function f are related by the inequalities*

$$\frac{\lambda}{\tau} \leqslant \varliminf_{n\to\infty} \frac{\ln|\lambda_n|}{\ln R_n} \leqslant \min\left\{\frac{\lambda}{\mu}, \frac{\rho}{\tau}\right\}, \tag{1.11}$$

$$\max\left\{\frac{\lambda}{\mu}, \frac{\rho}{\tau}\right\} \leqslant \varlimsup_{n\to\infty} \frac{\ln|\lambda_n|}{\ln R_n} \leqslant \frac{\rho}{\mu}, \tag{1.12}$$

where $R_n = \hat{f}_{n-1}/\hat{f}_n$ for all $n \in \mathbb{N}$.

Proof. The checking of the stated inequalities is based on comparing the formulas for the logarithmic densities of the sequence of zeros of the function with the corresponding formulas for coefficient calculation of the order and lower order of this function. For instance, applying formulas (1.4)–(1.6), we obtain

$$\varliminf_{n\to\infty} \frac{\ln|\lambda_n|}{\ln R_n} = \varliminf_{n\to\infty} \frac{\frac{\ln n}{\ln R_n}}{\frac{\ln n}{\ln|\lambda_n|}} \geqslant \frac{\varliminf\limits_{n\to\infty} \frac{\ln n}{\ln R_n}}{\varlimsup\limits_{n\to\infty} \frac{\ln n}{\ln|\lambda_n|}} = \frac{\lambda}{\tau}.$$

An upper bound for the lower limit of the quotient via the lower limits of the numerator and denominator gives

$$\varliminf_{n\to\infty} \frac{\ln|\lambda_n|}{\ln R_n} \leqslant \frac{\varliminf\limits_{n\to\infty} \frac{\ln n}{\ln R_n}}{\varliminf\limits_{n\to\infty} \frac{\ln n}{\ln|\lambda_n|}} = \frac{\lambda}{\mu}.$$

A similar estimate by the upper limits leads to the relation

$$\varliminf_{n\to\infty} \frac{\ln|\lambda_n|}{\ln R_n} \leqslant \frac{\varlimsup\limits_{n\to\infty} \frac{\ln n}{\ln R_n}}{\varlimsup\limits_{n\to\infty} \frac{\ln n}{\ln|\lambda_n|}} = \frac{\rho}{\tau}.$$

This proves inequalities (1.11). In the same way we get

$$\varlimsup_{n\to\infty} \frac{\ln|\lambda_n|}{\ln R_n} \leqslant \frac{\varlimsup\limits_{n\to\infty} \frac{\ln n}{\ln R_n}}{\varliminf\limits_{n\to\infty} \frac{\ln n}{\ln|\lambda_n|}} = \frac{\rho}{\mu},$$

$$\varlimsup_{n\to\infty} \frac{\ln|\lambda_n|}{\ln R_n} \geqslant \frac{\varlimsup\limits_{n\to\infty} \frac{\ln n}{\ln R_n}}{\varlimsup\limits_{n\to\infty} \frac{\ln n}{\ln|\lambda_n|}} = \frac{\rho}{\tau}, \qquad \varlimsup_{n\to\infty} \frac{\ln|\lambda_n|}{\ln R_n} \geqslant \frac{\varliminf\limits_{n\to\infty} \frac{\ln n}{\ln R_n}}{\varliminf\limits_{n\to\infty} \frac{\ln n}{\ln|\lambda_n|}} = \frac{\lambda}{\mu},$$

and this proves inequalities (1.12). The proof is complete. □

As a corollary we get the following statement.

Theorem 1.2. *Let the assumptions of Theorem 1.1 be satisfied.*

I. If a function has a Borel regular growth, then the identities

$$\varliminf_{n\to\infty} \frac{\ln|\lambda_n|}{\ln R_n} = \frac{\rho}{\tau}, \qquad \varlimsup_{n\to\infty} \frac{\ln|\lambda_n|}{\ln R_n} = \frac{\rho}{\mu}$$

hold.

[1]A value $a \in \mathbb{C}$ is called Borel exceptional for an entire function f, if the growth category of the counting function of the sequence of its a–points is below the growth category of the logarithm of the absolute value of this function, for more details see [3].

II. *If the sequence of the zeroes of a function is logarithmically measurable, then the identities*

$$\underline{\lim}_{n\to\infty} \frac{\ln|\lambda_n|}{\ln R_n} = \frac{\lambda}{\tau}, \qquad \overline{\lim}_{n\to\infty} \frac{\ln|\lambda_n|}{\ln R_n} = \frac{\rho}{\tau}$$

hold true.

III. *If the assumptions of the items I and II, then there exists the limit*

$$\lim_{n\to\infty} \frac{\ln|\lambda_n|}{\ln R_n} = \frac{\rho}{\tau}.$$

If in addition $a = 0$ is not a Borel exceptional value, then the asymptotics

$$\ln|\lambda_n| \sim \ln R_n, \quad n \to \infty$$

is valid.

Proof. It is sufficient to apply Theorem 1.1 and take into consideration that $\rho = \lambda$ when the function has a Borel regular growth, $\tau = \mu$ when the sequence of the zeros of a function is logarithmically measurable, and finally $\rho = \tau$ when the value $a = 0$ is not Borel exceptional for an entire function. The proof is complete. □

Theorem 1.2 indicates that the estimates in Theorem 1.1 are sharp and provide the classes of functions, on which these estimates are attained. We recall that entire functions of non–integer and zero order have no Borel exceptional values. We also point out that in the case of entire function of zero order, that is, as $\rho = 0$, the above result make no sense. This case is considered in the next section.

2. Entire functions of logarithmic growth

First asymptotics relating the zeros of entire function with its Taylor coefficients was given by Valiron [9]. He proved that if the Taylor coefficients of an entire function are non–zero and satisfy the condition

$$\frac{f_{n-1}f_{n+1}}{f_n^2} \to 0, \qquad n \to \infty, \tag{2.1}$$

then the asymptotic formula

$$\lambda_n \sim -\frac{f_{n-1}}{f_n}, \qquad n \to \infty, \tag{2.2}$$

holds true. We note that a rather restrictive condition (2.1) is not necessary for the validity of asymptotic relation (2.2). Indeed, an entire function

$$f(z) = \prod_{n=1}^{\infty}\left(1 + \frac{z}{q^n}\right), \qquad |q| > 1, \tag{2.3}$$

satisfy the equation

$$f(qz) = (1+z)\,f(z), \qquad z \in \mathbb{C}.$$

Expanding $f(z)$ into the power series and equating the Taylor coefficients, we successively obtain

$$f(0) = f_0 = 1, \qquad q^n f_n = f_n + f_{n-1}, \qquad f_n = \frac{f_{n-1}}{q^n - 1}, \quad n \in \mathbb{N}.$$

This yields

$$\lambda_n = -q^n \sim -(q^n - 1) = -\frac{f_{n-1}}{f_n}, \qquad n \to \infty,$$

and condition (2.2) holds, while condition (2.1) fails since

$$\frac{f_{n-1}f_{n+1}}{f_n^2} = \frac{q^n - 1}{q^{n+1} - 1} \to \frac{1}{q} \neq 0, \qquad n \to \infty.$$

Entire functions, the coefficients of which obey restriction (2.1), have a slow growth, more precisely, they satisfy the condition

$$\lim_{r\to+\infty} \frac{\ln M_f(r)}{\ln^2 r} = 0.$$

The function defined by product (2.3) satisfies a weaker than (2.1) condition

$$\frac{|f_n|^2}{|f_{n-1}||f_{n+1}|} \geqslant A > 0, \qquad n \geqslant n_0(A)\,. \tag{2.4}$$

Restriction (2.4) for the Taylor coefficients of an entire function is equivalent to a weakened restriction (the symbol o is replaced by the symbol O)

$$\ln M_f(r) = O(\ln^2 r), \qquad r \to +\infty$$

for the growth of the function itself.

Here we consider wider classes of entire functions of logarithmic growth fixed by the condition

$$\ln M_f(r) = O(h(r)), \qquad r \to +\infty,$$

in which the function $h(r)$, called a weight in what follows, is defined, grows unboundedly and differentiable on $(0, +\infty)$ and is such that

$$\lim_{r\to+\infty} \frac{rh'(r)\ln r}{h(r)} = q, \qquad 1 \leqslant q < +\infty\,, \tag{2.5}$$

as, for instance, a model weight $h(r) = \ln^q r$ with $q \geqslant 1$ or the Lindelöf weight, which is finite products of form $h(r) = \ln^q r \cdot \ln^s(\ln r) \cdot \ldots$ involving the powers of iterations of the logarithm.

We introduce the growth characteristics of entire functions (and sequences of its zeros) from the classes defined by weights with property (2.5). We first give auxiliary definitions. A logarthmic order (briefly, ln–order) and lower logarithmic order (briefly, lower ln–order) of an entire function are quantities

$$\gamma = \gamma_f = \varlimsup_{r\to+\infty} \frac{\ln\ln M_f(r)}{\ln\ln r}, \qquad \eta = \eta_f = \varliminf_{r\to+\infty} \frac{\ln\ln M_f(r)}{\ln\ln r}.$$

If these quantities coincide, that is, $\gamma_f = \eta_f$, then we say that f has a regular logarithmic growth. It follows from the well–known Liouville theorem that an entire function with a lower logarithmic order $\eta_f < 1$ is constant. This is why a transcendental (not coinciding with a polynomial) entire function of logarithmic order $\gamma_f = 1$ possesses a regular logarithmic growth. A logarithmic growth and lower logarithmic growth of an entire function represented by the Taylor series (1.1) can be found by formulas, see [6], [10],

$$\gamma - 1 = \varlimsup_{n\to\infty} \frac{\ln n}{\ln\ln |f_n|^{-\frac{1}{n}}} = \varlimsup_{n\to\infty} \frac{\ln n}{\ln\ln \hat{f}_n^{\,-\frac{1}{n}}}\,, \tag{2.6}$$

$$\eta - 1 = \varliminf_{n\to\infty} \frac{\ln n}{\ln\ln \hat{f}_n^{\,-\frac{1}{n}}}\,. \tag{2.7}$$

Let us define density growth characteristics for the sequence of the zeros $\{\lambda_n\}$ of an entire function of a finite logarithmic growth. An upper bilogarithmic density (briefly, an upper $\ln_2$–density) of the sequence $\{\lambda_n\}$ is the upper limit

$$\overline{\Delta}_{\ln_2} = \varlimsup_{n\to\infty} \frac{\ln n}{\ln\ln |\lambda_n|} = \varlimsup_{r\to+\infty} \frac{\ln n(r)}{\ln\ln r}, \tag{2.8}$$

and a lower bilogarithmic density of this sequence (lower $\ln_2$–density) is the corresponding lower limit

$$\underline{\Delta}_{\ln_2} = \varliminf_{n\to\infty} \frac{\ln n}{\ln\ln |\lambda_n|} = \varliminf_{r\to+\infty} \frac{\ln n(r)}{\ln\ln r}. \tag{2.9}$$

We saythat a sequence is bilogarithmically measurable if its upper and lower densities coincide, that is, $\underline{\Delta}_{\ln_2} = \overline{\Delta}_{\ln_2}$.

Averaged upper and lower bilogarithmic densities of a sequence are defined by formulas similar to (2.8), (2.9), but with replacing the counting function $n(r)$ by its averaged counting function $N(r)$.

Namely,

$$\overline{\Delta}^*_{\ln_2} = \varlimsup_{r\to+\infty} \frac{\ln N(r)}{\ln\ln r}, \tag{2.10}$$

$$\underline{\Delta}^*_{\ln_2} = \varliminf_{r\to+\infty} \frac{\ln N(r)}{\ln\ln r}. \tag{2.11}$$

Quantities (2.8) and (2.10), as well as (2.9) and (2.11) are pairwise related by simple identities

$$\overline{\Delta}_{\ln_2} = \overline{\Delta}^*_{\ln_2} - 1, \qquad \underline{\Delta}_{\ln_2} = \underline{\Delta}^*_{\ln_2} - 1. \tag{2.12}$$

The derivation of the above formulas is based on estimates (1.7), (1.8), see also [10]. In order to understand the matter, it is useful to compare (2.8)–(2.12) with (1.5)–(1.10).

Applying formulas (2.6)–(2.9), we arrive at the following result.

Theorem 2.1. *Let f be an entire function of logarithmic order γ and lower logarithmic order $\eta > 1$, while $\underline{\Delta}_{\ln_2}$ and $\overline{\Delta}_{\ln_2}$ be the upper and lower logarithmic densities of the sequence of its zeros. Then the Hadamard rectified Taylor coefficients and zeros of this function are related by the inequalities*

$$\frac{\eta-1}{\overline{\Delta}_{\ln_2}} \leqslant \varliminf_{n\to\infty} \frac{\ln\ln|\lambda_n|}{\ln\ln \hat{f}_n^{-\frac{1}{n}}} \leqslant \min\left\{\frac{\eta-1}{\underline{\Delta}_{\ln_2}}, \frac{\gamma-1}{\overline{\Delta}_{\ln_2}}\right\}, \tag{2.13}$$

$$\max\left\{\frac{\eta-1}{\underline{\Delta}_{\ln_2}}, \frac{\gamma-1}{\overline{\Delta}_{\ln_2}}\right\} \leqslant \varlimsup_{n\to\infty} \frac{\ln\ln|\lambda_n|}{\ln\ln \hat{f}_n^{-\frac{1}{n}}} \leqslant \frac{\gamma-1}{\underline{\Delta}_{\ln_2}}. \tag{2.14}$$

I. If a function has a regular logarithmic growth, then the identities

$$\varliminf_{n\to\infty} \frac{\ln\ln|\lambda_n|}{\ln\ln \hat{f}_n^{-\frac{1}{n}}} = \frac{\gamma-1}{\overline{\Delta}_{\ln_2}}, \qquad \varlimsup_{n\to\infty} \frac{\ln\ln|\lambda_n|}{\ln\ln \hat{f}_n^{-\frac{1}{n}}} = \frac{\gamma-1}{\underline{\Delta}_{\ln_2}}$$

hold true.

II. If a sequence of zeros of the function is bilogarithmically measurable, then the identities

$$\varliminf_{n\to\infty} \frac{\ln\ln|\lambda_n|}{\ln\ln \hat{f}_n^{-\frac{1}{n}}} = \frac{\eta-1}{\overline{\Delta}_{\ln_2}}, \qquad \varlimsup_{n\to\infty} \frac{\ln\ln|\lambda_n|}{\ln\ln \hat{f}_n^{-\frac{1}{n}}} = \frac{\gamma-1}{\overline{\Delta}_{\ln_2}}$$

hold true.

III. If the assumptions of Items I and II are satisfied, then there exists the limit

$$\lim_{n\to\infty} \frac{\ln\ln|\lambda_n|}{\ln\ln \hat{f}_n^{-\frac{1}{n}}} = \frac{\gamma-1}{\overline{\Delta}_{\ln_2}}.$$

Formulas (2.13), (2.14) are appropriate analogues of formulas (1.11), (1.12). The proof of Theorem 2.1 reproduces the arguing from the proofs of Theorems 1.1, 1.2 and this is why we omit it.

Let us introduce more gentle characteristics of an entire function of logarithmic growth and of the sequence of its zeros. Let a weight $h(r)$ satisfies condition (2.5). A type and a lower type of entire function with respect to $h(r)$ (briefly, h–type and lower h–type) are defined respectively by the formulas

$$T_h = T_h(f) = \varlimsup_{r\to+\infty} \frac{\ln M_f(r)}{h(r)}, \qquad t_h = t_h(f) = \varliminf_{r\to+\infty} \frac{\ln M_f(r)}{h(r)}. \tag{2.15}$$

Following the Valiron terminology [9], we say that an entire function has a perfectly regular growth or, more precisely, perfectly regular h–growth if its h–type and lower h-type coincide, that is, if $T_h = t_h$.

The following quantities characterise the growth of the sequence Λ of the zeros of an entire function f. Upper and lower h–densities of Λ, as well as averaged upper and lower h–densities Λ are respectively defined by the formulas

$$\overline{\Delta}_h = \overline{\Delta}_h(\Lambda) = \varlimsup_{r\to+\infty} \frac{n(r)}{rh'(r)}, \qquad \underline{\Delta}_h = \underline{\Delta}_h(\Lambda) = \varliminf_{r\to+\infty} \frac{n(r)}{rh'(r)},$$

$$\overline{\Delta}^*_h = \overline{\Delta}^*_h(\Lambda) = \varlimsup_{r\to+\infty} \frac{N(r)}{h(r)}, \qquad \underline{\Delta}^*_h = \underline{\Delta}^*_h(\Lambda) = \varliminf_{r\to+\infty} \frac{N(r)}{h(r)}.$$

Let $q \geqslant 1$ be the value of the limit in (2.5). For $q > 1$ the introduced characteristics satisfy the inequalities

$$a_1 \,\overline{\Delta}_h^* \leqslant \underline{\Delta}_h \leqslant \underline{\Delta}_h^*, \qquad \overline{\Delta}_h^* \leqslant \overline{\Delta}_h \leqslant a_2 \,\overline{\Delta}_h^*,$$

where a_1, a_2 are the roots of the equation

$$qa + (1-q)\, a^{q/(q-1)} = \underline{\Delta}_h^* \,/\, \overline{\Delta}_h^*. \tag{2.16}$$

For $q = 1$ we have

$$\underline{\Delta}_h = \underline{\Delta}_h^*, \qquad \overline{\Delta}_h = \overline{\Delta}_h^*.$$

If $\overline{\Delta}_h(\Lambda) = \underline{\Delta}_h(\Lambda)$, which is equivalent to $\overline{\Delta}_h^*(\Lambda) = \underline{\Delta}_h^*(\Lambda)$, we say that the sequence Λ is h–measurable. We note that an idea to measure density growth characteristics of the sequence of zeros by comparing the counting and averaged counting function with a single growth function $h(r)$ turns out to be inappropriate. Indeed, considering the upper limit $\overline{\lim\limits_{r\to+\infty}} \frac{n(r)}{h(r)} = \delta$ similarly to the averaged h–density $\overline{\Delta}_h^*$, for entire functions of zero order with a finite h–type we have the identity $\delta = 0$. Indeed, for each $k > 1$ we successively obtain (cf. [11]) the relations

$$N(kr) \geqslant \int\limits_r^{kr} \frac{n(t)}{t}\,dt \geqslant n(r)\ln k, \qquad \frac{N(kr)}{h(kr)}\,\frac{h(kr)}{h(r)} \geqslant \frac{n(r)}{h(r)}\,\ln k.$$

Since $h(kr) \sim h(r)$ as $r \to +\infty$, the passage to the upper limit gives $\overline{\Delta}_h^* \geqslant \delta \ln k$ with an arbitrary $k > 1$ and this implies $\delta = 0$.

The next inequalities establishing relations between the growth of a function and the growth of its sequence of zeros are immediately implied by the Jensen formula. Here we mean the relations

$$\overline{\Delta}_h^* \leqslant T_h, \qquad \underline{\Delta}_h^* \leqslant t_h.$$

As it was shown in thesis [12, Thms. 2.11, 2.12], for the functions of zero order with a finite h–type both estimates become identities. This implies the following fact.

Proposition 2.1. *Let the weight $h(r)$ satisfies condition (2.5), and the sequence of zeros of an entire function of a finite h–type T_h and lower h–type t_h has averaged upper and lower h–densities $\overline{\Delta}_h^*$ and $\underline{\Delta}_h^*$, respectively. Then the identities*

$$\overline{\Delta}_h^* = T_h, \qquad \underline{\Delta}_h^* = t_h \tag{2.17}$$

hold true. Moreover, if $q = 1$ in condition (2.5), then we additionally have

$$\overline{\Delta}_h = T_h, \qquad \underline{\Delta}_h = t_h. \tag{2.18}$$

Identities (2.17) and (2.18) play a key role in establishing the results on joint variation of zeros and Taylor coefficients of an entire function. Here we need formulas for calculating the h–type and lower h–type of an entire function (the definitions of which are given in (2.15)) by its Taylor coefficients, see, for instance, [6].

Theorem A. *Let a weight $h(r)$ satisfy condition (2.5) with a constant $q > 1$ and $k(\zeta)$ be an inverse function for $h(e^r)/r$. Let the entire function f represented by series (1.1) has h–type $T_h = T \in (0, +\infty)$ and lower h–type $t_h = t$. Then its Hadamard rectified Taylor coefficients satisfy the identities*

$$\overline{\lim_{n\to\infty}} \frac{nk(n)}{\ln \hat{f}_n^{\,-1}} = \frac{q}{q-1}\,(Tq)^{\frac{1}{q-1}}, \tag{2.19}$$

$$\underline{\lim_{n\to\infty}} \frac{nk(n)}{\ln \hat{f}_n^{\,-1}} = \frac{q}{q-1}\,(tq)^{\frac{1}{q-1}}. \tag{2.20}$$

The next two theorems are the main results of this paper.

Theorem 2.2. *Let the assumptions of Theorem A be satisfied. If $q > 1$ in (2.5), then the inequalities*

$$\left(\frac{t}{T}\right)^{\frac{1}{q-1}} \leqslant \underline{\lim}_{n\to\infty} \frac{\ln \hat{f}_n^{\,-1}}{\ln|\lambda_1\lambda_2\,\ldots\,\lambda_n|} \leqslant 1 \leqslant \overline{\lim}_{n\to\infty} \frac{\ln \hat{f}_n^{\,-1}}{\ln|\lambda_1\lambda_2\,\ldots\,\lambda_n|} \leqslant \left(\frac{T}{t}\right)^{\frac{1}{q-1}}, \tag{2.21}$$

$$\left(\frac{a_1}{a_2}\right)^{\frac{1}{q-1}} \leqslant \underline{\lim}_{n\to\infty} \frac{\ln|\lambda_n|}{\ln R_n} \leqslant \overline{\lim}_{n\to\infty} \frac{\ln|\lambda_n|}{\ln R_n} \leqslant \left(\frac{a_2}{a_1}\right)^{\frac{1}{q-1}}, \tag{2.22}$$

hold true, where $R_n = \hat{f}_{n-1}/\hat{f}_n$ and a_1, a_2, $a_1 \leqslant 1 \leqslant a_2$, are the roots of equation (2.16), which reads as

$$qa + (1-q)\,a^{q/(q-1)} = t/T.$$

If the function has a perfectly regular h–growth, or, equivalently, the sequence of its zeros is h–measurable, then the asymptotic identities

$$\ln|\lambda_1\lambda_2\,\ldots\,\lambda_n| \sim \ln \hat{f}_n^{\,-1}, \qquad n\to\infty,$$
$$\ln|\lambda_n| \sim \ln R_n, \qquad n\to\infty,$$

hold.

Proof. It was shown in work [2] that the upper and lower averaged h–densities $\overline{\Delta}_h^* = \overline{\Delta}^*$ and $\underline{\Delta}_h^* = \underline{\Delta}^*$ of the sequence of zeros of the function f coincide respectively with the upper and lower h–type of an auxiliary function

$$F(z) = \sum_{n=0}^{\infty} \frac{z^n}{\lambda_1\lambda_2\ldots\lambda_n}$$

constructed by the zeros of the original function $f(z)$. This is why for calculating the needed quantities we use formulas similar to (2.19), (2.20), where $\hat{f}_n^{\,-1}$ is replaced by the logarithmically convex sequence $|\lambda_1\lambda_2\,\ldots\,\lambda_n|$. Namely,

$$\overline{\lim}_{n\to\infty} \frac{nk(n)}{\ln|\lambda_1\lambda_2\,\ldots\,\lambda_n|} = \frac{q}{q-1}\left(\overline{\Delta}^* q\right)^{\frac{1}{q-1}}, \tag{2.23}$$

$$\underline{\lim}_{n\to\infty} \frac{nk(n)}{\ln|\lambda_1\lambda_2\,\ldots\,\lambda_n|} = \frac{q}{q-1}\left(\underline{\Delta}^* q\right)^{\frac{1}{q-1}}. \tag{2.24}$$

Let us estimate the upper and lower limits of the quotient in (2.21) in the usual way by using formulas (2.19), (2.20), (2.23), (2.24) and identity (2.17) from Proposition 2.1. We get

$$\overline{\lim}_{n\to\infty} \frac{\ln \hat{f}_n^{\,-1}}{\ln|\lambda_1\lambda_2\,\ldots\,\lambda_n|} = \overline{\lim}_{n\to\infty} \frac{\frac{nk(n)}{\ln|\lambda_1\lambda_2\ldots\lambda_n|}}{\frac{nk(n)}{\ln \hat{f}_n^{\,-1}}} \leqslant \frac{\overline{\lim}_{n\to\infty} \frac{nk(n)}{\ln|\lambda_1\lambda_2\ldots\lambda_n|}}{\underline{\lim}_{n\to\infty} \frac{nk(n)}{\ln \hat{f}_n^{\,-1}}} = \left(\frac{T}{t}\right)^{\frac{1}{q-1}},$$

$$\overline{\lim}_{n\to\infty} \frac{\ln \hat{f}_n^{\,-1}}{\ln|\lambda_1\lambda_2\,\ldots\,\lambda_n|} \geqslant \frac{\underline{\lim}_{n\to\infty} \frac{nk(n)}{\ln|\lambda_1\lambda_2\ldots\lambda_n|}}{\underline{\lim}_{n\to\infty} \frac{nk(n)}{\ln \hat{f}_n^{\,-1}}} = \frac{t}{t} = 1.$$

The lower limit in (2.21) can be treated in the same way and this is why we omit the details.

In order to obtain (2.22), we employ estimates

$$\left(\overline{\Delta}^* q\right)^{\frac{1}{q-1}} \leqslant \overline{\lim}_{n\to\infty} \frac{k(n)}{\ln|\lambda_n|} \leqslant \left(a_2\,\overline{\Delta}^* q\right)^{\frac{1}{q-1}}, \tag{2.25}$$

$$\left(a_1\,\underline{\Delta}^* q\right)^{\frac{1}{q-1}} \leqslant \underline{\lim}_{n\to\infty} \frac{k(n)}{\ln|\lambda_n|} \leqslant \left(\underline{\Delta}^* q\right)^{\frac{1}{q-1}}. \tag{2.26}$$

which were proved in [6]. In accordance with the Valiron formula [9], the logarithm of the maximal term $\mu(r)$ of the Taylor series of an entire function has the same representations via the central index

$\nu(r)$ as the averaged counting function of the sequence of zeros $N(r)$ via the counting function $n(r)$. More precisely,

$$\ln \mu(r) = \int\limits_0^r \frac{\nu(t)}{t}\, dt.$$

Since the central index $\nu(r)$ is the counting function of the sequence R_n, it satisfies analogues of formulas (2.25), (2.26), in which the averaged h–densities of the sequence of the zeros of the entire function are to be replaced by h–types. Thus,

$$(Tq)^{\frac{1}{q-1}} \leqslant \overline{\lim_{n\to\infty}} \frac{k(n)}{\ln R_n} \leqslant (a_2Tq)^{\frac{1}{q-1}}\,, \tag{2.27}$$

$$(a_1Tq)^{\frac{1}{q-1}} \leqslant \underline{\lim_{n\to\infty}} \frac{k(n)}{\ln R_n} \leqslant (tq)^{\frac{1}{q-1}}\,. \tag{2.28}$$

Now result (2.22) can be easily obtained by using estimates (2.25)–(2.28). Indeed, the right inequality in (2.22) is derived by the scheme

$$\overline{\lim_{n\to\infty}} \frac{\ln|\lambda_n|}{\ln R_n} \leqslant \frac{\overline{\lim\limits_{n\to\infty}}\, \frac{k(n)}{\ln R_n}}{\underline{\lim\limits_{n\to\infty}}\, \frac{k(n)}{\ln|\lambda_n|}} \leqslant \frac{(a_2Tq)^{\frac{1}{q-1}}}{\left(a_1\,\overline{\Delta}^*q\right)^{\frac{1}{q-1}}} = \left(\frac{a_2}{a_1}\right)^{\frac{1}{q-1}}.$$

In order to get the left estimate in (2.22), we write

$$\underline{\lim_{n\to\infty}} \frac{\ln|\lambda_n|}{\ln R_n} \geqslant \frac{\underline{\lim\limits_{n\to\infty}}\, \frac{k(n)}{\ln R_n}}{\overline{\lim\limits_{n\to\infty}}\, \frac{k(n)}{\ln|\lambda_n|}} \geqslant \frac{(a_1Tq)^{\frac{1}{q-1}}}{\left(a_2\,\overline{\Delta}^*q\right)^{\frac{1}{q-1}}} = \left(\frac{a_1}{a_2}\right)^{\frac{1}{q-1}}.$$

We have taken into consideration Proposition 2.1, according to which the identity $T = \overline{\Delta}^*$ holds true.

Under the made assumptions the identities $T = t = \underline{\Delta}^* = \overline{\Delta}^*$ hold and the roots of equation (2.16) coincide and are equal to one. This implies the last statement of the theorem. And finally we observe that the last asymptotic identity in the statement of the theorem implies the previous identity by the Stolz theorem [7]. □

It remains to consider the case of a very slow growth of an entire function, when its logarithmic order is equal to one. Here we need an independent study since the previous arguing fails. In this case the weight reads as $h(r) = \ln r \cdot h_1(r)$, where for the function $h_1(e^r)$ the limit in (2.5) is greater than zero. For the sake of clarity we choose a model weight $h(r) = \ln r \cdot \ln^s(\ln r)$ with an exponent $s > 0$.

Theorem 2.3. *Let $h(r) = \ln r \cdot \ln^s(\ln r)$, where $s > 0$, and an entire function f has a finite h–type $T_h = T$ and lower h–type $t_h = t > 0$. Then its Hadamard rectified Taylor coefficients and zero satisfy the inequalities*

$$\left(\frac{t}{T}\right)^{\frac{1}{s}} \leqslant \underline{\lim_{n\to\infty}} \frac{\ln(\ln R_n)}{\ln(\ln|\lambda_n|)} \leqslant 1 \leqslant \overline{\lim_{n\to\infty}} \frac{\ln(\ln R_n)}{\ln(\ln|\lambda_n|)} \leqslant \left(\frac{T}{t}\right)^{\frac{1}{s}}. \tag{2.29}$$

If an entire function f has a perfectly regular h–growth or the sequence of its zeros is h–measurable, then the Hadamard rectified Taylor coefficients $\{\hat{f}_n\}$ and zeros $\{\lambda_n\}$ are related by the asymptotic formula

$$\ln(\ln|\lambda_n|) \sim \ln(\ln R_n), \qquad n \to \infty.$$

As above, here we denote $R_n = \hat{f}_{n-1}/\hat{f}_n$.

Proof. Since the weight $h(r)$ satisfies condition (2.5) with constant $q = 1$, then according to Proposition 2.1 the identities $T = \overline{\Delta}$, $t = \underline{\Delta}$ hold. In the consider case, when $rh'(r) \sim \frac{h(r)}{\ln r}$ as $r \to +\infty$, they can be written as

$$T = \overline{\lim_{r\to+\infty}} \frac{\ln M_f(r)}{\ln r \ln^s(\ln r)} = \overline{\lim_{r\to+\infty}} \frac{n(r)}{\ln^s(\ln r)} = \overline{\lim_{n\to\infty}} \frac{n}{\ln^s(\ln|\lambda_n|)} = \overline{\Delta}\,,$$

$$t = \underline{\lim_{r\to+\infty}} \frac{\ln M_f(r)}{\ln r \ln^s(\ln r)} = \underline{\lim_{r\to+\infty}} \frac{n(r)}{\ln^s(\ln r)} = \underline{\lim_{n\to\infty}} \frac{n}{\ln^s(\ln|\lambda_n|)} = \underline{\Delta}\,.$$

The calculation of h–types of the function by the Taylor coefficients lead to the formulas

$$T = \varlimsup_{n\to\infty} \frac{n}{\ln^s(\ln R_n)}, \qquad t = \varliminf_{n\to\infty} \frac{n}{\ln^s(\ln R_n)}.$$

Now it is easy to estimate

$$T = \varlimsup_{n\to\infty} \frac{n}{\ln^s(\ln|\lambda_n|)} \geqslant \varliminf_{n\to\infty} \frac{n}{\ln^s(\ln R_n)} \varlimsup_{n\to\infty} \frac{\ln^s(\ln R_n)}{\ln^s(\ln|\lambda_n|)} = t \varlimsup_{n\to\infty} \frac{\ln^s(\ln R_n)}{\ln^s(\ln|\lambda_n|)}.$$

Therefore,

$$\varlimsup_{n\to\infty} \frac{\ln^s(\ln R_n)}{\ln^s(\ln|\lambda_n|)} \leqslant \frac{T}{t}.$$

Estimating from above, we get

$$T = \varlimsup_{n\to\infty} \frac{n}{\ln^s(\ln|\lambda_n|)} \leqslant \varlimsup_{n\to\infty} \frac{n}{\ln^s(\ln R_n)} \varlimsup_{n\to\infty} \frac{\ln^s(\ln R_n)}{\ln^s(\ln|\lambda_n|)} = T \varlimsup_{n\to\infty} \frac{\ln^s(\ln R_n)}{\ln^s(\ln|\lambda_n|)},$$

and this implies

$$\varlimsup_{n\to\infty} \frac{\ln^s(\ln R_n)}{\ln^s(\ln|\lambda_n|)} \geqslant 1.$$

Swapping the sequences R_n and $|\lambda_n|$ in the above arguing, we get the estimates in the left hand side (2.29). Finally, the last statement of the theorem is implied by its main part for $t = T$. The proof is complete. □

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