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SOLVABILITY OF NONLINEAR BOUNDARY VALUE PROBLEMS FOR NON-SLOPING TIMOSHENKO-TYPE ISOTROPIC SHELLS OF ZERO PRINCIPAL CURVATURE

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Abstract. We study the solvability of boundary value problem for a system of second order partial differential equations under boundary given conditions describing the equilibrium of elastic non-sloping isotropic inhomogeneous shells with free boundary in the framework of the Timoshenko shear model. The base of the study method are the integral representations for generalized motions involving arbitrary functions, which also involve arbitrary holomorphic functions. The arbitrary functions ate determined so that the generalized motions satisfy a linear system of equations and linear boundary conditions extracted from the original boundary value problem. The holomorphic functions are sought as Cauchy type integrals with real densities. The integral representations allow us to reduce the initial boundary value problem to a nonlinear operator equations for generalized motions in the Sobolev space. While studying the solvability of this operator equation, the most essential point is to invert it with respect to the linear part. As a result, the work is reduced to an equation, the solvability of which is established on the base of the contracting mapping principle.

Keywords: non-sloping isotropic inhomogeneous Timoshenko-type shell of zero principal curvature, nonlinear boundary value problem, partial differential equations, generalized solution, holomorphic function, operator equation, existence theorem.

Mathematics Subject Classification: 35G30, 74G25

1. Introduction

At present the solvability of nonlinear boundary value problems of equilibria of elastic non-sloping shells is studied rather completely in the framework of the simplest Kirchoff–Love, see [1]–[5] and the references therein. At the same time a topical problem is to study similar boundary value problems in the framework of more complicated models in the shell theory not relying on the Kirchoff–Love assumptions [1]. Nowadays there is a series of works [6]–[12], in which in the framework of the Timoshenko shear model the solvability of nonlinear boundary value problems for sloping shells was studied. The base of the studies in [6]–[12] were integral representations for generalized displacements involving arbitrary holomorphic functions. The latter are defined so that the generalized displacement satisfy given boundary conditions. In the present work the method of works [6]–[12] is developed for the case of non-sloping inhomogeneous isotropic Timoshenko type shells of zero principal curvature referred to the Euclidean coordinate systems. The passage to non-sloping shells complicates essentially the study of the boundary value problem.

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2. Formulation of problem

In a planar simply connected bounded domain Ω we consider a system of nonlinear differential equations of form

$$\begin{aligned} &T^{j\lambda}_{\alpha^\lambda} - B_{j\lambda}T^{\lambda\mu}\omega_\mu - B_{j\lambda}T^{\lambda 3} + R^j = 0, \qquad j = 1,2,\\ &(T^{\lambda\mu}\omega_\mu)_{\alpha^\lambda} + T^{\lambda 3}_{\alpha^\lambda} + B_{\lambda\mu}T^{\lambda\mu} + R^3 = 0,\\ &M^{j\lambda}_{\alpha^\lambda} - T^{j3} + L^j = 0, \qquad j = 1,2, \end{aligned} \tag{2.1}$$

with the following conditions on the boundary Γ of the domain Ω

$$\begin{aligned} &T^{j1}d\alpha^2/ds - T^{j2}d\alpha^1/ds = P^j(s), \qquad j = 1,2,\\ &T^{13}d\alpha^2/ds - T^{23}d\alpha^1/ds + (T^{1\lambda}d\alpha^2/ds - T^{2\lambda}d\alpha^1/ds)\omega_\lambda = P^3(s),\\ &M^{j1}d\alpha^2/ds - M^{j2}d\alpha^1/ds = N^j(s), \qquad j = 1,2. \end{aligned} \tag{2.2}$$

In (2.1), (2.2) and below we use the following notations:

$$\begin{aligned} &T^{ij} \equiv T^{ij}(\gamma) = D^{ijkn}_{\lambda-1}\gamma^{\lambda-1}_{kn},\\ &M^{ij} \equiv M^{ij}(\gamma) = D^{ijkn}_{\lambda}\gamma^{\lambda-1}_{kn},\\ &\gamma = (\gamma^0,\gamma^1), \qquad \gamma^k = (\gamma^k_{11},\gamma^k_{12},\gamma^k_{13},\gamma^k_{22},\gamma^k_{23},\gamma^k_{33}), \quad k = 0,1;\\ &D^{ijkn}_m = D^{ijkn}_m(\alpha^1,\alpha^2) = \int_{-h_0/2}^{h_0/2} B^{ijkn}(\alpha^1,\alpha^2,\alpha^3)(\alpha^3)^m d\alpha^3, \qquad m = \overline{0,2}, \quad i,j,k,n = \overline{1,3};\\ &B^{1111} = B^{2222} = E/(1-\nu^2),\\ &B^{1212} = E/(2(1+\nu)), \qquad B^{1122} = \nu E/(1-\nu^2),\\ &B^{1313} = B^{2323} = E\kappa^2/(2(1+\nu));\\ &\omega_j = w_{3\alpha^j} + B_{j1}w_1 + B_{j2}w_2, \quad j = 1,2;\\ &\gamma^0_{jj} = w_{j\alpha^j} - B_{jj}w_3 + \omega_j^2/2, \quad j = 1,2,\\ &\gamma^0_{12} = w_{1\alpha^2} + w_{2\alpha^1} - 2B_{12}w_3 + \omega_1\omega_2,\\ &\gamma^1_{jj} = \psi_{j\alpha^j}, \quad j = 1,2,\\ &\gamma^1_{12} = \psi_{1\alpha^2} + \psi_{2\alpha^1},\\ &\gamma^0_{j3} = \omega_j + \psi_j, \quad j = 1,2,\\ &\gamma^0_{33} = \gamma^1_{k3} \equiv 0, \quad k = \overline{1,3}; \end{aligned} \tag{2.3}$$

other B^{ijkn} are zero, $\alpha^j = \alpha^j(s)$, $j = 1,2$, is the equation of the curve Γ and s is the arc length on the curve Γ. The subscript α^λ in (2.1)–(2.3) and later denotes the differentiation in α^λ, $\lambda = 1,2$.

System (2.1) with boundary conditions (2.2) describe the equilibrium of an elastic non-sloping isotropic inhomogeneous shell with free boundary in the framework of Timoshenko shear model [13] referred to the Euclidean coordinate system. Here T^{ij} are forces, M^{ij} are momenta γ^k_{ij}, $i,j = \overline{1,3}$, $k = 0,1$, are the components of the deformation of thee middle surface S_0 of the shell identified with the domain Ω; w_j, $j = 1,2$, and w_3 are respectively tangential and normal displacement of the points S_0; ψ_i, $i = 1,2$, are the rotation angles of normal sections S_0; B_{ij}, $i,j = 1,2$, are the entries of the curvature tensor for the surface S_0; R^j, P^j, $j = \overline{1,3}$, L^k, N^k, $k = 1,2$, are the components of external forces acting on the shell; ν is the Poisson coefficient, E is the Young's module, κ^2 is the translation coefficient, $h_0 = const$ is the width of the shell; α^1, α^2 are Cartesian coordinates of the points in the domain Ω.

In (2.1)–(2.3) and later we suppose the summation over the repeating Latin indices from 1 to 3, while the summation over Greek indices is from 1 to 2.

Problem (2.1)**,** (2.2)**.** We need to find a solution to problem (2.1) satisfying boundary conditions (2.2).

We study boundary value problem (2.1), (2.2) in the generalized setting. We suppose the following conditions:

(a) $B^{ijkn}(\alpha^1,\alpha^2,\alpha^3) \in (W_p^{(1)}(\Omega) \cap C_\beta(\overline{\Omega})) \times L_1[-h_0/2, h_0/2]$, $i,j,k,n=\overline{1,3}$;

(b) $B_{\lambda\mu}(\alpha^1,\alpha^2) \in C^1(\overline{\Omega})$, $\lambda,\mu = 1,2$, and at the same time $B_{11}B_{22} - B_{12}^2 = 0$, $B_{12} \neq 0$ in $\overline{\Omega}$;

(c) The components of external forces R^j, $j=\overline{1,3}$, and L^k, $k=1,2$, belong to the space $L_p(\Omega)$, the components P^j, $j=\overline{1,3}$, N^k, $k=1,2$, belong to the space $C_\beta(\Gamma)$, and the external forces are self-balanced;

(d) Ω is an arbitrary simply connected domain with the boundary $\Gamma \in C^1_\beta$.

Hereinafter

$$2 < p < 4/(2-\beta), \qquad 0 < \beta < 1.$$

Definition 2.1. *A vector of generalized displacements $a = (w_1, w_2, w_3, \psi_1, \psi_2)$ is called a generalized solution of problem (2.1), (2.2) if a belongs to the space $W_p^{(2)}(\Omega)$, satisfies system* (2.1) *almost everywhere and boundary conditions* (2.2) *pointwise.*

Here $W_p^{(j)}(\Omega)$, $j=1,2$, are the Sobolev spaces. By the embedding theorems for the Sobolev spaces $W_p^{(2)}(\Omega)$ with $p>2$, the generalized solution a belongs to the space $C^1_\alpha(\overline{\Omega})$. Hereinafter

$$\alpha = (p-2)/p.$$

We note that as $2 < p < 4/(2-\beta)$, the inequality $\alpha < \beta/2$ holds.

For the sake of convenience in further study, we write relations for the component of deformations in (2.3) as

$$\gamma_{ij}^k = e_{sij}^k + e_{cij}^k + \chi_{ij}^k, \quad i,j=\overline{1,3}, \quad k=0,1, \tag{2.4}$$

where we have adopted the notations

$$\begin{aligned}
&e_{sjj}^0 = w_{j\alpha^j}, \qquad e_{sj3}^0 = w_{3\alpha^j} + \psi_j, \qquad e_{sjj}^1 = \psi_{j\alpha^j}, \quad j=1,2,\\
&e_{s12}^0 = w_{1\alpha^2} + w_{2\alpha^1}, \qquad e_{s12}^1 = \psi_{1\alpha^2} + \psi_{2\alpha^1},\\
&e_{cjj}^0 = -B_{jj}w_3, \qquad e_{cij}^1 \equiv 0, \quad j=1,2,\\
&e_{c12}^0 = -2B_{12}w_3, \qquad e_{cj3}^0 = B_{j1}w_1 + B_{j2}w_2, \quad j=1,2,\\
&\chi_{jj}^0 = \omega_j^2/2, \quad i,j=1,2, \qquad \chi_{12}^0 = \omega_1\omega_2,\\
&\chi_{ij}^1 = \chi_{j3}^0 = e_{s33}^0 = e_{sj3}^1 = e_{c33}^k \equiv 0, \quad i,j=\overline{1,3}, \quad k=0,1.
\end{aligned} \tag{2.5}$$

3. Construction of integral representations for generalized displacements

We introduce two complex functions

$$\begin{aligned}
v_j = v_j(z) &= D_{j-1}^{1111}(w_{1\alpha^1} + w_{2\alpha^2}) + D_j^{1111}(\psi_{1\alpha^1} + \psi_{2\alpha^2})\\
&+ i[D_{j-1}^{1212}(w_{2\alpha^1} - w_{1\alpha^2}) + D_j^{1212}(\psi_{2\alpha^1} - \psi_{1\alpha^2})], \quad j=1,2, \quad z=\alpha^1 + i\alpha^2.
\end{aligned} \tag{3.1}$$

In system (2.1), we replace the forces T^{jk}, momenta M^{jk} and the components of the deformations γ_{jk}^n by their expressions from (2.3), (2.4). After that we add the second equation in (2.1) multiplied by the imaginary unit i to the first equation and the fifth equation multiplied by i to the forth equation. In this way, by means of the functions $v_j(z)$ from (3.1), we represent system in a form convenient for further study:

$$\begin{aligned}
&v_{j\bar z} + h^j(a) = f_c^j(a) + f_\chi^j(a) - F^j(z), \quad j=1,2,\\
&D_0^{1313}(w_{3\alpha^1\alpha^1} + w_{3\alpha^2\alpha^2}) + h^3(a) = f_c^3(a) + f_\chi^3(a) - F^3(z), \quad z\in\Omega,
\end{aligned} \tag{3.2}$$

where we have adopted the following notations:

$$
\begin{aligned}
&\upsilon_{j\bar z}=(\upsilon_{j\alpha^1}+i\upsilon_{j\alpha^2})/2,\quad j=1,2,\\
&h^j(a)=(-1)^{\mu-1}(D^{1212}_{j+\lambda-2\alpha^{3-\mu}}\nu_{\lambda 2\alpha^\mu}+iD^{1212}_{j+\lambda-2\alpha^\mu}\nu_{\lambda 1\alpha^{3-\mu}})-(j-1)D_0^{1313}(e^0_{s13}+ie^0_{s23})/2,\\
&\nu_{1j}=w_j,\qquad \nu_{2j}=\psi_j,\quad j=1,2;\\
&h^3(a)=D^{1313}_{0\alpha^\lambda}w_{3\alpha^\lambda}+(D_0^{1313}\psi_\lambda)_{\alpha^\lambda};\\
&f_c^j(a)=(f_{c3j-2}+if_{c3j-1})/2,\quad j=1,2,\\
&f_\chi^j(a)=(f_{\chi 3j-2}+if_{\chi 3j-1})/2,\quad j=1,2,\\
&f_c^3(a)=f_{c3}(a),\qquad f_\chi^3(a)=f_{\chi 3}(a),\\
&f_{cj}(a)=-T^{j\lambda}_{\alpha^\lambda}(e_c)+B_{j\lambda}T^{\lambda 3}(\gamma),\quad j=1,2,\\
&f_{c3+j}(a)=-M^{j\lambda}_{\alpha^\lambda}(e_c)+T^{j3}(e_c),\quad j=1,2,\\
&f_{c3}(a)=-T^{\lambda 3}_{\alpha^\lambda}(e_c)-B_{\lambda\mu}T^{\lambda\mu}(e),\\
&f_{\chi j}(a)=-T^{j\lambda}_{\alpha^\lambda}(\chi)+B_{j\lambda}T^{\lambda\mu}(\gamma)\omega_\mu,\\
&f_{\chi 3+j}(a)=-M^{j\lambda}_{\alpha^\lambda}(\chi),\quad j=1,2,\\
&f_{\chi 3}(a)=-(T^{\lambda\mu}\omega_\mu)_{\alpha^\lambda}-B_{\lambda\mu}T^{\lambda\mu}(\chi),\\
&F^1=(R^1+iR^2)/2,\qquad F^2=(L^1+iL^2)/2,\qquad F^3=R^3;\\
&e=e_s+e_c,\qquad e_s=(e_s^0,e_s^1),\qquad e_c=(e_c^0,e_c^1),\\
&e_s^k=(e^k_{s11},e^k_{s12},e^k_{s13},e^k_{s22},e^k_{s23},e^k_{s33}),\\
&e_c^k=(e^k_{c11},e^k_{c12},e^k_{c13},e^k_{c22},e^k_{c23},e^k_{c33}),\quad k=0,1;\\
&\chi=(\chi^0_{11},\chi^0_{12},\chi^0_{22});
\end{aligned}
\tag{3.3}
$$

e^k_{sij}, e^k_{cij}, χ^k_{ij} were defined in (2.5). We note that by e and χ we define respectively linear and nonlinear parts of the components of deformations γ and this ensures the representation $\gamma=e+\chi$.

Similarly, we write boundary conditions (2.2) in the form

$$
\begin{aligned}
&\mathrm{Re}[(-i)^j t'\upsilon_k(t)]+2(-1)^jD^{1212}_{k+\delta-2}\nu_{\delta 3-j\alpha^\lambda}d\alpha^\lambda/ds\\
&\qquad=\varphi_{c3(k-1)+j}(a)(t)+\varphi_{\chi 3(k-1)+j}(a)(t)-F^{3k+j}(s),\quad k,j=1,2,\\
&D_0^{1313}[(w_{3\alpha^2}+\psi_2)d\alpha^1/ds-(w_{3\alpha^1}+\psi_1)d\alpha^2/ds]=\varphi_{c3}(a)(t)+\varphi_{\chi 3}(a)(t)-F^6(s),
\end{aligned}
\tag{3.4}
$$

where

$$
\begin{aligned}
&\varphi_{cj}(a)(t)=T^{j2}(e_c)d\alpha^1/ds-T^{j1}(e_c)d\alpha^2/ds,\\
&\varphi_{c3+j}(a)(t)=M^{j2}(e_c)d\alpha^1/ds-M^{j1}(e_c)d\alpha^2/ds,\\
&\varphi_{c3}(a)(t)=T^{13}(e_c)d\alpha^2/ds-T^{23}(e_c)d\alpha^1/ds;\\
&\varphi_{\chi j}(a)(t)=T^{j2}(\chi)d\alpha^1/ds-T^{j1}(\chi)d\alpha^2/ds,\\
&\varphi_{\chi 3+j}(a)(t)=M^{j2}(\chi)d\alpha^1/ds-M^{j1}(\chi)d\alpha^2/ds,\quad j=1,2,\\
&\varphi_{\chi 3}(a)(t)=[(T^{11}(\gamma)\omega_1+T^{12}(\gamma)\omega_2]d\alpha^2/ds-[T^{22}(\gamma)\omega_2+T^{12}(\gamma)\omega_1]d\alpha^1/ds;\\
&F^{3+j}=-P^j,\quad j=1,2,\\
&F^6(s)=P^3(s),\qquad F^{6+k}=-N^k,\quad k=1,2;
\end{aligned}
\tag{3.5}
$$

the forces T^{jk} and the moments M^{jk} wrre defined in (2.3).

The base for studying system of equations (3.2) with boundary conditions (3.4) is integral representations for the generalized displacements w_j, $j=\overline{1,3}$, ψ_k, $k=1,2$. In order to derive them, we introduce the equations

$$\upsilon_{j\bar z} = \rho^j \quad (j=1,2), \qquad D_0^{1313}(w_{3\alpha^1\alpha^1} + w_{3\alpha^2\alpha^2}) = \rho^3, \tag{3.6}$$

where

$$\rho^1 = \rho_1 + i\rho_2, \qquad \rho^2 = \rho_4 + i\rho_5, \qquad \rho^3 = \rho_3$$

are arbitrary fixed functions belongs to the space $L_p(\Omega)$.

The first two equations in (3.6) are inhomogeneous Cauchy–Riemann equations. Their general solutions are given by the formulas [14]

$$\begin{aligned} &\upsilon_j(z) = \Phi_j(z) + T\rho^j(z) \equiv \upsilon_j(\Phi_j;\rho^j)(z), \\ &T\rho^j(z) = -\frac{1}{\pi}\iint\limits_{\Omega} \frac{\rho^j(\zeta)}{\zeta - z} d\xi d\eta, \quad j=1,2, \quad \zeta = \xi + i\eta, \end{aligned} \tag{3.7}$$

where $\Phi_j(z)$ are arbitrary holomorphic functions belonging to the space $C_\alpha(\overline{\Omega})$.

It is known [14] that T is a completely continuous operator in the spaces $L_p(\Omega)$ and $C_\alpha^k(\overline{\Omega})$, which maps these spaces into the spaces $C_\alpha(\overline{\Omega})$ and $C_\alpha^{k+1}(\overline{\Omega})$, respectively. Moreover, there exist generalized derivatives

$$\begin{aligned} &\frac{\partial Tf}{\partial \bar z} = f, \\ &\frac{\partial Tf}{\partial z} \equiv Sf = -\frac{1}{\pi}\iint\limits_{\Omega} \frac{f(\zeta)}{(\zeta - z)^2}\, d\xi\, d\eta, \end{aligned} \tag{3.8}$$

where S is a bounded linear operator in $L_p(\Omega)$, $p>1$, and $C_\alpha^k(\overline{\Omega})$.

By means of the functions $\upsilon_1^0 = w_2 + iw_1$, $\upsilon_2^0 = \psi_2 + i\psi_1$ we rewrite representations (3.7) as inhomogeneous Cauchy–Riemann equations

$$\upsilon_{j\bar z}^0 = i(d_{2j-1}[\upsilon_1] + d_{2j}[\upsilon_2]) \equiv iT_j\upsilon, \quad j=1,2, \quad \upsilon = (\upsilon_1, \upsilon_2), \tag{3.9}$$

the general solutions of which read as

$$\upsilon_j^0(z) = \Psi_j(z) + iTT_j\upsilon(z) \equiv \upsilon_j^0(\Psi_j;\upsilon)(z), \quad j=1,2. \tag{3.10}$$

In (3.9), (3.10) we have adopted the notations

$$\begin{aligned} &d_{2j+\lambda-2}[\upsilon_\lambda] = d^1_{2j+\lambda-2}\upsilon_\lambda + (-1)^{j+\lambda} d^2_{2j+\lambda-2}\overline{\upsilon_\lambda}, \quad j,\lambda = 1,2, \\ &d^j_{3k-2} = \frac{1}{4}\left(\frac{D^{1111}_{4-2k}}{\delta_0} + (-1)^j \frac{D^{1212}_{4-2k}}{\delta_1}\right), \\ &d^j_2 = d^j_3 = \frac{1}{4}\left(\frac{D^{1212}_1}{\delta_1} + (-1)^j\frac{D^{1111}_1}{\delta_0}\right), \quad k,j=1,2, \\ &\delta_0 = D_0^{1111}D_2^{1111} - \left(D_1^{1111}\right)^2, \\ &\delta_1 = D_0^{1212}D_2^{1212} - \left(D_1^{1212}\right)^2; \end{aligned} \tag{3.11}$$

$\Psi_j(z) \in C^1_\alpha(\overline{\Omega})$ are arbitrary holomorphic functions.

We represent the third equation in (3.6) as

$$w_{3z\bar z} = \widetilde{\rho}_3/4, \quad \widetilde{\rho}_3 = \rho_3/D_0^{1313}, \quad w_{3z} = (w_{3\alpha^1} - iw_{3\alpha^2})/2,$$

which yields

$$\begin{aligned} &w_3(z) = \operatorname{Re}\Psi_3(z) - \widetilde{T}\widetilde{\rho}_3 \equiv w_3(\Psi_3;\rho_3)(z), \\ &\widetilde{T}\widetilde{\rho}_3 = -\frac{1}{2\pi}\iint\limits_{\Omega} \widetilde{\rho}_3(\zeta)\ln\left|1 - \frac{z}{\zeta}\right| d\xi\, d\eta, \end{aligned} \tag{3.12}$$

where $\Psi_3(z) \in C^1_\alpha(\overline{\Omega})$ is an arbitrary holomorphic function.

Relations (3.10), (3.12) are sought integral representations for the generalized displacements. By means of formulas (3.7)–(3.12) and formula (8.20) in [14] for their partial derivatives of first and second orders we obtain the representations

$$
\begin{aligned}
&\nu_{jk\alpha^k} = \mathrm{Im}[\upsilon^0_{j\bar z} - (-1)^k \upsilon^0_{jz}],\\
&\nu_{jk\alpha^n} = \mathrm{Re}[\upsilon^0_{jz} + (-1)^k \upsilon^0_{j\bar z}], \quad k \neq n, \quad j,k,n = 1,2;\\
&\upsilon^0_{jz} = \Psi'_j(z) + iST_j\upsilon(z),\\
&\upsilon^0_{j\bar z} = iT_j\upsilon,\\
&w_{3\alpha^j} = 2\,\mathrm{Re}\left(i^{j-1}w_{3z}\right), \quad j = 1,2,\\
&w_{3z} = \Psi'_3(z)/2 + T\widetilde{\rho}_3(z)/4;\\
&\nu_{kn\alpha^j\alpha^j} = -\,\mathrm{Re}\{i^n[\upsilon^0_{kz\bar z} + (-1)^j(\upsilon^0_{kzz} + \upsilon^0_{k\bar z\bar z})]\},\\
&\nu_{kn\alpha^1\alpha^2} = \mathrm{Re}\{i^{n-1}(\upsilon^0_{kzz} - \upsilon^0_{k\bar z\bar z})\},\\
&w_{3\alpha^j\alpha^j} = 2[w_{3z\bar z} + (-1)^{j-1}\,\mathrm{Re}\,w_{3zz}], \quad k,n,j = 1,2,\\
&w_{3\alpha^1\alpha^2} = -2\,\mathrm{Im}\,w_{3zz};\\
&\upsilon^0_{kz\bar z} = T_{k1}\upsilon + S_{k1}(\Phi'_0;\rho_0),\\
&\upsilon^0_{k\bar z\bar z} = T_{k2}\upsilon + S_{k2}(\Phi'_0;\rho_0),\\
&\upsilon^0_{kzz} = \Psi''_k(z) + S\upsilon^0_{k\zeta\bar\zeta}(z) - \frac{1}{2\pi}\int\limits_\Gamma \frac{T_k\upsilon(\tau)}{(\tau - z)^2}d\overline{\tau}, \quad k = 1,2,\\
&\Phi'_0 = (\Phi'_1, \Phi'_2), \quad \rho_0 = (\rho^1, \rho^2),\\
&w_{3zz} = \Psi''_3(z)/2 + S\widetilde{\rho}_3/4,\\
&w_{3z\bar z} = \widetilde{\rho}_3/4,\\
&T_{jk}\upsilon = i[d^1_{2j+\mu-2,k}\upsilon_\mu + (-1)^{j+\mu}d^2_{2j+\mu-2,k}\overline{\upsilon_\mu}], \quad j,k = 1,2,\\
&S_{jk}(\Phi'_0;\rho_0) = i[d^1_{2j+\mu-2}\upsilon_{\mu,k} + (-1)^{j+\mu}d^2_{2j+\mu-2}\overline{\upsilon_{\mu,3-k}}], \quad j,k = 1,2,\\
&\upsilon_{j,1} \equiv \upsilon_{jz} = \Phi'_j(z) + S\rho^j(z), \quad j = 1,2,\\
&\upsilon_{j,2} \equiv \upsilon_{j\bar z} = \rho^j, \quad j = 1,2,\\
&d^j_{m,1} \equiv d^j_{mz}, \qquad d^j_{m,2} \equiv d^j_{m\bar z}, \quad j,k = 1,2, \quad m = \overline{1,4}.
\end{aligned}
\tag{3.13}
$$

4. Solution of problem (2.1), (2.2)

Integral representations (3.10), (3.12) for generalized displacements $a = (w_1, w_2, w_3, \psi_1, \psi_2)$ involve arbitrary holomorphic functions $\Phi_j(z)$, $j = 1,2$, $\Psi_k(z)$, $k = \overline{1,3}$ and arbitrary functions $\rho^j(z)$, $j = \overline{1,3}$. We find them so that the generalized displacements satisfy system (3.2) and boundary conditions (3.4) assuming temporarily that the right hand sides of equations (3.2) and boundary conditions (3.4) are known. In order to do this, we substitute relations (3.10), (3.12), (3.13) into the left hand sides of system (3.2) and boundary conditions (3.4). Then system of equations (3.2) is written as

$$\rho^j(z) + h^j_1(\rho)(z) + h^j_2(\Phi)(z) = f^j_c(a)(z) + f^j_\chi(a)(z) - F^j(z), \quad j = \overline{1,3}, \quad z \in \Omega, \tag{4.1}$$

where $h^j_1(\rho)(z)$ and $h^j_2(\Phi)(z)$ we denote the parts of the expression for the operator $h^j(a)$ in (3.3), which involve the functions $\rho = (\rho^1, \rho^2, \rho^3)$ and $\Phi = (\Phi_1, \Phi_2, \Psi_1, \Psi_2, \Psi_3)$, respectively.

The representations

$$
\begin{aligned}
&S(T_j\Phi_0)^+(t) = -(\overline{t}')^2[d^1_{2j-1}(t)\Phi_1(t) + d^1_{2j}(t)\Phi_2(t)] + K_{0j}(\Phi_0)(t), \qquad \Phi_0 = (\Phi_1, \Phi_2),\\
&K_{0j}(\Phi_0)(t) = -\frac{d^1_{2j+\mu-2}(t)}{2\pi i}\int\limits_\Gamma \frac{\psi(\tau,t)-\psi(t,t)}{\tau - t}\Phi_\mu(\tau)d\tau\\
&\qquad - (-1)^{j+\mu}\frac{d^2_{2j+\mu-2}(t)}{2\pi i}\int\limits_\Gamma \frac{\psi(\tau,t)-\psi(t,t)}{\overline{\tau}-\overline{t}}\overline{\Phi_\mu(\tau)}d\overline{\tau}\\
&\qquad - \frac{1}{\pi}\iint\limits_\Omega \frac{d^1_{2j+\mu-2}(\zeta) - d^1_{2j+\mu-2}(t)}{(\zeta - t)^2}\Phi_\mu(\zeta)d\xi d\eta\\
&\qquad - \frac{(-1)^{j+\mu}}{\pi}\iint\limits_\Omega \frac{d^2_{2j+\mu-2}(\zeta) - d^2_{2j+\mu-2}(t)}{(\zeta - t)^2}\overline{\Phi_\mu(\zeta)}d\xi d\eta, \quad j = 1,2,\\
&\psi(\tau,t) = (\overline{\tau}-\overline{t})/(\tau - t), \quad \psi(t,t) = (\overline{t}')^2,
\end{aligned} \tag{4.2}
$$

obtained by means of relations (3.7)–(3.9), formulas (4.7), (4.9) from [14] and Sokhotskii formulas [15] allow us to rewrite boundary conditions (3.4) in the form

$$
\begin{aligned}
&(-1)^j d_{k\lambda}(t)\operatorname{Re}[i^j t'\Phi_\lambda(t)] - 2D^{1212}_{\lambda+k-2}(t)\operatorname{Re}[i^{j-1}t'\Psi'_\lambda(t)] - 2D^{1212}_{\lambda+k-2}(t)\operatorname{Re}[i^j t' K_{0\lambda}(\Phi_0)(t)]\\
&\qquad + H_{3(k-1)+j}\rho(t) = \varphi_{c3(k-1)+j}(a)(t) + \varphi_{\chi 3(k-1)+j}(a)(t) - F^{3k+j}(s), \quad k,j = 1,2, \\
&D^{1313}_0(t)\operatorname{Re}[it'\Psi'_3(t)] + K_{03}(\Phi)(t) + H_3\rho(t) = \varphi_{c3}(a)(t) + \varphi_{\chi 3}(a)(t) - F^6(s),
\end{aligned} \tag{4.3}
$$

where we have adopted the following notations

$$
\begin{aligned}
&H_{3(k-1)+j}\rho(t) = \operatorname{Re}[(-i)^j t' T\rho^k(t)]\\
&\qquad\qquad - 2D^{1212}_{k+\lambda-2}(t)\operatorname{Re}\{i^j t'(I+S)(T_\lambda T\rho_0)^+(t)\}, \quad k,j = 1,2,\\
&H_3\rho(t) = D^{1313}_0(t)\operatorname{Re}[it'(T\widetilde{\rho}_3(t)/2 + TT_2T\rho_0(t))],\\
&K_{03}(\Phi)(t) = D^{1313}_0(t)\operatorname{Re}\{t'[\Psi_2(t) + iTT_2\Phi_0(t)]\};\\
&d_{kj}(t) = (-1)^{j-1}[2(-1)^\lambda D^{1212}_{\lambda+k-2}(t)d^2_{2\lambda+j-2}(t) + 3 - k - j], \quad k,j = 1,2,
\end{aligned} \tag{4.4}
$$

I is the identity mappings, the operators T, S, T_λ and the functions $d^k_j(t)$ are defined in (3.7), (3.8), (3.9), (3.11), respectively; $\Phi_\lambda(t) \equiv \Phi^+_\lambda(t)$, $t \in \Gamma$. Hereinafter the symbol $\Phi^+_\lambda(t)$ denotes the limit of the function $\Phi_\lambda(z)$ as $z \to t \in \Gamma$ inside the domain Ω.

Thus, to determine the functions

$$
\begin{aligned}
&\rho^j \in L_p(\Omega), \qquad j = \overline{1,3}), \qquad \Phi_k(z) \in C_\alpha(\overline{\Omega}), \qquad k = 1,2,\\
&\Psi_j(z) \in C^1_\alpha(\overline{\Omega}), \quad j = \overline{1,3},
\end{aligned}
$$

we have obtained system of equations (4.1), (4.3). We seek holomorphic functions as Cauchy type integrals with real densities

$$
\begin{aligned}
&\Phi_k(z) = \Theta(\mu_{2k})(z) \equiv \Phi_k(\mu_{2k})(z), \quad k = 1,2,\\
&\Psi'_j(z) = i^{(j-1)(j-2)/2}\Theta(\mu_{2j-1})(z) \equiv \Psi'_j(\mu_{2j-1})(z), \quad j = \overline{1,3},\\
&\Theta(f)(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{f(\tau)\,d\tau}{\tau'(\tau - z)},
\end{aligned} \tag{4.5}
$$

where $\mu_j(t) \in C_\alpha(\Gamma)$, $j = \overline{1,5}$, are arbitrary real functions, $\tau' = d\tau/d\sigma$, $d\sigma$ is a differential of the arc length of the curve Γ.

For the functions $\Psi_j(z)$, $j=\overline{1,3}$, we have the representations

$$\Psi_j(z)=i^{(j-1)(j-2)/2}\Theta^0(\mu_{2j-1})(z)+c_{2j-1}+ic_{2j}\equiv\Psi_j(\mu_{2j-1})(z)+c_{2j-1}+ic_{2j},\quad j=\overline{1,3},$$
$$\Theta^0(f)(z)=-\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{f(\tau)}{\tau'}\,ln\left(1-\frac{z}{\tau}\right)d\tau,\tag{4.6}$$

where c_j, $j=\overline{1,6}$, are arbitrary real constants, while by $\ln(1-z/\tau)$ we mean a univalent branch vanishing as $z=0$.

Using the Sokhotskii formulas [15], we find $\Phi_k(t)$ $(k=1,2)$, $\Psi'_j(t)$, $j=\overline{1,3}$, $t\in\Gamma$. Substituting their expressions and representations (4.6) into system (4.1), (4.3), after simple transformations we arrive at the following system of equations for the functions $\rho\in L_p(\Omega)$ and $\mu=(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5)\in C_\alpha(\Gamma)$:

$$\rho^j(z)+h_1^j(\rho)(z)+h_2^j(\mu)(z)=f_c^j(a)(z)+f_\chi^j(a)(z)+g_c^j(z)-F^j(z),\ z\in\Omega,\quad j=\overline{1,3},$$
$$\sum_{n=1}^{5}\left[a_{jn}(t)\mu_n(t)+b_{jn}(t)\int\limits_{\Gamma}\frac{\mu_n(\tau)}{\tau-t}\,d\tau\right]+K_j\mu(t)+H_j\rho(t)\tag{4.7}$$
$$=\varphi_{cj}(a)(t)+\varphi_{\chi j}(a)(t)+g_c^{3+j}(t)-F^{3+j}(t),\quad t\in\Gamma,\quad j=\overline{1,5},$$

in which we have adopted the notations

$$\begin{aligned}K_{3(n-1)+j}\mu(t)&=(-1)^jd_{n\lambda}(t)\{\mathrm{Re}[i^jt'\Theta(\mu_{2\lambda})(t)]-i\,\mathrm{Re}(i^{j-1})\Theta(\tau'\mu_{2\lambda})(t)\}\\&\quad+2D^{1212}_{\lambda+n-2}(t)\{\mathrm{Re}[i^{j+1}t'\Theta(\mu_{2\lambda-1})(t)]-i\,\mathrm{Re}(i^j)\Theta(\tau'\mu_{2\lambda-1})(t)\\&\quad-\mathrm{Re}[i^jt'K_{0\lambda}(\mu_0)(t)]\},\quad n,j=1,2,\\K_3\mu(t)&=K_{03}(\mu)(t)-D_0^{1313}(t)\,\mathrm{Re}[t'\Theta(\mu_5)(t)];\\g_c^2(z)&=D_0^{1313}(c_4+ic_3)/2,\\g_c^3(z)&=-c_4D^{1313}_{0\alpha^1}-c_3D^{1313}_{0\alpha^2},\\g_c^6(t)&=D_0^{1313}(t)(c_4d\alpha^2/ds-c_3d\alpha^1/ds),\\g_c^1(z)&=g_c^{3+j}(t)\equiv0,\quad j=1,2,4,5;\\a_{3(k-1)+j\ 2\lambda}(t)&=(-1)^jd_{k\lambda}(t)\,\mathrm{Re}(i^j)/2,\\b_{3(k-1)+j\ 2\lambda}(t)&=(-1)^jd_{k\lambda}(t)\,\mathrm{Re}(i^{j-1})/(2\pi),\\a_{3(k-1)+j\ 2\lambda-1}(t)&=-D^{1212}_{\lambda+k-2}(t)\,\mathrm{Re}(i^{j-1}),\\b_{3(k-1)+j\ 2\lambda-1}(t)&=D^{1212}_{\lambda+k-2}(t)\,\mathrm{Re}(i^j)/\pi,\quad k,j,\lambda=1,2,\\a_{35}(t)&=-D_0^{1313}(t)/2;\end{aligned}\tag{4.8}$$

other a_{jk},b_{jk} are zero. Here

$$\begin{aligned}&h_2^j(\mu)(z)\equiv h_2^j(\Phi(\mu))(z),\\&K_{0j}(\mu_0)(t)\equiv K_{0j}(\Phi_0(\mu_0))(t),\quad j=1,2;\\&K_{03}(\mu)(t)\equiv K_{03}(\Phi(\mu))(t),\\&\Phi(\mu)=(\Phi_1(\mu_2),\Phi_2(\mu_4),\Psi_1(\mu_1),\Psi_2(\mu_3),\Psi_3(\mu_5)),\\&\mu_0=(\mu_2,\mu_4).\end{aligned}$$

Lemma 4.1. *Let Conditions (a), (b), (c), (d) hold true. Then*

1) $h_1^j(\rho)$, $j=\overline{1,3}$, *are linear completely continuous operators in* $L_p(\Omega)$;
2) $h_2^j(\mu)$, $j=\overline{1,3}$, *are linear completely continuous operators from* $C_\nu(\Gamma)$ *into* $L_p(\Omega)$ *for all* $\nu\in(0,1)$;

3) $K_j\,\mu$, $j=\overline{1,5}$, *are linear completely continuous operators from* $C_\nu(\Gamma)$ *into* $C_\gamma(\Gamma)$ *for all* $\nu\in(0,1)$ *and* $\gamma<\beta/2$;
4) $H_j\,\rho$, $j=\overline{1,5}$), *are linear completely continuous operators from* $L_p(\Omega)$ *into* $C_{\alpha'}(\Gamma)$ *for all* $\alpha'<\alpha$ *and are bounded operators from* $L_p(\Omega)$ *into* $C_\alpha(\Gamma)$;
5) *The belongings hold:*

$$f_c^j(a)(z),\ f_\chi^j(a)(z),\ F^j(z),\ g_c^j(z)\in L_p(\Omega),\quad j=\overline{1,3};$$
$$\varphi_{cj}(a)(t),\ \varphi_{\chi j}(a)(t)\in C_\alpha(\Gamma),\ F^{3+j}(t),\ g_c^6(t),\ a_{jk}(t),\ b_{jk}(t)\in C_\beta(\Gamma),\quad j,k=\overline{1,5}.$$

Proof. It is known that [14] the Cauchy type integral $\theta(f)$ in (4.5) is a bounded operator from $C_\alpha(\Gamma)$ into $C_\alpha(\overline{\Omega})$, and its derivative $\theta'(f)$ is a bounded operator from $C_\alpha(\Gamma)$ into $L_q(\Omega)$, $1<q<2/(1-\alpha)$. Moreover, it is easy to show that $\theta(f)$ is a completely continuous operator from $C_\alpha(\Gamma)$ into $L_p(\Omega)$ for all $p>1$ and into $C_{\alpha'}(\overline{\Omega})$ for all $\alpha'<\alpha$. Taking this fact into consideration as well as properties of the operators T, S defined in (3.7), (3.8) and using the representations for the first derivatives of the generalized displacements in (3.13) and expressions for the operators $h^j(a)$ in (3.3), we conclude that the first two statements of the lemma are true.

Since $\psi(\tau,t)\in C_\beta(\Gamma)\times C_\beta(\Gamma)$ [15], $d_{k\lambda}(t)\in C_\beta(\Gamma)$, $d_j^k(z), D_0^{1313}(z)\in C_\beta(\overline{\Omega})$, then taking into consideration Corollary 4.3 from [16], we easily confirm that first two terms in the right hand side for the operators $K_{0j}(\mu_0)$ in (4.2) are completely continuous operators from $C_\nu(\Gamma)$ into $C_\gamma(\Gamma)$ for all $\nu\in(0,1)$ and $\gamma<\beta$. It is also easy to show that the third and forth terms of this representation in (4.2) are completely continuous operators from $C_\nu(\Gamma)$ into $C_\gamma(\Gamma)$ for all $\nu\in(0,1)$ and $\gamma<\beta$. We then obtain that $K_{0j}(\mu_0)$, $j=1,2$, are linear completely continuous operators from $C_\nu(\Gamma)$ into $C_\gamma(\Gamma)$ for all $\nu\in(0,1)$ and $\gamma<\beta$. Similarly to the representation of the operator $K_{03}(\mu)$ in (4.4) it follows that $K_{03}(\mu)$ is a linear completely continuous operator from $C_\nu(\Gamma)$ into $C_\beta(\Gamma)$ for all $\nu\in(0,1)$.

We transform the first two terms in the right hand side of the formula for the operator $K_{3(n-1)+j}\mu$ in (4.8) to the form

$$\frac{(-i)^j}{2}d_{n\lambda}(t)\left\{\frac{(-1)^j-1}{2\pi i}\int\limits_\Gamma\frac{\mu_{2\lambda}(\tau)}{\tau'}\frac{\tau'-t'}{\tau-t}d\tau+\frac{(-1)^j}{\pi}\int\limits_\Gamma\frac{\mu_{2\lambda}(\tau)}{\tau'}\,\mathrm{Im}\left(\frac{t'}{\tau-t}\right)d\tau\right\}.$$

Therefore, in view of the belongings $\tau', d_{n\lambda}\in C_\beta(\Gamma)$ and the identity

$$\mathrm{Im}[t'/(\tau-t)]=k_*(\tau,t)/|\tau-t|^{1-\beta/2},$$

where $k_*(\tau,t)\in C_{\beta/2}(\Gamma)\times C_{\beta/2}(\Gamma)$ [15], and also by Corollaries 4.4, 4.5 in [16] we obtain that the first two terms in the expression for the operator $K_{3(n-1)+j}\mu$ in (4.8) define a linear completely continuous operator from $C_\nu(\Gamma)$ into $C_\gamma(\Gamma)$ for all $\nu\in(0,1)$ and $\gamma<\beta/2$. In the same way we show that the third and fourth terms in the expression for the operator $K_{3(n-1)+j}\mu$ in (4.8) possess the same properties. Then the representations for the operators $K_j\mu$, $j=\overline{1,5}$, in (4.8) imply the third statement of the lemma. The validity of its fourth statement is implied by the representations for the operators $H_j\rho$, $j=\overline{1,5}$, in (4.4) due to the properties of the operators T, S, of the Cauchy type integral and the relations

$$S(T_\lambda T\rho_0)^+(t)=T\left(\frac{\partial}{\partial\zeta}T_\lambda T\rho_0\right)(t)-\frac{1}{2}(\bar t')^2T_\lambda T\rho_0(t)-\frac{1}{2\pi i}\int\limits_\Gamma\frac{T_\lambda T\rho_0(\tau)}{\tau-t}d\overline{\tau},\quad\lambda=1,2,$$

which are obtained by using formulas (8.20) from [14] and the Sokhotskii formulas. The validity of the fifth statement of the lemma follows directly from formulas (3.3), (3.5), (4.8). The proof is complete. □

We are going to study the solvability of the system of equations (4.7) in the space $L_p(\Omega) \times C_{\alpha'}(\Gamma)$, $\alpha' < \alpha$. We observe that each solution $(\rho,\mu) \in L_p(\Omega) \times C_{\alpha'}(\Gamma)$ of system (4.7) by Lemma 4.1 belongs to the space $L_p(\Omega)\times C_\alpha(\Gamma)$. Using the expressions for $a_{jk}(t)$, $b_{jk}(t)$ in (4.8), we calculate the determinant

$$\det[A(t)-\pi iB(t)] = D_0^{1313}\delta_1(a_1^2-a_0a_2)/(32\delta_0), \qquad a_n = D_n^{1111}+D_n^{1122}, \quad n=0,1,2,$$

where δ_0, δ_1 are defined in (3.11), while $A=(a_{jk})$, $B=(b_{jk})$ are square matrices of fifth order. Thus, $\det[A(t)-\pi iB(t)]\neq 0$ on Γ for the index of system (4.7) we obtain

$$\chi = \frac{1}{2\pi}\left[\arg\frac{\det(A-\pi iB)}{\det(A+\pi iB)}\right]_\Gamma = 0;$$

here the symbol $[\arg\varphi]_\Gamma$ means the increment of the argument of the function φ while passing the curve Γ once in the positive direction. Hence, we can apply the Fredholm alternative to system (4.7). Let $(\rho,\mu)\in L_p(\Omega)\times C_{\alpha'}(\Gamma)$ be a solution of system (4.7) for the zero right hand side. By formulas (4.5), (4.6) with constants $c_j=0$, $j=\overline{1,6}$, to these solutions, the holomorphic functions $\Phi_k(z)$, $\Psi_j(z)$ correspond to, which by formulas (3.10), (3.12) determine the function w_j, $j=\overline{1,3}$, ψ_k, $k=1,2$. As we see easily, these functions satisfy homogeneous system of linear equations (3.2),

$$f_c^j + f_\chi^j - F^j \equiv 0, \qquad j=\overline{1,3},$$

and homogeneous linear boundary conditions (3.4),

$$\varphi_{cj}+\varphi_{\chi j} - F^{3+j}\equiv 0, \qquad j=\overline{1,5}.$$

We multiply the real and imaginary part of the first equation in homogeneous system (3.2) respectively by w_1 and w_2. For the second equation we make a similar multiplication by respectively ψ_1 and ψ_2, while the third equation is multiplied by w_3. After that we integrate over the domain Ω and sum the obtained identities. In view of homogeneous boundary conditions (3.4) we obtain that w_j, $j=\overline{1,3}$, ψ_k, $k=1,2$, satisfy the system

$$\nu_{j1\alpha^1}=0, \qquad \nu_{j2\alpha^2}=0, \qquad \nu_{j1\alpha^2}+\nu_{j2\alpha^1}=0, \qquad w_{3\alpha^j}+\psi_j=0, \quad j=1,2,$$

a solution of which reads as

$$\begin{aligned} w_1 &= -c_0\alpha^2+c_1, \qquad w_2 = c_0\alpha^1+c_2,\\ w_3 &= -c_4\alpha^1-c_5\alpha^2+c_6, \qquad \psi_1=c_4, \qquad \psi_2=c_5, \end{aligned} \tag{4.9}$$

where c_j are arbitrary real constants.

Since $\Psi_j(0)=0$, $j=\overline{1,3}$, $w_3(0)=0$, by (4.9) we get

$$w_1=-c_0\alpha^2+c_1, \qquad w_2=c_0\alpha^1+c_2, \qquad w_3=\psi_1=\psi_2\equiv 0.$$

Then $\upsilon_j(z)=2ic_0D_{j-1}^{1212}$, $j=1,2$ and equations (3.6) imply the identities

$$\rho^j(z)=2ic_0D_{j-1\bar z}^{1212}, \quad j=1,2, \quad \rho^3(z)\equiv 0, \quad z\in\Omega. \tag{4.10}$$

Using formulas (3.7), (3.10), (3.12) and representation for υ_{jz}^0 in (3.13), we find $\Phi_k(z)$, $k=1,2$, $\Psi_j'(z)$, $j=\overline{1,3}$. Substituting them into (4.5), we obtain

$$\begin{aligned} &\mu_1(t)/t' - c_0(\bar t')^2 = F_1^-(t),\\ &\mu_{2j}(t)/t' - 2ic_0D_{j-1}^{1212}(t) = F_{2j}^-(t), \quad j=1,2,\\ &\mu_{2j-1}(t)/t' = F_{2j-1}^-(t), \quad j=2,3, \end{aligned}$$

where $F_j^-(t)$ are the boundary values of the function $F_j^-(z)$, which is holomorphic outside Ω and decays at infinity. Therefore, we obtain the Riemann–Hilbert problem for the function $F_j^-(z)$ in the exterior of the domain Ω with the boundary condition

$$\mathrm{Re}[it' F_j^-(t)] = f_j^-(t), \qquad j = \overline{1,5},$$

where

$$f_1^-(t) = c_0 \,\mathrm{Re}(it'), \qquad f_{2j}^-(t) = 2c_0 D_{j-1}^{1212}(t)\,\mathrm{Re}\, t', \quad j = 1,2, \qquad f_{2j-1}^-(t) = 0, \quad j = 2,3.$$

Using the solution of this problem [17], we obtain representations for the functions $\mu_j(t)$:

$$\mu_j(t) = c_0\mu_j^0(t) + \beta_{0j}\mu_j^1(t), \quad j = 1,2,4, \qquad \mu_j(t) = \beta_{0j}\mu_j^1(t), \quad j = 3,5, \tag{4.11}$$

where $\mu_j^k(t)$ are some known real functions belonging to the space $C_\alpha(\Gamma)$; c_0, β_{0j} are arbitrary real constants.

Solutions (4.10), (4.11) show that homogeneous system of equations (4.7) possesses six linearly independent solutions. Then the adjoint system of equations also has six linearly independent solutions. In order to derive the adjoint system, we multiply the real and imaginary parts of the left hand sides of the equations in (4.1) respectively by functions $v_1, v_2, v_3, v_4, v_5 \in L_q(\Omega)$, $1/p + 1/q = 1$, and integrate over the domain Ω, while the left hand sides of the equations in (4.3) are multiplied by real functions $\nu_1, \nu_2, \nu_3, \nu_4, \nu_5 \in C_\alpha(\Gamma)$ and then we integrate over the curve Γ. After that we sum them up and equate to zero. Replacing the holomorphic functions $\Phi_j(z)$, $\Psi_k(z)$, $\Psi_k'(z)$ by their expressions in (4.5), (4.6) with constants equalling to zero, interchanging the integration order in the obtained iterated integrals, by means of traditional arguing we make simple but rather bulky calculations, and this leads us to the adjoint system

$$\begin{aligned}
&\overline{v^j(z)} - T_{3+j}v(z) + 2\Theta(\tau'\overline{\nu^j})(z) = 0, \quad j = 1,2, \quad z \in \Omega,\\
&\mathrm{Re}\, T_3 v(z) = 0, \quad z \in \Omega,\\
&\mathrm{Re}\{i[T_{3+j}v(t) - 2\Theta^-(\tau'\overline{\nu^j})(t)]\} = 0, \quad j = 1,2,\\
&\mathrm{Re}[Tg(v)(t) + \Theta^-(\tau' D_0^{1313}\nu_3)(t)] = 0,\\
&\mathrm{Re}\{T[D_{\lambda+j-2\bar\zeta}^{1212}v^\lambda](t) - 2\Theta^-(\tau' D_{\lambda+j-2}^{1212}\nu^\lambda)(t)\\
&\qquad + (j-1)[iT^0 g(v)(t) - T_\Gamma^0(D_0^{1313}\tau'\nu_3)(t)]\} = 0, \quad t \in \Gamma, \quad j = 1,2;\\
&v^j = v_{3j-2} + iv_{3j-1}, \qquad \nu^j = \nu_{3j-2} + i\nu_{3j-1}, \quad j = 1,2,\\
&v^3 = v_3, \qquad \nu^3 = \nu_3.
\end{aligned} \tag{4.12}$$

In equations (4.12) we have adopted the notations

$$\begin{aligned}
&T_3 v(z) = -2Tg(v)(z) + 2D_0^{1313}(z)v_3(z) - 2\Theta(\tau' D_0^{1313}\nu_3)(z),\\
&T_{3+j}v(z) = 2Td_{j+2\lambda-2}[S_\lambda v](z) + Td_{2+j}[T_3 v](z),\\
&v = (v_1, v_2, v_3, v_4, v_5),\\
&S_j v(z) = S[D_{j+\lambda-2\bar\zeta}^{1212}v^\lambda](z) - D_{j+\lambda-2z}^{1212}v^\lambda(z) - 2\Theta'(\tau' D_{j+\lambda-2}^{1212}\nu^\lambda)(z), \quad j = 1,2,\\
&g(v)(z) = D_{0\bar z}^{1313}(z)v_3(z) - D_0^{1313}(z)v^2(z)/4,\\
&T^0 f(z) = -\frac{1}{\pi i}\iint\limits_{\Omega} f(\zeta) ln\left(1 - \frac{\zeta}{z}\right) d\xi d\eta,\\
&T_\Gamma^0 f(z) = -\frac{1}{2\pi i}\int\limits_{\Gamma} f(\tau) ln\left(1 - \frac{\tau}{z}\right) d\sigma,\\
&\Theta'(f)(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{f(\tau)d\tau}{\tau'(\tau - z)^2};
\end{aligned} \tag{4.13}$$

$\Theta^-(f)(t)$ are the boundary values of the function $\Theta(f)(z)$ as $z \to t \in \Gamma$ outside Ω; the operators Tf, Sf, $d_j[f]$, $\Theta(f)$ are defined in (3.7), (3.8), (3.11), (4.5), respectively.

As it was noted above, system (4.12) possesses six linearly independent solutions; we are going to obtain their explicit expressions. Belon in (4.12) by $v \in L_q(\Omega)$, $1/p + 1/q = 1$, $\nu \in C_\alpha(\Gamma)$, we mean some its solution.

We observe that the operators T, T^0, T^0_Γ introduced in (3.7), (4.13), define the functions $Tf(z)$, $T^0 f(z)$, T^0_Γ, $f(z)$, which are holomorphic in the exterior of the domain Ω and vanish at infinity. The function $\theta(f)(z)$ possesses the same property. This is why the five latter identities on the curve Γ in (4.12) are boundary conditions for the Riemann–Liouville problem with the zero index for the functions holomorphic outside Ω and decaying at infinity. As it is known, such problem possesses only the zero solution. Therefore, these five identities on the curve Γ are transformed to the form

$$\begin{aligned} &T_{3+j}v(z) - 2\Theta(\tau'\overline{\nu^j})(z) = 0, \quad j = 1,2,\\ &Tg(v)(z) + \Theta(\tau' D_0^{1313}\nu_3)(z) = 0,\\ &T[D^{1212}_{\lambda+j-2\bar{\zeta}}v^\lambda](z) - 2\Theta(\tau' D^{1212}_{\lambda+j-2}\nu^\lambda)(z)\\ &\quad + (j-1)[iT^0 g(v)(z) - T^0_\Gamma(D_0^{1313}\tau'\nu_3)(z)] = 0, \quad j = 1,2, \quad z \in \Omega_1 \equiv \mathbb{C}\setminus\overline{\Omega}, \end{aligned} \tag{4.14}$$

$\mathbb{C}$ is the complex plane.

It follows from the first three identities in (4.12) that the functions v_j, $j = \overline{1,5}$, belong to the space $W^{(1)}_{q_1}(\Omega)\cap C_\alpha(\overline{\Omega})$, $1 < q_1 < 2/(1-\alpha)$. In these identities we pass to the limit as $z \to t \in \Gamma$ inside the domain Ω, while in the first three identities in (4.14) we do the same outside the domain Ω. Then the latter identities are added to the former three identities, respectively. Taking into consideration the continuity of the functions outside of form $Tf(z)$ as $f \in L_p(\Omega)$ on $\mathbb{C}$ and using the Sokhotskii formulas, we obtain

$$v^j(t) = -2\nu^j(t), \quad j = 1,2, \qquad v_3(t) = \nu_3(t), \quad t \in \Gamma. \tag{4.15}$$

We differentiate first two identities in (4.12) in $\overline{z}$. In view of (3.8) we get the identities

$$\overline{v^j}_{\overline{z}} = 2d_{j+2\lambda-2}[S_\lambda v](z) + d_{2+j}[T_3 v](z), \quad j = 1,2, \quad z \in \Omega.$$

Considering them as a system for $X_1 = 2S_1 v$, $X_2 = 2S_2 v + T_3 v$ and solving it, we have

$$X_j = (D^{1111}_{j+\lambda-2} - D^{1212}_{j+\lambda-2})\overline{v^\lambda_z} + (D^{1111}_{j+\lambda-2} + D^{1212}_{j+\lambda-2})v^\lambda_z, \quad j = 1,2, \quad z \in \Omega. \tag{4.16}$$

We additionally suppose that the conditions

$$D^{1212}_j (j = 0,1,2), \quad D^{1313}_0 \in W^{(2)}_p(\Omega) \tag{4.17}$$

hold true. Using the relations for the functions $T_3 v(z)$, $S_j v(z)$, $j = 1,2$, in (4.13), we find $X_{j\overline{z}}$, $j = 1,2$, which, as we see easily, belong to the space $L_{q_1}(\Omega)$, $1 < q_1 < 2/(1-\alpha)$. Now we substitute these expressions $X_{j\overline{z}}$, $j = 1,2$, into the left hand sides of the relations obtained by differentiating of identities (4.16) with respect to $\overline{z}$. We differentiate the third identity in (4.12) in z and $\overline{z}$. By means of simple transformations of the obtained relations we confirm that the vector function $\widetilde{v} = (v_1, v_2, 2v_3, v_4, v_5)$ is a solution of system of linear equations (3.2) with zero right hand side.

Then for the solution (v,ν) of adjoint system of equations (4.12) we require $\nu(t) \in C^1_\alpha(\Gamma)$. Then, as we see easily, $v(z) \in C^1_\alpha(\overline{\Omega})$. Now we pass to limit as $z \to t \in \Gamma$ inside the domain Ω in identities (4.16) and the left hand side $X^+_j(t)$ is replaced by the expression obtained by using the representations for $(S_j v)(z)$, $T_3 v(z)$ in (4.13). Then we deduct respectively the identities obtained by differentiating in z of the latter two relations (4.14) followed by the passage to the limit as $z \to t \in \Gamma$ outside Ω. Then the third identities in (4.12) and (4.14) are differentiated in z, in the obtained identities we pass to the limit as $z \to t \in \Gamma$ respectively inside and outside the domain Ω and we deduct one from another. By means of the obtained in this way identities on the curve Γ, using relations (4.15), the formulas

$$(Sf)^+(t) - (Sf)^-(t) = -f(t)\cdot(\bar{t}')^2, \quad \theta'^+(\tau' f)(t) - \theta'^-(\tau' f)(t) = f_t + f_{\bar{t}}\cdot(\bar{t}')^2, \quad t\in\Gamma,$$

in which the operators Sf, $\Theta'(f)$ are defined in (3.8), (4.13), and assuming without loss of generality that $t=0\in\Gamma$, after simple transformations we see that the functions v_1, v_2, $2v_3$, v_4, v_5 satisfy also homogeneous boundary conditions in (3.4). Thus, the vector $\widetilde{v}=(v_1,v_2,2v_3,v_4,v_5)$ is a solution of homogeneous system of linear equations in (3.2) satisfying homogeneous boundary conditions in (3.4). Therefore, in accordance with (4.9), for the components of the vector $\widetilde{v}$ we obtain the following representations:

$$v_1 = -c_0\alpha^2 + c_1, \qquad v_2 = c_0\alpha^1 + c_2, \qquad v_3 = (-c_4\alpha^1 - c_5\alpha^2 + c_6)/2, \qquad v_4 = c_4, \qquad v_5 = c_5,$$

where c_j are arbitrary real constants.

The functions $\nu_j(t)$ and v_k are related by formulas (4.15). Therefore, the solution $(v,\nu)^T$, $v=(v_1,v_2,v_3,v_4,v_5)$, $\nu=(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$ of adjoint system (4.12) can be represented as

$$(v,\nu)^T = c_0\gamma_1 + c_1\gamma_2 + c_2\gamma_3 + c_4\gamma_4 + c_5\gamma_5 + c_6\gamma_6,$$

where $\gamma_k = (\gamma_{k1},\gamma_{k2},\dots,\gamma_{k10})$, $k=\overline{1,6}$, are linearly independent solutions of system (4.12). Then system (4.7) is solvable if and only the conditions

$$\iint\limits_{\Omega} \big\{\mathrm{Re}\,\big[(f_c^1 + f_\chi^1 + g_c^1 - F^1)(z)(\gamma_{k1} - i\gamma_{k2})(z)$$
$$+(f_c^2 + f_\chi^2 + g_c^2 - F^2)(z)(\gamma_{k4} - i\gamma_{k5})(z)\big] + (f_c^3 + f_\chi^3 + g_c^3 - F^3)(z)\gamma_{k3}(z)\big\}\, d\alpha^1 d\alpha^2$$
$$+\sum_{j=1}^{5}\int\limits_{\Gamma}(\varphi_{cj} + \varphi_{\chi j} + g_c^{3+j} - F^{3+j})(t)\gamma_{k,5+j}(t)ds = 0, \quad k=\overline{1,6},$$

hold true, which, after simple transformations, become

$$\begin{aligned}
&\iint\limits_{\Omega} R^j d\alpha^1 d\alpha^2 + \int\limits_{\Gamma} P^j ds - \iint\limits_{\Omega} B_{j\lambda}[T^{\lambda 3}(\gamma) + T^{\lambda\mu}(\gamma)\omega_\mu] d\alpha^1 d\alpha^2 = 0, \quad j=1,2,\\
&\iint\limits_{\Omega}(R^1\alpha^2 - R^2\alpha^1) d\alpha^1 d\alpha^2 + \int\limits_{\Gamma}(P^1\alpha^2 - P^2\alpha^1) ds\\
&\qquad + \iint\limits_{\Omega}(\alpha^1 B_{2\lambda} - \alpha^2 B_{1\lambda})[T^{\lambda\mu}(\gamma)\omega_\mu + T^{\lambda 3}(\gamma)] d\alpha^1 d\alpha^2 = 0,\\
&\iint\limits_{\Omega}(\alpha^j R^3 - L^j) d\alpha^1 d\alpha^2 + \int\limits_{\Gamma}(\alpha^j P^3 - N^j) ds + \iint\limits_{\Omega}\alpha^j B_{\lambda\mu} T^{\lambda\mu}(\gamma) d\alpha^1 d\alpha^2\\
&\qquad - \iint\limits_{\Omega} T^{j\mu}(\gamma)\omega_\mu d\alpha^1 d\alpha^2 = 0, \quad j=1,2,\\
&\iint\limits_{\Omega} R^3 d\alpha^1 d\alpha^2 + \int\limits_{\Gamma} P^3 ds + \iint\limits_{\Omega} B_{\lambda\mu} T^{\lambda\mu}(\gamma) d\alpha^1 d\alpha^2 = 0,
\end{aligned} \tag{4.18}$$

where R^j, P^j $(j=\overline{1,3})$, L^k, N^k, $k=1,2$, are the components of external forces, γ is an arbitrarily fixed vector of deformation, ω_μ is an arbitrarily fixed function.

Under conditions (4.18), the general solution of system (4.7) can be represented as

$$\begin{aligned}
&(\rho,\mu) = (\rho_c,\mu_c)(a) + (\rho_\chi,\mu_\chi)(a) + (\rho_*,\mu_*) + (\rho_F,\mu_F), \quad (\rho_c,\mu_c)(a) = \mathfrak{R} f_c(a),\\
&(\rho_\chi,\mu_\chi)(a) = \mathfrak{R} f_\chi(a), \quad (\rho_*,\mu_*) = \mathfrak{R} g_c + (\widetilde{\rho},\widetilde{\mu}), \quad (\rho_F,\mu_F) = -\mathfrak{R} F,
\end{aligned} \tag{4.19}$$

where

$$f_c(a) = (f_c^1, f_c^2, f_c^3, \varphi_{c1}, \dots, \varphi_{c5}), \quad f_\chi(a) = (f_\chi^1, f_\chi^2, f_\chi^3, \varphi_{\chi 1}, \dots, \varphi_{\chi 5}), \quad g_c = (g_c^1, \dots, g_c^8),$$
$$F = (F^1, \dots, F^8); \quad \mathfrak{R} = (\mathfrak{R}_1, \dots, \mathfrak{R}_8);$$

$\mathfrak{R}_j$, $j = \overline{1,3}$, and $\mathfrak{R}_k$, $k = \overline{4,8}$, are linear bounded operators from $L_p(\Omega) \times C_\alpha(\Gamma)$ into $L_p(\Omega)$ and into $C_\alpha(\Gamma)$, respectively; the functions $\widetilde{\rho} = (\widetilde{\rho}_1, \widetilde{\rho}_2, \widetilde{\rho}_3)$, $\widetilde{\mu} = (\widetilde{\mu}_1, \dots, \widetilde{\mu}_5)$ are defined by formulas (4.10), (4.11), while f_c^j, f_χ^j, φ_{ck}, $\varphi_{\chi k}$, g_c^n, F^n are defined by formulas in (3.3), (3.5), (4.8).

If we substitute the expression for the vector function $\mu(t)$ in (4.19) into relations (4.5), (4.6), then for a holomorphic vector function $\Phi(z) = (\Phi_0, \Psi)$, $\Phi_0 = (\Phi_1, \Phi_2)$, $\Psi = (\Psi_1, \Psi_2, \Psi_3)$, we obtain a representation:

$$\Phi(z) = \Phi_c(a)(z) + \Phi_\chi(a)(z) + \Phi_*(z) + \Phi_F(z), \quad z \in \Omega, \tag{4.20}$$

where

$$\Phi_c(a)(z) = \Phi(\mu_c(a))(z), \quad \Phi_\chi(a)(z) = \Phi(\mu_\chi(a))(z), \quad \Phi_F(z) = \Phi(\mu_F)(z),$$
$$\Phi_*(z) = \Phi(\mathfrak{R} g_c)(z) + \widetilde{\Phi}(z), \quad \widetilde{\Phi}(z) = (c_0\beta_0(z), c_0\beta_1(z), c_0\gamma_0(z) + c_1 + ic_2, 0, 0),$$
$$\beta_j(z) = 2i\Theta(t' D_j^{1212})(z), \quad j = 0, 1, \quad \gamma_0(z) = \Theta(t'\overline{t})(z);$$

the function $\Theta(f)(z)$ is defined in (4.5) and c_j are arbitrary real constants.

Now we substitute the expressions for $\rho(z)$ in (4.19) and for the holomorphic functions in (4.20) into (3.10), (3.12). Then problem (2.1), (2.2) is reduced to a system of nonlinear equations for the vector function $a = (w_1, w_2, w_3, \psi_1, \psi_2)$, which we represent as

$$\begin{aligned} \upsilon_j^0(z) &= \upsilon_{jc}^0(a) + \upsilon_{j\chi}^0(a) + \upsilon_{j*}^0(z) + \upsilon_{jF}^0(z), \qquad j = 1, 2, \\ w_3(z) &= w_{3c}(a) + w_{3\chi}(a) + w_{3*}(z) + w_{3F}(z), \qquad z \in \Omega, \end{aligned} \tag{4.21}$$

where

$$\upsilon_{jc}^0(a) = \upsilon_j^0(\Psi_{jc}(a); \upsilon_{jc}(a)), \qquad \upsilon_c(a) = (\upsilon_{1c}, \upsilon_{2c}),$$
$$\upsilon_{jc}(a) = \upsilon_j(\Phi_{jc}(a); \rho_c^j(a)), \quad j = 1, 2, \qquad w_{3c}(a) = w_3(\Psi_{3c}(a); \rho_c^3(a)).$$

Other terms in (4.21) are defined similarly, while the operators $\upsilon_j(\Phi_j; \rho^j)$, $\upsilon_j^0(\Psi_j; \upsilon_j)$ and $w_3(\Psi_3; \rho^3)$ are defined in (3.7), (3.10), (3.12), respectively.

We observe that the functions $\upsilon_{j*}^0(z)$, $w_{3*}(z)$ and $\upsilon_{jF}^0(z)$, $w_{3F}(z)$ depend respectively on arbitrary constants and external forces acting on the shell. At the same time, as we see easily, the functions $\upsilon_{1*}^0(z) = w_{2*} + iw_{1*}$, $\upsilon_{2*}^0(z) = \psi_{2*} + i\psi_{1*}$, $w_{3*}(z)$ satisfy representations (4.9).

We proceed to studying the solvability of system (4.21) in the space $W_p^{(2)}(\Omega)$.

Lemma 4.2. *Let Conditions (a), (b), (c), (d) hold. Then*

1) $\upsilon_{jc}^0(a)$ $(j = 1, 2)$, $w_{3c}(a)$ *are linear completely continuous operators in* $W_p^{(2)}(\Omega)$;
2) $\upsilon_{j\chi}^0(a)$, $j = 1, 2$, $w_{3\chi}(a)$ *are nonlinear bounded operators in* $W_p^{(2)}(\Omega)$, *and for all* $a^j = (w_1^j, w_2^j, w_3^j, \psi_1^j, \psi_2^j) \in W_p^{(2)}(\Omega)$, $j = 1, 2$, *the estimates*

$$\begin{aligned} \|\upsilon_{j\chi}^0(a^1) - \upsilon_{j\chi}^0(a^2)\|_{W_p^{(2)}(\Omega)} &\leqslant c\Big(\|a^1\|_{W_p^{(2)}(\Omega)} + \|a^2\|_{W_p^{(2)}(\Omega)} \\ &\qquad + \|w^1\|^2_{W_p^{(2)}(\Omega)} + \|w^2\|^2_{W_p^{(2)}(\Omega)}\Big)\|a^1 - a^2\|_{W_p^{(2)}(\Omega)}, \\ \|w_{3\chi}(a^1) - w_{3\chi}(a^2)\|_{W_p^{(2)}(\Omega)} &\leqslant c\Big(\|a^1\|_{W_p^{(2)}(\Omega)} + \|a^2\|_{W_p^{(2)}(\Omega)} \\ &\qquad + \|w^1\|^2_{W_p^{(2)}(\Omega)} + \|w^2\|^2_{W_p^{(2)}(\Omega)}\Big)\|a^1 - a^2\|_{W_p^{(2)}(\Omega)} \end{aligned} \tag{4.22}$$

hold, where

$$\|w^j\|^2_{W_p^{(2)}(\Omega)} = \|w_1^j\|^2_{W_p^{(2)}(\Omega)} + \|w_2^j\|^2_{W_p^{(2)}(\Omega)} + \|w_3^j\|^2_{W_p^{(2)}(\Omega)}, \quad j=1,2,$$

c is a known positive constant depending on physical and geometrical characteristics of the shell;

3) $\upsilon^0_{j*}(z), \upsilon^0_{jF}(z) \in W_p^{(2)}(\Omega)$, $j=1,2$.

Proof. The representations for $f_c^j(a)$, $f_\chi^j(a)$ in (3.3) and those for $\varphi_{cj}(a)$, $\varphi_{\chi j}(a)$ in (3.5) imply that $f_c^j(a)$ and $\varphi_{cj}(a)$ are linear completely continuous, while $f_\chi^j(a)$ and $\varphi_{\chi j}(a)$ are nonlinear bounded operators from $W_p^{(2)}(\Omega)$ into $L_p(\Omega)$ and into $C_\alpha(\Gamma)$, respectively; for $f_\chi^3(a)$, $\varphi_{\chi 3}(a)$ we have estimates of form (4.22), while $f_\chi^j(a)$, $\varphi_{\chi j}(a)$, $j=1,2$, satisfy estimates of form

$$\begin{aligned} \|f_\chi^j(a^1) - f_\chi^j(a^2)\|_{L_p(\Omega)}, \quad \|\varphi_{\chi j}(a^1) - \varphi_{\chi j}(a^2)\|_{C_\alpha(\Gamma)} \leqslant c\Big(\|w^1\|_{W_p^{(2)}(\Omega)} \\ +\|w^2\|_{W_p^{(2)}(\Omega)}\Big)\|a^1-a^2\|_{W_p^{(2)}(\Omega)}, \quad j=1,2. \end{aligned} \tag{4.23}$$

Then in view of the boundedness of the operators $\mathfrak{R}_j$ by (4.19) we obtain that $\rho_c^j(a)$ and $\mu_{kc}(a)$ are linear completely continuous, while $\rho_\chi^j(a)$ and $\mu_{k\chi}(a)$ are nonlinear bounded operators from $W_p^{(2)}(\Omega)$ into $L_p(\Omega)$ and into $C_\alpha(\Gamma)$, respectively, and for $\rho_\chi^j(a)$, $\mu_{k\chi}(a)$ estimates (4.22) hold. Therefore, in view of the properties of the Cauchy type integral in in (4.20) we conclude that $\Phi_{kc}(a)$, $\Psi'_{jc}(a)$ are linear completely continuous, $\Phi_{k\chi}(a)$, $\Psi'_{j\chi}(a)$ are nonlinear bounded operators from $W_p^{(2)}(\Omega)$ into $C_\alpha(\overline{\Omega})$ and for nonlinear operators $\Phi_{k\chi}(a)$, $\Psi'_{j\chi}(a)$ estimates (4.22) hold.

Let us study the properties of the operators

$$\Phi'_{kc}(a) = \Theta'(\mu_{2kc}(a)), \quad k=1,2, \quad \Psi''_{jc}(a) = i^{(j-1)(j-2)/2}\Theta'(\mu_{2j-1c}(a)), \quad j=\overline{1,3}, \tag{4.24}$$

where the operator $\Theta'(f)$ is defined in (4.13).

We observe that the functions $\rho_c^j(a)(z)$, $\mu_{kc}(a)(t)$ defined in (4.19) are solutions of system (4.7) with the right hand side $f_c^j(a)(z)$, $j=\overline{1,3}$, $\varphi_{ck}(a)(t)$ $k=\overline{1,5}$. This is why the vector $\mu_c = (\mu_{1c},\dots,\mu_{5c})$ can be represented as

$$\mu_c(a)(t) = A^{-1}(t)\left[\varphi_c(a)(t) - B(t)\int\limits_\Gamma \frac{\mu_c(a)(\tau)}{\tau - t}d\tau - K\mu_c(a)(t) - H\rho_c(a)(t)\right], \tag{4.25}$$

where $A^{-1}(t) \in C_\beta(\Gamma)$ is the matrix inverse to the matrix $A(t)$, $\varphi_c = (\varphi_{c1},\dots,\varphi_{c5})$, $K = (K_1,\dots,K_5)$, $H=(H_1,\dots,H_5)$, $\rho_c = (\rho_c^1,\rho_c^2,\rho_c^3)$.

We substitute expression (4.25) for $\mu_c(a)(t)$ into (4.24), interchange the integration order in the iterated integrals and use the aforementioned properties of the Cauchy type integral and of the operators T, S, as well as relations (4.7), (4.9) from [14] and Lemma 4.1. Then after simple but rather bulky transformations we obtain that the operators $\Phi'_{kc}(a)$, $k=1,2$, $\Psi''_{jc}(a)$, $j=\overline{1,3}$, are linear completely continuous operators from $W_p^{(2)}(\Omega)$ into $L_p(\Omega)$. By means of similar arguing we also see that $\Phi'_{k\chi}(a)$, $k=1,2$, $\Psi''_{j\chi}(a)$, $j=\overline{1,3}$, are nonlinear bounded operators from $W_p^{(2)}(\Omega)$ into $L_p(\Omega)$ and they satisfy estimates (4.22). Once we employ now relations (3.9), (3.10), (3.13) and estimates (4.23), then the statement of the lemma becomes obvious. The proof is complete. □

We write system (4.21) in the form

$$a - L(a) - G(a) = a_* + \widetilde{a}_F, \tag{4.26}$$

where

$$L = (L_1,\dots,L_5), \qquad G = (G_1,\dots,G_5),$$

$$a_* = (w_{1*}, w_{2*}, w_{3*}, \psi_{1*}, \psi_{2*}), \quad \widetilde{a}_F = (\widetilde{w}_{1F}, \widetilde{w}_{2F}, \widetilde{w}_{3F}, \widetilde{\psi}_{1F}, \widetilde{\psi}_{2F}),$$
$$\upsilon^0_{1*} = w_{2*} + iw_{1*}, \quad \upsilon^0_{2*} = \psi_{2*} + i\psi_{1*},$$
$$L_{3(n-1)+j}(a) = -\operatorname{Re}[i^j \upsilon^0_{nc}(a)], \quad n, j = 1, 2;$$
$$G_{3(n-1)+j}(a) = -\operatorname{Re}[i^j \upsilon^0_{n\chi}(a))], \quad n, j = 1, 2;$$
$$L_3(a) = w_{3c}(a), \quad G_3(a) = w_{3\chi}(a),$$
$$\widetilde{w}_{jF} = -\operatorname{Re}[i^j \upsilon^0_{1F}], \quad j = 1, 2,$$
$$\widetilde{\psi}_{jF} = -\operatorname{Re}[i^j \upsilon^0_{2F}], \quad j = 1, 2,$$
$$\widetilde{w}_{3F} = w_{3F}.$$

We note that $L(a)$ is a linear completely continuous and $G(a)$ is a nonlinear bounded operator in $W^{(2)}_p(\Omega)$ and $G(a)$ satisfies estimate (4.22); $\widetilde{a}_F \in W^{(2)}_p(\Omega)$ is a known function depending on the external forces; the components of the vector a_* are given by formulas (4.9).

The equation $a - L(a) = 0$ has only trivial solution in $W^{(2)}_p(\Omega)$. Indeed if $a \in W^{(2)}_p(\Omega)$ is its nonzero solution then, as one can easily see a is a solution of system of linear equations of equilibrium obeying linear homogeneous boundary conditions. Arguing then as in the case of system (4.7), we conclude that the vector a satisfies the system

$$\begin{aligned} &w_{j\alpha^j} - B_{jj}w_3 = 0, \quad j = 1, 2,\\ &w_{1\alpha^2} + w_{2\alpha^1} - 2B_{12}w_3 = 0,\\ &\psi_{j\alpha^j} = 0, \quad j = 1, 2,\\ &\psi_{1\alpha^2} + \psi_{2\alpha^1} = 0,\\ &w_{3\alpha^j} + B_{j\lambda}w_\lambda + \psi_j = 0, \quad j = 1, 2. \end{aligned} \tag{4.27}$$

We proceed to solving system (4.27). By means of the fourth, fifth and sixth identities for ψ_1, ψ_2 we obtain the representations

$$\psi_1 = c_0\alpha^2 + c_1, \qquad \psi_2 = -c_0\alpha^1 + c_2, \tag{4.28}$$

where c_0, c_1, c_2 are arbitrary real constants.

We multiply the first identity in (4.27) by B_{22}, the second is multiplied by B_{11}, the third is multiplied by B_{12}. After that we sum first two identities and deduct the third one. As a result, in view of condition (b) and the relations

$$B_{11\alpha^2} = B_{12\alpha^1}, \qquad B_{12\alpha^2} = B_{22\alpha^1} \tag{4.29}$$

implied by the Gauss–Peterson–Codazzi formulas [1] we obtain the identity

$$(B_{22}w_1 - B_{12}w_2)_{\alpha^1} + (B_{11}w_2 - B_{12}w_1)_{\alpha^2} = 0.$$

Then we easily see that there exists a function $u(\alpha^1, \alpha^2) \in C^2(\overline{\Omega})$ such that the relations

$$B_{12}w_1 - B_{11}w_2 = u_{\alpha^1}, \qquad B_{22}w_1 - B_{12}w_2 = u_{\alpha^2} \tag{4.30}$$

hold. We multiply the first identity in (4.30) by B_{12}, the second identity is multiplied by B_{11} and then we deduct one from the other. Then we obtain an equation for the function $u(\alpha^1, \alpha^2)$:

$$B_{12}u_{\alpha^1} - B_{11}u_{\alpha^2} = 0,$$

the general solution of which is given by the formula [18]

$$u(\alpha^1, \alpha^2) = \Lambda_1(x), \quad x = x(\alpha^1, \alpha^2) = \int\limits_{(\alpha^1_0, \alpha^2_0)}^{(\alpha^1, \alpha^2)} B_{11}(\beta^1, \beta^2)d\beta^1 + B_{12}(\beta^1, \beta^2)d\beta^2, \tag{4.31}$$

where $\Lambda_1(x)$ is an arbitrary real function belonging to the space C^2 and (α_0^1, α_0^2) is an arbitrarily fixed point in $\overline{\Omega}$.

Now we multiply the seventh identity in (4.27) by B_{22}, the eighth identity is implied by B_{12} and we deduct one identity from the other. This gives

$$B_{22}w_{3\alpha^1} - B_{12}w_{3\alpha^2} = B_{12}\psi_2 - B_{22}\psi_1,$$

and in view of relations (4.28), (4.29) this gives a representation for the function w_3 [18]:

$$\begin{aligned}
&w_3(\alpha^1,\alpha^2) = \Lambda_2(y) + w_3^*(\alpha^1,\alpha^2),\\
&y = y(\alpha^1,\alpha^2) = \int\limits_{(\alpha_0^1,\alpha_0^2)}^{(\alpha^1,\alpha^2)} B_{12}(\beta^1,\beta^2)d\beta^1 + B_{22}(\beta^1,\beta^2)d\beta^2,\\
&w_3^*(\alpha^1,\alpha^2) = c_0a_1(\alpha^1,\alpha^2) + c_1a_2(\alpha^1,\alpha^2) + c_2a_3(\alpha^1,\alpha^2),\\
&\widetilde{a}_1(\alpha^1,y) = -\int\limits_{\alpha_0^1}^{\alpha^1} [\beta^1\widetilde{B}_{12}(\beta^1,y) + \alpha^2(\beta^1,y)\widetilde{B}_{22}(\beta^1,y)]/\widetilde{B}_{22}(\beta^1,y)d\beta^1,\\
&a_2(\alpha^1,\alpha^2) = \alpha_0^1 - \alpha^1, \quad \widetilde{a}_3(\alpha^1,y) = \int\limits_{\alpha_0^1}^{\alpha^1} \widetilde{B}_{12}(\beta^1,y)/\widetilde{B}_{22}(\beta^1,y)d\beta^1,\\
&\widetilde{B}_{\lambda\mu}(\beta^1,y) \equiv B_{\lambda\mu}(\beta^1,\alpha^2), \quad \lambda,\mu = 1,2, \quad \widetilde{a}_j(\alpha^1,y) \equiv a_j(\alpha^1,\alpha^2), \quad j = 1,3,
\end{aligned} \tag{4.32}$$

where $\alpha^2 = \alpha^2(\alpha^1, y)$ is a solution of the equation $y(\alpha^1,\alpha^2) = y$ with respect to α^2; this solution exists due to the condition $y_{\alpha^2} = B_{22} \neq 0$ in $\overline{\Omega}$; c_j, $j = 0,1,2$, are arbitrary real constants.

In order to derive representations for w_1, w_2 from the seventh identity in (4.27) and the first identity in (4.30), we form the system

$$B_{11}w_1 + B_{12}w_2 = -w_{3\alpha^1} - \psi_1, \qquad B_{12}w_1 - B_{11}w_2 = u_{\alpha^1}.$$

Solving this system with respect to w_1, w_2, we obtain

$$\begin{aligned}
&w_1 = b_1[\Lambda_1'(x) - \Lambda_2'(y)] + w_1^*(\alpha^1,\alpha^2),\\
&w_2 = b_2[\Lambda_1'(x) - \Lambda_2'(y)] - \Lambda_1'(x) + w_2^*(\alpha^1,\alpha^2),\\
&w_j^* = -B_{1j}(w_{3\alpha^1}^* + \psi_1)/(B_{11}^2 + B_{12}^2), \quad j = 1,2,\\
&b_1 = B/(1+B^2),\\
&b_2 = 1/(1+B^2),\\
&B = B_{11}/B_{12} = B_{12}/B_{22},
\end{aligned} \tag{4.33}$$

where ψ_1, w_3^* are defined in (4.28), (4.32). We note that by Condition (b) we have w_j^*, $b_j \in C^1(\overline{\Omega})$, $j = 1,2$.

Excluding the function w_3 from the first two identities in (4.27), we obtain

$$B_{22}w_{1\alpha^1} - B_{11}w_{2\alpha^2} = 0. \tag{4.34}$$

We differentiate the seventh identity in (4.27) in the variable α^2, while the eighth identity is differentiated in the variable α^1; then we deduct one identity from the other. In view of relations (4.28), (4.29) we then get

$$B_{12}(w_{1\alpha^1} - w_{2\alpha^2}) + B_{22}w_{2\alpha^1} - B_{11}w_{1\alpha^2} = 2c_0. \tag{4.35}$$

Now we substitute the expressions for w_1, w_2, w_3 from (4.32), (4.33) into (4.34), (4.35) and into the last identity in system (4.27). As a result, in view of relations $x_{\alpha^1} = B_{11}$, $x_{\alpha^2} = y_{\alpha^1} = B_{12}$,

$y_{\alpha^2} = B_{22}$ implied by the representations for the functions $x(\alpha^1, \alpha^2)$, $y(\alpha^1, \alpha^2)$ in (4.31), (4.32), we obtain a system of form

$$\begin{aligned} &B_{11}B_{12}\Lambda_1''(x) + b_3[\Lambda_1'(x) - \Lambda_2'(y)] = d_1,\\ &b_4[\Lambda_1'(x) - \Lambda_2'(y)] = d_2,\\ &w_{3\alpha^2}^* - (w_{3\alpha^1}^* + \psi_1)/B + \psi_2 = 0, \end{aligned} \tag{4.36}$$

where we have adopted the notations

$$\begin{aligned} &b_3 = B_{22}b_{1\alpha^1} - B_{11}b_{2\alpha^2}, && b_4 = B_{22}b_{2\alpha^1} - B_{11}b_{1\alpha^2} + B_{12}(b_{1\alpha^1} - b_{2\alpha^2}),\\ &d_1 = B_{11}w_{2\alpha^2}^* - B_{22}w_{1\alpha^1}^*, && d_2 = 2c_0 + B_{11}w_{1\alpha^2}^* - B_{22}w_{2\alpha^1}^* - B_{12}(w_{1\alpha^1}^* - w_{2\alpha^2}^*), \end{aligned} \tag{4.37}$$

the functions $w_j^*(\alpha^1, \alpha^2)$, $j = \overline{1,3}$, are defined in (4.32), (4.33); d_j, $b_{2+j} \in C(\overline{\Omega})$, $j = 1, 2$.

Suppose that the components of the curvature tensor of the middle surface of the shell satisfy the conditions

$$BB_{\alpha^2} - B_{\alpha^1} \neq 0, \qquad B_{\alpha^2} \neq 0, \qquad 1 + \alpha^1 B_{\alpha^2} \neq 0, \qquad (\alpha^1, \alpha^2) \in \Omega, \tag{4.38}$$

where B is defined in (4.33).

In view of the expressions for the functions w_3^*, ψ_1, ψ_2 in (4.28), (4.32), we rewrite the third identity in (4.36) as

$$c_2 B_{\alpha^2} - c_0(1 + \alpha^1 B_{\alpha^2}) = 0,$$

and by conditions (4.38) this implies $c_0 = c_2 = 0$. Then

$$w_1^* = w_2^* \equiv 0, \qquad \psi_2 \equiv 0, \qquad w_3^* = -c_1\alpha^1, \qquad \psi_1 = c_1,$$

and hence, by (4.37) we get: $d_1 = d_2 \equiv 0$. We note that by the first condition in (4.38) we have $b_4 \neq 0$ in Ω. This is why by the second equation in (4.36) we get

$$\Lambda_1'(x) - \Lambda_2'(y) = 0.$$

Then the first equation in (4.36) implies $\Lambda_1''(x) = 0$ and hence, $\Lambda_1'(x) = c_3 = \Lambda_2'(y)$, where c_3 is an arbitrary real constant. By formulas (4.28), (4.33) we have $w_1 \equiv 0$, $w_2 = -c_3$, and it follows from the first identity in (4.27) that $w_3 \equiv 0$ in $\overline{\Omega}$. Therefore, taking into consideration representation (4.32) for w_3, we get the identity $\Lambda_2(y) - c_1\alpha^1 = 0$. Differentiating this identity in the variable α^1 and using the formula $y_{\alpha^1} = B_{12}$, we arrive at the identity $c_3 B_{12}(\alpha^1, \alpha^2) - c_1 = 0$. Then in view of Condition (b) we have $c_1 = c_3 = 0$, that is, $w_j = 0$, $j = \overline{1,3}$, $\psi_k = 0$, $k = 1, 2$, in $\overline{\Omega}$. Thus, the equation $a - L(a) = 0$ has only trivial solution in $W_p^{(2)}(\Omega)$. Hence, there exists the inverse operator $(I - L)^{-1}$ bounded in $W_p^{(2)}(\Omega)$, by means of which equation (4.26) is reduced to the equivalent equation

$$a - G_*(a) = a_F, \tag{4.39}$$

where

$$G_*(a) = (I - L)^{-1}G(a), \qquad a_F = (I - L)^{-1}\widetilde{a}_F.$$

We observe that the vector $a_c = (I - L)^{-1}a_*$ is a solution of the homogeneous system of linear equilibrium equations satisfying homogeneous linear boundary conditions. This is why by the above proven facts we have $a_c \equiv 0$, which has been taken into consideration while passing to equation (4.39).

We also note that the vector a_F in (4.39) depends only on external forces and $a_F = 0$ once the external forces are absent.

Lemma 4.3. *Let Conditions (a), (b), (c), (d) hold. Then*

1) *$G_*(a)$ is a nonlinear bounded operator in $W_p^{(2)}(\Omega)$ and for all $a^j = (w_1^j, w_2^j, w_3^j, \psi_1^j, \psi_2^j)$, $j = 1, 2$, the estimate*

$$\|G_*(a^1) - G_*(a^2)\|_{W_p^{(2)}(\Omega)} \leqslant c_*\Big(\|a^1\|_{W_p^{(2)}(\Omega)} + \|a^2\|_{W_p^{(2)}(\Omega)}$$

$$+ \|w^1\|^2_{W^{(2)}_p(\Omega)} + \|w^2\|^2_{W^{(2)}_p(\Omega)}\Big)\|a^1 - a^2\|_{W^{(2)}_p(\Omega)},$$

$$\|w^j\|^2_{W^{(2)}_p(\Omega)} = \|w^j_1\|^2_{W^{(2)}_p(\Omega)} + \|w^j_2\|^2_{W^{(2)}_p(\Omega)} + \|w^j_3\|^2_{W^{(2)}_p(\Omega)}, \quad j = 1, 2,$$

holds true, where c_* is a known positive constant depending on physical and geometrical characteristics of the shell;

2) $a_F \in W^{(2)}_p(\Omega)$.

The validity of the lemma is implied by Lemma 4.2 in view of the aforementioned properties of the operators $(I - L)^{-1}$ and G.

We proceed to studying the solvability of equation (4.39) in the space $W^{(2)}_p(\Omega)$. Using Lemma 4.3, for all $a^j \in W^{(2)}_p(\Omega)$, $j = 1, 2$, belonging to the ball $\|a\|_{W^{(2)}_p(\Omega)} < r$, we obtain

$$\|G_*(a^1) - G_*(a^2)\|_{W^{(2)}_p(\Omega)} \leqslant q_*\|a^1 - a^2\|_{W^{(2)}_p(\Omega)}, \qquad q_* = 2c_*r(1 + r).$$

Suppose that the radius r of the ball and external forces are such that the inequalities

$$q_* < 1, \qquad \|a_F\|_{W^{(2)}_p(\Omega)} < (1 - q_*)r \tag{4.40}$$

hold. Then we can apply the contracting mapping principle to equation (4.39) [19], according to which equation (4.39) in the ball $\|a\|_{W^{(2)}_p(\Omega)} < r$ possesses a unique solution of form $a = \mathcal{R}(a_F) \in W^{(2)}_p(\Omega)$, where $\mathcal{R}$ is the resolvent of the operator G_*. We note that if the external load is absent, then problem (2.1), (2.2) possesses only the zero solution.

We return back to solvability conditions (4.18), in which by $a = (w_1, w_2, w_3, \psi_1, \psi_2) \in W^{(2)}_p(\Omega)$ we mean a solution to problem (2.1), (2.2) and ω_μ, $\mu = 1, 2$, are defined in (4.3). Using identities (2.1) and (2.2), we confirm that solvability conditions (4.18) are satisfied.

Thus, we have proved the following theorem.

Theorem 4.1. *Let Conditions (a), (b), (c), (d),* (4.17), (4.38) *and* (4.40) *be satisfied. Then problem (2.1), (2.2) possesses a unique generalized solution* $a = (w_1, w_2, w_3, \psi_1, \psi_2) \in W^{(2)}_p(\Omega)$, $2 < p < 4/(2 - \beta)$.

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