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GENERALIZED COMPOSITION OPERATORS ON WEIGHTED FOCK SPACES

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Abstract. The generalized composition operators J_g^Φ and C_g^Φ, induced by analytic functions g and Φ on the complex plane $\mathbb{C}$, are defined by

$$J_g^\Phi(f)(z) = \int\limits_0^z f'(\Phi(\omega))g(\omega)d\omega \text{ and } C_g^\Phi(f)(z) = \int\limits_0^{\Phi(z)} f'(\omega)g(\omega)d\omega.$$

In this paper, we consider these operators on weighted Fock spaces $\mathcal{F}_p^\Psi$, consisting of entire functions, which are $\mathcal{L}^p(\mathbb{C})$-integrable with respect to the measure $d\lambda(z) = e^{-\Psi(z)}d\Lambda(z)$, where $d\Lambda$ is the usual Lebesgue area measure in $\mathbb{C}$. We assume that the weight function Ψ in the spaces satisfies certain smoothness conditions, in particular, this weight function grows faster than the Gaussian weight $\frac{|z|^2}{2}$ defining the classical Fock spaces.

We first consider bounded and compact properties of J_g^Φ and C_g^Φ, and characterize these properties in terms function theory of inducing functions g and Φ, given by

$$\mathcal{M}_g^\Phi(z) := \frac{|g(z)|\Psi'(\Phi(z))}{1+\Psi'(z)}e^{\Psi(\Phi(z))-\Psi(z)}.$$

Our characterization is simpler to use than the Berezin type integral transform characterization. In some cases, our result shows that these operators experience poorer boundedness and compactness structures when acting between such spaces than the classical Fock spaces. For instance, for $\Phi(z) = z$, there is no nontrivial bounded J_g^Φ and C_g^Φ on weighted Fock spaces. In the case of classical Fock spaces, they are bounded if and only if g is constant.

In the second part of this paper, we apply our simpler characterization of boundedness and compactness to further study the Schatten-class membership of these operators. In particular, we express the Schatten $S_p(\mathcal{F}_2^\Psi)$ class membership property in terms of $\mathcal{L}^p(\mathbb{C}, \Delta\Psi d\Lambda)$-integrability of $\mathcal{M}_g^\Phi$.

Keywords: weighted Fock spaces, generalized composition operator, Schatten-class, boundedness, compactness.

Mathematics Subject Classification: 47B32, 30H20, 46E22, 46E20, 47B33

1. Introduction

Given analytic functions g and Φ on the complex plane $\mathbb{C}$, generalized composition operators J_g^Φ and C_g^Φ, induced by g and Φ, are defined by

$$J_g^\Phi(f)(z) = \int\limits_0^z f'(\Phi(\omega))g(\omega)d\omega \text{ and } C_g^\Phi(f)(z) = \int\limits_0^{\Phi(z)} f'(\omega)g(\omega)d\omega. \tag{1.1}$$

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Specifically, for the case $\varPhi$ is an identity function, that is, $\varPhi(z) = z$,

$$J_g^{\varPhi}(f)(z) = C_g^{\varPhi}(f)(z) = \int_0^z f'(\omega)g(\omega)d\omega$$

is the well-known Volterra companion operator denoted by J_g. Operators in 1.1 are first introduced by S. Li and S. Stević [2], [3], on some spaces of analytic functions defined on a unit disc, and then considered by several authors on different function spaces. In particular, on classical Fock spaces, T. Mengestie [4], [5] characterized bounded, compact and Schatten-class membership properties in terms of the properties of inducing symbols g and $\varPhi$. Later in [7], together with the first author of this paper, they studied some topological properties of these operators. In [6], T. Mengestie and S. Ueki studied boundedness and compactness of the particular operator J_g, on weighted Fock spaces $\mathcal{F}_p^{\varPsi}$ (defined below), with a weight function $\varPsi$ satisfying the following conditions (as in [1]):

(1) $\varPsi : [0, \infty) \to \mathbb{R}^+$ is twice continuously differentiable function and $\varPsi(z) = \varPsi(|z|)$, $z \in \mathbb{C}$.
(2) The Laplacian of $\varPsi$ is positive and there exists a function $\nu(z)$ obeying[1]

$$\nu(z) \asymp \begin{cases} 1, & 0 \leqslant |z| < 1 \\ (\Delta\varPsi(|z|))^{-\frac{1}{2}}, & |z| \geqslant 1, \end{cases}$$

has the following properties;
- (I) ν is a radial positive differentiable function and decreases to zero as $|z| \to \infty$.
- (II) $\lim\limits_{r\to\infty} \nu'(r) = 0$.
- (III) either there exists a constant $\alpha > 0$ such that $\nu(r)r^{\alpha}$ increases for large r or $\lim\limits_{r\to\infty} \nu'(r) \log \frac{1}{\nu(r)} = 0$.

Throughout the manuscript we assume that the weight function $\varPsi$ satisfies the above conditions. We note that this kind of weight function grows faster than the Gaussian weight $\frac{|z|^2}{2}$ defining the classical Fock spaces.

Next we define a corresponding weighted Fock space. Let $d\lambda(z) = e^{-\varPsi(z)}d\Lambda(z)$, where $d\Lambda$ is the Lebesgue area measure in $\mathbb{C}$. Then, for $0 < p < \infty$, the weighted Fock space $\mathcal{F}_p^{\varPsi}$ is space of analytic functions on $\mathbb{C}$, which are $\mathcal{L}^p(\mathbb{C}, d\lambda)$-integrable, that is,

$$\|f\|_{\mathcal{F}_p^{\varPsi}}^p := \int_{\mathbb{C}} |f(z)|^p e^{-p\varPsi(z)} d\Lambda(z) < \infty. \tag{1.2}$$

The growth type weighted Fock space $\mathcal{F}_\infty^{\varPsi}$ consists of analytic functions on $\mathbb{C}$ such that

$$\|f\|_{\mathcal{F}_\infty^{\varPsi}} := \sup_{z\in\mathbb{C}} |f(z)|e^{-\varPsi(z)} < \infty. \tag{1.3}$$

Recently, in [8], Z. Yang and Z. Zhou characterized boundedness and compactness of the generalized composition operator $J_g^{\varPhi}$, acting between the weighted Fock spaces $\mathcal{F}_p^{\varPsi}$ and $\mathcal{F}_q^{\varPsi}$, for $0 < p, q < \infty$, in terms of the Berezin type integral transform,

$$\int_{\mathbb{C}} |k_{(w,\varPsi)}(\varPhi(z))|^q \frac{(1 + \varPsi'(\varPhi(z))^q}{(1 + \varPsi'(z)^q} |g(z)|^q e^{-q\varPsi(z)} d\Lambda(z),$$

where $k_{(w,\varPsi)}$ is normalized kernel function of $\mathcal{F}_2^{\varPsi}$. Motivated by this research, the purpose of the present work is to further characterize boundedness and compactness of $J_g^{\varPhi}$ on weighted

[1]The notation $U(z) \asymp V(z)$ means both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$, where $U(z) \lesssim V(z)$ (or $V(z) \gtrsim U(z)$) means that there exists a constant C such that $U(z) \leqslant CV(z)$ holds.

Fock spaces in terms of a simpler function

$$\mathcal{M}_g^{\Phi}(z) := \frac{|g(z)|\Psi'(\Phi(z))}{1+\Psi'(z)} e^{\Psi(\Phi(z))-\Psi(z)},$$

by using the notion of embedding map. We also do the same for the similar operator C_g^{Φ}. Moreover, applying the simplified condition, we characterize the Schatten class $S_p(\mathcal{F}_2^{\Psi})$ membership property of J_g^{Φ} and C_g^{Φ}.

2. Preliminaries

We begin the section with some preliminary results that will be used in the proof of our main results. In [1], [6], weighted Fock spaces were described in terms of a derivative, which expresses (1.2) and (1.3) as

$$\|f\|_{\mathcal{F}_p^{\Psi}}^p \asymp |f(0)|^p + \int_{\mathbb{C}} \frac{|f'(z)|^p}{(1+\Psi'(z))^p} e^{-p\Psi(z)} d\Lambda(z) \tag{2.1}$$

for finite p and

$$\|f\|_{\mathcal{F}_\infty^{\Psi}} \asymp |f(0)| + \sup_{z\in\mathbb{C}} \frac{|f'(z)|}{1+\Psi'(z)} e^{-\Psi(z)}. \tag{2.2}$$

This type of estimate is usually called Littlewood-Paley type estimate, and it plays an important role specially in studying integral operators. Let $D(\alpha, r)$ be a disc with a center α and a radius r. By Lemma 7 in [1], for $0 < p < \infty$ and subharmonic functions Ψ and f we have the pointwise estimate

$$|f(z)|^p e^{-p\Psi(z)} \lesssim \frac{1}{\sigma^2\nu(z)^2} \int_{D(z,\sigma\nu(z))} |f(w)|^p e^{-p\Psi(w)} d\Lambda(w) \lesssim \nu(z)^{-2}\|f\|_{\mathcal{F}_p^{\Psi}}^p, \tag{2.3}$$

for a small positive number σ, which shows that the space $\mathcal{F}_2^{\Psi}$ is a reproducing kernel Hilbert space. At the same time, an explicit formula for the kernel $K_{(w,\Psi)}$ is not known yet. However, if $\{e_n\}$ is an orthonormal basis of $\mathcal{F}_2^{\Psi}$, then

$$K_{(w,\Psi)}(z) = \sum_n e_n(z)\overline{e_n(w)} \qquad \text{and} \qquad \frac{\partial}{\partial \overline{w}} K_{(w,\Psi)}(z) = \sum_n e_n(z)\overline{e_n'(w)}.$$

Moreover, by Corollary 8 and Lemma 22 in [1] we have

$$\|K_{(w,\Psi)}\|_{\mathcal{F}_2^{\Psi}}^2 = \sum_n |e_n(w)|^2 \asymp \nu(w)^{-2} e^{2\Psi(w)} \tag{2.4}$$

$$\left\|\frac{\partial}{\partial \overline{w}} K_{(w,\Psi)}\right\|_{\mathcal{F}_2^{\Psi}}^2 = \sum_n |e_n'(w)|^2 \asymp \|K_{(w,\Psi)}\|_{\mathcal{F}_2^{\Psi}}^2 (\Psi'(w))^2 \asymp (\Psi'(w))^2 \nu(w)^{-2} e^{2\Psi(w)}. \tag{2.5}$$

In [1], there were constructed test functions, which played the role of kernel function, as it is stated by the following lemma.

Lemma 2.1. *For a large number R there exists a number $\eta(R)$ such that for any $w \in \mathbb{C}$ with $|w| > \eta(R)$, there exists an entire test function $F_{(w,R)}$ in $\mathcal{F}_p^{\Psi}$ with*

$$\|F_{(w,R)}\|_{\mathcal{F}_p^{\Psi}} \asymp \nu(w)^{\frac{2}{p}}$$

for $0 < p < \infty$ and $\|F_{(w,R)}\|_{\mathcal{F}_\infty^{\Psi}} \asymp 1$. In particular, for $z \in D(w, R\nu(w))$,

$$\frac{|F_{(w,R)}'(z)|}{1+\Psi'(z)} e^{-\Psi(z)} \asymp 1. \tag{2.6}$$

We also need the following statement called covering lemma to prove our main results; this lemma was proved in [1].

Lemma 2.2. *Assume that* $t:\mathbb{C}\to(0,\infty)$ *is a continuous function such that*

$$|t(z)-t(w)|\leqslant\frac{|z-w|}{4},\qquad z,w\in\mathbb{C},\qquad \lim_{|z|\to\infty}t(z)=0.$$

Then there exists a sequence $\{z_n\}$ *such that*

(i) $z_i\notin D(z_n,t(z_n))$ *for* $i\neq n$, *and* $\cup_{n\geqslant 1}D(z_n,t(z_n))=\mathbb{C}$;

(ii) $\cup_{z\in D(z_n,t(z_n))}D(z,t(z))\subset D(z_n,3t(z_n))$ *and a sequence* $\{D(z_n,3t(z_n))\}$ *is a covering of* $\mathbb{C}$ *of finite multiplicity.*

3. Boundedness and compactness

As it has been noted in the Introduction, the boundedness and compactness of the Volterra companion operator J_g on the weighted Fock spaces was studied in [6]. It was shown that, in some cases, the structure of J_g becomes poorer when the operator acts between such spaces in comparison with the case of the classical Fock spaces. Our result in this paper shows that this is also the case for the generalized composition operator J_g^{Φ} in general.

Before proving our main results, we state the next lemma, which gives the form of Φ whenever the function $\mathcal{M}_g^{\Phi}(z)$ or $\mathcal{M}_{g(\Phi)}^{\Phi}(z)$ is uniformly bounded over $\mathbb{C}$.

Lemma 3.1. *Let* g *and* Φ *be nonconstant entire functions. If there exist a positive constant* C *such that* $\mathcal{M}_g^{\Phi}(z)\leqslant C$ *or* $\mathcal{M}_{g(\Phi)}^{\Phi}(z)\leqslant C$ *for all* $z\in\mathbb{C}$, *then* $\Phi(z)=az+b$ *for some* $a,b\in\mathbb{C}$ *with* $|a|\leqslant 1$.

Proof. We first assume that $\mathcal{M}_g^{\Phi}$ is uniformly bounded over $\mathbb{C}$. Then there exists a positive constant C such that

$$|g(z)|\leqslant C\Big(\frac{1+\Psi'(z)}{\Psi'(\Phi(z))}\Big)\big(e^{\Psi(\Phi(z))-\Psi(z)}\big)^{-1},$$

and since g is nonconstant, we have $\Psi(\Phi(z))-\Psi(z)<0$. This implies

$$\limsup_{|z|\to\infty}\Psi(\Phi(z))-\Psi(z)\leqslant 0. \tag{3.1}$$

Since Φ has its own power series expansion, by (3.1) we conclude that Φ reads as $\Phi(z)=az+b$ for some $a,b\in\mathbb{C}$ with $|a|\leqslant 1$. The case when $\mathcal{M}_{h(\Phi)}^{\Phi}$ is uniformly bounded can be treated in the same way. The proof is complete. ☐

Theorem 3.1. *Let* $0<p\leqslant q\leqslant\infty$, *and let* g *and* Φ *be nonconstant entire functions. Then*

(i) $J_g^{\Phi}:\mathcal{F}_p^{\Psi}\to\mathcal{F}_q^{\Psi}$ *is*
  (a) *bounded if and only if* $\sup_{z\in\mathbb{C}}\Delta\Psi(z)^s\mathcal{M}_g^{\Phi}(z)<\infty$;
  (b) *compact if and only if* $\lim_{|z|\to\infty}\Delta\Psi(z)^s\mathcal{M}_g^{\Phi}(z)=0$,
  where

$$s=\begin{cases}\dfrac{q-p}{pq}, & p\leqslant q<\infty\\[2mm] \dfrac{1}{p}, & p<q=\infty\\[2mm] 0, & p=q=\infty.\end{cases}$$

(ii) $C_g^{\Phi}:\mathcal{F}_p^{\Psi}\to\mathcal{F}_q^{\Psi}$ *is*
  (a) *bounded if and only if* $\sup_{z\in\mathbb{C}}\Delta\Psi(z)^s\mathcal{M}_{g(\Phi)}^{\Phi}(z)<\infty$;
  (b) *compact if and only if* $\lim_{|z|\to\infty}\Delta\Psi(z)^s\mathcal{M}_{g(\Phi)}^{\Phi}(z)=0$.

Proof. (i) The proof for the case $p \leqslant q = \infty$ was given in Theorem 2.1 in [8]. Here we study the case $p \leqslant q < \infty$. For $f \in \mathcal{F}_p^\Psi$, an application of the Littlewood-Paley type formula in (2.1) to $J_g^\Phi f$ gives

$$\|J_g^\Phi f\|_{\mathcal{F}_q^\Psi}^q \asymp \int_{\mathbb{C}} \frac{|f'(\Phi(z))|^q |g(z)|^q}{(1+\Psi'(z))^q} e^{-q\Psi(z)} d\Lambda(z) = \int_{\mathbb{C}} |f'(z)|^q d\mu_{(q,\Phi,\Psi)}(z),$$

where $d\mu_{(q,\Phi,\Psi)}$ is a pull-back measure given by

$$\mu_{(q,\Phi,\Psi)}(B) = \int_{\Phi^{-1}(B)} \frac{|g(w)|^q}{(1+\Psi'(w))^q} e^{-q\Psi(w)} d\Lambda(w)$$

for every Borel subset B of $\mathbb{C}$. Then it is easy to observe that the operator $J_g^\Phi : \mathcal{F}_p^\Psi \to \mathcal{F}_q^\Psi$ is bounded (respectively, compact) if and only if the embedding operator $E : \mathcal{O}_{(p,\Psi)} \to \mathcal{F}_q^\Psi$ is bounded (respectively, compact), where $\mathcal{O}_{(p,\Psi)}$ is space of entire functions such that

$$\int_{\mathbb{C}} \frac{|f(z)|^p}{(1+\Psi'(z))^p} e^{-p\Psi(z)} d\Lambda(z)$$

is finite. But, by proposition 3.8 of [8], E is bounded if and only if for some $\delta > 0$

$$\sup_{w\in\mathbb{C}} \frac{1}{\nu(w)^{\frac{2q}{p}}} \int_{D(w,\delta\nu(w))} (\Psi'(z))^q e^{q\Psi(z)} d\mu_{(q,\Phi,\Psi)}(z) \tag{3.2}$$

is finite, and E is compact if and only if

$$\lim_{|w|\to\infty} \frac{1}{\nu(w)^{\frac{2q}{p}}} \int_{D(w,\delta\nu(w))} (\Psi'(z))^q e^{q\Psi(z)} d\mu_{(q,\Phi,\Psi)}(z) = 0. \tag{3.3}$$

Substituting back $\mu_{(q,\Phi,\Psi)}$ into (3.2) and (3.3), we obtain that J_g^Φ is bounded if and only if

$$\sup_{w\in\mathbb{C}} \frac{1}{\nu(w)^{\frac{2q}{p}}} \int_{D(w,\delta\nu(w))} (\mathcal{M}_g^\Phi)^q(z) d\Lambda(z) < \infty, \tag{3.4}$$

and the operator is compact if and only if

$$\lim_{|w|\to\infty} \frac{1}{\nu(w)^{\frac{2q}{p}}} \int_{D(w,\delta\nu(w))} (\mathcal{M}_g^\Phi)^q(z) d\Lambda(z) = 0. \tag{3.5}$$

Our next step to simplify these conditions.

(a) First assume $\sup_{z\in\mathbb{C}} \Delta\Psi(z)^{\frac{q-p}{pq}} \mathcal{M}_g^\Phi(z)$ is finite. From (3.4), using the fact that $\nu(w) \asymp \nu(z)$ for $z \in D(w, \delta\nu(w))$, we obtain

$$\begin{aligned}
&\sup_{w\in\mathbb{C}} \frac{1}{\nu(w)^{\frac{2q}{p}}} \int_{D(w,\delta\nu(w))} (\mathcal{M}_g^\Phi)^q(z) d\Lambda(z) \\
&\qquad \leqslant \Big(\sup_{w\in\mathbb{C}} \Delta\Psi(w)^{\frac{q-p}{pq}} \mathcal{M}_g^\Phi(w)\Big)\Big(\sup_{w\in\mathbb{C}} \frac{1}{\nu(w)^{\frac{2q}{p}}} \int_{D(w,\delta\nu(w))} \frac{1}{(\nu(z))^{\frac{2(p-q)}{p}}} d\Lambda(z)\Big) \\
&\qquad \lesssim \sup_{w\in\mathbb{C}} \frac{1}{\nu(w)^{\frac{2q}{p}}} \int_{D(w,\delta\nu(w))} \frac{1}{(\nu(z))^{\frac{2(p-q)}{p}}} d\Lambda(z) < \infty.
\end{aligned} \tag{3.6}$$

On the other hand, if (3.4) holds, then using the estimate in (2.3), the subharmonicity of $g\Psi'(\Phi)e^{\Psi(\Phi)}$, and the fact that $1+\Psi'(w) \asymp 1+\Psi'(z)$ for $w \in D(z, \delta\nu(z))$, we get

$$\begin{aligned}
&\sup_{z\in\mathbb{C}} \Delta\Psi(z)^{\frac{q-p}{p}} (\mathcal{M}_g^{\Phi})^q(z) \\
&\quad\lesssim \sup_{z\in\mathbb{C}} \Big(\frac{\nu(z)^{\frac{-2q}{p}}}{(1+\Psi'(z))^q} \int\limits_{D(z,\delta\nu(z))} |g(w)|^q (\Psi'(\Phi(w)))^q e^{q\Psi(\Phi(w))-q\Psi(w)} d\Lambda(w)\Big) \\
&\quad\asymp \sup_{z\in\mathbb{C}} \frac{1}{\nu(z)^{\frac{2q}{p}}} \int\limits_{D(z,\delta\nu(z))} (\mathcal{M}_g^{\Phi})^q(w) d\Lambda(w) < \infty.
\end{aligned} \tag{3.7}$$

These relations and (3.6) yield that J_g^{Φ} is bounded if and only if the function $\Delta\Psi(z)^{\frac{q-p}{pq}} \mathcal{M}_g^{\Phi}(z)$ is uniformly bounded over $\mathbb{C}$.

(*b*) From (3.5), (3.6) and (3.7), it is easy to see that the operator is compact if and only if the function $\Delta\Psi(z)^{\frac{q-p}{pq}} \mathcal{M}_g^{\Phi}(z)$ goes to zero as $|z| \to \infty$.

(*ii*) The proof of this part is very similar to the proof of part (*i*) above and this is why we omit it. The proof is complete. □

The above theorem shows that, if p is strictly less than q and $J_g^{\Phi} : \mathcal{F}_p^{\Psi} \to \mathcal{F}_q^{\Psi}$ (or $C_g^{\Phi} : \mathcal{F}_p^{\Psi} \to \mathcal{F}_q^{\Psi}$), then the unboundedness of the Laplacian of Ψ forces the operator to have poorer structure compared with the classical Fock spaces case, cf. [7]. In particular, the operator $J_g : \mathcal{F}_p^{\Psi} \to \mathcal{F}_q^{\Psi}$ is bounded if and only if

$$\begin{cases} \Delta\Psi(z)^{\frac{q-p}{pq}} |g(z)|, & q < \infty \\ \Delta\Psi(z)^{\frac{1}{p}} |g(z)|, & q = \infty \end{cases}$$

is bounded. This holds true if and only if g is the zero function and hence there is no nontrivial bounded J_g in this case. But, in the classical Fock spaces case the operator is bounded if and only if g is constant (see [4]). We may now proceed to the case $0 < q < p \leqslant \infty$, in this case, our next result shows that boundedness and compactness of the operator $J_g^{\Phi} : \mathcal{F}_p^{\Psi} \to \mathcal{F}_q^{\Psi}$ (respectively, $C_g^{\Phi} : \mathcal{F}_p^{\Psi} \to \mathcal{F}_q^{\Psi}$) are equivalent.

Theorem 3.2. *Let $0 < q < p \leqslant \infty$, and let g and Φ be nonconstant entire functions. Then*

(i) $J_g^{\Phi} : \mathcal{F}_p^{\Psi} \to \mathcal{F}_q^{\Psi}$ *is bounded or compact if and only if* $\int_{\mathbb{C}} (\mathcal{M}_g^{\Phi})^r(z) d\Lambda(z)$ *is finite, where*

$$r = \begin{cases} \frac{pq}{p-q}, & p < \infty \\ q, & p = \infty. \end{cases} \tag{3.8}$$

(ii) $C_g^{\Phi} : \mathcal{F}_p^{\Psi} \to \mathcal{F}_q^{\Psi}$ *is bounded or compact if and only if* $\int_{\mathbb{C}} (\mathcal{M}_{g(\Phi)}^{\Phi})^r(z) d\Lambda(z)$ *is finite, where r is as defined above.*

Proof. (*i*) As it has been shown in the proof of Theorem 3.1, $J_g^{\Phi} : \mathcal{F}_p^{\Psi} \to \mathcal{F}_q^{\Psi}$ is bounded (respectively, compact) if and only if the embedding operator $E : \mathcal{O}_{(p,\Psi)} \to \mathcal{F}_q^{\Psi}$ is bounded (respectively, compact). But, by Proposition 3.8 in [8], the boundedness and compactness of E are equivalent to the condition that the function

$$T(z) := \frac{1}{\nu(w)^2} \int\limits_{D(w,\delta\nu(w))} (\mathcal{M}_g^{\Phi})^q(z) d\Lambda(z),$$

belongs to $\mathcal{L}^{\frac{r}{q}}(\mathbb{C}, d\Lambda)$ for some $\delta > 0$. Now we consider two different cases.

Case 1: $p < \infty$**.** Suppose T is in $\mathcal{L}^{\frac{r}{q}}(\mathbb{C}, d\Lambda)$, where $r = \frac{pq}{p-q}$. Then using the estimate in (2.3) and the fact that $1+\Psi'(z) \asymp 1+\Psi'(w)$, for $z \in D(w, \delta\nu(w))$, we obtain

$$\int_{\mathbb{C}} (\mathcal{M}_g^{\Phi})^r(z)d\Lambda(z) = \int_{\mathbb{C}} \frac{|g(z)|^r(\Psi'(\Phi(z)))^r}{(1+\Psi'(z))^r} e^{r\Psi(\Phi(z))-r\Psi(z)}d\Lambda(z)$$
$$\lesssim \int_{\mathbb{C}} \left(\frac{1}{\nu(z)^2(1+\Psi'(z))^q} \int_{D(z,\delta\nu(z))} |g(w)|^q(\Psi'(\Phi(z)))^q e^{q\Psi(\Phi(w))-q\Psi(w)}d\Lambda(w)\right)^{\frac{p}{p-q}} d\Lambda(z)$$
$$\lesssim \int_{\mathbb{C}} \left(\frac{1}{\nu(z)^2} \int_{D(z,\delta\nu(z))} \frac{|g(w)|^q(\Psi'(\Phi(z)))^q}{(1+\Psi'(w))^q} e^{q\Psi(\Phi(w))-q\Psi(w)}d\Lambda(w)\right)^{\frac{p}{p-q}} d\Lambda(z) < \infty.$$

On the other hand, if $\mathcal{M}_g^{\Phi}$ is in $\mathcal{L}^r(\mathbb{C}, d\Lambda)$, $r = \frac{pq}{p-q}$, then (2.1) and the Hölder inequality give

$$\|J_g^{\Phi} f\|_{\mathcal{F}_q^{\Psi}}^q \asymp \int_{\mathbb{C}} \frac{|f'(\Phi(z))|^q|g(z)|^q}{(1+\Psi'(z))^q} e^{-q\Psi(z)}d\Lambda(z)$$
$$\lesssim \left(\int_{\mathbb{C}} \frac{|f'(\Phi(z))|^p}{(1+\Psi'(\Phi(z))^p} e^{-p\Psi(\Phi(z))}d\Lambda(z)\right)^{\frac{q}{p}} \times \left(\int_{\mathbb{C}} \frac{|g(z)|^r(\Psi'(\Phi(z)))^r}{(1+\Psi'(z))^r} e^{r\Psi(\Phi(z))-r\Psi(z)}d\Lambda(z)\right)^{\frac{q}{r}}$$
$$\lesssim \left(\int_{\mathbb{C}} \frac{|f'(\Phi(z))|^p}{(1+\Psi'(\Phi(z))^p} e^{-p\Psi(\Phi(z))}d\Lambda(z)\right)^{\frac{q}{p}} \lesssim \|f\|_{\mathcal{F}_p^{\Psi}}^q,$$

where in the latter estimate we have employed a change of variable and the identity $\Phi(z) = az+b$ with $0 < |a| \leqslant 1$ due to Lemma 3.1. Therefore, J_g^{Φ} is bounded and hence the function T belongs to $\mathcal{L}^{\frac{r}{q}}(\mathbb{C}, d\Lambda)$.

Case 2: $p = \infty$**.** Suppose T is in $\mathcal{L}(\mathbb{C}, d\Lambda)$. Proceeding then as in Case 1, we get

$$\int_{\mathbb{C}} (\mathcal{M}_g^{\Phi})^q(z)d\Lambda(z) = \int_{\mathbb{C}} \frac{|g(z)|^q(\Psi'(\Phi(z)))^q}{(1+\Psi'(z))^q} e^{q\Psi(\Phi(z))-q\Psi(z)}d\Lambda(z)$$
$$\lesssim \int_{\mathbb{C}} \frac{1}{\nu(z)^{2q}(1+\Psi'(z))^q} \int_{D(z,\delta\nu(z))} |g(w)|^q(\Psi'(\Phi(z)))^q e^{q\Psi(\Phi(w))-q\Psi(w)}d\Lambda(w)d\Lambda(z)$$
$$\lesssim \int_{\mathbb{C}} \frac{1}{\nu(z)^{2q}} \int_{D(z,\delta\nu(z))} \frac{|g(w)|^q(\Psi'(\Phi(z)))^q}{(1+\Psi'(w))^q} e^{q\Psi(\Phi(w))-q\Psi(w)}d\Lambda(w)d\Lambda(z) < \infty.$$

On the other hand, if $\mathcal{M}_g^{\Phi} \in \mathcal{L}^q(\mathbb{C}, d\Lambda)$, then $\mathcal{M}_g^{\Phi}$ is bounded and hence $\Phi(z) = az + b$ with $0 < |a| \leqslant 1$ (see Lemma 3.1). Using this and the estimate in (2.2), we find:

$$\|J_g^{\Phi} f\|_{\mathcal{F}_q^{\Psi}}^q \asymp \int_{\mathbb{C}} \frac{|f'(\Phi(z))|^q|g(z)|^q}{(1+\Psi'(z))^q} e^{-q\Psi(z)}d\Lambda(z)$$
$$\lesssim \left(\sup_{z\in\mathbb{C}} \frac{|f'(\Phi(z))|^q}{(1+\Psi'(\Phi(z))^q} e^{-q\Psi(\Phi(z))}\right)\left(\int_{\mathbb{C}} \frac{|g(z)|^q(\Psi'(\Phi(z)))^q}{(1+\Psi'(z))^q} e^{q\Psi(\Phi(z))-q\Psi(z)}d\Lambda(z)\right)$$
$$\lesssim \|f\|_{\mathcal{F}_\infty^{\Psi}}^q.$$

Therefore, J_g^{Φ} is bounded and hence T is in $\mathcal{L}(\mathbb{C}, d\Lambda)$. In view of Cases 1 and 2 we conclude that the operator is bounded or compact if and only if $\int_{\mathbb{C}} (\mathcal{M}_g^{\Phi})^r(z)d\Lambda(z)$ is finite.

(ii) Here proof is very similar to the above proof and we omit it. The proof is complete. □

4. Schatten-class membership

The singular values of a compact operator T on a Hilbert space $\mathcal{H}$ are the square roots of the positive eigenvalues of the operator T^*T, where T^* denotes the adjoint of T. Given $0 < p < \infty$, the Schatten class of a Hilbert space $\mathcal{H}$, denoted by $S_p(\mathcal{H})$, is the space of all compact operators T on $\mathcal{H}$ with its singular value sequence $\{\beta_n\}$ belonging to the sequence space l^p. The space $S_p(\mathcal{H})$ is Banach space for $1 \leqslant p < \infty$ with the norm

$$\|T\|_{S_p} = \Big(\sum_n |\beta_n|^p\Big)^{\frac{1}{p}}.$$

In particular, $S_1(\mathcal{H})$ is called the trace class and $S_2(\mathcal{H})$ is called the Hilbert-Schmidt class. We next characterize the Schatten class membership of J_g^Φ and C_g^Φ on $\mathcal{F}_2^\Psi$. Our next theorem gives necessary condition for these operators to belong to $S_p(\mathcal{F}_2^\Psi)$ and sufficient condition will be provided later.

Theorem 4.1. *Let $0 < p < \infty$ and (g, Φ) be pair of nonconstant entire functions. If J_g^Φ (respectively, C_g^Φ) is in the class $S_p(\mathcal{F}_2^\Psi)$, then $\mathcal{M}_g^\Phi$ (respectively, $\mathcal{M}_{g(\Phi)}^\Phi$) belongs to $\mathcal{L}^p(\mathbb{C}, \Delta\Psi d\Lambda)$.*

Proof. We will prove the statement for the operator J_g^Φ, and a similar procedure works for the operator C_g^Φ. Firs, we define a scalar product $\langle \cdot, \cdot \rangle_*$ on $\mathcal{F}_2^\Psi$ by

$$\langle f, g\rangle_* := f(0)\overline{g(0)} + \int_{\mathbb{C}} \frac{f(z)\overline{g(z)}}{(1+\Psi'(z))^2} e^{-2\Psi(z)} d\Lambda(z), \tag{4.1}$$

which by (2.1) gives an equivalent norm on $\mathcal{F}_2^\Psi$, and divide the proof into two cases.

Case 1: $0 < p < 2$. Since the operator J_g^Φ is in $S_p(\mathcal{F}_2^\Psi)$, then the operator $(J_g^\Phi)^*(J_g^\Phi)$ is in $S_{\frac{p}{2}}(\mathcal{F}_2^\Psi)$ and has a canonical decomposition:

$$(J_g^\Phi)^*(J_g^\Phi) f = \sum_n \beta_n \langle f, e_n\rangle_* e_n,$$

where $\{e_n\}$ is an orthonormal basis in $\mathcal{F}_2^\Psi$ and $\{\beta_n\}$ is the sequence of singular values of a positive operator $(J_g^\Phi)^*(J_g^\Phi)$. Moreover,

$$\|(J_g^\Phi)^*(J_g^\Phi)\|_{S_{\frac{p}{2}}}^{\frac{p}{2}} = \sum_n |\beta_n|^{\frac{p}{2}}.$$

Using the estimate in (2.5) and the Hölder's inequality, we then obtain

$$\begin{aligned}
\int_{\mathbb{C}} (\mathcal{M}_g^\Phi)^p(w)\Delta\Psi(w)d\Lambda(w) &= \int_{\mathbb{C}} \frac{|g(w)|^p(\Psi'(\Phi(w)))^p}{(1+\Psi'(w))^p} e^{p\Psi(\Phi(w))-p\Psi(w)}\Delta\Psi(w)d\Lambda(w)\\
&\asymp \int_{\mathbb{C}} \frac{|g(w)|^p(\Psi'(\Phi(w)))^{p-2}\|\frac{\partial}{\partial\overline{\Phi(w)}}K_{(\Phi(w),\Psi)}(\Phi(w))\|_{\mathcal{F}_2^\Psi}^2}{(1+\Psi'(w))^p} e^{(p-2)\Psi(\Phi(w))-p\Psi(w)}d\Lambda(w)\\
&= \sum_n \int_{\mathbb{C}} \frac{|g(w)|^p(\Psi'(\Phi(w)))^{p-2}|e_n'(\Phi(w))|^2}{(1+\Psi'(w))^p} e^{(p-2)\Psi(\Phi(w))-p\Psi(w)}d\Lambda(w)\\
&\leqslant \sum_n \Big(\int_{\mathbb{C}} \frac{|g(w)|^2|e_n'(\Phi(w))|^2}{(1+\Psi'(w))^2} e^{-2\Psi(w)}d\Lambda(w)\Big)^{\frac{p}{2}}\\
&\quad\cdot \Big(\int_{\mathbb{C}} \frac{|e_n'(\Phi(w))|^2}{(\Psi'(\Phi(w)))^2} e^{-2\Psi(\Phi(w))}d\Lambda(w)\Big)^{\frac{2-p}{2}}.
\end{aligned} \tag{4.2}$$

Since J_g^Φ is compact, by Theorem 3.1 and Lemma 3.1 the function Φ has the form $\Phi(z) = az+b$ for some $a, b \in \mathbb{C}$ with $0 < |a| \leqslant 1$. Then a change of variable gives

$$\int_{\mathbb{C}} \frac{|e_n'(\Phi(w))|^2}{(\Psi'(\Phi(w)))^2} e^{-2\Psi(\Phi(w))} d\Lambda(w) \lesssim \int_{\mathbb{C}} \frac{|e_n'(\Phi(w))|^2}{(1+\Psi'(\Phi(w)))^2} e^{-2\Psi(\Phi(w))} d\Lambda(w)$$
$$= \frac{1}{|a|^2} \int_{\mathbb{C}} \frac{|e_n'(\zeta)|^2}{(1+\Psi'(\zeta))^2} e^{-2\Psi(\zeta)} d\Lambda(\zeta) \asymp \|e_n\|^2_{\mathcal{F}_2^\Psi} = 1,$$

and

$$\sum_n \left(\int_{\mathbb{C}} \frac{|g(w)|^2 |e_n'(\Phi(w))|^2}{(1+\Psi'(w))^2} e^{-2\Psi(w)} d\Lambda(w) \right)^{\frac{p}{2}} = \sum_n \langle (J_g^\Phi)^*(J_g^\Phi) e_n, e_n \rangle_*^{\frac{p}{2}}$$
$$= \sum_n |\beta_n|^{\frac{p}{2}} = \|(J_g^\Phi)^*(J_g^\Phi)\|^{\frac{p}{2}}_{S_{\frac{p}{2}}}.$$

By (4.2) and the above two estimates we obtain

$$\int_{\mathbb{C}} (\mathcal{M}_g^\Phi)^p(w) \Delta\Psi(w) d\Lambda(w) \lesssim \|(J_g^\Phi)^*(J_g^\Phi)\|^{\frac{p}{2}}_{S_{\frac{p}{2}}} < \infty$$

and this is the desired inequality.

Case 2: $2 \leqslant p < \infty$. Let $\{z_n\}$ be the sequence as in Lemma 2.2 and $\{e_n\}$ be an orthonormal basis in $\mathcal{F}_2^\Psi$. Let T be an operator taking $e_n(z)$ to $f_{(z_n,R)}(z) = \frac{F_{(z_n,R)}(z)}{\nu(z_n)}$. By Proposition 9 in [1], T is bounded operator and $J_g^\Phi T$ is in $S_p(\mathcal{F}_2^\Psi)$, and by Theorem 1.33 of [9],

$$\sum_n \|J_g^\Phi f_{(z_n,R)}\|^p_{\mathcal{F}_2^\Psi} = \sum_n \|J_g^\Phi T e_n\|^p_{\mathcal{F}_2^\Psi} < \infty. \tag{4.3}$$

Using (2.3) and the estimate $\nu(z) \asymp \nu(z_n)$ for $z \in D(z_n, \delta\nu(z_n))$, we get

$$\int_{\mathbb{C}} (\mathcal{M}_g^\Phi)^p(z) \Delta\Psi(z) d\Lambda(z) = \sum_n \int_{D(z_n,\delta\nu(z_n))} \frac{|g(z)|^p (\Psi'(\Phi(z)))^p}{(1+\Psi'(z))^p} e^{p\Psi(\Phi(z)) - p\Psi(z)} \Delta\Psi(z) d\Lambda(z)$$
$$\lesssim \sum_n \nu(z_n)^{-p} \int_{D(z_n,\delta\nu(z_n))} (H(z))^{\frac{p}{2}} \nu(z)^{-2} d\Lambda(z),$$

where

$$H(z) = \int_{D(z,\delta\nu(z))} \frac{|g(w)|^2 (\Psi'(\Phi(w)))^2}{(1+\Psi'(w))^2} e^{2\Psi(\Phi(w)) - 2\Psi(w)} d\Lambda(w).$$

Using the estimate in (2.6), we proceed as follows:

$$\sum_n \nu(z_n)^{-p} \left(\int_{D(z_n,\delta\nu(z_n))} \frac{|g(z)|^2 (\Psi'(\Phi(z)))^2}{(1+\Psi'(z))^2} e^{2\Psi(\Phi(z)) - 2\Psi(z)} d\Lambda(z) \right)^{\frac{p}{2}}$$
$$\lesssim \sum_n \left(\int_{D(z_n,\delta\nu(z_n))} \frac{|g(z)|^2 |F'_{(z_n,R)}(\Phi(z))|^2}{(1+\Psi'(z))^2} e^{-2\Psi(z)} d\Lambda(z) \right)^{\frac{p}{2}} \lesssim \sum_n \|J_g^\Phi f_{(z_n,R)}\|^p_{\mathcal{F}_2^\Psi}$$

which, by (4.3), is finite, and therefore, $\mathcal{M}_g^\Phi$ belongs to $\mathcal{L}^p(\mathbb{C}, \Delta\Psi d\Lambda)$. The proof is complete. □

Theorem 4.2. *Let $1 < p < \infty$ and (g, Φ) be a pair of nonconstant entire functions. If $\mathcal{M}_g^\Phi$ (respectively, $\mathcal{M}_{g(\Phi)}^\Phi$) is in $\mathcal{L}^p(\mathbb{C}, \Delta\Psi(\Phi) d\Lambda)$, then J_g^Φ (respectively, C_g^Φ) is in the class $S_p(\mathcal{F}_2^\Psi)$.*

Proof. The proof of the theorem for the two operators is very similar. This is why we provide the proof only for the operator J_g^Φ. We consider two cases.

Case 1: $1 < p < 2$**.** Let $\{e_n\}$ be an orthonormal basis of $\mathcal{F}_2^\Psi$. Then by Theorem 1.27 in [9], J_g^Φ is in the class $S_p(\mathcal{F}_2^\Psi)$ if and only if

$$\sum_n |\langle J_g^\Phi e_n, e_n\rangle_*|^p < \infty,$$

where $\langle \cdot, \cdot\rangle_*$ was defined in (4.1). Since $p > 1$, the Hölder inequality yields

$$\begin{aligned}
\sum_n |\langle J_g^\Phi e_n, e_n\rangle_*|^p &\lesssim \sum_n \Big(\int_{\mathbb{C}} \frac{|g(z)e_n'(\Phi(z))\overline{e_n'(z)}|}{(1+\Psi'(z))^2} e^{-2\Psi(z)} d\Lambda(z)\Big)^p \\
&\leqslant \sum_n \int_{\mathbb{C}} \frac{|g(z)|^p}{(1+\Psi'(z))^{2p}} |e_n'(\Phi(z))|^p |e_n'(z)|^{2-p} e^{-2\Psi(z)} d\Lambda(z) \\
&\quad \cdot \Big(\int_{\mathbb{C}} \frac{|e_n'(z)|^2}{(1+\Psi'(z))^2} e^{-2\Psi(z)} d\Lambda(z)\Big)^{p-1} \\
&\asymp \int_{\mathbb{C}} \frac{|g(z)|^p}{(1+\Psi'(z))^{2p}} \Big(\sum_n |e_n'(\Phi(z))|^p |e_n'(z)|^{2-p}\Big) e^{-2\Psi(z)} d\Lambda(z).
\end{aligned} \tag{4.4}$$

Since $p < 2$, applying the Hölder inequality, using (2.5) and the estimate

$$\big(\Psi'(z)\big)^{2(1-p)} \lesssim \frac{\nu(z)^{2-p}}{\nu(\Phi(z))^{2-p}},$$

which follows from definition of ν, we obtain

$$\begin{aligned}
\sum_n |e_n'(\Phi(z))|^p |e_n'(z)|^{2-p} &\leqslant \Big(\sum_n |e_n'(\Phi(z))|^2\Big)^{\frac{p}{2}} \Big(\sum_n |e_n'(z)|^2\Big)^{\frac{2-p}{2}} \\
&= \|\frac{\partial}{\partial \overline{z}} K_{(\Phi(z),\Psi)}\|_{\mathcal{F}_2^\Psi}^p \|\frac{\partial}{\partial \overline{z}} K_{(z,\Psi)}\|_{\mathcal{F}_2^\Psi}^{2-p} \\
&\asymp \|K_{(\Phi(z),\Psi)}\|_{\mathcal{F}_2^\Psi}^p \big(\Psi'(\Phi(z))\big)^p \|K_{(z,\Psi)}\|_{\mathcal{F}_2^\Psi}^{2-p} \big(\Psi'(z)\big)^{2-p} \\
&\asymp \frac{\big(\Psi'(\Phi(z))\big)^p \big(\Psi'(z)\big)^{2-p} e^{p\Psi(\Phi(z))+(2-p)\Psi(z)}}{\nu(\Phi(z))^p \nu(z)^{2-p}} \\
&\lesssim \frac{\big(\Psi'(\Phi(z))\big)^p \big(\Psi'(z)\big)^p e^{p\Psi(\Phi(z))+(2-p)\Psi(z)}}{\nu(\Phi(z))^2}.
\end{aligned}$$

Substituting this estimate into (4.4), we get

$$\begin{aligned}
\sum_n |\langle J_g^\Phi e_n, e_n\rangle_*|^p &\lesssim \int_{\mathbb{C}} \frac{|g(z)|^p \big(\Psi'(\Phi(z))\big)^p (\Psi'(z))^p}{(1+\Psi'(z))^{2p} \nu(\Phi(z))^2} e^{p\Psi(\Phi(z))-p\Psi(z)} d\Lambda(z) \\
&\asymp \int_{\mathbb{C}} \Big(\mathcal{M}_g^\Phi(z)\Big)^p \Delta\Psi(\Phi(z)) d\Lambda(z) < \infty.
\end{aligned}$$

Case 2: $2 \leqslant p < \infty$**.** Similarly, let $\{e_n\}$ be an orthonormal basis of $\mathcal{F}_2^\Psi$. By Theorem 1.33 [9], it is sufficient to show that $\sum_n \|J_g^\Phi e_n\|_{\mathcal{F}_2^\Psi}^p < \infty$. For $p = 2$, by estimates (2.1) and (2.5) we find

$$\begin{aligned}\sum_n \|J_g^\Phi e_n\|^2_{\mathcal{F}_2^\Psi} &\asymp \sum_n \int_{\mathbb{C}} \frac{|e_n'(\Phi(z))|^2|g(z)|^2}{(1+\Psi'(z))^2} e^{-2\Psi(z)} d\Lambda(z)\\ &\asymp \int_{\mathbb{C}} \frac{|g(z)|^2(\Psi'(\Phi(z)))^2}{(1+\Psi'(z))^2\nu(\Phi(z))^2} e^{2\Psi(\Phi(z))-2\Psi(z)} d\Lambda(z)\\ &= \int_{\mathbb{C}} \Big(\mathcal{M}_g^\Phi(z)\Big)^p \Delta\Psi(\Phi(z)) d\Lambda(z).\end{aligned} \tag{4.5}$$

For $p>2$, by the estimate in (2.1), the Hölder's inequality and the norm estimate in (2.5) we obtain

$$\begin{aligned}\sum_n \|J_g^\Phi e_n\|^p_{\mathcal{F}_2^\Psi} &\asymp \sum_n \Big(\int_{\mathbb{C}} \frac{|e_n'(\Phi(z))|^2|g(z)|^2}{(1+\Psi'(z))^2} e^{-2\Psi(z)} d\Lambda(z)\Big)^{\frac{p}{2}}\\ &\leqslant \sum_n \Big(\int_{\mathbb{C}} \frac{|e_n'(\Phi(z))|^2|g(z)|^p(1+\Psi'(\Phi(z)))^{p-2}}{(1+\Psi'(z))^2} e^{-p\Psi(z)+(p-2)\Psi(\Phi(z))} d\Lambda(z)\Big)\\ &\times \Big(\int_{\mathbb{C}} \frac{|e_n'(\Phi(z))|^2}{(1+\Psi'(\Phi(z)))^2} e^{-2\Psi(\Phi(z))} d\Lambda(z)\Big)^{\frac{p-2}{2}}\\ &\lesssim \sum_n \Big(\int_{\mathbb{C}} \frac{|e_n'(\Phi(z))|^2|g(z)|^p(1+\Psi'(\Phi(z)))^{p-2}}{(1+\Psi'(z))^2} e^{-p\Psi(z)+(p-2)\Psi(\Phi(z))} d\Lambda(z)\Big)\\ &\lesssim \int_{\mathbb{C}} \frac{\big(\sum_n |e_n'(\Phi(z))|^2\big)|g(z)|^p(\Psi'(\Phi(z)))^{p-2}}{(1+\Psi'(z))^2} e^{-p\Psi(z)+(p-2)\Psi(\Phi(z))} d\Lambda(z)\\ &\asymp \int_{\mathbb{C}} \frac{|g(z)|^p(\Psi'(\Phi(z)))^p}{(1+\Psi'(z))^2\nu(\Phi(z))^2} e^{-p\Psi(z)+p\Psi(\Phi(z))} d\Lambda(z)\\ &= \int_{\mathbb{C}} \Big(\mathcal{M}_g^\Phi(z)\Big)^p \Delta\Psi(\Phi(z)) d\Lambda(z) \lesssim \int_{\mathbb{C}} \Big(\mathcal{M}_g^\Phi(z)\Big)^p \Delta\Psi(z) d\Lambda(z).\end{aligned} \tag{4.6}$$

By the above relations and (4.5) we conclude that $J_g^\Phi \in S_p(\mathcal{F}_2^\Psi)$. The proof is complete. □

Our next proposition gives another characterization of Schatten class membership of J_g^Φ and C_g^Φ on $\mathcal{F}_2^\Psi$. It holds for any compact operator T on $\mathcal{F}_2^\Psi$.

Proposition 4.3. *Let $0<p<\infty$ and (g,Φ) be a pair of nonconstant entire functions such that J_g^Φ (respectively, C_g^Φ) is compact on $\mathcal{F}_2^\Psi$. Then*

- (i) *if $2\leqslant p<\infty$ and J_g^Φ (respectively, C_g^Φ) is in $S_p(\mathcal{F}_2^\Psi)$, then the function $\|J_g^\Phi k_{(w,\Psi)}\|_{\mathcal{F}_2^\Psi}$ (respectively, $\|C_g^\Phi k_{(w,\Psi)}\|_{\mathcal{F}_2^\Psi}$) belongs to $\mathcal{L}^p(\mathbb{C},\Delta\Psi d\Lambda)$.*
- (ii) *if $0<p<2$ and the function $\|J_g^\Phi k_{(w,\Psi)}\|_{\mathcal{F}_2^\Psi}$ (respectively, $\|C_g^\Phi k_{(w,\Psi)}\|_{\mathcal{F}_2^\Psi}$) belongs to $\mathcal{L}^p(\mathbb{C},\Delta\Psi d\Lambda)$, then J_g^Φ (respectively, C_g^Φ) is in $S_p(\mathcal{F}_2^\Psi)$.*

Proof. We again prove for the operator J_g^Φ only. Since J_g^Φ is compact, it has a canonical decomposition,

$$J_g^\Phi f = \sum_n \beta_n \langle f, e_n\rangle e_n, \tag{4.7}$$

where $\{\beta_n\}$ is a sequence of real numbers tending to 0 as n goes to infinity and $\{e_n\}$ is an orthonormal basis of $\mathcal{F}_2^{\Psi}$ (see, Theorem 1.20 in [9]). Moreover,

$$\|J_g^{\Phi}\|_{S_p}^p = \sum_n |\beta_n|^p.$$

Applying (4.7) to the normalized kernel function $k_{(w,\Psi)}$, we obtain

$$\|J_g^{\Phi} k_{(w,\Psi)}\|_{\mathcal{F}_2^{\Psi}}^2 = \|K_{(w,\Psi)}\|_{\mathcal{F}_2^{\Psi}}^{-2} \sum_n |\beta_n|^2 |e_n(w)|^2 \asymp \nu(w)^2 e^{-2\Psi(w)} \sum_n |\beta_n|^2 |e_n(w)|^2. \tag{4.8}$$

(i) If $p = 2$, then using the estimate in (4.8), we find

$$\begin{aligned}\int_{\mathbb{C}} \|J_g^{\Phi} k_{(w,\Psi)}\|_{\mathcal{F}_2^{\Psi}}^p \Delta\Psi(w) d\Lambda(w) &\asymp \int_{\mathbb{C}} e^{-2\Psi(w)} \Big(\sum_n |\beta_n|^2 |e_n(w)|^2\Big) d\Lambda(w) \\ &= \sum_n |\beta_n|^2 \int_{\mathbb{C}} |e_n(w)|^2 e^{-2\Psi(w)} d\Lambda(w) = \sum_n |\beta_n|^2 = \|J_g^{\Phi}\|_{S_2}^2 < \infty.\end{aligned}$$

If $p > 2$, then again using the estimate in (4.8), applying Hölder's inequality and using

$$\sum_n |e_n(w)|^2 = K_{(w,\Psi)}(w) = \|K_{(w,\Psi)}\|_{\mathcal{F}_2^{\Psi}}^2$$

together with estimate (2.4), we get

$$\begin{aligned}\int_{\mathbb{C}} \|J_g^{\Phi} k_{(w,\Psi)}\|_{\mathcal{F}_2^{\Psi}}^p \Delta\Psi(w) d\Lambda(w) &\asymp \int_{\mathbb{C}} \nu(w)^{p-2} e^{-p\Psi(w)} \Big(\sum_n |\beta_n|^2 |e_n(w)|^2\Big)^{\frac{p}{2}} d\Lambda(w) \\ &\leqslant \int_{\mathbb{C}} \nu(w)^{p-2} e^{-p\Psi(w)} \Big(\sum_n |\beta_n|^p |e_n(w)|^2\Big)\Big(\sum_n |e_n(w)|^2\Big)^{\frac{p-2}{2}} d\Lambda(w) \\ &\asymp \sum_n |\beta_n|^p \int_{\mathbb{C}} |e_n(w)|^2 e^{-2\Psi(w)} d\Lambda(w) \\ &= \sum_n |\beta_n|^p = \|J_g^{\Phi}\|_{S_p}^p < \infty.\end{aligned}$$

(ii) Using the estimate in (2.4), applying Hölder's inequality and the estimate in (4.8), we obtain

$$\begin{aligned}\|J_g^{\Phi}\|_{S_p}^p = \sum_n |\beta_n|^p = \sum_n |\beta_n|^p \|e_n\|_{\mathcal{F}_2^{\Psi}}^2 &= \sum_n |\beta_n|^p \int_{\mathbb{C}} |e_n(z)|^2 e^{-2\Psi(z)} d\Lambda(z) \\ &\asymp \sum_n |\beta_n|^p \int_{\mathbb{C}} |e_n(z)|^2 \frac{\|K_{(z,\Psi)}\|_{\mathcal{F}_2^{\Psi}}^{-2}}{\nu(z)^2} d\Lambda(z) \\ &\leqslant \int_{\mathbb{C}} \Big(\sum_n |\beta_n|^2 |e_n(z)|^2\Big)^{\frac{p}{2}} \Big(\sum_n |e_n(z)|^2\Big)^{\frac{2-p}{2}} \frac{\|K_{(z,\Psi)}\|_{\mathcal{F}_2^{\Psi}}^{-2}}{\nu(z)^2} d\Lambda(z) \\ &= \int_{\mathbb{C}} \Big(\sum_n |\beta_n|^2 |e_n(z)|^2\Big)^{\frac{p}{2}} \frac{\|K_{(z,\Psi)}\|_{\mathcal{F}_2^{\Psi}}^{-p}}{\nu(z)^2} d\Lambda(z) \\ &\asymp \int_{\mathbb{C}} \|J_g^{\Phi} k_{(z,\Psi)}\|_{\mathcal{F}_2^{\Psi}}^p \Delta\Psi(z) d\Lambda(z) < \infty,\end{aligned}$$

from which the conclusion follows. The proof is complete. □

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