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ON EMBEDDING INTO LORENTZ SPACES (A DISTANT CASE)

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Abstract. In the work we study an upper bound for a non–increasing non–negative function in the space $L^p(0,1)$ by the modulus of continuity of a variable increment $\omega_{p,\alpha,\psi}(f,\delta)$. We show that for the increment of the function of form $f(x)-f(x+hx^\alpha\psi(x))$ in the bound the modulus of continuity casts into the form $\omega_{p,\alpha,\psi}\left(f,\frac{\delta}{\delta^\alpha\psi(\frac{1}{\delta})}\right)$. We also study the embedding $\tilde{H}^\omega_{p,\alpha,\psi}\subset L(\mu,\nu)(\mu\neq\nu)$ (a distant case). We obtained necessary and sufficient conditions for the parameters p, α, μ, ν and the functions ψ, ω for this embedding.

Keywords: classes of functions, modulus of continuity of variable increment, non–increasing permutation of the function, Lorentz spaces.

Mathematics Subject Classification: 34B45, 81Q15

1. Introduction

Let $\omega(\delta)$ be a continuous on $[0,1]$ function obeying the conditions

$$0=\omega(0)\leqslant\omega(\delta)\leqslant\omega(\eta)\leqslant\omega(\delta+\eta)\leqslant\omega(\delta)+\omega(\eta),\quad 0\leqslant\delta\leqslant\eta\leqslant\delta+\eta\leqslant1.$$

Such functions are called *continuity moduluses*.

Let $\omega_p(f,\delta)$ be a modulus of continuity of a function f in the space $L^p(0,1)$, that is,

$$\omega_p(f,\delta)=\sup_{0<h\leqslant\delta}\left(\int\limits_0^{1-h}|f(x+h)-f(x)|^p\,dx\right)^{\frac{1}{p}},\quad 0<\delta\leqslant1. \tag{1.1}$$

We let

$$H^\omega_p=\{f\in L^p(0,1):\omega_p(f,\delta)\leqslant\omega(\delta)\},\quad 0<\delta\leqslant1, \tag{1.2}$$

where $\omega(\delta)$ is a given modulus of continuity .

A positive function $\psi(x)$ defined for $x>x_0$ is called weakly oscillating if for each $\delta>0$ the function $x^\delta\psi(x)$ increases for sufficiently large x, while $x^{-\delta}\psi(x)$ decreases [1].

Let $1\leqslant p<\infty$, $0\leqslant\alpha<1$ and $f\in L^p(0,1)$, then the function

$$\omega_{p,\alpha,\psi}(f,\delta)=\sup_{0<h\leqslant\delta}\left\{\int\limits_{E_{h,\alpha,\psi}}|f(x+hx^\alpha\psi(x))-f(x)|^p\,dx\right\}^{\frac{1}{p}}\quad(0<\delta<1), \tag{1.3}$$

where $E_{h,\alpha,\psi}=\{x\in(0,1):x+hx^\alpha\psi(x)\in(0,1)\}$, is called a *modulus of continuity of a variable of special type increment for the function f in $L^p(0,1)$.*

We note that Z. Ditzian and V. Totik in [7] introduced and studied a general case, which is obtained by replacing the function $x^\alpha\psi(x)$ by a continuous on $[0,1]$ function $\phi(x)$ in Definition (1.3).

A.T. Baidaulet, K.M. Suleimenov, On embedding into Lorentz spaces (a distant case).

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It is clear that as $\alpha = 0$, $\psi(x) = 1$ we have $\omega_{p,0,1}(f, \delta) = \omega(f, \delta)$.

Let $1 \leqslant p < \infty$, $0 \leqslant \alpha < 1$, $\psi(\delta)$ be a weakly oscillating function and $\omega(\delta)$ be a given modulus of continuity. By $\tilde{H}^{\omega}_{p,\alpha,\psi}$ we denote the class of all non–increasing non–negative functions $f \in L^p(0, 1)$ such that

$$\omega_{p,\alpha,\psi}(f, \delta) \leqslant \omega(\delta).$$

We note that as $\alpha = 0$, $\psi(x) = 1$ we obtain $\tilde{H}^{\omega}_{\alpha,p,\psi} \subset H^{\omega}_p$.

Let $0 < \nu$, $\mu < \infty$. A Lorentz space $L(\mu, \nu)$ is defined as the set of all Lebesgue measurable on $[0, 1]$ functions f, for which the quantity

$$\| f \|^{\nu}_{\mu,\nu} = \left\{ \int_0^1 x^{\frac{\nu}{\mu}-1} [f]^{\nu}\, dx \right\}$$

is finite. A detailed study of the Lorentz space $L(\mu, \nu)$ is given in [3] as well as in work [5].

In what follows by $C(\alpha, \beta, \ldots) = C_{\alpha,\beta,\ldots}$ we denote positive quantities depending only on the parameters $\alpha, \beta, \ldots$ and generally speaking, these constants are different in various formulas. Let A and B be some scalar products, and A is non–negative. Then the writing $B = 0_{\alpha,\beta,\ldots}(A)$, $B \underset{\alpha,\beta,\ldots}{\ll} A$ stands for $|B| \leqslant C(\alpha, \beta, \ldots)A$.

In work [4] there was obtained an upper bound for non–increasing non–negative functions $f \in L^p(0, 1)$:

$$f(x) \ll \left\{ \int_x^1 \frac{\omega_{p,\alpha}\left(f, 2^{-1}\left(2^{\frac{2}{1-\alpha}} - 1\right) t^{1-\alpha}\right)}{t^{\frac{1}{p}+1}}\, dt + \| f \|_p \right\},$$

$$\text{for} \quad 0 \leqslant x \leqslant 1, \quad 1 \leqslant p < \infty, \quad 0 \leqslant \alpha < 1.$$

The present paper is a continuation of work [5]. In this work we obtain an upper bound for a non–negative non–increasing function by a modulus of continuity of a variable of special type increment as well as the embedding theorem for the classes of functions $\tilde{H}^{\omega}_{p,\alpha,\psi}$ into the Lorentz space $L(\mu, \nu)$.

The application of methods based on estimates of non–increasing permutations in the embedding theory for classes of functions go back to works by P.L. Ulyanov, see, for instance, [6].

In work [4] there was applied an upper bound for a non–increasing non–negative function by modulus of continuity of a variable of special type increment.

The first general embedding theorem providing necessary and sufficient conditions in terms of an arbitrary modulus of continuity reads as follows.

Theorem A. [6] *Given a modulus of continuity $\omega(\delta)$ and numbers $1 \leqslant p < q < \infty$, we have*

$$H^{\omega}_p \subset L^q(0, 1) \Leftrightarrow \sum_{n=1}^{\infty} n^{\frac{q}{p}-2} \omega^q \left(\frac{1}{n}\right) < \infty.$$

Let us provide a criterion of embedding the classes of functions H^{ω}_p into the Lorentz space $L(\mu, \nu)$, which is related with the formulation of the problem in the present work.

Theorem B. [5] *Given a modulus of continuity $\omega(\delta)$ and numbers $1 \leqslant p < \infty$, $0 < \nu < \infty$, $0 < \mu < \infty$, we have the following statements:*

1) *If $\mu > p$, $0 < \nu < \infty$, then*

$$H^{\omega}_p \subset L(\mu, \nu) \Leftrightarrow \sum_{n=1}^{\infty} n^{\nu\left(\frac{1}{p}-\frac{1}{\mu}-1\right)} \omega^q \left(\frac{1}{n}\right) < +\infty;$$

2) *If* $\mu = p$ *and* $0 < \nu < p$, *then*

$$H_p^\omega \subset L(p,\nu) \Leftrightarrow \sum_{n=10}^{\infty} \frac{1}{n(\ln n)^{\frac{\nu}{p}}} \omega^\nu \left(\frac{1}{n}\right) < +\infty,$$

where, if this is necessary, we let $\omega(\delta) = O\{\omega(\delta^2)\}$, $0 < \delta < 1$.

Theorem C. [4] *Let* $1 \leqslant p < \mu$, $0 \leqslant \alpha < 1 - \left(\frac{1}{p} - \frac{1}{\mu}\right)$, $0 < \nu < \infty$ *be some numbers and* $\omega(\delta)$ *be a modulus of continuity. Then*

$$\tilde{H}_{p,\alpha}^\omega \subset L(\mu,\nu) \Leftrightarrow \sum_{n=1}^{\infty} n^{\frac{\nu}{1-\alpha}\left(\frac{1}{p}-\frac{1}{\mu}\right)-1} \omega^\nu \left(\frac{1}{n}\right) < \infty.$$

Remark 1.1. *As* $\alpha = 0$, *the embedding condition coincides with the embedding condition in Ulyanov theorem, see Theorem A.*

2. Auxiliary statements

Lemma 2.1. [2]. *For all positive numbers* τ, q *and each sequence* $\{a_t\}_{t=0}^\infty$, $a_t \geqslant 0$, *the inequality*

$$\sum_{l=0}^{\infty} 2^{-l\tau} \left(\sum_{t=0}^{l} a_t\right)^q \ll \sum_{t=0}^{\infty} 2^{-l\tau} a_t^q$$

holds true.

Lemma 2.2. [6] *Let a finite non–negative function* $\beta(x)$ *be non–increasing on* $[1; +\infty)$ *and* $\tau \in (-\infty, +\infty)$ *be some real number. Then*

$$\sum_{n=2}^{\infty} 2^{n(1-\tau)} \beta(2^n) \ll_\tau \sum_{n=3}^{\infty} n^{-\tau} \beta(n) \ll_\tau \sum_{n=1}^{\infty} 2^{n(1-\tau)} \beta(2^n).$$

Lemma 2.3. [6] *Let* $\nu > 0$, $r \in (1-\nu, 1)$ *be some numbers and* $\omega(\delta)$ *be a modulus of continuity. If*

$$\sum_{n=1}^{\infty} n^{-r} \omega^\nu \left(\frac{1}{n}\right) = +\infty,$$

then there exist numbers B_n, $n = 1, 2, \dots$, *such that*

1) $B_n \downarrow 0$ *as* $n \uparrow \infty$ *and* $B_n \leqslant \omega\left(\frac{1}{n}\right)$ *for all* n;
2) $\sum\limits_{n=1}^{N} B_n = O\left\{N\omega\left(\frac{1}{N}\right)\right\}$ *as* $N \to +\infty$;
3) $\sum\limits_{n=1}^{\infty} 2^{n(1-r)} \left[B_{2^n} - B_{2^{n+1}}\right]^\nu = +\infty$.

Lemma 2.4. *Let* $\nu > 0$, $r \in (1-\nu, 1)$ *be some number,* $\omega(\delta)$ *be a modulus of continuity, and* $\{\tau_n\}_{n=1}^\infty : \tau_n = n^{1-\alpha}\psi\left(\frac{1}{n}\right)$ *be a non–decreasing sequence. If*

$$\sum_{n=3}^{\infty} n^{-r} \omega^\nu \left(\frac{1}{\tau_n}\right) = +\infty$$

then there exist numbers B_n, $n = 1, 2, \dots$, *such that*

1) $B_n \downarrow 0$ *as* $n \uparrow +\infty$ *and* $B_n \leqslant \omega\left(\frac{1}{\tau_n}\right)$ *for all* n;
2) $\sum\limits_{n=1}^{N} B_n = O\left\{\tau_N \omega\left(\frac{1}{\tau_N}\right)\right\}$ *as* $N \to +\infty$;

3) $\sum\limits_{n=1}^{\infty} 2^{n(1-r)} \left[B_{2^n} - B_{2^{n+1}}\right]^{\nu} = +\infty.$

Proof. We follow the lines of the proof of Lemma 2.3. Owing to Stechkin lemma, we can suppose that $\omega(\delta)$ is a convex modulus of continuity and this is why $\delta^{-1}\omega(\delta) \uparrow$ as $\delta \downarrow 0$.

Let $n_0 = 0$, $n_1 = 1$. If the numbers $n_1 < n_2 < \ldots < n_k$ are chosen, then in view of the non–decreasing sequence $\{\tau_n\}$ we let m_{k+1} to be the smalles among non–integer numbers N, for which

$$\tau_N \omega\left(\frac{1}{\tau_N}\right) > 2\tau_{nk}\omega\left(\frac{1}{\tau_{nk}}\right).$$

Thus,

$$\tau_n \omega\left(\frac{1}{\tau_n}\right) \leqslant 2\tau_{nk}\omega\left(\frac{1}{\tau_{nk}}\right) \quad \text{as} \quad n_k \leqslant n < n_{k+1} \tag{2.1}$$

and

$$\tau_{mk+1}\omega\left(\frac{1}{\tau_{mk+1}}\right) > 2\tau_{nk}\omega\left(\frac{1}{\tau_{nk}}\right). \tag{2.2}$$

Since $\omega(\delta) \downarrow 0$ as $\delta \downarrow 0$ and the sequence $\{\tau_n\}$ is non–decreasing,

$$\tau_n \omega\left(\frac{1}{\tau_n}\right) \leqslant 2\tau_{nk}\omega\left(\frac{1}{\tau_{nk}}\right) \quad \text{as} \quad n_k \leqslant n < 2n_k$$

and then

$$m_{k+1} > 2n_k. \tag{2.3}$$

If

$$\omega\left(\frac{1}{\tau_{mk+1}}\right) \leqslant \frac{1}{2}\omega\left(\frac{1}{\tau_{nk}}\right), \tag{2.4}$$

then we let

$$n_{k+1} = m_{k+1}. \tag{2.5}$$

If

$$\omega\left(\frac{1}{\tau_{mk+1}}\right) > \frac{1}{2}\omega\left(\frac{1}{\tau_{nk}}\right),$$

then we let n_{k+1} to be the smallest among all integer numbers N, for which

$$\omega\left(\frac{1}{\tau_{mN}}\right) \leqslant \frac{1}{2}\omega\left(\frac{1}{\tau_{nk}}\right).$$

In this case

$$n_{k+1} > m_{k+1} > 2n_k, \qquad \omega\left(\frac{1}{\tau_{mk+1}}\right) \leqslant \frac{1}{2}\omega\left(\frac{1}{\tau_{nk}}\right)$$

and

$$\omega\left(\frac{1}{\tau_n}\right) > \frac{1}{2}\omega\left(\frac{1}{\tau_{nk}}\right) \quad \text{as} \quad n_k \leqslant n < n_{k+1}. \tag{2.6}$$

We let

$$B_1 = \omega(1), B_n = \omega\left(\frac{1}{\tau_{nk+1}}\right), \quad n_k \leqslant n < n_{k+1} (k = 1, 2, \ldots). \tag{2.7}$$

Since $\omega(\delta) \downarrow 0$ as $\delta \downarrow 0$, then (2.7) implies Statement 1).

Let N be an integer number and $n_{p-1} \leqslant n < n_p$, $p \geqslant 2$. Then in view of the lacunarity of $\tau_{nk}B_{nk}$ we have

$$\sum_{n=1}^{N} B_n = \sum_{k=1}^{p-1} \sum_{n=n_{k-1}+1}^{n_k} B_n + \sum_{n=n_{p-1}+1}^{N} B_n \sum_{k=1}^{p-1} \tau_{nk}B_{nk} + \tau_{nN}B_{nN}$$

$$\ll 2\tau_{np-1}B_{np-1} + \tau_{nN}B_N \ll 2\tau_{np-1}\omega\left(\frac{1}{\tau_{np-1}}\right) + \tau_{nN}\omega\left(\frac{1}{\tau_{nN}}\right) \ll 3\tau_{nN}\omega\left(\frac{1}{\tau_{nN}}\right).$$

The proof of Statement 3) is similar to Lemma 2.3. The proof is complete. □

3. Upper bound for non–increasing non–negative function

The following theorem is true.

Theorem 3.1. *Let $1 \leqslant p < \infty$, $0 \leqslant \alpha < 1$ be some numbers and $\psi = \psi(x)$ is a weakly oscillating function on $[0,1]$. Then for each non–increasing non–negative function $f \in L^p(0,1)$ the inequality*

$$f(x) \leqslant C(p,\alpha,\psi)\left\{\int_x^1 \frac{\omega_{p,\alpha,\psi}\left(f, C(\alpha)\frac{t}{t^\alpha\psi(t)}\right)}{t^{\frac{1}{p}+1}}\,dt + \|f\|_p\right\}, \qquad 0 < x < 1, \tag{3.1}$$

holds.

Proof. Let $f \in L^p(0,1)$. We choose a number $k_0 \in N$ so that

$$k_0(1-\alpha) \geqslant \log_2(2^{1+2\alpha}+1) \quad \text{and} \quad \psi\left(\frac{1}{2^{k_0}}\right) \geqslant 1.$$

Then we have $k_0 \geqslant \log_2(2^{1+2\alpha}+1)$. We define a sequence h_k as follows:

$$h_k = \frac{C(\alpha)}{2^{k(1-\alpha)}\psi\left(\frac{1}{2^k}\right)}, \tag{3.2}$$

where $C(\alpha) = 2^{2\alpha+1}$ and $k \geqslant k_0$ is integer.

Let $k \geqslant k_0$ and $\frac{1}{2^{k+1}} < x \leqslant \frac{1}{2^k}$, then

$$\frac{1}{2^{(k+1)\alpha}} < x^\alpha \leqslant \frac{1}{2^{k\alpha}}.$$

Then the following inequalities hold:

$$0 < h_k < 1, \tag{3.3}$$

$$x + hx^\alpha\psi(x) \leqslant 1, \tag{3.4}$$

$$x + hx^\alpha\psi(x) \geqslant \frac{1}{2^{k-1}}. \tag{3.5}$$

Indeed, as $k \geqslant k_0$ we have

$$h_k = \frac{C(\alpha)}{2^{k(1-\alpha)}\psi\left(\frac{1}{2^k}\right)} = \frac{2^{2\alpha+1}}{2^{k(1-\alpha)}\psi(\frac{1}{2^k})} \leqslant \frac{2^{2\alpha+1}}{2^{k_0(1-\alpha)}\psi(\frac{1}{2^{k_0}})} \leqslant \frac{2^{2\alpha+1}}{2^{1+2\alpha}+1} < 1,$$

and this proves inequality (3.3).

Let $k \geqslant k_0$ and $\frac{1}{2^{k+1}} < x \leqslant \frac{1}{2^k}$, then

$$x + hx^\alpha\psi(x) \leqslant \frac{1}{2^k} + \frac{C(\alpha)}{2^{k(1-\alpha)}\psi\left(\frac{1}{2^k}\right)}\frac{1}{2^{k\alpha}}\psi\left(\frac{1}{2^k}\right) = \frac{1}{2^k} + \frac{C(\alpha)}{2^{k-k\alpha}2^{k\alpha}}$$
$$= \frac{1}{2^k} + \frac{C(\alpha)}{2^k} = \frac{1+C(\alpha)}{2^k} \leqslant \frac{1+C(\alpha)}{2^{k_0}} \leqslant \frac{1}{2^{k_0}}\left(1+2^{2\alpha+1}\right) \leqslant \frac{2^{2\alpha+1}+1}{2^{2\alpha+1}+1} = 1,$$

and this proves inequality (3.4).

For $k \geqslant k_0$ and $\frac{1}{2^{k+1}} < x \leqslant \frac{1}{2^k}$ we obtain

$$x + hx^\alpha\psi(x) \geqslant \frac{1}{2^{k+1}} + \frac{C(\alpha)}{2^{k(1-\alpha)}\psi\left(\frac{1}{2^k}\right)}\frac{1}{2^{(k+1)\alpha}}\psi\left(\frac{1}{2^{k+1}}\right)$$

$$= \frac{1}{2^{k+1}} + \frac{C(\alpha)}{2^{k(1-\alpha)}\psi\left(\frac{1}{2^k}\right)} \frac{1}{2^{(k+1)\alpha}} \frac{1}{2^{(k+1)\alpha}} \frac{1}{2^{-(k+1)\alpha}} \psi\left(\frac{1}{2^k}\right)$$
$$= \frac{1}{2^{k+1}} + \frac{C(\alpha)}{2^{k(1-\alpha)}} \frac{1}{2^{2(k+1)\alpha}} \frac{1}{2^{-k\alpha}} = \frac{1}{2^{k+1}} + \frac{C(\alpha)}{2^{k+2\alpha}}$$
$$= \frac{1}{2^{k-1}}\left[\frac{1}{2^{k-1-k+1}} + \frac{C(\alpha)}{2^{k+2\alpha-k+1}}\right] = \frac{1}{2^{k-1}}\left[\frac{1}{4} + \frac{C(\alpha)}{2^{2\alpha+1}}\right]$$
$$= \frac{1}{2^{k-1}}\left[\frac{1}{4} + \frac{2^{2\alpha+1}}{2^{2\alpha+1}}\right] = \frac{1}{2^{k-1}}\left[\frac{1}{4} + 1\right] \geqslant \frac{1}{2^{k-1}},$$

and this proves inequality (3.5).

We proceed to a lower bound for the modulus of continuity $\omega_{p,\alpha,\psi}(f,t)$, $0 \leqslant t < 1$. First we are going to prove that

$$E_{h_k,\alpha,\psi} \supset \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right]. \tag{3.6}$$

Indeed, as $\frac{1}{2^{k+1}} < x \leqslant \frac{1}{2^k}$ and $C(\alpha) = 2^{2\alpha+1}$ we have

$$x + h_k x^\alpha \psi(x) \geqslant \frac{1}{2^{k-1}} = \frac{2}{2^k} > \frac{1}{2^k},$$

and this proves (3.6).

It follows from relation (3.6) that

$$\omega_{p,\alpha,\psi}\left(f, \frac{C(\alpha)}{2^{k(1-\alpha)}\psi\left(\frac{1}{2^k}\right)}\right) \geqslant \left\{\int_{\frac{1}{2^{k+1}}}^{\frac{1}{2^k}} |f(x + hx^\alpha\psi(x)) - f(x)|^p \, dx\right\}^{\frac{1}{p}}.$$

Since

$$\frac{1}{2^k} - \frac{1}{2^{k+1}} = \frac{1}{2^{k+1}}(2-1) = \frac{1}{2^{k+1}},$$

then by (2.5) we obtain

$$\omega_{p,\alpha,\psi}\left(f, \frac{C(\alpha)}{2^{k(1-\alpha)}\psi\left(\frac{1}{2^k}\right)}\right) \gg \frac{1}{2^{\frac{k}{p}}}\left[f\left(\frac{1}{2^k}\right) - f\left(\frac{1}{2^{k-1}}\right)\right].$$

This yields

$$\left[f\left(\frac{1}{2^k}\right) - f\left(\frac{1}{2^{k-1}}\right)\right] \leqslant 2^{\frac{k}{p}} \omega_{p,\alpha,\psi}\left(f, \frac{C(\alpha)}{2^{k(1-\alpha)}\psi\left(\frac{1}{2^k}\right)}\right). \tag{3.7}$$

Let us show that the inequality

$$2^{\frac{k}{p}} \omega_{p,\alpha,\psi}\left(f, \frac{C(\alpha)}{2^{k(1-\alpha)}\psi\left(\frac{1}{2^k}\right)}\right) \ll \int_{\frac{1}{2^k}}^{\frac{1}{2^{k-1}}} \frac{\omega_{p,\alpha,\psi}\left(f, C(\alpha)\frac{t}{t^\alpha\psi(t)}\right)}{t^{\frac{1}{p}+1}} \, dt \tag{3.8}$$

holds. Indeed, by the monotonicity of the modulus of continuity we obtain

$$\int_{\frac{1}{2^k}}^{\frac{1}{2^{k-1}}} \frac{\omega_{p,\alpha,\psi}\left(f, C(\alpha)\frac{t}{t^\alpha\psi(t)}\right)}{t^{\frac{1}{p}+1}} \, dt \gg \omega_{p,\alpha,\psi}\left(f, \frac{C(\alpha)}{2^{k(1-\alpha)}\psi\left(\frac{1}{2^k}\right)}\right) \int_{\frac{1}{2^k}}^{\frac{1}{2^{k-1}}} t^{-\frac{1}{p}-1} \, dt,$$

and then

$$\int\limits_{\frac{1}{2^k}}^{\frac{1}{2^{k-1}}} t^{-\frac{1}{p}-1}\,dt = \frac{1}{-\frac{1}{p}}\frac{1}{t^{\frac{1}{p}}}\Bigg|_{\frac{1}{2^k}}^{\frac{1}{2^{k-1}}} = p\left[2^{\frac{k}{p}} - 2^{\frac{k-1}{p}}\right] = p2^{\frac{k}{p}}\left[1-\frac{1}{2^{\frac{1}{p}}}\right] \underset{p,\alpha,\psi}{\gg} 2^{\frac{k}{p}}.$$

This proves relation (3.8).

By (3.7) and (3.8) we have

$$\left[f\left(\frac{1}{2^k}\right) - f\left(\frac{1}{2^{k-1}}\right)\right] \ll \int\limits_{\frac{1}{2^k}}^{\frac{1}{2^{k-1}}} \frac{\omega_{p,\alpha,\psi}\left(f, C(\alpha)\frac{t}{t^{\alpha}\psi(t)}\right)}{t^{\frac{1}{p}+1}}\,dt. \tag{3.9}$$

Using the estimate

$$\| f \|_p^p = \int\limits_0^1 [f(x)]^p\,dx \geqslant \int\limits_0^{\lambda} [f(x)]^p\,dx \geqslant [f(x)]^p\lambda$$

for $\lambda = \frac{1}{2^{k_0}}$, we get

$$f\left(\frac{1}{2^n}\right) \ll \left\{\int\limits_{\frac{1}{2^k}}^{\frac{1}{2^{k+1}}} \frac{\omega_{p,\alpha,\psi}\left(f, C(\alpha)\frac{t}{t^{\alpha}\psi(t)}\right)}{t^{\frac{1}{p}+1}}\,dt + \| f \|\right\}.$$

Hence, for each $0 \leqslant x < 1$ we have

$$f(x) \ll \left\{\int\limits_x^1 \frac{\omega_{p,\alpha,\psi}\left(f, C(\alpha)\frac{t}{t^{\alpha}\psi(t)}\right)}{t^{\frac{1}{p}+1}}\,dt + \| f \|\right\}. \tag{3.10}$$

This completes the proof. □

Remark 3.1. *As $\psi(t) \equiv 1$, by (3.10) we get the estimate*

$$f(x) \ll \left\{\int\limits_x^1 \frac{\omega_{p,\alpha}\left(f, 2^{-1}\left(2^{\frac{2}{1-\alpha}}-1\right)t^{1-\alpha}\right)}{t^{\frac{1}{p}+1}}\,dt + \| f \|_p\right\}$$

for

$$0 \leqslant x \leqslant 1, \quad 1 \leqslant p < \infty, \quad 0 \leqslant \alpha < 1.$$

4. On embedding $\tilde{H}^{\omega}_{\alpha,p,\psi} \subset L(\mu,\nu)$, $\mu \neq p$

Theorem 4.1. *Let $1 \leqslant p < \mu < \infty$, $0 \leqslant \alpha \leqslant 1 - (1/p - 1/\mu)$, $0 < \nu < \infty$ be some number, $\omega(\delta)$ be a modulus of continuity, $\psi = \psi(x)$, $x \in [0,1]$, be a weakly oscillating function. The embedding*

$$\tilde{H}^{\omega}_{p,\alpha,\psi} \subset L(\mu,\nu) \tag{4.1}$$

is true if and only if

$$\sum_{k=0}^{\infty} 2^{k\nu\left(\frac{1}{p}-\frac{1}{\mu}\right)}\omega^{\nu}\left(\frac{1}{2^{k(1-\alpha)}\psi\left(\frac{1}{2^k}\right)}\right) < +\infty. \tag{4.2}$$

Proof. Sufficiency. Let condition (4.2) be satisfied and $f \in \tilde{H}^{\omega}_{p,\alpha,\psi}$. Then

$$\omega_{p,\alpha,\psi}\left(f, C(\alpha,\psi)\frac{t}{t^{\alpha}\psi(t)}\right) \ll \omega_{p,\alpha,\psi}\left(C(\alpha,\psi)\frac{t}{t^{\alpha}\psi(t)}\right)\omega_{p,\alpha,\psi}\left(\frac{t}{t^{\alpha}\psi(t)}\right) \ (0<t<1).$$

By Theorem 3.1 for a non–negative non–increasing function $f(x) \in L^p(0,1)$ we have

$$f(x) \ll \left\{\int\limits_x^1 \frac{\omega_{p,\alpha,\psi}\left(f, C(\alpha,\psi)\frac{t}{t^{\alpha}\psi(t)}\right)}{t^{\frac{1}{p}+1}}\,dt + \|f\|_p\right\}, \quad 0<x<1.$$

There can the following cases.

1) The inequality

$$\lim_{x\to+0}\int\limits_x^1 \frac{\omega_{p,\alpha,\psi}\left(f, C(\alpha,\psi)\frac{t}{t^{\alpha}\psi(t)}\right)}{t^{\frac{1}{p}+1}}\,dt < +\infty$$

holds. Then there exists $C_1>0$ such that $0 \leqslant f(x) \leqslant C_1$ for all $0 \leqslant x \leqslant 1$,

$$\| f \|^{\nu}_{(\mu,\nu)} \ll \int\limits_0^1 x^{\frac{\nu}{\mu}-1} C_1^{\nu}\,dx < +\infty$$

and the belonging $f \in L(\mu,\nu)$ holds for all $0<\mu,\ \nu<\infty$.

2) The identity

$$\lim_{x\to+0}\int\limits_x^1 \frac{\omega_{p,\alpha,\psi}\left(f, C(\alpha,\psi)\frac{t}{t^{\alpha}\psi(t)}\right)}{t^{\frac{1}{p}+1}}\,dt = +\infty$$

holds. Then by applying Theorem 3.1 we obtain

$$f(x) \ll \int\limits_x^1 \frac{\omega_{p,\alpha,\psi}\left(f, C(\alpha,\psi)\frac{t}{t^{\alpha}\psi(t)}\right)}{t^{\frac{1}{p}+1}}\,dt, \quad 0<x\leqslant 1.$$

Applying Lemma 2.1 with

$$\tau = \frac{\nu}{\mu}, \qquad a_n = 2^{\frac{n}{p}}\omega\left(\frac{1}{2^{n(1-\alpha)}\psi\left(\frac{1}{2^n}\right)}\right),$$

we have

$$\begin{aligned}
\| f \|^{\nu}_{(\mu,\nu)} &= \int\limits_0^1 x^{\frac{\nu}{\mu}-1}[f(x)]^{\nu}\,dx \\
&\ll \sum_{k=0}^{\infty}\int\limits_{\frac{1}{2^{k+1}}}^{\frac{1}{2^k}} x^{\frac{\nu}{\mu}-1}\left[\int\limits_x^1 \frac{\omega_{p,\alpha,\psi}\left(f, C(\alpha,\psi)\frac{t}{t^{\alpha}\psi(t)}\right)}{t^{\frac{1}{p}+1}}\,dt\right]^{\nu} dx \\
&\ll \sum_{k=0}^{\infty}\int\limits_{\frac{1}{2^{k+1}}}^{\frac{1}{2^k}} x^{\frac{\nu}{\mu}-1}\left[\int\limits_x^1 \frac{\omega\left(\frac{t}{t^{\alpha}\psi(t)}\right)}{t^{\frac{1}{p}+1}}\,dt\right]^{\nu} dx
\end{aligned}$$

$$\ll \sum_{k=0}^{\infty} \int\limits_{\frac{1}{2^{k+1}}}^{\frac{1}{2^k}} 2^{-k\left(\frac{\nu}{\mu}-1\right)} \left[\sum_{n=0}^{k} \int\limits_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} \frac{\omega\left(\frac{t}{t^\alpha \psi(t)}\right)}{t^{\frac{1}{p}+1}}\, dt \right]^\nu dx$$

$$\ll \sum_{k=0}^{\infty} 2^{-k\frac{\nu}{\mu}} \left[\sum_{n=0}^{k} 2^{n\left(\frac{1}{p}+1\right)} \frac{1}{2^n} \omega\left(\frac{1}{2^{n(1-\alpha)}\psi\left(\frac{1}{2^n}\right)} \right) \right]^p$$

$$\ll \sum_{k=0}^{\infty} 2^{-k\frac{\nu}{\mu}} \left[\sum_{n=0}^{k} 2^{\frac{n}{p}} \omega\left(\frac{1}{2^{n(1-\alpha)}\psi\left(\frac{1}{2^n}\right)} \right) \right]^p$$

$$\ll \sum_{k=0}^{\infty} 2^{k\nu\left(\frac{1}{p}-\frac{1}{\mu}\right)} \omega^\nu \left(\frac{1}{2^{k(1-\alpha)}\omega\left(\frac{1}{2^k}\right)} \right).$$

By (4.2) we have $\tilde{H}^{\omega}_{p,\alpha,\psi} \subset L(\mu,\nu)$. The proves embedding (4.1).

Necessity. Let condition (4.2) be violated, that is,

$$\sum_{k=0}^{\infty} 2^{k\nu\left(\frac{1}{p}-\frac{1}{\mu}\right)} \omega^\nu \left(\frac{1}{2^{k(1-\alpha)}\psi\left(\frac{1}{2^k}\right)} \right) = \infty. \tag{4.3}$$

By Lemma 2.2 we have

$$\sum_{k=0}^{\infty} 2^{\nu\left(\frac{1}{p}-\frac{1}{\mu}\right)} \omega^\nu \left(\frac{1}{2^{k(1-\alpha)}\psi\left(\frac{1}{2^k}\right)} \right) \succ\prec \sum_{k=1}^{\infty} k^{\nu\left(\frac{1}{p}-\frac{1}{\mu}\right)-1} \omega^\nu \left(\frac{1}{k^{1-\alpha}\psi\left(\frac{1}{k}\right)} \right),$$

$$\sum_{k=1}^{\infty} k^{\nu\left(\frac{1}{p}-\frac{1}{\mu}\right)-1} \omega^\nu \left(\frac{1}{k^{1-\alpha}\psi\left(\frac{1}{k}\right)} \right) \succ\prec \sum_{k=1}^{\infty} k^{\frac{\nu}{1-\alpha}\left(\frac{1}{p}-\frac{1}{\mu}\right)-1} \omega^\nu \left(\frac{1}{k\psi\left(\frac{1}{k^{\frac{1}{1-\alpha}}}\right)} \right).$$

As $0 \leqslant \alpha \leqslant 1 - \left(\frac{1}{p} - \frac{1}{\mu}\right)$ we have

$$r = 1 - \nu\left(\frac{1}{p} - \frac{1}{\mu}\right) \subset (1-\nu, 1]. \tag{4.4}$$

Conditions (4.3) and (4.4) provide an opportunity to apply Lemma 2.4.

We defined the sought function by the identity

$$f(x) = \int\limits_{x}^{1} \frac{\eta(t)}{t^{\frac{1}{p}+1}}\, dx, \quad x \in [0,1), \tag{4.5}$$

$$\eta(t) = \begin{cases} B_{2^k} - B_{2^{k+1}}, & x \in \left(\frac{1}{2^{k+1}}, \frac{1}{2^k}\right], \\ 0, & x \notin \left(\frac{1}{2^{k+1}}, \frac{1}{2^k}\right], \end{cases} \tag{4.6}$$

where $\{B_n\}$ is a scalar sequence from Lemma 2.4. The function $f(x)$ is non–negative and is non–increasing on $[0,1]$. Let us prove the embedding $f \in \tilde{H}_{p,\alpha,\psi}$. In order to this, it is sufficient to ensure that the function

$$I(h) = \int\limits_{E_{h,\alpha,\psi}} [f(x) - f(x + hx^\alpha \psi(x))]^p\, dx, \tag{4.7}$$

where $E_{h,\alpha,\psi} = \{x \in (0,1) :\ x + hx^\alpha\psi(x) \in (0,1)\}$, satisfies the estimate

$$I(h) = O\{\omega^p(h)\}, \quad h \to 0. \tag{4.8}$$

This in particular implies that $f \in L^p(0,1)$.

Given a number n, $n = 0, 1, 2, \ldots$, and

$$h_n = \frac{1}{2^{n(1-\alpha)}\psi\left(\frac{1}{2^n}\right)},$$

as

$$0 \leqslant k \leqslant n \quad \text{and} \quad \frac{1}{2^{k+1}} < x \leqslant \frac{1}{2^k}$$

we have the relation

$$\frac{1}{2^{k+1}} < x + hx^\alpha\psi(x) \leqslant \frac{1}{2^{k-1}} \tag{4.9}$$

Indeed, as $0 \leqslant k \leqslant n$, in view of the relation

$$2^{n(1-\alpha)}\psi\left(\frac{1}{2^n}\right) \geqslant 2^{k(1-\alpha)}\psi\left(\frac{1}{2^k}\right),$$

we obtain

$$\begin{aligned}
\frac{1}{2^{k+1}} < x + hx^\alpha\psi(x) &\leqslant \frac{1}{2^k} + \frac{1}{2^{n(1-\alpha)}\psi\left(\frac{1}{2^n}\right)}\frac{1}{2^{k\alpha}}\psi\left(\frac{1}{2^k}\right) \\
&= \frac{1}{2^{k-1}}\left[\frac{1}{2^k 2^{-k+1}} + \frac{1}{2^{n(1-\alpha)}\psi\left(\frac{1}{2^n}\right)2^{-k+1}}\frac{1}{2^{k\alpha}}\psi\left(\frac{1}{2^k}\right)\right] \\
&\leqslant \frac{1}{2^{k-1}}\left[\frac{1}{2} + \frac{1}{2^{k(1-\alpha)}2^{-k+1}\psi\left(\frac{1}{2^k}\right)}\frac{1}{2^{k\alpha}}\psi\left(\frac{1}{2^k}\right)\right] \\
&= \frac{1}{2^{k-1}}\left[\frac{1}{2} + \frac{1}{2}\right] \leqslant \frac{1}{2^{k-1}},
\end{aligned}$$

and this proves relation (4.9). As

$$k \geqslant n \quad \text{and} \quad \frac{1}{2^{k+1}} < x \leqslant \frac{1}{2^k}$$

we have

$$\frac{1}{2^{k+1}} < x + hx^\alpha\psi(x) \leqslant \frac{1}{2^{n-1}}. \tag{4.10}$$

Since for $k \geqslant n$ the inequality

$$\frac{1}{2^k}\psi\left(\frac{1}{2^k}\right) \leqslant \frac{1}{2^n}\psi\left(\frac{1}{2^n}\right),$$

then

$$\begin{aligned}
\frac{1}{2^{k+1}} < x + hx^\alpha\psi(x) &\leqslant \frac{1}{2^k} + \frac{1}{2^{n(1-\alpha)}\psi\left(\frac{1}{2^n}\right)}\frac{1}{2^{k\alpha}}\psi\left(\frac{1}{2^k}\right) \\
&= \frac{1}{2^{n-1}}\left[\frac{1}{2^k 2^{-n+1}} + \frac{1}{2^{-n+1}2^{n(1-\alpha)}\psi\left(\frac{1}{2^n}\right)}\frac{1}{2^{k\alpha}}\psi\left(\frac{1}{2^k}\right)\right] \\
&= \frac{1}{2^{n-1}}\left[\frac{1}{2^{k-n+1}} + \frac{1}{2^{-n+1}2^{n(1-\alpha)}\psi\left(\frac{1}{2^n}\right)}\frac{1}{2^{k\alpha}}\frac{2^{-k\alpha}\psi\left(\frac{1}{2^k}\right)}{2^{-k\alpha}}\right] \\
&\leqslant \frac{1}{2^{n-1}}\left[\frac{1}{2^{k-n+1}} + \frac{2^{-n\alpha}\psi\left(\frac{1}{2^n}\right)}{2\cdot 2^{-n\alpha}\psi\left(\frac{1}{2^n}\right)}\right] = \frac{1}{2^{n-1}}\left[\frac{1}{2^{k-n+1}} + \frac{1}{2}\right] \\
&\leqslant \frac{1}{2^{n-1}}\left[\frac{1}{2} + \frac{1}{2}\right] \leqslant \frac{1}{2^{n-1}},
\end{aligned}$$

and this proves relation (4.10).

Let us estimate the modulus of continuity $\omega_{p,\alpha,\psi}(f,t)$, $0<t\leqslant 1$:

$$\begin{aligned}
\omega_{p,\alpha,\psi}(f,h_n) &\leqslant \int_0^1 \left[f(x)-f\left(x+h_n x^\alpha \psi(x)\right)\right]^p\,dx\\
&=\int_0^1 \left[\int_x^1 \frac{\eta(t)}{t^{\frac{1}{p}+1}}\,dt-\int_{x+h_n x^\alpha\psi(x)}^1 \frac{\eta(t)}{t^{\frac{1}{p}+1}}\,dt\right]^p dx=\int_0^1 \left[\int_x^{x+h_n x^\alpha\psi(x)} \frac{\eta(t)}{t^{\frac{1}{p}+1}}\,dt\right]^p dx\\
&=\sum_{k=0}^{n}\int_{\frac{1}{2^{k+1}}}^{\frac{1}{2^k}} \left[\int_x^{x+h_n x^\alpha\psi(x)} \frac{\eta(t)}{t^{\frac{1}{p}+1}}\,dt\right]^p dx+\sum_{k=n+1}^{\infty}\int_{\frac{1}{2^{k+1}}}^{\frac{1}{2^k}} \left[\int_x^{x+h_n x^\alpha\psi(x)} \frac{\eta(t)}{t^{\frac{1}{p}+1}}\,dt\right]^p dx=I_1+I_2.
\end{aligned}$$

We first estimate I_1. As

$$0\leqslant k\leqslant n \quad\text{and}\quad \frac{1}{2^{k+1}}<x\leqslant\frac{1}{2^k}$$

we have, see (3.10):

$$[x,x+h_n x^\alpha\psi(x)]\subset\left(\frac{1}{2^{k+1}},\frac{1}{2^{k-1}}\right]$$

and this is why

$$\eta(x)=\eta_k+\eta_{k+1}.$$

Hence,

$$\begin{aligned}
I_1&=\sum_{k=0}^{n}\int_{\frac{1}{2^{k+1}}}^{\frac{1}{2^k}} \left[\int_x^{x+h_n x^\alpha\psi(x)} \frac{\eta(t)}{t^{\frac{1}{p}+1}}\,dt\right]^p dx\\
&\leqslant\sum_{k=0}^{n}\int_{\frac{1}{2^{k+1}}}^{\frac{1}{2^k}} \left[(\eta_k+\eta_{k+1})\int_x^{x+h_n x^\alpha\psi(x)} \frac{1}{t^{\frac{1}{p}+1}}\,dt\right]^p dx\\
&\ll\sum_{k=0}^{n}\int_{\frac{1}{2^{k+1}}}^{\frac{1}{2^k}} \left[(\eta_k+\eta_{k+1})\left(h_n x^\alpha\psi(x)\right)2^{\frac{k}{p}}2^k\right]^p dx\\
&\ll\sum_{k=0}^{n}2^{-k}\left[(\eta_k+\eta_{k+1})\left(h_n\frac{1}{2^{k\alpha}}\psi\left(\frac{1}{2^k}\right)\right)2^{\frac{k}{p}}2^k\right]^p dx\\
&\ll\sum_{k=0}^{n}\left[\eta_k\left(\frac{1}{2^{n(1-\alpha)}\psi\left(\frac{1}{2^n}\right)}\frac{1}{2^{k\alpha}}\psi\left(\frac{1}{2^k}\right)2^k\right)\right]^p dx\\
&\ll\frac{1}{2^{n(1-\alpha)}\psi^p\left(\frac{1}{2^n}\right)}\sum_{k=0}^{n}\left[\eta_k\left(2^{k(1-\alpha)}\psi\left(\frac{1}{2^k}\right)\right)\right]^p\\
&=\frac{1}{2^{n(1-\alpha)}\psi^p\left(\frac{1}{2^n}\right)}\sum_{k=0}^{n}2^{kp(1-\alpha)}\psi^p\left(\frac{1}{2^k}\right)\eta_k^p.
\end{aligned}$$

Applying (4.6), the definition of the sequence η_k and Lemma 2.4 for

$$\tau_{2^k} = 2^{k(1-\alpha)}\psi\left(\frac{1}{2^k}\right),$$

we find:

$$\begin{aligned}
I_1 &\ll \frac{1}{2^{n(1-\alpha)}\psi^p\left(\frac{1}{2^n}\right)}\sum_{k=0}^{n} 2^{kp(1-\alpha)}\psi^p\left(\frac{1}{2^k}\right)\eta_k^p \\
&\ll \frac{1}{2^{n(1-\alpha)}\psi^p\left(\frac{1}{2^n}\right)}\sum_{k=0}^{n} 2^{kp(1-\alpha)}\psi^p\left(\frac{1}{2^k}\right)[B_{2^k}-B_{2^{k+1}}]^p \\
&\ll \frac{1}{2^{n(1-\alpha)}\psi^p\left(\frac{1}{2^n}\right)}\sum_{k=0}^{n}\left[2^{k(1-\alpha)}\psi^p\left(\frac{1}{2^k}\right)B_{2^k}\right]^p \\
&\ll \frac{1}{2^{n(1-\alpha)}\psi^p\left(\frac{1}{2^n}\right)}\sum_{k=0}^{n}\left[2^{k(1-\alpha)}\psi^p\left(\frac{1}{2^k}\right)B_{2^k}\right]^{p-1}\left[2^{k(1-\alpha)}\psi\left(\frac{1}{2^k}\right)B_{2^k}\right] \\
&\ll \frac{\left[2^{n(1-\alpha)}\psi^p\left(\frac{1}{2^n}\right)B_{2^n}\right]^{p-1}}{2^{n(1-\alpha)}\psi^p\left(\frac{1}{2^n}\right)}\left(2^{n(1-\alpha)}\psi\left(\frac{1}{2^n}\right)\right)\sum_{k=0}^{n}[B_{2^k}] \\
&\ll [B_{2^n}]^{p-1}\sum_{k=0}^{n}\sum_{s=2^{k-1}+1}^{2^k} B_s \ll [B_{2^n}]^p \ll \omega\left(\frac{1}{2^{n(1-\alpha)}\psi^p\left(\frac{1}{2^n}\right)}\right).
\end{aligned}$$

In view of the identity

$$h_n = \frac{1}{2^{n(1-\alpha)}\psi^p\left(\frac{1}{2^n}\right)},$$

we then obtain

$$I_1 \ll \omega(h).$$

We proceed to estimating I_2. For

$$k \geqslant n \quad \text{and} \quad \frac{1}{2^{k+1}} < x \leqslant \frac{1}{2^k},$$

in view of relation (4.9) and Lemma 2.1 we have

$$\begin{aligned}
I_2 &= \sum_{k=n+1}^{\infty}\int_{\frac{1}{2^{k+1}}}^{\frac{1}{2^k}}\left[\int_x^{x+h_n x^\alpha\psi(x)}\frac{\eta(t)}{t^{\frac{1}{p}+1}}\,dt\right]^p dx \leqslant \sum_{k=n+1}^{\infty}\int_{\frac{1}{2^{k+1}}}^{\frac{1}{2^k}}\left[\int_{\frac{1}{2^{k+1}}}^{\frac{1}{2^{n-1}}}\frac{\eta(t)}{t^{\frac{1}{p}+1}}\,dt\right]^p dx \\
&\ll \sum_{k=n+1}^{\infty}\int_{\frac{1}{2^{k+1}}}^{\frac{1}{2^k}}\left[\sum_{m=n-1}^{k}\int_{\frac{1}{2^{m+1}}}^{\frac{1}{2^m}}\frac{\eta(t)}{t^{\frac{1}{p}+1}}\,dt\right]^p dx \ll \sum_{k=n+1}^{\infty}\frac{1}{2^k}\left[\sum_{m=n-1}^{k}\int_{\frac{1}{2^{m+1}}}^{\frac{1}{2^m}}\frac{\eta(t)}{t^{\frac{1}{p}+1}}\,dt\right]^p dx \\
&\leqslant \sum_{k=n+1}^{\infty}\frac{1}{2^k}\left[\sum_{m=n-1}^{k}\eta_m 2^{m\left(\frac{1}{p}+1\right)}2^{-m}\right]^p \ll \sum_{k=n+1}^{\infty}\frac{1}{2^k}2^k\eta_k^p = \sum_{k=n+1}^{\infty}\eta_k^p.
\end{aligned}$$

Applying definition (4.6) of sequence η_k, we get

$$I_2 = \sum_{k=n+1}^{\infty}\eta_k^p = \sum_{k=n+1}^{\infty}[B_{2^k}-B_{2^{k+1}}]^p$$

$$
\begin{aligned}
&\ll \sum_{k=n+1}^{\infty} [B_{2^k}]^{p-1} [B_{2^k} - B_{2^{k+1}}]^p \\
&\ll \left[\omega\left(\frac{1}{2^{n(1-\alpha)}\psi^p\left(\frac{1}{2^n}\right)}\right)\right]^{p-1} \sum_{k=n+1}^{\infty} [B_{2^k} - B_{2^{k+1}}] \\
&\ll \left[\omega\left(\frac{1}{2^{n(1-\alpha)}\psi^p\left(\frac{1}{2^n}\right)}\right)\right]^{p-1} B_{2^{n+1}} \\
&\ll \left[\omega\left(\frac{1}{2^{n(1-\alpha)}\psi^p\left(\frac{1}{2^n}\right)}\right)\right]^{p-1} \omega\left(\frac{1}{2^{n(1-\alpha)}\psi^p\left(\frac{1}{2^n}\right)}\right) = \omega^p(h),
\end{aligned}
$$

and this proves the relation $I_2 \ll \omega^p(h)$. By (4.8) we then get

$$I(h) = O\{\omega^p(h)\}, \quad h \to 0,$$

that is, $f \in \tilde{H}^{\omega}_{p,\alpha,\psi}$.

It remains to show that $f \notin L(\mu, \nu)$. By Lemma 2.4 we get

$$\sum_{n=1}^{\infty} 2^{n(1-r)} [B_{2^n} - B_{2^{n+1}}]^{\nu} = +\infty.$$

As $x = \frac{1}{2^k}$ we have

$$f\left(\frac{1}{2^k}\right) = \int_{\frac{1}{2^k}}^{1} \eta(t) t^{-\left(\frac{1}{p}+1\right)}\, dt \gg \int_{\frac{1}{2^k}}^{\frac{1}{2^{k-1}}} \eta(t) t^{-\left(\frac{1}{p}+1\right)}\, dt \gg \eta_k 2^{k\left(\frac{1}{p}+1\right)} 2^{-k} = \eta_k 2^{\frac{k}{p}}.$$

This and the definition of the Lorentz space, the definition of the sequence η_k and Lemma 2.4 imply that

$$
\begin{aligned}
&r = 1 - \nu\left(\frac{1}{p} - \frac{1}{\mu}\right), \\
&\| f \|^{\nu}_{(\mu,\nu)} = \int_0^1 x^{\frac{\nu}{\mu}-1} [f(x)]^{\nu}\, dx \gg \sum_{k=0}^{\infty} \int_{\frac{1}{2^k}}^{\frac{1}{2^{k-1}}} x^{\frac{\nu}{\mu}-1} [f(x)]^{\nu}\, dx \\
&\sum_{k=0}^{\infty} 2^{k\nu\left(\frac{1}{p}-\frac{1}{\mu}\right)} [B_{2^n} - B_{2^{n+1}}]^{\nu} = \sum_{k=0}^{\infty} 2^{n(1-r)} [B_{2^n} - B_{2^{n+1}}]^{\nu} = +\infty.
\end{aligned}
$$

This proves $f \notin L(\mu, \nu)$ and the completes the proof of the theorem. □

Corollary 4.1. *Let $\omega(\delta) = \delta^{\tau}$, $0 < \tau < 1$. As*

$$0 < \alpha < 1 - \frac{1}{\tau}\left(\frac{1}{p} - \frac{1}{\mu}\right)$$

for each function $\psi(x)$ we have

$$\tilde{H}^{\delta^{\tau}}_{p,\alpha,\psi} \subset L(\mu, \nu).$$

Corollary 4.2. *Let $\omega(\delta) = \delta^{\tau}$, $0 < \tau < 1$, $\psi(x) = \ln^{\beta} \frac{1}{x}$ and*

$$\alpha = 1 - \frac{1}{\tau}\left(\frac{1}{p} - \frac{1}{\mu}\right).$$

Then

$$\tilde{H}^{\delta^{\tau}}_{p,\alpha,\psi} \subset L(\mu,\nu) \Leftrightarrow \beta > \frac{1-\alpha}{\nu\left(\frac{1}{p}-\frac{1}{\mu}\right)}.$$

Corollary 4.3. *Let $\mu=\nu=q$ and $1 \leqslant p < q < \infty$, $0 \leqslant \alpha \leqslant 1$ be some number, while $\omega(\delta)$ be some modulus of continuity. Then*

$$\tilde{H}^{\omega}_{p,\alpha,\psi} \subset L^q(0,1) \Leftrightarrow \sum_{k=0}^{\infty} 2^{kq\left(\frac{1}{p}-\frac{1}{q}\right)} \omega^q \left(\frac{1}{2^{k(1-\alpha)}\psi^p\left(\frac{1}{2^k}\right)}\right) < +\infty.$$

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