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INFLUENCE OF WINKLER–STEKLOV CONDITIONS ON NATURAL OSCILLATIONS OF ELASTIC WEIGHTY BODY

S.A. NAZAROV

Abstract. We consider the spectral problem for the spacial system of equations of the elasticity theory. Small parts of the body surface are supplied with the Winkler–Steklov conditions, which model spring mount, while the remaining part of the boundary is traction-free. In several cases (the relative stiffness of springs and their positions are varied) we construct asymptotics for eigenfrequencies of the body and for corresponding eigenmodes. The limiting problems are ones for the body (spectral or stationary in some case) and problems of the elasticity theory for the half-spaces with the Winkler–Steklov conditions on flat sets (separated or joined into a single spectral theory in some cases). The discreteness of the spectrum of the problem in the half-space is ensured by a polynomial property of the system of equations of the elasticity theory. We study particular cases, formulate open questions and discuss patological situations, in which the spectrum loses usual properties. We construct asymptotic models of the problem, which provide two-terms asymptotics for the eigenpairs of the initial problem and which use the technique of self-adjoint extensions of differential operators or Hilbers spaces with separated asymptotics.

Keywords: elastic body, Winkler–Steklov conditions of spring mount, singular perturbation, asymptotics of eigenfrequencies.

Mathematics Subject Classification: 35P05, 74B05, 35J47

1. Introduction

1.1. Formulation of problem. Let Ω be a convex domain in the Euclidean space $\mathbb{R}^3$ with a smooth (of class C^∞ for simplicity; cf. Subsection 4.1) boundary $\Gamma = \partial\Omega$ and a compact closure $\overline{\Omega} = \Omega \cup \partial\Omega$. On the surface $\partial\Omega$ we choose pairwise different points $P^1, \dots, P^J$ and introduce fine sets

$$\omega_j^\varepsilon = \{x = (x_1, x_2, x_3) \in \partial\Omega : (\varepsilon^{-1} s_1^j,\, \varepsilon^{-1} s_2^j) \in \varpi_j\}, \quad j = 1, \dots, J. \tag{1.1}$$

Here $\varepsilon > 0$ is a small parameter, ϖ_j are domains in the plane $\mathbb{R}^2$ enveloped by simple smooth closed contours $\gamma_j = \partial\varpi_j$, while $x^j = \theta^j(x - P^j)$ are local Cartesian coordinates and θ^j is an orthogonal (3×3)-matrix introduced in order to direct the axis x_3^j along the outer normal $n(P^j)$ to the surface Γ at the point P^j, and the axes x_1^j and x_2^j are located in the tangential plane $\Pi^j \ni P^j$. Finally, (s_1^j, s_2^j, n^j) are curvilinear coordinates in a neighbourhood $\mathcal{V}^j \ni P^j$, n^j is an oriented distance to Γ, $n^j < 0$ in $\Omega \cap \mathcal{V}^j$, and s_i^j is an oriented distance to the point P^j measured along the projection of the axis x_i^j on Γ, $i = 1, 2$. The set of the points $P^1, \dots, P^J$ is denoted by $\mathcal{P}$.

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In the domain Ω we consider a problem of the elasticity theory:

$$L(\nabla)u(x) := D(-\nabla)^\top AD(\nabla)u(x) = \lambda\rho u(x), \quad x \in \Omega, \tag{1.2}$$

$$N(x,\nabla)u(x) := D(n(x))^\top AD(\nabla)u(x) = 0, \quad x \in \Omega \setminus \overline{\omega^\varepsilon}, \tag{1.3}$$

$$N(x,\nabla)u(x) = \lambda\rho_\varepsilon Q(x)u(x), \quad x \in \omega^\varepsilon = \omega_1^\varepsilon \cup \ldots \omega_J^\varepsilon. \tag{1.4}$$

At the same time we use a matrix[1] form of constitutive relations of the linear elasticity theory, that is, the displacement vector $u = (u_1, u_2, u_3)^\top$ is interpreted as a column in $\mathbb{R}^3$ ($\top$ is the transposition sign), $N(x,\nabla)u(x)$ is a vector of normal stresses determined by the column of stresses

$$\sigma(u) = \Big(\sigma_{11}(u), \sigma_{22}(u), \sigma_{33}(u), \sqrt{2}\sigma_{23}(u), \sqrt{2}\sigma_{31}(u), \sqrt{2}\sigma_{12}(u)\Big)^\top, \tag{1.5}$$

where $\sigma_{pq}(u)$ are the Cartesian coordinates of the stress tensor generated by the displacements u and obeying the Hooke's law

$$\sigma(u) = AD(\nabla)u,$$

$\nabla = (\partial_1, \partial_2, \partial_3)^\top$ is a gradient operator, $D(\nabla)u$ is the column of strains of the same structure as (1.5) and

$$D(\nabla)^\top = \begin{pmatrix} \partial_1 & 0 & 0 & 0 & 2^{-1/2}\partial_3 & 2^{-1/2}\partial_2 \\ 0 & \partial_2 & 0 & 2^{-1/2}\partial_3 & 0 & 2^{-1/2}\partial_1 \\ 0 & 0 & \partial_2 & 2^{-1/2}\partial_2 & 2^{-1/2}\partial_1 & 0 \end{pmatrix}, \quad \partial_p = \frac{\partial}{\partial x_p}. \tag{1.6}$$

The factors $2^{\pm 1/2}$ are introduced in formulas (1.5) and (1.6) in order to equate the natural norms of the tensor of the two rank and of the corresponding column of height six. Finally, equations (1.2) for the oscillations of the body Ω involve a symmetric positive definite (6×6)-matrix A of elastic moduli, a constant density $\rho > 0$ of a material and a spectral parameter λ, that is, the square of the oscillation frequency. Spectral conditions (1.4), called Winkler-Steklov conditions and modelling [4] dense sets of fine stiff springs, which react only to normal displacements of the surface Γ, involve an orthogonal projector in the Euclidean space $\mathbb{R}^3$

$$Q(x) = n(x)n(x)^\top \tag{1.7}$$

and the stiff compliance coefficient of the spring

$$\rho_\varepsilon = \varepsilon^\alpha \rho_0, \quad \rho_0 > 0, \quad \alpha \in \mathbb{R}. \tag{1.8}$$

In the following sections the coefficient α varies for achieving various asymptotic regimes. Finally, conditions (1.3) mean that the surface $\Gamma \setminus \overline{\omega^\varepsilon}$ is traction-free.

A variational formulation of problem (1.2)–(1.4) appeals to integral identity [5], [6]

$$E(u,\psi;\Omega) = \lambda\left(\rho(u,\psi)_\Omega + \rho_\varepsilon(u,\psi)_{\omega^\varepsilon}\right), \quad \psi \in H^1(\Omega)^3, \tag{1.9}$$

where $(\ ,\ )_\Omega$ is a natural scalar product in the Lebesgue space $L^2(\Omega)$, scalar or vector, while an eigenfunction u is sought in the Sobolev space $H^1(\Omega)^3$ and the superscript 3 indicates the number of the components of the vector but such superscript is absent in the notation of norm and scalar products. Moreover, $E(u,u;\Omega)$ is a doubled elastic energy kept by the body Ω and generating a bilinear form

$$E(u,\psi;\Omega) = (AD(\nabla)u, D(\nabla)\psi)_\Omega. \tag{1.10}$$

Owing to the Korn's inequality, see, for instance, [7],

$$\|u; H^1(\Omega)\|^2 \leqslant K(E(u,u;\Omega) + \rho\|u; L^2(\Omega)\|^2),$$

[1]In the English literature it is called Voigt-Mendel notation, while in the Russian literature it is related with the name of S.G. Lekhnitskii, see, respectively, monographs [1] and [2], [3].

in which the factor K depends on the parameters of the problem, and the bilinear form

$$\langle u, \psi\rangle = E(u, \psi; \Omega) + \rho(u, \psi)_\Omega + \rho_\varepsilon(u, \psi)_{\omega^\varepsilon} \tag{1.11}$$

can serve as the scalar product in the Sobolev space $\mathcal{H} = H^1(\Omega)^3$. We introduce also a positive symmetric continuous and hence self-adjoint operator $\mathcal{K}$ in $\mathcal{H}$ by means of the identity

$$\langle \mathcal{K}u, \psi\rangle = \rho(u, \psi)_\Omega + \rho_\varepsilon(u, \psi)_{\omega^\varepsilon}, \quad u, \psi \in \mathcal{H}. \tag{1.12}$$

This operator is compact and according to Theorems 10.1.5 and 10.2.2 [8], its essential spectrum consists of a single point $\kappa = 0$, while the discrete spectrum composes a monotone infinitesimal sequence

$$1 \geqslant \kappa_1 \geqslant \kappa_2 \geqslant \kappa_3 \geqslant \ldots \geqslant \kappa_\ell \geqslant \cdots \to +0. \tag{1.13}$$

By definitions (1.11) and (1.12) integral identity (1.9) is equivalent to an abstract equation

$$\mathcal{K}u = \kappa u \quad \text{in} \quad \mathcal{H}, \tag{1.14}$$

and the spectral parameters are related by the identity

$$\kappa = (1 + \lambda)^{-1}, \tag{1.15}$$

which transforms sequence (1.13) into a monotone unbounded sequence of eigenvalues of problem (1.2)–(1.4)

$$0 = \lambda_1 = \cdots = \lambda_6 < \lambda_7 \leqslant \lambda_8 \leqslant \ldots \leqslant \lambda_m \leqslant \cdots \to +\infty. \tag{1.16}$$

The corresponding eigenvectors $u^\varepsilon_{(1)}, \ldots u^\varepsilon_{(m)}, \dots \in \mathcal{H}$ are chosen as obeying the orthogonality and normalization conditions

$$\langle u_{(m)}, u_{(p)}\rangle = \delta_{m,p}, \quad m, p \in \mathbb{N}, \tag{1.17}$$

where $\delta_{m,p}$ is the Kronecker delta and $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the natural series.

A root subspace for $\lambda = 0$ is a six-dimensional linear space of rigid motions

$$\mathcal{R} = \{u(x) = d(x)c \,|\, c = (c_1, \ldots, c_6)^\top \in \mathbb{R}^6\}, \tag{1.18}$$

where

$$d(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 2^{-1/2}x_3 & -2-1/2x_2 \\ 0 & 1 & 0 & -2^{1/2}x_3 & 0 & 2{-}1/2x_1 \\ 0 & 0 & 1 & 2^{-1/2}x_2 & -2^{1/2}x_1 & 0 \end{pmatrix}. \tag{1.19}$$

The columns $a^t = (a_1, a_2, a_3)^\top$ and $a^r = (a_4, a_5, a_6)^\top$ correspond to translation and rotation displacements. The columns of the matrices (1.19) and (1.6) form a basis in a twelve-dimensional space of linear vector functions in $\mathbb{R}^3$.

Form (1.10) possesses polynomial property [9], that is, for each domain $\Xi \subset \mathbb{R}^3$ with a Lipschitz boundary and a compact closure we have the implication

$$u \in H^1(\Xi)^3, \quad E(u, u; \Xi) = 0 \quad \Leftrightarrow \quad u \in \mathcal{R}|_\Xi. \tag{1.20}$$

This property provides a useful information on solvability and properties of solutions of the considered problems, see survey [10].

1.2. Content of paper. In the paper we study the behavior of spectrum (1.16) as $\varepsilon \to +0$, $\rho > 0$ and as $\rho \to +0$, $\varepsilon > 0$, and the corresponding objects are equipped with superscripts ε and ρ, respectively. In Subsection 1.3 we derive a Korn's inequality, which is asymptotically sharp with respect to the mentioned parameters.

In Section 2 we construct an asymptotics for eigenpairs $\{\lambda^\varepsilon_m; u^\varepsilon_{(m)}\}$ of problem (1.9) (or (1.2)–(1.4) in the differential form), while the estimates for the asymptotics errors are provided in Section 3. These results require an additional description. Namely, for a fixed density $\rho > 0$ we consider three cases: $\alpha > -1$, $\alpha < -1$ and $\alpha = -1$. In the first case spectrum (1.16) is obtained by perturbing the spectrum of the problem in the domain Ω and here Neumann condition (1.3) are extended to the entire boundary $\partial\Omega$. In the second case positive eigenvalues of problem (1.9) become

$$\lambda^\varepsilon_{6+m} = \varepsilon^{-1-\alpha}\mu^\varepsilon_m,$$

and the sequence $\{\mu^0_m\}_{m\in\mathbb{N}}$ of the limits of factors μ^ε_m is the discrete spectrum of the family ($j = 1,\dots,J$) of problems on boundary layers near the points $P^1,\dots,P^J$. The problems consist of static (without spectral parameter) systems of differential equations in the half-space $\mathbb{R}^3_-$ with Winkler-Steklov conditions on the subdomain $\varpi_j \subset \partial\mathbb{R}^3_-$ and the Neumann conditions on the rest $\partial\mathbb{R}^3_- \setminus \overline{\varpi_j}$ of the plane. A remarkable point is that each Winkler–Steklov condition on ϖ_j becomes integro-differential and involves mean values of the eigenfunctions over the sets ϖ_k, $k = 1,\dots,J$, joining in this way the problems with superscripts $j = 1,\dots,J$ into a single spectral problem. Such far-field interaction of small singular spectral perturbations already appeared in other problems, see [11], [12] and other publications. In the special case $\alpha = 1$ the discussed interaction of the limiting problems disappears, but the sequence $\{\lambda^0_m\}_{m\in\mathbb{N}}$ of thr limits of the eigenvalues Λ^ε_m of problems (1.2)–(1.4) becomes the union of the spectra of $J+1$ problems, namely, of J copies of independent problems in the half-space $\mathbb{R}^3_-$ and of one problem in the domain Ω.

The estimates for the asymptotic errors in the obtained representations for the eigenpairs $\{\lambda^\varepsilon_m; u^\varepsilon_m\}$ are based on asymptotically shapr Korn's inequality derived in Subsection 1.3 and also on Proposition 3.1 about the convergence and classical lemma 3.1 on almost eigenvalues and eigenvectors. However, Theorems 3.1 and 3.2 concern the most representative but particular case discussed in Subsection 2.3, but their adaption to other cases, for instance, considered in Section 1 and Subsection 2.2, as well as for studying partial sums of infinite asymptotic series (cf. Subsection 4.1) is easy and rather traditional. Anyway, the justification of the asymptotics for $\alpha < -1$ or $\alpha > -1$ can be extracted from publications [13, Ch. 4] and [12].

In the final forth section we provide an accompanying information. First we discuss various generalizations like piece-wise smooth boundary, infinite series and so forth. Then we make an asymptotic analysis of the spectrum of problem (1.2)–(1.4) for an infinitesimal density of the body Ω, that is, as $\rho \to +0$. Moreover, we study a limiting case $\rho = 0$, when the spectral parameter is absent in system (1.2). A feature of such problem is that in some situations its spectrum fills entire complex plane $\mathbb{C}$ since the elements of some non-trivial subspace $\mathcal{R}_0 \subset \mathcal{R}$ satisfy relations (1.2)–(1.4) for each $\lambda \in \mathbb{C}$. Let us provide several such situations.

1^0. *Suppose that* $\Upsilon = \{x \in \Gamma : x_3 = 0\}$ *is a non-empty domain in the plane and* $\mathcal{P} \subset \Upsilon$, *that is,* $\omega^\varepsilon \subset \Upsilon$. *Then* $\mathcal{R}_0 = \{d(x)c \,|\, c_3 = c_4 = c_5 = 0\}$ *and* $\dim \mathcal{R}_0 = 3$.

2^0. *If a piece* $\Upsilon \supset \mathcal{P}$ *of the surface* Γ *is located on the sphere* $\{x : |x| = R\}$ *and* $\omega^\varepsilon \subset \Upsilon$, *then* $\mathcal{R}_0 = \{d(x)c \,|\, c_1 = c_2 = c_3 = 0\}$ *and* $\dim \mathcal{R}_0 = 3$.

3^0. *Let* Ω *be a cylinder* $\{x : x_1^2 + x_2^2 < R^2, |x_3| < L\}$. *Then* $\mathcal{R}_0 \subset \{d(x)c \,|\, c_1 = \cdots = c_5 = 0\}$, *but in the case* $\mathcal{P} \subset \{x \in \partial\Omega : |x_3| < L\}$ *(the points* $P^1,\dots,P^J$ *are located on a cylindrical surface) the dimension* $\dim \mathcal{R}_o$ *is equal to two since* $\mathcal{R}_0$ *also contains forward displacements along the axis* x_3.

In many sections of the paper we introduce a condition excluding the aforementioned pathology: a linear span $\mathcal{L}$ of columns (cf. survey [14, Sect. 2]),

$$d(P^1)n(P^1)^\top,\dots,d(P^J)n(P^J)^\top \tag{1.21}$$

has the dimension six, that is, it coincides with the space $\mathbb{R}^6$ and, in particular, $J \geqslant 6$.

Finally, in Subsection 4.3 we discuss the questions on modelling singularly perturbed problem (1.2)–(1.4). The first way is traditional and consists in constructing an appropriate self-adjoint extension $\mathfrak{S}^\varepsilon$ of a symmetric closed unbounded operator $\mathfrak{S}$ in the Hilbert space $L^2(\Omega)^3$ with the differential expression $L(\nabla_x)$ and the domain

$$\mathcal{D}(\mathfrak{S}) = \{u \in H^2(\Omega)^3 \,:\, N(x,\nabla_x)u(x) = 0,\ x \in \partial\Omega,\ u(P^j) = 0, j = 1\dots J\}. \tag{1.22}$$

Unfortunately, the characteristics of the required extension depend on the spectral parameter and this lessens an application value of the first model. The second way corresponds to a problem on the space of vector functions with detached asymptotics (singularities $O(|x-P^j|^{-1})$ at the points $P^j, j = 1,\dots,J$ are admitted) and posing at these points asymptotic conditions (algebraic relations for the coefficients of the expansions of the eigenfunctions). As it was demonstrated in [15], [16] and [17, Ch. 7], both approaches are closely related with the method of matching asymptotic expansions, see [18], [19], [13, Ch. 2] and other monographs.

1.3. Korn's inequality. We first suppose $\rho \in (0,\rho_*]$ and $\rho_* > 0$. We represent the field $u \in H^1(\Omega)^3$ in the form

$$u(x) = d(x)u^0 + u_\perp(x), \qquad \int\limits_\Omega d(x)^\top u_\perp(x)\,dx = 0 \in \mathbb{R}^6, \tag{1.23}$$

where

$$u^0 = d_\Omega^{-1}\int\limits_\Omega d(x)^\top u(x)dx \in \mathbb{R}^6, \quad d_\Omega = \int\limits_\Omega d(x)^\top d(x)dx. \tag{1.24}$$

Herewith the Gram (6×6)-matrix d_Ω is symmetric and positive definite since the columns of matrix (1.19) are linearly independent in the Lebesgue space $L^2(\Omega)^3$. In view of the last orthogonality conditions in list (1.23), the following Korn's inequality holds [7]:

$$\|u_\perp; H^1(\Omega)\|^2 \leqslant CE(u_\perp,u_\perp;\Omega) = CE(u,u;\Omega). \tag{1.25}$$

Here the factor C depends on Ω and A but, of course, is independent of ρ and ε. Moreover,

$$d_\Omega u^0 = \int\limits_\Omega d(x)^\top(u(x) - u_\perp(x))dx \quad\Rightarrow\quad \|u^0;\mathbb{R}^6\| \leqslant c(\|u;L^2(\Omega)\|^2 + \|u_\perp;L^2(\Omega)\|^2),$$

and hence,

$$\|du^0; H^1(\Omega)\|^2 \leqslant c(\|u;L^2(\Omega)\|^2 + E(u,u;\Omega)).$$

We finally get:

$$\|u;H^1(\Omega)\|^2 \leqslant c\rho^{-1}\|u;\mathcal{H}\|^2. \tag{1.26}$$

Now we consider the case $\rho = 0$ under an additional condition $\dim\mathcal{L} = 6$ for the linear span $\mathcal{L}$ of columns (1.21). To formula (1.25), we add the relation

$$\|r_j^{-1}u_\perp; L^2(\Omega)\|^2 + \varepsilon^{-1}\|u_\perp; L^2(\omega_j^\varepsilon)\|^2 \leqslant c_j\|u_\perp;H^1(\Omega)\|^2, \tag{1.27}$$

where $r_j = |x-P^j| = |x^j|$ and $j = 1,\dots,J$. An estimate for the first weight norm in the left hand side is ensured by the classical Hardy inequality

$$\int\limits_0^{+\infty} |U(r)|^2\,dr \leqslant 4\int\limits_0^{+\infty}\left|\frac{dU}{dr}(r)\right|^2 r^2dr, \qquad U \in C_c^\infty[0,+\infty), \tag{1.28}$$

applied to the product $\chi_j u_\perp$; this inequality is to be written in the spherical coordinates (r_j,φ^j) and integrated in the angular variables φ^j. Hereinafter χ_j is a smooth cut-off function with a support in the neighbourhood $\mathcal{V}^j$, equalling to one in the vicinity of the point P^j, and $\operatorname{supp}\chi_j \cap \operatorname{supp}\chi_k = \varnothing$ as $j \neq k$, see (2.5). The estimate for the second norm on the left hand side of (1.27) is obtained by means of the dilatation of the coordinates $x \mapsto \xi^J = \varepsilon^{-1}x^J$ and using a usual trace inequality, see, for instance, [5, Ch. 1].

We multiply the first inequality in (1.23) from the left by $d(x)^\top n(x)n(x)^\top$ and integrate over ω^ε. Summing up the results over $j = 1, \dots, J$, we arrive to a system of algebraic equations

$$M^\varepsilon u^0 = H^\varepsilon := \sum_{j=1}^{J} \int\limits_{\omega_j^\varepsilon} d(x)^\top n(x)n(x)^\top (u(x) - u_\perp(x))\, dx, \tag{1.29}$$

where the (6×6) matrix M^ε and the column $H^\varepsilon \in \mathbb{R}^6$ satisfy the relations

$$\begin{aligned} &M^\varepsilon = M^\varepsilon_{(1)} + \dots + M^\varepsilon_{(J)}, \quad \|M^\varepsilon_{(j)} - \varepsilon^2 M^0_{(j)}; \mathbb{R}^{6\times 6}\| \leqslant c\varepsilon^3, \\ &M^0_{(j)} = |\varpi_j| d(P^j)^\top n(P^j) n(P^j)^\top d(P^j), \quad j = 1, \dots, J, \end{aligned} \tag{1.30}$$

and

$$\|H^\varepsilon; \mathbb{R}^6\|^2 \leqslant c \sum_{j=1}^{J} |\omega_j^\varepsilon| \left(\|n^\top u; L^2(\omega_j^\varepsilon)\|^2 + \|u_\perp; L^2(\omega_j^\varepsilon)\|^2 \right), \tag{1.31}$$

while $|\omega_j^\varepsilon| = O(\varepsilon^2)$ is the area of domain (1.1). The matrix $M^0_{(1)} + \dots + M^0_{(J)}$ is symmetric and positive definite due to the restriction $\dim \mathcal{L} = 6$. Indeed, the symmetricity and positivity of the matrices M^j are obvious. Moreover, in view of the restrictions imposed for columns (1.21) we have

$$\begin{aligned} b^\top M b = 0 \quad &\Rightarrow \quad b^\top M_{(j)} b = 0, \quad j = 1, \dots, J, \\ &\Rightarrow \quad n(P^j)^\top d(P^j) b = 0, \quad j = 1, \dots, J, \quad \Rightarrow \quad b = 0 \in \mathbb{R}^6. \end{aligned}$$

Thus, from formulas (1.27)–(1.31) and (1.25) we derive the estimate

$$\begin{aligned} \|u^0; \mathbb{R}^6\|^2 &\leqslant c\varepsilon^2 (\|n^\top u; L^2(\omega^\varepsilon)\|^2 + \varepsilon \|u_\perp; H^1(\Omega)\|^2) \\ &\leqslant c\varepsilon^{-1} (E(u, u; \Omega) + \varepsilon^{-1} \|n^\top u; L^2(\omega^\varepsilon)\|^2), \end{aligned} \tag{1.32}$$

and then the Korn's inequality

$$\|u; H^1(\Omega)\|^2 \leqslant C(\|u_\perp; H^1(\Omega)\|^2 + \|u^0; \mathbb{R}^6\|^2) \leqslant C\varepsilon^{-1}(E(u, u; \Omega) + \varepsilon^{-1}\|n^\top u; L^2(\omega^\varepsilon)\|^2), \tag{1.33}$$

in which the factor C is independent of the variable $\varepsilon \in (0, \varepsilon_0]$ for some $\varepsilon_0 > 0$.

Relations (1.33) and (1.26) provide estimates for the eigenvalues of problem (1.2)–(1.4), however, in Section 2 and Subsection 4.2 we shall obtain a more detailed information on their behavior as $\varepsilon \to +0$ and $\rho \to +0$, respectively.

2. Formal asymptotics

2.1. Preliminaries. In this subsection for various values of the exponent α in representation (1.8) we construct asymptotics for the eigenpairs $\{\lambda_m^\varepsilon, u_{(m)}^\varepsilon\}$ of problem (1.2)–(1.4). In Subsections 2.2 and 2.4 the position of sets (1.1) on the boundary $\partial\Omega$ plays no role, but in Subsection 2.3 we suppose that the restriction $\dim \mathcal{L} = 6$ holds for a linear combination of columns (1.21). Moreover, to simplify the presentation in Subsection 2.4 we suppose that the parts $\Gamma^j = \partial\Omega \cap \mathcal{V}^j$ are flat (cf. Subsection 4.2). In the construction the leading asymptotic terms this assumptions plays no essential role, but for nontirival curvatures at points P^j the next-to-leading terms are to be properly interpreted, see Subsection 2.5.

2.2. Simplest case $\alpha > -1$. We suppose the following asymptotic ansätze for the eigenpairs of problem (1.2)–(1.4):

$$\lambda_m^\varepsilon = \lambda_m^0 + \varepsilon^{2+\alpha} \lambda_m' + \dots, \tag{2.1}$$

$$u_{(m)}^\varepsilon(x) = U_{(m)}^0(x) + \varepsilon^{1+\alpha} \sum_{j=1} \chi_j(x) w_{(m)}^j(\xi^j) + \varepsilon^{2+\alpha} u_{(m)}'(x) + \dots. \tag{2.2}$$

Here dots replace higher order asymptotic terms, which are not essential in our analysis, $\{\lambda_m^0, u_{(m)}^0\}$ is the eigenpair of the limiting problem

$$L(\nabla_x)u^0(x) = \lambda^0 \rho u^0(x), \quad x \in \Omega, \tag{2.3}$$

$$N(x, \nabla_x)u^0(x) = 0, \quad x \in \partial\Omega, \tag{2.4}$$

while the pair $\{\lambda_m', u_{(m)}'\}$ is to be determined together with the terms of the boundary layers $w_{(m)}^1, \dots, w_{(m)}^J$ written in terms of the rescaled variables $\xi^j = \varepsilon^{-1}x^j$. Moreover, the cut-off functions $\chi_j \in C_c^\infty(\mathcal{V}^j)$ are introduced to localize the boundary layers and

$$\chi_j = 1 \quad \text{near the point} \quad P^j \quad \text{and} \quad \operatorname{supp}\chi_j \cap \operatorname{supp}\chi_k = \varnothing \quad \text{as} \quad j \neq k. \tag{2.5}$$

Finally, the factor $\varepsilon^{1+\alpha}$ at the sum over $j = 1, \dots, J$ in (2.2) is chosen so that the change $x \mapsto \xi^j$ and the formal passage to $\varepsilon = 0$, which straighten the boundary Γ and transform the domain Ω into the half-space $\mathbb{R}_-^3 = \{\xi^j = (\xi_1^j, \xi_2^j, \xi_3^j) : \xi_3^j < 0\}$, after substitution of ansätze (2.2) and (2.1) into problem (1.2)–(1.4) and collecting coefficients at like powers of the small parameter ε give the system of differential equations

$$L^j(\nabla_{\xi^j})w_{(m)}^j(\xi^j) = 0, \quad \xi^j \in \mathbb{R}_-^3, \tag{2.6}$$

subject to the boundary conditions

$$N^j(\nabla_{\xi^j})w_{(m)}^j(\xi_\natural^j, 0), \quad \xi_\natural^j := (\xi_1^j, \xi_2^j) \in \mathbb{R}^2 \setminus \overline{\omega_j}, \tag{2.7}$$

$$N^j(\nabla_{\xi^j} w_{(m)}^j(\xi_\natural^j, 0) = g^j(\xi_\natural^j) := \lambda_m^0 \rho_0 n(P^j) n(P^j)^\top u_{(m)}^0(P^j), \quad \xi_\natural^j \in \varpi_j. \tag{2.8}$$

We stress that the right hand side of boundary condition (2.8) appeared as a result of the freezing of orthogonal projector (1.7) at the point P^j, formula (1.8) with the exponent $\alpha = -1$ and taking into consideration definition (1.1) of small sets ω_j^ε. At the same time the passage to local coordinates is accompanied by transformation of differential operators

$$\begin{aligned} &L^j(\nabla_{\xi^j}) = D^j(-\nabla_{\xi^j})^\top A D^j(\nabla_{\xi_j}), && N^j(\nabla_{\xi^j}) = D^j(e_{(3)})^\top A D^j(\nabla_{\xi^j}), \\ &D^j(\nabla_{\xi^j}) = D((\theta^j)^{-1}\nabla_{\xi^j}), && e_{(3)} = (0, 0, 1)^\top, \end{aligned} \tag{2.9}$$

but opposite to the rules of mechanics, we do not change the displacement fields. Owing to polynomial property (1.20), general results [10, Item 3, Sect. 5] and [17, Chs. 3, 6] show that problem (2.6)–(2.8) possesses a unique decaying at infinity solution

$$w_{(m)}^j(\xi^j) = X(\xi^j)\Phi^j(\xi^j)b^j + \widetilde{w}^j(\xi^j), \tag{2.10}$$

where the remainder $\widetilde{w}_{(m)}^j \in H^1(\mathbb{R}_-^3)^3$ can be estimated as

$$|\nabla_{\xi^j}^p w_{(m)}^j(\xi^j)| \leqslant c_{mp}(1 + \rho_j)^{-2-p}, \quad \rho_j > R_\varpi, \quad p \in \mathbb{N}_0 = \{0\} \cup \mathbb{N},$$

the radius R_ϖ is fixed so that $\overline{\varpi_j} \subset \mathbb{B}^2(R_\varpi) = \{\xi_\sharp^j : \rho_j < R_\varpi\}$, while the cut-off function $X \in C^\infty(\mathbb{R}^3)$ is defined by the formulas

$$X(\xi^j) = 0 \quad \text{as} \quad \rho_j \leqslant R_\varpi \quad \text{and} \quad X(\xi^j) = 1 \quad \text{as} \quad \rho_j \geqslant 2R_\varpi.$$

Moreover, the (3×3)-matrix $\Phi^j = \big(\Phi_{(1)}^j, \Phi_{(2)}^j, \Phi_{(3)}^j\big)$ is formed by the columns, which are solutions of the three-dimensional Flamant problem (forces concentrated on the boundary of the half-space) satisfying the relations

$$\Phi^j(\xi^j) = \rho_j^{-1}\Phi^j(\rho_j^{-1}\xi^j), \tag{2.11}$$

and

$$-\int\limits_{\mathbb{S}_-^2(R_\varpi)} N_\cup^j(\xi^j, \nabla_{\xi^j}\Phi^j(\xi^j)\, ds_{\xi^j} = \mathbb{I}_3, \tag{2.12}$$

where the unit $(n \times n)$-matrix $\mathbb{I}_n$ appears as well as the operator

$$N^j_{\cup}(\xi^j, \nabla_{\xi^j}) = D^j(\rho_j^{-1}\xi^j)^\top A D^j(\nabla_{\xi^j})$$

of boundary conditions on the surface of semi-sphere $\mathbb{S}^2_-(R_\varpi) = \{\xi^j : \rho_j = R_\varpi, \xi^j_3 < 0\}$. Finally, the column of coefficients $b^j \in \mathbb{R}^3$ is calculated by the formula

$$b^j = \int\limits_{\varpi_j} g^j(\xi^j_\natural) d\xi^j_\natural = \lambda^0_m \rho_0 n(P^j) n(P^j)^\top u^0_{(m)}(P^j) |\varpi_j|, \tag{2.13}$$

and, as above, $|\varpi_j|$ is the area of the domain $\varpi_j \subset \Pi^j$. Representation (2.13) is derived by means of relation (2.12) via integration by parts in the semi-ball $\{\xi^j \in \mathbb{R}^3 : \rho_j < 0\}$ and passing to the limit as $R \to +\infty$.

Let us find a smooth corrector in ansatz (2.2). We once again substitute expansions (2.2) and (2.1) into problem (1.2)–(1.4) and collect the coefficients at $\varepsilon^{2+\alpha}$ written in the coordinates x taking into account the identity $\Phi^j(\xi^j) = \varepsilon \Phi^j(x^j)$ implied by (2.11). As a result we arrive at the problem

$$L(\nabla_x) u'_{(m)}(x) - \lambda^0_m \rho u'_{(m)}(x) = \lambda'_m \rho u^0_{(m)}(x) - f'(x), \quad x \in \Omega, \tag{2.14}$$

$$N(x, \nabla_x) u'_{(m)}(x) = -g'_{(m)}(x), \quad x \in \partial\Omega, \tag{2.15}$$

in which

$$\begin{pmatrix} c f'_{(m)}(x) \\ g'_{(m)}(x) \end{pmatrix} = \sum_{j=1}^{J} \begin{pmatrix} c[L(\nabla_x), \chi_j(x)] \\ [N(x, \nabla_x), \chi_j(x)] \end{pmatrix} \Phi^j(x^j) b^j, \tag{2.16}$$

and $[\mathsf{P}, \mathsf{Q}] = \mathsf{PQ} - \mathsf{QP}$ is the commutator of the operators P and Q.

In order to determine the pair $\{\lambda'_m, u'_{(m)}\}$, we specify the information on the initial pair $\{\lambda^0_m, u^0_{(m)}\}$, namely, we suppose that $\lambda^0_m = \boldsymbol{\lambda}_q$ is an eigenvalue of problem (2.3), (2.4) with multiplicity $\varkappa_q$, that is,

$$\boldsymbol{\lambda}_{q-1} < \boldsymbol{\lambda}_q = \cdots = \boldsymbol{\lambda}_{q+\varkappa_q-1} < \boldsymbol{\lambda}_{q+\varkappa_q}. \tag{2.17}$$

We represent the vector function $u^0_{(m)}$ in the form

$$u^0_{(m)}(x) = a^m_q \mathbf{u}_{(q)}(x) + \cdots + a^m_{q+\varkappa_q-1} \mathbf{u}_{q+\varkappa_q-1}(x), \tag{2.18}$$

where the basis $\mathbf{u}_{(q)}, \ldots, \mathbf{u}_{q+\varkappa_q-1}$ in the root subspace obeys the identities

$$\rho(\mathbf{u}_{(m)}, \mathbf{u}_{(p)})_\Omega = \delta_{m,p}, \quad m, p = q, \ldots, q + \varkappa_q - 1, \tag{2.19}$$

while the columns of the coefficients $a^m = (a^m_q, \ldots, a^m_{q+\varkappa q-1})^\top \in \mathbb{R}^{\varkappa_q}$ satisfy the identities

$$(a^p)^\top a^m = \delta_{m,p}, \quad m, p = q, \ldots, q + \varkappa_q - 1. \tag{2.20}$$

In such situation problem (2.14), (2.15) has $\varkappa_q$ compatibility conditions, which are satisfied as follows:

$$\begin{aligned}
\lambda'_m a^m_k =& \lambda'_m \rho(u^0_{(m)}, \mathbf{u}_{(k)})_\Omega \\
& - \lim_{R \to +0} \int\limits_{\Omega(R)} \mathbf{u}_{(k)}(x)^\top (f'_{(m)}(x) + (L(\nabla_x) - \lambda^0_m \rho \mathbb{I}_3) u'_{(m)}(x))\, dx \\
=& \sum_{j=1}^{J} \lim_{R \to +0} \int\limits_{\Sigma^j(R)} (\mathbf{u}_{(k)}(x)^\top N_{\cup}(x^j \nabla_{x^j}) \Phi^j(x^j) \\
& \qquad - (N^j_{\cup}(x^j, \nabla_{x^j}) \mathbf{u}_{(k)}(x))^\top \Phi^j(x^j))\, ds_x b^j \\
=& - \sum_{j=1}^{J} \mathbf{u}_{(k)}(P^j)^\top b^j.
\end{aligned} \tag{2.21}$$

These calculations are to be clarified. We first calculate the scalar product of system (2.14) with $\mathbf{u}_{(k)}(x)$ and, recalling relations (2.19) and (2.20), we apply the Green's formula in the domain $\Omega(R) = \{x \in \Omega : r_j = |x^j| > R\}$. Then the remaining integrals over the spherical sets $\Sigma^j(R) = \{x \in \Omega : r_j = R\}$ of a small radius $R > 0$ are calculated in accordance with formula (2.12). Here we have taken several facts into consideration. First, the vector-function f' is smooth since the domain Ω is convex, we have $L^j(\nabla_{x^j})\Phi^j(x^j) = 0$ as $x \in \Omega \cap \mathcal{V}^j$, while the vector function g' is bounded owing to the smoothness of the surfaces $\Gamma \cap \mathcal{V}^j$, that is, the integrals converge. Second, the set $\Sigma^j(R)$ differs from the semi-sphere $\{x : r_j = R, x_3^j < 0\}$ only inside a strip of width $O(R^2)$ along the equator, which makes no influence under the passage to the limit as $R \to +0$. Finally, formulas (2.18) and (2.13) allow us to transform relations (2.21) with indices $k = q, \dots, q + \varkappa_q - 1$ into the system of algebraic equations

$$M^q a_m = \lambda'_m a^m,$$

where the entries of symmetric $(\varkappa_1 \times \varkappa_q)$-matrix M^q read as

$$M^q_{kp} = -\lambda^0_m \rho_0 \sum_{j=1}^{J} |\varpi_j| \mathbf{u}_{(k)}(P^j)^\top n(P^j) n(P^j)^\top \mathbf{u}_{(p)}(P^j)^\top. \tag{2.22}$$

This negative matrix possesses eigenvalues

$$\lambda'_q \leqslant \lambda'_{q+1} \leqslant \ldots \leqslant \lambda'_{q+\varkappa_q-1} \leqslant 0, \tag{2.23}$$

which together with corresponding eigenvectors $a^q, \dots, a^{q+\varkappa_q-1} \in \mathbb{R}^{\varkappa_q}$ obeying orthogonality and normalization conditions (2.19) and with the smooth correctors $u'_{(q)}, \dots, u'_{(q+\varkappa_q-1)}$ found from already solvable problems (2.14), (2.15) specify detached terms of asymptotic ansätze (2.1) and (2.2). However, the mentioned correctors are defined up to linear combinations of form (2.18), the coefficients $a^{m\prime}$ of which can be calculated at the next steps of asymptotic procedure (cf. Subsection 4.1).

Let us formulate estimates for errors in asymptotic representations (2.1) for the eigenvalues, which are implied by general results [13, Ch. 4, 9, 10].

Theorem 2.1. *Let $\alpha > -1$ and $\boldsymbol{\lambda}_q$ be an eigenvalue of problem* (2.3), (2.4) *in the domain Ω with multipicity $\varkappa_q$, see relation* (2.17)*, and $q > 6$. Then there exist positive ε_q and c_q such that the eigenvalues $\lambda^\varepsilon_q, \dots, \lambda^\varepsilon_{q+\varkappa_q-1}$ of problem* (1.2)–(1.4) *satisfy the inequality*

$$|\lambda^\varepsilon_m - \boldsymbol{\lambda}_q - \varepsilon^{2-\alpha}\lambda'_m| \leqslant c_q \varepsilon^{3-\alpha} \quad as \quad \varepsilon \in (0, \varepsilon_q], \tag{2.24}$$

where the eigenvalues (2.23) *of the matrix M^q with entries* (2.22) *are involved. The first six eigenvalues $\lambda^\varepsilon_1, \dots, \lambda^\varepsilon_6$ are zero.*

We observe that the number and position of sets (1.1) on the surface Γ play no role. The iteration processes developed in monograph [13] allow one to construct infinite asymptotic series for the eigenpairs $\{\lambda^\varepsilon_m, u^\varepsilon_{(m)}\}$.

2.3. Interaction of boundary layers as $\alpha < -1$. According to general results [12], for such exponent α in formula (1.8) asymptotic ansätze change essentially:

$$\lambda^\varepsilon_{6+m} = \varepsilon^{-1-\alpha}\mu_m + \dots, \tag{2.25}$$

$$u^\varepsilon_{(6+m)}(x) = d(x)c^0_{(m)} + \sum_{j=1}^{J} \chi_j(x) w^j_{(m)}(\xi^j) + \varepsilon u'_{(m)}(x) + \dots. \tag{2.26}$$

Here the number μ_m, the column $c^0 \in \mathbb{R}^6$ and the vector functions $w^1_{(m)}, \dots, w^J_{(m)}$ satisfy problems in the half-space, which consist of the differential equations (2.6), boundary conditions

(2.7) and

$$N^j(\nabla_{\xi^j})w^j_{(m)}(\xi^j_\natural,0)=\mu_m\rho_0 n(P^j)n(P^j)^\top\big(w^j_{(m)}(\xi^j_\natural;0)+d(P^j)c^0_{(m)}\big),\quad \xi^j_\natural\in\varpi_j. \tag{2.27}$$

We stress that the right hand side of boundary condition (2.27) includes a constant term $d(P^j)c^0_{(m)}$ generated by the first term in ansätze (2.26).

The solution to problem (2.6), (2.7), (2.27) admits representation (2.10), in which the column of coefficients $b^j\in\mathbb{R}^3$ looks as

$$b^j=\mu_m\rho_0 n(P^j)n(P^j)^\top\bigg(\int\limits_{\varpi_j} w^j_{(m)}(\xi^j_\natural,0)\,d\xi^j_\natural+d(P^j)C^0|\varpi_j|\bigg), \tag{2.28}$$

cf. formula (2.13). Since for $\alpha<-1$ the factor $\varepsilon^{-1-\alpha}$ at μ_m in ansatz (2.25) is small, we arrive at a problem for the smooth corrector, which is a stationary system of equations

$$L(\nabla_x)u'_{(m)}(x)=-f'_{(m)}(x),\quad x\in\Omega, \tag{2.29}$$

and we also get boundary conditions (2.15). The right hand sides $f'_{(m)}$ and $g'_{(m)}$ are calculated by formula (2.16). However, the problem itself, being free of the spectral parameter, has six solvability conditions

$$\int\limits_\Omega d(x)^\top f'_{(m)}(x)\,dx+\int\limits_{\partial\Omega} d(x)^\top g'_{(m)}(x)\,ds_x=0\in\mathbb{R}^6. \tag{2.30}$$

Reproducing, with obvious modification, calculations (2.21), we find that according to (2.28) relation (2.30) becomes an algebraic system

$$Mc^0=-\sum_{k=1}^J|\varpi_k|d(P^k)^\top n(P^k)n(P^k)^\top\overline{w}^j_{(m)}, \tag{2.31}$$

where

$$M=\sum_{j=1}^J|\varpi_j|d(P^j)^\top n(P^j)n(P^j)^\top d(P^j)\in\mathbb{R}^{6\times6},\quad \overline{w}^j=\frac{1}{|\varpi_j|}\int\limits_{\varpi_j}w^j(\xi^j_\natural,0)d\xi^j_\natural\in\mathbb{R}^3. \tag{2.32}$$

The restriction $\dim\mathcal{L}=6$ imposed for the linear span of columns (1.21) ensures the positive definiteness of this symmetric (6×6)-matrix M and hence, having solved the system of linear algebraic equations (2.31) and substituting the result into (2.27), we obtain boundary conditions

$$\begin{aligned}N^j(\nabla^j_\xi)w^j_{(m)}(\xi^j_\natural,0)&=\mu_m\rho_0 n(P^j)n(P^j)^\top\bigg(w^j_{(m)}(\xi^j_\natural,0)\\&-d(P^j)M^{-1}\sum_{k=1}^J|\varpi_k|d(P^k)^\top n(P^k)n(P^k)^\top\overline{w}^k_{(m)}\bigg),\quad \xi^j_\natural\in\varpi_j,\quad j=1,\dots,J,\end{aligned} \tag{2.33}$$

completing the set of problems (2.6), (2.7).

The presence of all means $\overline{w}^k_{(m)}$ of the vector functions $w^k_{(m)},\dots,w^J_{(m)}$ on the right hand side of (2.33) joins problems (2.6), (2.7), (2.33) into a single spectral problem, the variational formulation of which becomes the integral identity

$$\begin{aligned}\sum_{j=1}^{J} E(w^j,\psi^j;\mathbb{R}^3_-) = \mu\rho_0 \sum_{j=1}^{J}\bigg(&(n(P^j)^\top w^j, n(P^j)^\top \psi^j)_{\varpi_j} \\ &- \Big(d(P^j)^\top n(P^j) n(P^j)\overline{\psi}^j\Big)^\top \\ &\cdot M^{-1}\sum_{k=1}^{J} d(P^k)^\top n(P^k) n(P^k)^\top \overline{w}^j\bigg),\end{aligned} \tag{2.34}$$

$$\overrightarrow{\psi} = \big(\psi^1,\dots,\psi^J\big) \in \mathcal{E}\big(\mathbb{R}^3_-\big)^J.$$

At the same time, $\mathcal{E}(\mathbb{R}^3_-)$ is the space obtained by completing the linear space $C_c^\infty(\overline{\mathbb{R}^3_-})^3$ (infinitely differentiable compactly supported functions) with respect to the energy norm $E(w^j,w^j;\mathbb{R}^3_-)^{1/2}$. We stress that the Korn's inequality [7]

$$\|\nabla_{\xi^j} w^j; L^2(\mathbb{R}^3_-)\|^2 \leqslant c_A E\big(w^j,w^j;\mathbb{R}^3_-\big), \quad w^j\in C_c^\infty(\overline{\mathbb{R}^3_-})^3,$$

and a corollary of one-dimensional Hardy inequality (1.28)

$$\|\rho_j^{-1} w^j; L^2(\mathbb{R}^3_-)\|^2 \leqslant c\|\nabla_{\xi^j} w^j; L^2(\mathbb{R}^3_-)\|^2, \quad w^j\in C_c^\infty(\overline{\mathbb{R}^3_-})^3, \tag{2.35}$$

show that the left hand side of (2.34) is a scalar product in the space $\mathcal{E}(\mathbb{R}^3_-)^J$, which consists of the vectors $w\in H^1_{loc}(\mathbb{R}^3_-)^{3\times J}$ possessing finite energy norms and the weight norm from the left hand side of (2.35). Owing to the Cauchy-Bunyakowsky-Schwartz inequality, both algebraic and integral, the factor $B(w,\psi)$ at $\mu\rho_0$ in the right hand side of (2.34) satisfies the relation $B(w,w)\geqslant 0$. It remains to mention that the rigid displacements from the linear space (1.18) annulling the left hand side of (2.34) are not in the space $\mathcal{E}(\mathbb{R}^3_-)$ since the integrals over $\mathbb{R}^3_-$ diverge in the norms from (2.35).

Theorem 2.2. *Under the condition* $\dim\mathcal{L}=6$ *imposed for columns* (1.21) *problem* (2.34) *possesses a discrete spectrum, which is a monotone positive unbounded sequence*

$$0<\mu_1\leqslant\mu_2\leqslant\ldots\leqslant\mu_m\leqslant\cdots\to+\infty. \tag{2.36}$$

The associated eigenvectors $w_{(1)},w_{(2)},\dots,w_{(m)},\dots\in\mathcal{E}(\mathbb{R}^3_-)^J$ *can be chosen satisfying orthogonality and normalization conditions*

$$B(w_{(m)},w_{(p)})=\delta_{m,p},\quad m,p\in\mathbb{N}.$$

The next statement was established in paper [12].

Theorem 2.3. *Under the conditions* $\alpha<-1$ *and* $\dim\mathcal{L}=6$, *for each* $m\in\mathbb{N}$ *there exist quantities* $\varepsilon_m>0$ *and* $c_m>0$, *for which positive eigenvalues of problem* (1.2)–(1.4) *satisfy the inequalities*

$$|\lambda^\varepsilon_{6+m}-\varepsilon^{-1-\alpha}\mu_m|\leqslant c_m\varepsilon^{-\alpha}\quad \text{as}\quad \varepsilon\in(0,\varepsilon_m],$$

where μ_m *are the terms of sequence* (2.36) *of the eigenvalues of problem* (2.7).

The restriction imposed on columns (1.21) played an essential role in the presented asymptotic analysis: once this restriction is omitted, the ansätze change dramatically, see works [12] and [20]. Once the leading terms of the asymptotic are constructed, by the procedures from monograph [13] one can produce infinite series for eigenpairs of problem (1.2)–(1.4). The constructions for correctors in ansätze (2.25) and (2.26) can be extracted from Subsection 2.2, however, under a simplifying assumption on flatness of the pieces $\Gamma^j=\Gamma\cap\mathcal{V}^j$ of the boundary $\partial\Omega$ of the domain Ω (cf. Subsection 4.1).

2.4. Overlapping of spectra in limiting problems as $\alpha = -1$. Previous asymptotic ansätze require essential changes. First of all, the asymptotics expansions are to be in the powers of the parameter $\sqrt{\varepsilon}$. Thus, we seek eigenfunctions of problem (1.2)–(1.4) in the form

$$\begin{aligned} u^\varepsilon_{(m)}(x) =& u^0_{(m)}(x) + \sqrt{\varepsilon}u'_{(m)}(x) + \varepsilon u''_{(m)}(x) \\ &+ \varepsilon^{-1/2}\sum_{j=1}^{J} \chi_j(x)\left(w^j_{(m)}(\xi^j) + \sqrt{\varepsilon}w^{j\prime}_{(m)}(\xi^j) + \varepsilon w^{j\prime\prime}_{(m)}(\xi^j)\right) + \dots . \end{aligned} \tag{2.37}$$

Here the $H^1(\Omega)$-norms of the leading terms $u^0_{(m)}$ and $\varepsilon^{-1/2}\chi_j w^j_{(m)}$ become of the same order as $\varepsilon \to +0$. The leading terms of the asymptotic ansatz for the eigenvalue

$$\lambda^\varepsilon_m = \boldsymbol{\mu}^0_m + \sqrt{\varepsilon}\boldsymbol{\mu}'_m + \varepsilon\boldsymbol{\mu}''_m + \dots \tag{2.38}$$

are the terms of the common sequence

$$0 = \boldsymbol{\mu}^0_1 = \dots = \boldsymbol{\mu}^0_6 < \boldsymbol{\mu}^0_7 \leqslant \boldsymbol{\mu}^0_8 \leqslant \ldots \leqslant \boldsymbol{\mu}^0_m \leqslant \dots \to +\infty \tag{2.39}$$

of the eigenvalues of problem (2.3), (2.4) in the bounded domain Ω and a family of independent problems in the half-space $\mathbb{R}^3_-$ consisting of differential equations (2.6) as well as of boundary conditions (2.7) and

$$N^j(\nabla_{\xi^j})\mathbf{w}^j_{(m)}(\xi^j_\natural) = \mu^j\rho_0 n(P^j)n(P^j)^\top\mathbf{w}^j_{(m)}(\xi^j_\natural, 0), \quad \xi^j_\natural \in \varpi_j. \tag{2.40}$$

In comparison with Subsection 2.3 boundary condition (2.40) involves no additional constant term because of the factor $\varepsilon^{-1/2}$ at the boundary layer.

As in Theorem 2.2, problem (2.6), (2.7), (2.40) possesses a discrete spectrum $\wp^j$, which forms the sequence

$$0 < \mu^j_1 \leqslant \mu^j_2 \leqslant \ldots \leqslant \mu^j_m \leqslant \dots \to +\infty, \tag{2.41}$$

and the associated eigenfunctions $\mathbf{w}^j_{(1)}, \mathbf{w}^j_{(2)}, \dots, \mathbf{w}^j_{(m)}, \dots \in \mathcal{E}(\mathbb{R}^3_-)$ can be chosen obeying the orthogonality and normalization conditions

$$\rho_0(\mathbf{w}^j_m, \mathbf{w}^j_p)_{\varpi_j} = \delta_{m,p} \quad m, p \in \mathbb{N}. \tag{2.42}$$

We recall that the eigenfunctions (2.3), (2.4) associated with its eigenvalues

$$0 = \boldsymbol{\lambda}_1 = \dots = \boldsymbol{\lambda}_6 < \boldsymbol{\lambda}_7 \leqslant \boldsymbol{\lambda}_8 \leqslant \ldots \leqslant \boldsymbol{\lambda}_m \leqslant \dots \to +\infty \tag{2.43}$$

satisfy relations (2.19). We denote spectrum (2.43) by $\wp^0$.

We are going to show how to determine the correctors in ansätze (2.38) and (2.37). In the general situation the formulas are too cumbersome and this is why we analyse only a few representative cases.

1°. *Problem in Ω.* Let $\boldsymbol{\mu}^0_m = \boldsymbol{\lambda}_q \in \wp^0$ be a simple eigenvalue, but

$$\boldsymbol{\mu}^0_m \notin \wp^j, \quad j = 1, \dots, J. \tag{2.44}$$

Then $u^0_{(m)} = \mathbf{u}_{(q)}$ is an associated eigenfunction of problem (2.3), (2.4) normalized by formula (2.19) and

$$\boldsymbol{\mu}'_m = 0, \quad u'_{(m)} = 0, \quad w^j_{(m)} = 0, \ j = 1, \dots, J.$$

Moreover, $w^j_{(m)}$ is solution (2.10) of problem (2.6)–(2.8), in which $\lambda^0_m = \boldsymbol{\lambda}_q$, and hence, in view of the supposed unique solvability of the problem (see condition (2.44)) formula (2.13) becomes

$$b^j = \boldsymbol{\lambda}_q\rho_0 n(P^j)n(P^j)^\top\mathbf{u}_{(q)}(P^j)|\varpi_j|, \quad j = 1, \dots, J.$$

As a result, the second correctors $u''_{(m)}$ and $\boldsymbol{\mu}''_m$ are found by problem (2.14), (2.15) with right hand side (2.16). Finally, by means of simplified calculation (2.21) we find that

$$\boldsymbol{\mu}''_m = -\boldsymbol{\lambda}_q \rho_0 \sum_{j=1}^{J} |\varpi_j| \mathbf{u}_{(q)}(P^j)^\top n((P^j) n(P^j)^\top \mathbf{u}_{(q)}(P^j) \leqslant 0.$$

2°. *Problems in half-space.* Let $\boldsymbol{\mu}^0_m = \mu^j_{mg} \in \wp^j$ be simple eigenvalues as $j = 1, \dots, K$, but $\boldsymbol{\mu}^0_m \notin \wp^0$ and $\boldsymbol{\mu}^0_m \notin \wp^j$ as $j = 1 + K, \dots, J$. Then

$$w^j_{(m)} = a^m_j \mathbf{w}^j_{(m(j))}, \quad j = 1, \dots, K, \tag{2.45}$$

where the eigenfunctions $\mathbf{w}^j_{(m(j))}$ of problems (2.6), (2.7), (2.27) and the columns $a^m = (a^m_1, \dots, a^m_k)^\top$ obey orthonormality conditions (2.42) and (2.20), respectively. Moreover,

$$w^j_{(m)} = 0, \quad j = 1 + K, \dots, J, \quad u^0_{(m)} = 0, \quad \boldsymbol{\mu}'_m = 0,$$

and $u'_{(m)}$ is a solution of problem (2.14), (2.15), in which $\lambda^0_m = \boldsymbol{\mu}^0_m$, $\lambda'_m = 0$ and right hand sides (2.16) involve the coefficients

$$\begin{aligned} &b^j_{(m)} = \mathbf{m}_j n(P^j) a^m_j, \quad \mathbf{m}_j = \boldsymbol{\mu}^0_m \rho_0 n(P^j)^\top |\varpi_j| \overline{\mathbf{w}}^j_{(m(j))}, \quad j = 1, \dots, K, \\ &b^j_{(m)} = 0, \quad j = 1 + K, \dots, J. \end{aligned} \tag{2.46}$$

Owing to the assumption $\boldsymbol{\mu}^0_m \notin \wp^0$ the formed problem in the domain Ω is uniquely solvable and its solution is represented as

$$u'_{(m)}(x) = \sum_{k=1}^{K} \widehat{G}^k(x) b^k, \tag{2.47}$$

where $\widehat{G}^j$ is the regular part of the Green's tensor G^j (matrix functions of size 3×3) with a singularity at the point P^j, that is, the solution of homogeneous ($f'_{(m)} = 0$ and $g'_{(m)} = 0$) problem (2.14), (2.15) in the domain Ω, which admits the representation

$$G^j(x) = \chi_j(x) \Phi^j(x^j) + \widehat{G}^j(x). \tag{2.48}$$

Substituting the matrices G^j and G^k into the Green's formula on the domain $\Omega(R)$ and passing to the limit as $R \to +0$ (cf. calculation (2.21)), we see that $\widehat{G}^j(P^k) = \widehat{G}^k(P^j)$ and hence, the $(K \times K)$-matrix

$$\mathbf{G} = \left(\mathbf{G}_{jk}\right)^K_{j,k=1} = (n(P^j)^\top \widehat{G}^k(P^j) n(P^k))^K_{j,k=1} \tag{2.49}$$

is symmetric.

Thus, by formulas (2.47) and (2.46) we obtain that

$$n(P^j)^\top u'_{(m)}(P^j) = \sum_{k=1}^{k} \mathbf{m}_k \mathbf{G}_{jk} a^m_k, \tag{2.50}$$

and therefore, the number $\boldsymbol{\mu}''_m$ and the vector function $w^{j\prime\prime}_{(m(j))}$ from ansätze (2.38) and (2.37), respectively, satisfy system of differential equations (2.6), as well as boundary conditions (2.7) and

$$\begin{aligned} N^j(\nabla_{\xi^j} w^{j\prime\prime}_{(m)}(\xi^j_\natural, 0) = {}&\rho_0 n(P^j) n(P^j)^\top \big(\boldsymbol{\mu}^0_m w^{j\prime\prime}_{(m)}(\xi^j_\natural, 0) \\ &+ \boldsymbol{\mu}''_m \mathbf{w}^j_{(m(j))}(\xi^j_\natural, 0) a^m_j + \boldsymbol{\mu}^0_m u\prime_{(m)}(P^j)\big), \quad \xi^j_\natural \in \varpi_j. \end{aligned}$$

Since $\boldsymbol{\mu}_m^0$ is a simple eigenvalue of problems (2.6), (2.7), (2.40) as $j = 1, \ldots, K$, each of the obtained problems for $w_{(m)}^{j\prime\prime}$ possesses only one compatibility conditions, and according to normalization (2.42) and formulas (2.50) and (2.46) we can express this condition in the form

$$\begin{aligned} \boldsymbol{\mu}_m'' a_j^m &= \boldsymbol{\mu}_m'' \rho_0 \| n(P^j)^\top \mathbf{w}_{(m(j))}^j ; L^2(\varpi_j) \|^2 a_j^m \\ &= -\boldsymbol{\mu}_m^0 |\varpi_j| \big(n(P^j)^\top \overline{w}_{(m(j))}^j \big)^\top n(P^j)^\top u_{(m)}'(P^j) \\ &= -\sum_{k=1}^K \mathbf{m}_j \mathbf{G}_{jk} \mathbf{m}_k a_k^m =: \sum_{k=1}^K M_{jk} a_k^m . \end{aligned} \tag{2.51}$$

Thus, the eigenvalues of (6×6)-matrix M with entries defined in (2.51) provide the second correctors in ansatz (2.38) for the eigenvalues, and the associated eigenvectors $a^m \in \mathbb{R}^K$ specify leading terms (2.45) in ansatz (2.37) for the eigenfunctions of problem (1.2)–(1.4).

3°. *Common eigenvalue of problems in* Ω *and* $\mathbb{R}_-^3$. Let $\boldsymbol{\mu}_m^0$ be a simple eigenvalue of problem (2.3), (2.4) in the domain Ω and of problem (2.6), (2.7), (2.40) in the half-space $\mathbb{R}_-^3$ with $j = 1, \ldots, K$, while the associated eigenfunctions $\mathbf{u}_{(m(0))}$ and $\mathbf{w}_{(m(1))}^1, \ldots, \mathbf{w}_{(m(k))}^k$ are orthonormalized in accordance with formulas (2.19) and (2.42).

In ansatz (2.37) we take

$$u_{(m)}^0 = a_0^m \mathbf{u}_{(m(0))}, \quad w_{(m)}^j = a_j^m \mathbf{w}_{(m(j))}^j, \ j = 1, \ldots, K, \quad \sum_{p=0}^K |a_p^m|^2 = 1, \tag{2.52}$$

while the columns $a^m = (a_0^m, \ldots, a_K^m)^\top$ obey orthogonality and normalization conditions similar to (2.20). As a result we find that the corrector $u_{(m)}'$ is to be sought by the system of differential equations

$$L(\nabla_x) u_{(m)}'(x) - \boldsymbol{\mu}_m^0 \rho u_{(m)}(x) = \boldsymbol{\mu}_m' \rho u_{(m)}^0(x) - f_{(m)}'(x), \quad x \in \Omega,$$

with boundary conditions (2.15), and in formula (2.16) for the right hands the summation is made over $j = 1, \ldots, K$, while the columns of the coefficients are given by formulas (2.46). Moreover, the correctors of boundary layer type satisfy system of equations (2.6) in the half-space $\mathbb{R}_-^3$ and boundary conditions (2.7) and

$$\begin{aligned} N^j(\nabla_{\xi^j}) w_{(m)}^{j\prime}(\xi_\natural^j, 0) =& \xi_0 n(P^j) n(P^j)^\top \big(\boldsymbol{\mu}_m^0 w_{(m)}^{j\prime}(\xi_\natural^j, 0) \\ &+ \boldsymbol{\mu}_m' w_{(m)}^j(\xi_\natural^j, 0) + \boldsymbol{\mu}_m^0 u_{(m)}^0(P^j) \big), \quad \xi_\natural^j \in \varpi_j . \end{aligned}$$

Now the compatibility conditions of the formed boundary value problems, in view of relations (2.52), become a system of algebraic equations for the column a^m:

$$M a^m = \mu_m' a^m \in \mathbb{R}^{1+K} \tag{2.53}$$

with a symmetric $((1+K) \times (1+K))$-matrix M, the first (for $j = 0$) row of which casts into the form

$$\big(0, -\mathbf{m}_1 n(P^1)^\top \mathbf{u}_{(m(0))}(P^1), \ldots, -\mathbf{m}_k n(P^k)^\top \mathbf{u}_{(m(0))}(P^k) \big) ,$$

while other rows with indices $j = 1, \ldots, K$ become

$$\big(-\mathbf{m}_j n(P^j)^\top \mathbf{u}_{(m(j))}(P^j), 0, \ldots, 0 \big) .$$

Simple calculations show that such matrix possesses a zero eigenvalue of multiplicity $K - 1$ and also extra two eigenvalues read as

$$\boldsymbol{\mu}_{m\pm}' = \pm \rho_0 \boldsymbol{\mu}_m^0 \Big(\sum_{j=1}^K |\varpi_j|^2 |n(P^j)^\top \overline{\mathbf{w}}_{(m(j))}^j|^2 |n(P^j)^\top \mathbf{u}_{(m(0))}(P^j)|^2 \Big)^{1/2} .$$

Thus, we have calculated the leading terms in ansatz (2.38) for the eigenvalues of problem (1.2)–(1.4). Eigencolumns of algebraic system (2.53) specify leading terms (2.37) of the ansatz for the eigenfunctions. The obtained formulas demonstrate that in the considered case the asymptotics expansions are indeed in the powers of the small parameter $\sqrt{\varepsilon}$.

2.5. Conclusion. The asymptotic procedures described for three particular situations can be easily adapted to other situations, in particular, for multiple eigenvalues of problems (2.6), (2.7), (2.40). The flatness requirements of the pieces $\Gamma_1, \dots, \Gamma_J$ was needed only in case 3° in Subsection 2.4, in which we had to construct a couple of terms of smooth type and boundary layer type. In the cases, when the leading terms of the asymptotics were defined at the first step, the dilatation of the coordinates

$$x \;\mapsto\; \xi^j = \varepsilon^{-1}(s_1^j, s_2^j, n^j)$$

straightens the boundary, while the variable coefficients of the transformed differential operators (2.9) appear only in lower asymptotic terms. This issue will be commented on in Subsection 4.1.

3. Justification of asymptotics

3.1. Preliminaries. As it has been mentioned, the justification of asymptotic expansions for the eigenpairs of problem (1.2)–(1.4) constructed in Subsections 2.1 and 2.2 is ensured by general results [13, Chs. 4, 10] and [12] (see also [20] for a similar problem of the elasticity theory). The developed schemes can also adapted to the situation $\alpha = -1$ considered in Subsection 2.3, but for the completeness of picture, in this section we provide the justification of the asymptotic construction, however, not completely but only for the leading terms since the correctors in ansätze (2.38) and (2.39) were constructed in Section 2 only under certain restrictions.

3.2. Convergence theorem. Let $u^\varepsilon_{(m)}$ be an eigenfunction of variational problem (1.9) associated with the eigenvalue

$$\lambda^\varepsilon_m \leqslant C_m \tag{3.1}$$

and normalized in accordance with formula (1.17), where $\langle\, ,\, \rangle$ is scalar product (1.11), in which $\rho_\varepsilon = \varepsilon^{-1}\rho_0$, and $\rho > 0$ and $\rho_0 > 0$ are fixed numbers. Then by Korn's inequality [7]

$$\|u^\varepsilon_{(m)}; H^1(\Omega)\|^2 \leqslant c\left(E(u^\varepsilon_{(m)}, u^\varepsilon_{(m)}; \Omega) + \rho\|u^\varepsilon_{(m)}; L^2(\Omega)\|^2\right)$$

there exists a positive infinitesimal sequence $\{\varepsilon_l\}_{l\in\mathbb{N}}$, along which the convergences hold

$$\lambda^{\varepsilon l}_m \to \boldsymbol{\lambda}^0_m, \tag{3.2}$$

$$u^{\varepsilon l}_{(m)} \to \mathbf{u}^0_{(m)} \quad \text{weakly in} \quad H^1(\Omega)^3 \quad \text{and strongly in} \quad L^2(\Omega)^3. \tag{3.3}$$

Relation (3.1) will be confirmed in Remark 3.1. We rewrite the vector function $u^\varepsilon_{(m)}$ in local curvilinear coordinates (see Subsection 1.1) and let

$$w^{j\varepsilon}_{(m)}(\xi^j) = \varepsilon^{1/2}\chi_j(x)u^\varepsilon_{(m)}(s^j, n^j), \tag{3.4}$$

where $\xi^j_i = \varepsilon^{-1}s^j_i$, $i = 1, 2$, and $\xi^j_3 = \varepsilon^{-1}n^j$. We have

$$\begin{aligned}\|w^\varepsilon_{(m)}; \mathcal{E}(\mathbb{R}^3_-)\|^2 &\leqslant c\varepsilon \int\limits_{\mathbb{R}^3_-} \left(|\nabla_{\xi^i}(\chi_j u^\varepsilon_{(m)})|^2 + (1+\rho_j)^{-2}|\chi_j u^\varepsilon_{(m)}|^2\right) d\xi^j \\ &\leqslant c\varepsilon \int\limits_{\Omega\cap\mathcal{V}^j} \left(\varepsilon^2|\nabla_x u^\varepsilon_{(m)}(x)|^2 + (1+\varepsilon^{-1}r_j)^{-2}|u^\varepsilon_{(m)}(x)|^2\right)\varepsilon^{-3}\, dx \leqslant c\|u^\varepsilon_{(m)}; H^1(\Omega)\|^2.\end{aligned} \tag{3.5}$$

We also clarify that while passing from the curvilinear coordinates ξ^j to initial Cartesian coordinates x, we have used the relations

$$\begin{aligned}&\nabla_{\xi^i} = T^\varepsilon(x)\nabla_x, \qquad d\xi^i = \varepsilon^{-3}t^\varepsilon(x)\,dx,\\ &\|T^\varepsilon_{(x)} - \theta^j; \mathbb{R}^{3\times 3}\| + |t^\varepsilon(x) - 1| \leqslant cr_j, \qquad x \in \Omega \cap \mathcal{V}^j,\end{aligned} \tag{3.6}$$

while in the latter estimate in (3.5) we have applied Corollary (1.27) of Hardy inequality (1.28).

Thus, along a subsequence (we keep the notation $\{\varepsilon_\ell\}$) the convergence holds

$$w^{j\varepsilon_\ell}_{(m)} \to \mathbf{w}^{j0}_{(m)} \qquad \text{weakly in} \quad \mathcal{E}(\mathbb{R}^3_-) \quad \text{and strongly in} \quad L^2(\varpi_j)^3. \tag{3.7}$$

Hereafter for the sake of brevity we omit the subscript ℓ for the symbol ε_ℓ.

We substitute a test vector function $\psi^0 \in C^\infty_c(\overline{\Omega} \setminus \mathcal{P})^3$ into integral identity (1.9). We note that $\psi = 0$ on the set ω^ε for small $\varepsilon > 0$ and by relations (3.2) and (3.3) we arrive at the formula

$$0 = E(u^\varepsilon_{(m)}, \psi; \Omega) - \lambda^\varepsilon_m \rho(u^\varepsilon_{(m)}, \psi)_\Omega \quad \to \quad E(\mathbf{u}^0_{(m)}, \psi; \Omega) - \boldsymbol{\lambda}^0_m \rho(\mathbf{u}^0_{(m)}, \psi)_\Omega = 0.$$

Since the subspace $C^\infty_c(\overline{\Omega} \setminus \mathcal{P})$ is dense in $H^1(\Omega)$, we obtain the integral identity

$$E(\mathbf{u}^0_{(m)}, \psi^0; \Omega) = \boldsymbol{\lambda}^0_m \rho(\mathbf{u}^0_{(m)}, \psi^0)_\Omega, \qquad \psi \in H^1(\Omega)^3, \tag{3.8}$$

which serves for spectral problem (2.3), (2.4).

As $j = 1, \dots, J$, similarly to formula (3.4) for $\psi^j \in C^\infty_c(\overline{\mathbb{R}}{}^3_-)^3$ we let

$$\phi^{j\varepsilon}(x) = \varepsilon^{-1/2}\psi^j(\xi^j).$$

We have

$$\lambda^\varepsilon_m \rho|(u^\varepsilon_{(m)}, \phi^{j\varepsilon})_\Omega| \leqslant C_m \|u^\varepsilon_{(m)}; L^2(\Omega)\| \varepsilon^{3/2} \|\varepsilon^{-1/2}\psi^j; L^2(\mathbb{R}^3_-)\| \leqslant c_m \varepsilon \left(\psi^j\right)$$

and

$$\begin{aligned}&E(u^\varepsilon_{(m)}, \phi^{j\varepsilon}; \Omega) \ \to \ E(\mathbf{w}^j_{(m)}, \psi^j; \mathbb{R}^3_-),\\ &\lambda^\varepsilon_m \rho_\varepsilon(u^\varepsilon_{(m)}, \phi^{j\varepsilon})_{\omega^\varepsilon_j} = \lambda^\varepsilon_m \varepsilon^{-1}\rho_0(\varepsilon^{-1/2}w^{j\varepsilon}_{(m)}, \varepsilon^{-1/2}\psi^j)_{\omega^\varepsilon_j} \ \to \ \boldsymbol{\lambda}^0_m \rho_0(\mathbf{w}^j_{(m)}, \psi^j)_{\varpi_j}.\end{aligned}$$

Therefore, passing to the limit as $\varepsilon \to +0$ in equation (1.9) with the mentioned ingredients, we get the integral identity

$$E(\mathbf{w}^j_{(m)}, \psi^j; \mathbb{R}^3_-) = \boldsymbol{\lambda}^0_m \rho_0(\mathbf{w}^j_{(m)}, \psi^j)_{\varpi_j}, \quad \psi^j \in \mathcal{E}(\mathbb{R}^3_-), \tag{3.9}$$

that is, a variational formulation of problem (2.6), (2.7), (2.40).

Proposition 3.1. *If $\alpha = -1$ in formula* (1.8), *then passage to limit* (3.2) *gives an eigenvalue $\boldsymbol{\lambda}^0_m$ of one of problems* (3.8) *and* (3.9), *$j = 1, \dots, J$, while passages to limit* (3.3) *and* (3.7) *give the vector functions $\mathbf{u}^0_{(m)} \in H^1(\Omega)^3$ and $\mathbf{w}^{10}_{(m)}, \dots \mathbf{w}^{J0}_{(m)} \in \mathcal{E}(\mathbb{R}^3_-)$ satisfying the mentioned problems and obeying the relationship*

$$\rho\|\mathbf{u}^0_{(m)}; L^2(\Omega)\|^2 + \rho_0 \sum_{j=1}^J \|\mathbf{w}^{j0}_{(m)}; L^2(\varpi_j)\|^2 = (1 + \boldsymbol{\lambda}^0_m)^{-1}. \tag{3.10}$$

Proof. It remains to confirm relationship (3.10), which in particular means that at least one of the aforementioned vector functions does not vanish, that is, $\boldsymbol{\lambda}^0_m$ is indeed an eigenvalue. According to formulas (1.17) and (1.11), (1.9) we have

$$\begin{aligned}1 = \langle u^\varepsilon_{(m)}, u^\varepsilon_{(m)} \rangle_\varepsilon = (\lambda^\varepsilon_m + 1)\Bigg(\rho\|u^\varepsilon_{(m)}; L^2(\Omega)\| + \varepsilon^{-1}\rho_0 \sum_{j=1}^J \|n(P^j)^\top u^\varepsilon; L^\varepsilon((\omega^\varepsilon_j)\|^2\Bigg)\\ \to \ (\boldsymbol{\lambda}^0_m + 1)\Bigg(\rho\|\mathbf{u}^0_{(m)}; L^2(\Omega)\|^2 + \rho_0 \sum_{j=1}^J \|\mathbf{w}^{j0}_{(m)}; L^2(\varpi_j)\|^2\Bigg).\end{aligned}$$

The proof is complete. □

3.3. Almost eigenvalues and eigenvectors. We reformulate problem (1.9) as abstract equation (1.14). The next statement, known as the lemma on almost eigenvalues and eigenvectors, is implied by the spectral decomposition of the resolvent (see primary source [21] and, for instance, book [8, Ch. 6]).

Lemma 3.1. *Let $U \in \mathcal{H}$ and $\Lambda \in \mathbb{R}_+$ be such that*

$$\|U; \mathcal{H}\| = 1, \quad \|\mathcal{K}U - \Lambda U; \mathcal{H}\| =: \delta \in (0, \Lambda). \tag{3.11}$$

Then the operator $\mathcal{K}$ possesses an eigenvalue κ_n in the segment $[\Lambda - \delta, \Lambda + \delta]$. Moreover, for each $\delta_* \in (\delta, \Lambda)$ there exist coefficients $C_{\mathcal{N}}, \dots, C_{\mathcal{N}+\mathcal{X}-1}$, for which the formulas

$$\|U - \sum_{k=\mathcal{N}}^{\mathcal{N}+\mathcal{X}-1} C_k u_{(k)}; \mathcal{H}\| \leqslant 2\frac{\delta}{\delta_*}, \qquad \sum_{k=\mathcal{N}}^{\mathcal{N}+\mathcal{X}-1} |C_k|^2 = 1 \tag{3.12}$$

hold, where $u_{(\mathcal{N})}, \dots, u_{(\mathcal{N}+\mathcal{X}-1)}$ is the set of all eigenvectors of the operator $\mathcal{K}$ associated with the eigenvalues in the segment $[\Lambda - \delta_*, \Lambda + \delta_*]$ and obeying orthogonality and normalization conditions (1.17).

Let $\boldsymbol{\mu}_q^0 > 0$ be an eigenvalue in the joint sequence (2.39) of multiplicity $\varkappa_q$, that is,

$$\boldsymbol{\mu}_{q-1}^0 < \boldsymbol{\mu}_q^0 = \dots = \boldsymbol{\mu}_{q+\varkappa_q-1}^0 < \boldsymbol{\mu}_{q+\varkappa_q}^0, \tag{3.13}$$

and

$$\boldsymbol{\mu}_q^0 = \dots = \boldsymbol{\mu}_{q+\varkappa_q^0-1}^0$$

is an eigenvalue of problem (2.3), (2.4), and $\varkappa_q^0 \geqslant 0$ is its multiplicity (the case $\boldsymbol{\mu}_q^0 \notin \wp^0$ and $\varkappa_q^0 = 0$ is not excluded). Moreover, $\boldsymbol{\mu}_l^0 = \mu_{m(\ell)}^{j(\ell)}$ is an eigenvalue of problem (2.6), (2.7), (2.40) with an index $j(\ell) \in \{1, \dots, J\}$. The corresponding eigenfunctions $\mathbf{u}_{(m)}$ and $\mathbf{w}_{m(\ell)}^{j(\ell)}$ obey orthogonality and normalization conditions (2.19) and (2.42), respectively. Moreover, $m(\ell) \neq m(k)$ as $j(\ell) = j(k)$ and $\ell \neq k$.

Relation (1.15) of spectral parameters hints that as almost eigenvalues of the operator $\mathcal{K}^\varepsilon$ one should take $\varkappa_q$ copies of the quantity

$$\Lambda_\ell^\varepsilon = (1 + \boldsymbol{\mu}_q^0)^{-1}, \quad \ell = q, \dots, q + \varkappa_q - 1. \tag{3.14}$$

In accordance with ansatz (2.37) almost eigenvectors

$$U_{(\ell)}^\varepsilon(x) = \|W_{(\ell)}^\varepsilon; \mathcal{H}^\varepsilon\|^{-1} W_{(\ell)}^\varepsilon(x), \tag{3.15}$$

obeying the first relation in (3.11), involve the vector functions

$$W_{(\ell)}^\varepsilon(x) = \mathbf{u}_{(\ell)}(x), \quad \ell = q, \dots, q + \varkappa_q^0 - 1, \tag{3.16}$$

$$W_{(\ell)}^\varepsilon(x) = \chi_{j(\ell)}(x)\varepsilon^{-1/2}\mathbf{w}_{(m(\ell))}^{j(\ell)}(\xi^{j(\ell)}), \quad \ell = q + \varkappa_q^0, \dots, q + \varkappa_q - 1. \tag{3.17}$$

Lemma 3.2. *Under the aforementioned condtions* (3.16) *and* (3.17) *the formulas*

$$\left|\langle W_{(\ell)}^\varepsilon, W_{(k)}^\varepsilon\rangle_\varepsilon - \delta_{\ell,k}(1 + \boldsymbol{\mu}_q)\right| \leqslant c_q\varepsilon \quad as \quad \varepsilon \in (0, \varepsilon_q], \quad \ell, k = q, \dots, q + \varkappa_q - 1 \tag{3.18}$$

hold, where $\varepsilon_q > 0$ and c_q are some numbers.

Proof. As $\ell, k = q, \dots, q + \varkappa_q^0 - 1$, inequality (3.18) is implied by relations (3.8), (2.22) and the estimate

$$\rho_\varepsilon \left|(\mathbf{u}_{(\ell)}, \mathbf{u}_{(k)})_{\omega^\varepsilon}\right| \leqslant \mathbf{c}_{(\ell k)}\varepsilon^{-1}\rho_0|\omega_\varepsilon| \leqslant c_q\varepsilon,$$

which is obvious for smooth vector functions $\mathbf{u}_{(l)}$ and $\mathbf{u}_{(k)}$.

Now let $\ell, k = q+\varkappa_q^0, \dots, q+\varkappa_q-1$. If $j(\ell) \neq j(k)$, then the supports of the vector functions $W^\varepsilon_{(\ell)}$ and $W^\varepsilon_{(k)}$ do not intersect and formula (3.18) is true even for $c_q = 0$. In the case $j(\ell) = j(k)$, a transformation similar to calculations (3.5) in view of (3.6) shows that

$$\begin{aligned}\langle W^\varepsilon_{(\ell)}, W^\varepsilon_{(k)}\rangle_\varepsilon =&E(\mathbf{w}^{j(\ell)}_{(m(\ell))}, \mathbf{w}^{j(k)}_{(m(k))}; \mathbb{R}^3_-) + \rho_0(\mathbf{w}^{j(\ell)}_{(m(\ell))}, \mathbf{w}^{j(k)}_{(m(k))})_{\varpi_j} + O(\varepsilon)\\ =&(\boldsymbol{\mu}^0_q + 1)\rho_0\big(\mathbf{w}^{j(\ell)}_{(m(\ell))}, \mathbf{w}^{j(k)}_{(m(k))}\big)_{\varpi_j} + O(\varepsilon)\\ =&(1+\boldsymbol{\mu}^0_q)\delta_{\ell,k} + O(\varepsilon).\end{aligned}$$

Here we have taken into consideration formulas (2.42) and (3.9). Finally, for $l = q, \dots, q+\varkappa_q^0-1$ and $k = q+\varkappa_q^0, \dots, q+\varkappa_q-1$ estimate (3.18) can be obtained easily:

$$\begin{aligned}\big|\langle W^\varepsilon_{(\ell)}, W^\varepsilon_{(k)}\rangle_\varepsilon\big| =& (1+\boldsymbol{\mu}^0_q)\rho\left(W^\varepsilon_{(\ell)}, W^\varepsilon_{(k)}\right)_\Omega + \varepsilon^{-1}\rho_0\left(W^\varepsilon_{(\ell)}, W^\varepsilon_{(k)}\right)_{\omega^\varepsilon_{j(k)}}\\ \leqslant& c\Big(\int\limits_{\Omega\cap\mathcal{V}^j}(1+\varepsilon^{-1}r_j)^{-1}\,dx + \varepsilon^{-1}\int\limits_{\omega^\varepsilon_j(k)} ds_x\Big) \leqslant c\varepsilon.\end{aligned}$$

The proof is complete. □

3.4. Treating of discrepancies. We are going to estimate the quantity δ^ε_ℓ found in accordance with the second formula in (3.11) by almost eigenvalue (3.14) and eigenvector (3.15). We have

$$\begin{aligned}\delta^\varepsilon_\ell =&\sup\big|\langle \mathcal{K}^\varepsilon U^\varepsilon_{(\ell)} - \Lambda^\varepsilon_\ell U^\varepsilon_{(\ell)}, V\rangle_\varepsilon\big|\\ =&(1+\boldsymbol{\mu}_q)^{-1}\|W^\varepsilon_{(\ell)}; \mathcal{H}^\varepsilon\|^{-1}\sup\big|E(W^\varepsilon_{(l)}, V; \Omega)\\ &-\boldsymbol{\mu}_q\big(\rho(W^\varepsilon_{(\ell)}), V\big)_\Omega + \varepsilon^{-1}\rho_0\big(n^\top W^\varepsilon_{(\ell)}, n^\top V\big)_{\omega^\varepsilon}\big|,\end{aligned} \tag{3.19}$$

where the supremum is calculated over the unit ball in the space $\mathcal{H}^\varepsilon$, that is, $\|V; \mathcal{H}^\varepsilon\| \leqslant 1$ and by formulas (1.11), (1.8), $\alpha = -1$, and (1.26), (1.32) the ingredients V^0 and $V_\perp$ in representation (1.23) for the test function V admit the estimates

$$\|V^0; \mathbb{R}^6\| \leqslant c\varepsilon^{-1/2} \quad \text{as} \quad \|V_\perp; H^1(\Omega)\| \leqslant c.$$

As $\ell = q, \dots, q+\varkappa_q^0-1$, we consider the expression $I^\varepsilon_\ell(V)$ between the last modulus signs in chain (3.19). Since $\{\boldsymbol{\mu}_q, \mathbf{u}_{(\ell)}\}$ is an eigenpair of problem (3.8), by means of weighted inequality (1.27) we derive the estimate

$$\begin{aligned}\big|I^\varepsilon_\ell(V)\big| =& \varepsilon^{-1}\rho_0|(n^\top\mathbf{u}_{(\ell)}, n^\top(dV^0 + V_\perp))_{\omega^\varepsilon}|\\ \leqslant& c_\ell\varepsilon^{-1}|\omega^\varepsilon|^{1/2}\left(|\omega^\varepsilon|\,\|V^0; \mathbb{R}^6\|^2 + \|V_\perp; L^2(\omega^\varepsilon)\|^2\right)^{1/2} \leqslant C_\ell\sqrt{\varepsilon}.\end{aligned}$$

Now let $\ell = q+\varkappa_q^0, \dots, q+\varkappa_q-1$. We make a usual procedure, namely, we pass to rescaled curvilinear variables with relations (3.6) taken into consideration, use integral identity (3.9) for the pair $\{\boldsymbol{\mu}_q, \mathbf{w}^{j(\ell)}_{(m(\ell))}\}$, remove the cut-off function $\chi_{j(\ell)}$ due to the decay of the eigenfunction and finally, we calculate

$$\begin{aligned}\varepsilon^{1/2}\rho\left|(\chi_{j(\ell)}\mathbf{w}^{j(\ell)}_{(m(\ell))}, V)_\Omega\right| \leqslant& c\varepsilon^{1/2}\big\|(\varepsilon + r_{j(\ell)})^{-1}\mathbf{w}^{j(\ell)}_{(m(\ell))}; L^2(\Omega\cap\mathcal{V}^{j(\ell)})\big\|\,\|(\varepsilon + r_{j(\ell)})V; L^2(\Omega)\|\\ \leqslant& c_\ell\varepsilon^{1/2}\varepsilon^{-1/2}\varepsilon^{3/2}\big\|(1+\rho_{j(\ell)})^{-1}\mathbf{w}^{j(\ell)}_{(m(\ell))}; L^2(\mathbb{R}^3_-)\big\|\\ &\cdot\big(\|V^0; \mathbb{R}^6\| + \|V_\perp; L^2(\Omega)\|\big)\\ \leqslant& C_\ell\sqrt{\varepsilon}.\end{aligned}$$

As a result we obtain the estimate

$$|I^\varepsilon_\ell(V)| \leqslant C_\ell\sqrt{\varepsilon}.$$

Moreover, Lemma 3.1 in particular means that $\|W^\varepsilon_{(\ell)};\mathcal{H}^\varepsilon\| \geqslant (1+\boldsymbol{\mu}_q)/2$ for small ε and therefore,

$$\delta^\varepsilon_\ell \leqslant c_q\sqrt{\varepsilon}, \quad \ell = q,\dots,q+\varkappa_q-1.$$

Thus, according to Lemma 3.1 we find the eigenvalues $\kappa^\varepsilon_{n(q)},\dots,\kappa^\varepsilon_{n(q+\varkappa_q-1)}$ of the operator $\mathcal{K}^\varepsilon$, for which the inequalities

$$|\kappa^\varepsilon_{n(\ell)} - (1+\boldsymbol{\mu}_q)^{-1}| \leqslant c_q\sqrt{\varepsilon}, \qquad \ell = q,\dots,q+\varkappa_q-1, \tag{3.20}$$

hold true.

3.5. Theorem on asymptotics of eigenvalues. We complete the above calculations by the following statement.

Theorem 3.1. *Let* $\alpha = -1$. *Positive terms in sequence* (1.16) *of eigenvalues of problem* (1.2)–(1.4) *and sequence* (2.39) *joining the spectra of problem* (2.3), (2.4) *in the domain* Ω *and of problems* (2.6), (2.7), (2.40) *in the half-space* $\mathbb{R}^3_-$, $j=1,\dots,J$, *satisfy the relation*

$$|\lambda^\varepsilon_m - \boldsymbol{\mu}^0_m| \leqslant \mathbf{c}_m\sqrt{\varepsilon} \quad as \quad \varepsilon \in (0,\boldsymbol{\varepsilon}_m], \tag{3.21}$$

where $m\in\mathbb{N}$, *while* $\boldsymbol{\varepsilon}_m$ *and* $\mathbf{c}_m$ *are some positive numbers.*

Proof. An immediate goal is to confirm that the indices $n(q),\dots,n(q-\varkappa_q-1)$ in formula (3.20) can be treated as fixed. To this end, we use the second part of Lemma 3.1, in which we take $\delta = c_q\sqrt{\varepsilon}$ and $\delta_* = \delta/\tau$, where $\tau\in(0,1)$. We denote by $C^\varepsilon_{(\ell)}\in\mathbb{R}^{\mathcal{X}^\varepsilon_q}$ and $S^\varepsilon_{(\ell)}$ the columns and sums over $k=\mathcal{N}^\varepsilon_q,\dots,\mathcal{N}^\varepsilon_q+\mathcal{X}^\varepsilon_q-1$ given by formulas (3.12) for almost eigenvectors (3.15), $\ell=q,\dots,q+\varkappa_q-1$. Owing to these formulas and orthogonality and normalization conditions (1.17) we find

$$\begin{aligned} |(C^\varepsilon_{(k)})^\top C^\varepsilon_{(\ell)} - \delta_{\ell,k}| &= \big|\langle S^\varepsilon_{(\ell)}, S^\varepsilon_{(k)}\rangle_\varepsilon - \delta_{\ell,k}\big| \\ &\leqslant \big|\langle S^\varepsilon_{(\ell)} - U^\varepsilon_{(\ell)}, S^\varepsilon_{(k)}\rangle_\varepsilon\big| + \big|\langle U^\varepsilon_{(\ell)}, S^\varepsilon_{(k)} - U^\varepsilon_{(k)}\rangle_\varepsilon\big| + \big|\langle U^\varepsilon_{(\ell)}, U^\varepsilon_{(k)}\rangle_\varepsilon - \delta_{\ell,k}\big|. \end{aligned} \tag{3.22}$$

Each of first two terms on the right hand side does not exceed 2τ, while the last term does not exceed $c_q\sqrt{\varepsilon}$, see formulas (3.12) and (3.18), and the latter is applied twice: first for $\ell=k$ to find out the asymptotics of the norm $\|W^\varepsilon_{(\ell)};\mathcal{H}^\varepsilon\|$ and then for the indices involved in (3.22). Thus, for small τ and ε the columns $C^\varepsilon_{(q)},\dots,C^\varepsilon_{(q+\varkappa_q-1)}$ are almost orthonormalized in the Euclidean space $\mathbb{R}^{\mathcal{X}^\varepsilon_q}$, which is possible only in the case

$$\varkappa_q \leqslant \mathcal{X}^\varepsilon_q.$$

Thus, having fixed an appropriate quantity τ and bounding the small parameter $\varepsilon\leqslant\varepsilon_q$, in the segment

$$\big[(1+\boldsymbol{\mu}^0_q)^{-1} - c_q\tau^{-1}\sqrt{\varepsilon}, (1+\boldsymbol{\mu}^0_q)^{-1} + c_q\tau^{-1}\sqrt{\varepsilon}\,\big]$$

we find eigenvalues $\kappa^\varepsilon_{\mathcal{N}^\varepsilon_q},\dots,\kappa^\varepsilon_{\mathcal{N}^\varepsilon_q+\varkappa_q-1}$, which owing to relation (1.16) for the spectral parameters transform into the terms

$$\lambda^\varepsilon_{\mathcal{N}^\varepsilon_q},\dots,\lambda^\varepsilon_{\mathcal{N}^\varepsilon_q+\varkappa_q-1}$$

of sequence (1.16). At the same time,

$$\begin{aligned} &\big|\kappa^\varepsilon_\ell - (1+\boldsymbol{\mu}^0_q)^{-1}\big| \leqslant c_q\tau^{-1}\sqrt{\varepsilon} \quad \text{as} \quad \varepsilon\in(0,\varepsilon_q] \\ &\quad\Rightarrow \left\{\begin{aligned} c1+\lambda^\varepsilon_\ell &\leqslant 1+\boldsymbol{\mu}^0_q + c_q\tau^{-1}\sqrt{\varepsilon}(1+\varkappa^0_q)(1+\lambda^\varepsilon_\ell) \\ |\lambda^\varepsilon - \boldsymbol{\mu}^0_q| &\leqslant c_q\tau^{-1}\sqrt{\varepsilon}(1+\boldsymbol{\mu}^0_q)(1+\lambda^\varepsilon_\ell) \end{aligned}\right. \qquad \text{as} \quad \varepsilon\in(0,\varepsilon_q] \\ &\quad\Rightarrow \left\{\begin{aligned} c1+\lambda^\varepsilon_\ell &\leqslant 2(1+\sqrt{\mu}^0_q) \quad \text{as} \quad c_q\tau^{-1}\sqrt{\varepsilon}(1+\boldsymbol{\mu}_q)\leqslant 1/2 \\ |\lambda^\varepsilon_\ell - \boldsymbol{\mu}^0_q| &\leqslant C_q\sqrt{\varepsilon} := 2c_q\tau^{-1}\sqrt{\varepsilon}(1+\boldsymbol{\mu}_q)^2 \end{aligned}\right. \qquad \text{as} \quad \varepsilon\in(0,\boldsymbol{\varepsilon}_q], \end{aligned} \tag{3.23}$$

where $\boldsymbol{\varepsilon}_q = \min\{\varepsilon_q, \tau^2(2c_q(1+\boldsymbol{\mu}^0_q))^{-2}\}$. $\square$

Remark 3.1. *For each $\varkappa_q$-multiple eigenvalue $\boldsymbol{\mu}_q^0$ (see formula* (3.13)*), in its neighbourhood we have found at least $\varkappa_q$ eigenvalues of problem* (1.2)–(1.4) *satisfying the last relation in* (3.23)*. In particular, this implies inequality* (3.1) *and this proves Proposition 3.1 completely.*

It remains to make sure that $\mathcal{N}_q^\varepsilon = q$. According to what was said in Remark 3.1, we have $\mathcal{N}_q^\varepsilon \geqslant q$. If $\mathcal{N}_q^\varepsilon > q$, then there exists an eigenvalue of problem (1.9), for which

$$\lambda^\varepsilon_{\mathcal{M}_q^\varepsilon} \leqslant \boldsymbol{\mu}_q^0 + c_q\sqrt{\varepsilon} < (\boldsymbol{\mu}_q^0 + \boldsymbol{\mu}_{q+\varkappa_q}^0)/2 < \varkappa^0_{q+\varkappa_q}, \quad \mathcal{M}_q^\varepsilon \geqslant q + \varkappa_q.$$

At the same time, the eigenfunction obeys the orthogonality conditions

$$\rho(u^\varepsilon_{(\mathcal{M}_q^\varepsilon)}, u^\varepsilon_{(m)})_\Omega + \varepsilon^{-1}\rho_0(n^\top u^\varepsilon_{(\mathcal{M}_q^\varepsilon)}, n^\top u^\varepsilon_{(m)})_{\omega^\varepsilon} = 0, \quad m = 1, \dots, q + \varkappa_q - 1.$$

Passages to limit (3.2) and (3.3), (3.7) provide the term

$$\boldsymbol{\mu}^0_{\mathcal{M}_q^0} \in [0, \boldsymbol{\mu}^0_{q+\varkappa_q}) \tag{3.24}$$

of sequence (2.39) and a nontrivial linear combination of eigenfunctions of limiting problems (2.3), (2.4) and (2.6), (2.7), (2.40), $j = 1, \dots, J$, under keeping orthogonality conditions (2.19) and (2.42). This contradicts the way of forming monotone sequence (2.39): eigenvalue (3.24) is redundant.

The proof of Theorem 3.1 is complete.

3.6. Conclusion. Several important aspects of the scheme of asymptotics justification were intentionally left out. The majorant in estimate (3.21) reflects the worst error in asymptotic formulas for eigenvalues λ_m^ε as $m > 6$ found in Situation 3° in Subsection 3.3. The construction of correctors in ansätze (2.38) and (2.39) and reproducing of the calculations in this section allow us to specify Theorem 3.1 both directly in situations 1° and 2°, where the majorant becomes equal to $\mathbf{c}'_q\varepsilon$, and after specifying the asymptotics, namely,

$$|\lambda_m^\varepsilon - \boldsymbol{\mu}_m^0 - \varepsilon\boldsymbol{\mu}''_m| \leqslant \mathbf{c}''_q\varepsilon^{3/2}, \quad m = q, \dots, q + K - 1, \tag{3.25}$$

in situations 1° (as $K = 1$) and 2°, as well as

$$|\lambda_m^\varepsilon - \boldsymbol{\mu}_m^0 - \sqrt{\varepsilon}\boldsymbol{\mu}'_m| \leqslant \mathbf{c}'_q\varepsilon, \quad m = q, \dots, q + K, \tag{3.26}$$

in situation 3^0. Here we adopt the notations from Subsection 3.3 and Theorem 3.1.

As usual, the justification of asymptotic representations for the eigenfunctions is made by means of Lemmas 3.1 and 3.2, more precisely, by the inequalities from lists (3.12) and (3.18). Since the correctors in ansatz (2.37) can be found by means of linear combinations of eigenfunctions of the limiting problems, the estimates for the error terms in formulas for $u^\varepsilon_{(m)}$ are worse than in formulas (3.21) or (3.25), (3.26) for the eigenvalues λ_m^ε. Moreover, in the case of the multiple eigenvalue λ_m^0 the spectrum of the matrix M in algebraic systems (2.51) or (2.53) can also be multiple and hence, there eigenvectors satisfying relations (2.20), are not determined uniquely and the same is true for the leading terms of ansatz (2.39). As a result, the statements on the asymptotics for eigenfunctions become too cumbersome and this is why we restrict ourselves by a particular case of a simple eigenvalue of limiting problems.

Theorem 3.2. *In the situation described in Subsection 2.3 (1°), in particular, as $\alpha = -1$, for an eigenfunction of problem* (1.2)–(1.4) *associated with its eigenvalue λ_m^ε in formula* (3.25) *and normalized in accordance with identity* (1.17) *satisfies the estimate*

$$\|u^\varepsilon_{(m)} - \mathbf{u}_{(q)}; H^1(\Omega)\| \leqslant c_m\sqrt{\varepsilon} \quad as \quad \varepsilon \in (0, \varepsilon_m],$$

where $\mathbf{u}_{(q)}$ is an eigenfunction of problem (2.3), (2.4) *for its simple eigenvalue $\boldsymbol{\lambda}_q = \boldsymbol{\mu}_m^0$, and relation* (2.19) *holds, while $\varepsilon_m > 0$ and c_m are some numbers.*

4. Versions, generalizations and open questions

4.1. Smoothness of boundary and infinite asymptotic series. Of course, all arguing and calculations remain true under the assumption that the pieces $\Gamma^j = \partial\Omega \cap \mathcal{V}^j$ are smooth, while the other part of the boundary of the domain Ω can be only Lipschitz. The flatness of these pieces simplifies the asymptotic analysis, when one needs at least two terms in the boundary layer (cf. Subsection 2.3 (1^0),), however, the iteration procedures developed in monograph [13] give an opportunity to extend this result to the case of non-zero curvatures. Moreover, the corresponding procedures allows us to construct infinite asymptotic series in the framework of the method of composite asymptotic expansions. However, such iteration processes, which involve a complicated procedure of redistribution of the discrepancies between limiting problems, are rather cumbersome and are rarely used in particular problems of mathematical physics oriented to applications. On the other hand, in some issues it is sufficient to know the information about the possibility of expanding the eigenvalues and eigenvectors into the discussed series.

In fact, the points $P^1, \dots, P^J$ can be conical, however, we have to reformulate the restriction $\dim \mathcal{L} = 6$ for the linear span of the columns for such points. For instance, for a spindle

$$\Omega = \{x : x_3 \in (-1,1), H(x_3)^{-1}(x_1, x_2) \in \Theta\}, \tag{4.1}$$

where $H \in C^\infty[-1,1]$, $H(z) > 0$ as $z \in (-1,1)$, $H(\pm 1) = 0$, $\mp\partial_z H(\pm 1) > 0$ and Θ is an ellipse with non-equal axis, while substituting the Winkler-Steklov conditions at two end zones

$$\omega_\pm^\varepsilon = \{x \in \partial\Omega : \pm x_3 \in (1-\varepsilon, 1)\},$$

the spectrum of problem (1.2)–(1.4) becomes discrete also in the case $\rho = 0$. At the same time, asymptotic ansätze for eigenpairs of problem (1.9) for body (4.1) remain unknown. One more question, which remained open, is on constructing the asymptotics under the condition that the spherical surface in Examples 2^0 and 3^0 in Subsection 1.3 is replaced by a polygonal surface and a polygon, respectively.

4.2. Weightless body. We fix a size ε of domains (1.1) on the surface $\partial\Omega$ and let the density ρ of the body Ω tend to zero. If a symmetric positive (6×6)-matrix

$$\int\limits_{\omega^\varepsilon} d(x)^T n(x) n(x)^\top d(x)\, dx \tag{4.2}$$

is non-degenerate, then the bilinear form

$$\langle u^\rho, \psi^\rho\rangle_{\varepsilon,\rho} = E(u^\rho, \psi^\rho; \Omega) + \rho_\varepsilon (n^\top u^\rho, n^\top \psi^\rho)_{\omega^\varepsilon} \tag{4.3}$$

can serve as a scalar product in the Sobolev space $H^1(\Omega)^3$. Under the mentioned restriction for matrix (4.2) problem (1.2)–(1.4) turns out to be a regular perturbation of the limiting ($\rho = 0$) problem, in which the system of equilibrium equations is

$$L(\nabla_x) u^{\varepsilon\rho}(x) = 0, \quad x \in \Omega, \tag{4.4}$$

with boundary conditions (1.3) and (1.4), while for the eigenvalues $\lambda_m^{\varepsilon\rho}$ of the original problem the asymptotic representations

$$\lambda_m^{\varepsilon\rho} = \lambda_m^{\varepsilon 0} + \rho\lambda_m^{\varepsilon\prime} + O(\rho^2)$$

hold, where $\{\lambda_m^{\varepsilon 0}\}_{m\in\mathbb{N}}$ is the spectrum of the limiting problem, while the correctors λ_m' can be easily calculated.

For Examples 1^0–3^0 provided in Subsection 1.3 and for many other situations matrix (4.2) and bilinear form (4.3) lose needed properties, while the spectrum of limiting problem (4.4), (1.3), (1.4) occupies the entire complex plane: for each $\lambda^{\varepsilon 0} \in \mathbb{C}$ the elements of the nontrivial subspace $\mathcal{R}^{\#}$ in linear space (1.18) of rigid displacement completely satisfy the limiting problem since $n(x)^{\top} r^{\#}(x) = 0$ as $x \in \omega^{\varepsilon}$ for $r^{\#} = da^{\#} \in \mathcal{R}^{\#}$. At the same time, the problem

$$E(u^{\varepsilon\#}, \psi^{\#}; \Omega) = \lambda^{\varepsilon\#} \rho_{\varepsilon} (n^{\top} u^{\varepsilon\#}, n^{\top} \psi^{\#})_{\omega^{\varepsilon}}, \quad \psi^{\#} \in H^1_{\#}(\Omega)^3,$$

restricted to the subspace $H^1_{\#}(\Omega)^3 = H^1(\Omega)^3 \ominus \mathcal{R}^{\#}$ takes a discrete spectrum $\wp^{\varepsilon}_{\#}$, in which a zero eigenvalue is of the multiplicity $6 - \dim \mathcal{R}^{\#}$. The author does not know a mechanical interpretation of such restriction.

It is easy to construct an asymptotics as $\rho \to +0$ for an inhomogeneous system of equations

$$L(\nabla_x) u^{\varepsilon\rho}(x) - \lambda^{\varepsilon 0} \rho u^{\varepsilon\rho}(x) = f(x), \quad x \in \Omega,$$

subject to boundary conditions (1.3), (1.4), which involves a parameter $\lambda^{\varepsilon 0} \in \mathbb{C} \setminus \wp^{\varepsilon}_{\#}$, namely,

$$u^{\varepsilon\rho}(x) = \rho^{-1} d(x) a^{\#} + u^{\varepsilon\prime}(x) + \ldots,$$

where the first term in the right hand side belongs to the subspace $\mathcal{R}^{\#}$, while the column $a^{\#}$ is determined by the compatibility conditions of the problem for $u^{\varepsilon\prime}$.

The matrix (4.2) is non-degenerate, for instance, under the condition $\dim \mathcal{L} = 6$ imposed for columns (1.21). At the same time, the passage to the limit $\varepsilon \to +0$ in problem (1.2)–(1.4) is possible also for $\rho = 0$: corresponding limiting problem (2.34) (or (2.6), (2.7), (2.33) in the differential form) is derived by means of the analysis presented in Subsection 2.2. We mention a result obtained in Subsection 1.3: the expression

$$(E(u, u; \Omega) + \|u^0; \mathbb{R}^6\|^2)^{1/2}$$

with the column u^0 given by the first formula in (1.24) is a norm in the Sobolev space $H^1(\Omega)^3$.

Under the discussed restriction $\dim \mathcal{L} = 6$ the simultaneous passage to the limit as $\varepsilon \to +0$ and $\rho \to +0$ can lead to a problem different from (2.34). For instance, let

$$\rho_{\varepsilon} = \varepsilon^{-1} \rho_0 \quad \text{and} \quad \rho = \varepsilon \rho_{\Omega}, \quad \rho_{\Omega} > 0. \tag{4.5}$$

Ansat (2.26) for the eigenfunction completely remains and since $\alpha = -1$ in definition (1.8), ansatz (2.25) for the eigenvalue casts into the form

$$\lambda^{\varepsilon}_{6+m} = \mu_m + \ldots .$$

According to assumption (4.5), in the right hand side f'_m in problem (2.29) for the corrector $u'_{(m)}$ of smooth type an additional term appears:

$$\mu_m \rho_{\Omega}\, d(x) c^0_{(m)}.$$

Hence, previous calculations in Subsection 2.2 show that the matrix M (see formula (2.32)) in limiting boundary conditions (2.33) becomes

$$M = \rho_{\Omega} \rho_0^{-1} \int\limits_{\Omega} d(x)^{\top} d(x)\, dx + \sum_{j=1}^{J} |\varpi_j| d(P^j)^{\top} n(P^j) n(P^j)^{\top} d(P^j). \tag{4.6}$$

Matrix (4.6) remains positive definite also once we omit the restriction $\dim \mathcal{L} = 6$, that is, Theorem 2.2 is true for problem (2.34) with a new matrix M for arbitrary number and position of the sets $\omega^{\varepsilon}_1, \ldots, \omega^{\varepsilon}_J \subset \partial\Omega$, on which the Winkler–Steklov conditions are imposed. As in Subsection 2.2, for $J > 1$ limiting problems (2.6), (2.7), (2.33) are joined in a single spectral problem. If $J = 1$, then the boundary condition on ϖ_1 for the only problem in the half-space

$\mathbb{R}^3_-$ becomes

$$N^1(\nabla_{\xi^1})w^1_{(m)}(\xi^1_\natural,0)=\mu_m\rho_0 n(P^1)\Big(n(P^1)^\top w^1_{(m)}(\xi^1_\natural,0)-$$
$$-d_{(1)}\Big(\frac{\rho_\Omega}{\rho_0}\int\limits_\Omega d(x)^\top d(x)\,dx+|\varpi^1|d^\top_{(1)}d_{(1)}\Big)^{-1}|\varpi^1|d^\top_{(1)}n(P^j)^\top\overline{w}^1_{(m)}\Big),\quad \xi^1_\natural\in\varpi^1,$$

where $d_{(1)}=n(P^1)^\top d(P^1)$ is a row of length six.

4.3. On modelling singularly perturbed problems. In mechanics and other applied fields many models obtained by means of partial asymptotic analysis preserve a small parameter. A very striking example is the shell theory (see monograph [22] and others), the equations in which, in contrast to the equations in the plate theory (see, for instance, books [23], [3]), involve the curvatures of the middle surface and a relative thickness of the shell, which is a natural small parameter. In the case of small singular perturbations the technique of self-adjoint extensions of differential operators (see [24]–[30], [15], [16] and others) turn out to be effective.

For the considered problem with Winkler–Steklov conditions on small parts of the boundary, the technique of self-adjoint extensions gains a feature: the extension parameters, in addition to the size, involve a sought eigenvalue. Let us demonstrate this feature for the exponent $\alpha=0$ in formula (1.8) and employ the results of asymptotic analysis from Section 2.1.

From ansatz (2.2) we extract the sum

$$u^\varepsilon(x)=u^0(x)+\varepsilon^2\Big(u'(x)+\sum_{j=1}^J\chi_j(x)\Phi^j(x^j)b^j\Big)\tag{4.7}$$

with coefficients (2.13) neglecting fast decaying terms in the boundary layer $\widetilde{w}^j(\xi^j)$ (we do not write the subscript m). We note that the vector function (4.7), being substituted into the system of differential equation

$$L(\nabla_x)\mathfrak{u}^\varepsilon(x)=\mathfrak{l}^\varepsilon\rho\mathfrak{u}^\varepsilon(x),\quad x\in\Omega,\tag{4.8}$$

with the parameter

$$\mathfrak{l}^\varepsilon=\lambda^0+\varepsilon^2\lambda',$$

from (2.1), satisfies it with an error, $L^2(\Omega)$-norm of which is equal to $O(\varepsilon^4)$. For a vector function $\mathfrak{u}^\varepsilon$, the boundary conditions

$$N(x,\nabla_x)\mathfrak{u}^\varepsilon(x)=0,\quad x\in\partial\Omega\setminus\mathcal{P},\tag{4.9}$$

hold everywhere except for the points $P^1,\dots,P^J$, at which it has singularities $O(r^{-1})$. A noticeable smallness of the error allows us to adopt problem (4.8), (4.9) as a model for original singularly perturbed problem and, as usually, (see primary source [24] and survey [26]), the Winkler–Steklov conditions on small neighbourhoods of the points $P^1,\dots,P^J$ is imitated by the Dirac delta-functions with approrpiate coefficients, that is, in the framework of the theory of distributions we have

$$N(x,\nabla_x)\mathfrak{u}^\varepsilon(x)=\mathfrak{l}^\varepsilon\rho_0\varepsilon^2\sum_{j=1}^J|\varpi_j|\delta(s^j)n(P^j)n(P^j)^\top\widehat{\mathfrak{u}}^\varepsilon(P^j),\qquad x\in\partial\Omega.\tag{4.10}$$

Here the coefficient at the Dirac function $\delta(s^j)$ with a singularity at the point $P^j\in\partial\Omega$ is found by formula (2.13) with admissible replacements $\lambda^0\mapsto\mathfrak{l}^\varepsilon$ and $u^0(P^j)\mapsto\widehat{\mathfrak{u}}^\varepsilon(P^j)$, namely,

$$\mathfrak{b}^{j\varepsilon}=\mathfrak{l}^\varepsilon\rho_0\varepsilon^2|\varpi_j|n(P^j)n(P^j)^\top\widehat{\mathfrak{u}}^\varepsilon(P^j),\tag{4.11}$$

where $\widehat{\mathfrak{u}}^\varepsilon \in H^2(\Omega)$ is a regular part of the vector function

$$\mathfrak{u}^\varepsilon(x) = \widehat{\mathfrak{u}}^\varepsilon(x) + \sum_{j=1}^{J} \chi_j(x)\Phi^j(x^j)\mathfrak{b}^{j\varepsilon}. \tag{4.12}$$

The right hand side of identity (4.10) involves unknowns $\mathfrak{l}^\varepsilon$ and $\mathfrak{u}^\varepsilon$, that is, it should be interpreted as a spectral boundary condition. However, its rigorous formulation requires the operator $\mathfrak{A}$ introduced in Subsection 1.2.

The next statement is a simple corollary of Kondratiev theory [31] (cf. arguging in work [27]).

Proposition 4.1. *The adjoint operator $\mathfrak{S}^*$ for the operator $\mathfrak{S}$ with domain* (1.22) *preserves the differential expression $L(\nabla_x)$ but gains the domain*

$$\begin{gathered}\mathfrak{D}(\mathfrak{S}^*) = \Big\{ \mathfrak{U} = \mathfrak{U}^0 + \sum_{j=1}^{J} G^j(x)\mathfrak{B}^j \,\Big|\, \mathfrak{U}^0 \in H^2(\Omega)^3, \\ N(x,\nabla_x)\mathfrak{U}^0(x) = 0, \quad x \in \partial\Omega, \quad \mathfrak{B}^j \in \mathbb{R}^3, \quad j = 1,\dots,J \Big\}.\end{gathered} \tag{4.13}$$

Here G^j is Green matrix (2.48) *with a singularity at the point P^j.*

Since

$$\big\{\mathfrak{U} \in \mathfrak{D}(\mathfrak{S}^*) : \mathfrak{B}^1 = \dots \mathfrak{B}^J = 0\big\} = H^2_N(\Omega)^3 := \{u \in H^2(\Omega)^3 : N(x,\nabla_x)u(x) = 0,\ x \in \partial\Omega\}$$

and the Fredholm mapping

$$H^2_N(\Omega)^3 \ni u \ \mapsto \ L(\nabla_x)u \in L^2(\Omega)^3$$

possesses a six-dimensinal kernel and co-kernel (cf. compatibility conditions (2.30) in problem (2.29) and polynomial property (1.18)), the defect index of the considered operator $\mathfrak{S}$ is equal to $3J : 3J$, and hence it admits a self-adjoint extension. As in papers [15], [16], [29], [30] and others, for modelling we need one of the extensions among the entire family of all possible ones. In order to make a proper choice, we compare the expansion

$$\mathfrak{U}(x) = \widetilde{\mathfrak{U}}(x) + \sum_{j=1}^{J} \chi_j(x)\Big(\Phi^j(x^j)\mathfrak{B}^j + \sum_{k=1}^{J} \widetilde{G}^k(P^j)\mathfrak{B}^k + \mathfrak{U}^0(P^j)\Big)$$

for an element of space (4.13) and the chosen expansion

$$\mathfrak{u}^\varepsilon(x) = \widetilde{\mathfrak{u}}^\varepsilon(x) + \sum_{j=1}^{J} \chi_j(x)\left(\Phi^j(x^j)\mathfrak{b}^{j\varepsilon} + \widehat{\mathfrak{u}}^\varepsilon(P^j)\right).$$

Here the errors $\widetilde{\mathfrak{U}}$ and $\widetilde{\mathfrak{u}}^\varepsilon$ belong to the subspace

$$H^2_\bullet(\Omega)^3 := \left\{\widetilde{u} \in H^2(\Omega)^3 : \widetilde{u}(P^1) = \dots = \widetilde{u}(P^J) = 0\right\}.$$

As a result we obtain the relations

$$\mathfrak{B}^j = \mathfrak{b}^{j\varepsilon}, \quad \mathfrak{U}^0(P^j) + \sum_{k=1}^{J} \widetilde{G}^k(P^j)\mathfrak{B}^k = \widehat{\mathfrak{u}}^\varepsilon(P^j), \quad j = 1,\dots,J,$$

which in view of formulas (4.11) and (2.49) imply the relations

$$\begin{gathered} n(P^j)\mathfrak{B}^j = \mathfrak{l}^\varepsilon\rho_0\varepsilon^2|\omega_j|\Big(n(P^j)^\top\mathfrak{U}^0(P^j) + \sum_{k=1}^{J} \mathbf{G}_{jk}n(P^k)\mathfrak{B}^k\Big), \\ \left(\mathbb{I}_3 - n(P^j)n(P^j)^\top\right)\mathfrak{B}^j = 0, \quad j = 1,\dots,J. \end{gathered} \tag{4.14}$$

Theorem 4.1. *For each $\mathfrak{l}^\varepsilon \in \mathbb{R}_+$ and small $\varepsilon > 0$ the restriction $\mathfrak{S}(\mathfrak{l}^\varepsilon)$ of the operator $\mathfrak{S}^*$ to the subspace*

$$\mathcal{D}(\mathfrak{S}(\mathfrak{l}^\varepsilon)) = \{\mathfrak{U}^\varepsilon \in \mathfrak{D}(\mathfrak{S}^*) : \text{ relations (4.14) are satisfied } \}$$

is a self-adjoint operator.

Proof. For the elements $\mathfrak{U}_{(i)}$, $i = 1, 2$, of space (4.13) with ingredients $\mathfrak{B}_{(i)}$ and $\mathfrak{U}^0_{(i)}$ a generalized Green's formula holds:

$$\big(L(\nabla_x)\mathfrak{U}_{(1)}, \mathfrak{U}_{(2)}\big)_\Omega - \big(\mathfrak{U}_{(1)}, L(\nabla_x)\mathfrak{U}_{(2)}\big)_\Omega = \sum_{j=1}^{J} \Big((\mathfrak{B}^j_{(2)})^\top \mathfrak{U}^0_{(1)}(P^j) - (\mathfrak{B}^j_{(1)})^\top \mathfrak{U}^0_{(2)}(P^j)\Big), \quad (4.15)$$

which is deduce by means of (2.21). It is easy to see that owing to relations (4.14) the right hand side of identity (4.15) vanishes for the vector functions $\mathfrak{U}_{(1)}, \mathfrak{U}_{(2)} \in \mathcal{D}(\mathfrak{S}(\mathfrak{l}^\varepsilon))$. It remains to note that for a small ε relations (4.14) impose exactly $3J$ conditions for the coefficients $\mathfrak{B}^j, \mathfrak{U}^0(P^j) \in \mathbb{R}^3$, $j = 1, \dots, J$, and hence, $\mathfrak{S}(\mathfrak{l}^\varepsilon)$ is a self-adjoint extension of the operator $\mathfrak{S}$ since its defect index is equal to $3J : 3J$. $\square$

Unfortunately, the domain of the self-adjoint extension $\mathfrak{S}(\mathfrak{l}^\varepsilon)$ depends on the spectral parameter, that is, the spectral problem

$$\mathfrak{S}(\mathfrak{l}^\varepsilon)\mathfrak{U}^\varepsilon = \mathfrak{l}^\varepsilon \mathfrak{U}^\varepsilon \quad (4.16)$$

in fact deals with an operator pencil and this is an obstacle for a mechanical interpretation of equation (4.16) and for creating numerical schemes for solving it.

We propose another model, which uses a Hilbert space of vector functions with a detached asymptotics

$$\mathfrak{D} = H^2_\bullet(\Omega)^3 \times \mathbb{R}^{3\times J} \times \mathbb{R}^{3\times J} \quad (4.17)$$

equipped with the norm

$$\|\mathfrak{u}^\varepsilon; \mathfrak{D}\| = \big(\|\widetilde{\mathfrak{u}}^\varepsilon; H^2(\Omega)\|^2 + \|\mathfrak{a}^\varepsilon; \mathbb{R}^{3\times J}\|^2 + \|\mathfrak{b}^\varepsilon; \mathbb{R}^{3\times J}\|^2\big)^{1/2},$$

which involves the remainder $\widetilde{u}^\varepsilon$ and the columns $\mathfrak{a}^\varepsilon = (\mathfrak{a}^{\varepsilon 1}, \dots, \mathfrak{a}^{\varepsilon J})^\top$, $\mathfrak{b}^\varepsilon = (\mathfrak{b}^{\varepsilon 1}, \dots, \mathfrak{b}^{\varepsilon J})^\top$ of the following representation for an element in space (4.17):

$$\mathfrak{U}^\varepsilon(x) = \widetilde{\mathfrak{U}}^\varepsilon(x) + \sum_{j=1}^{J} \chi_j(x)(\Phi^j(x^j)\mathfrak{b}^{\varepsilon j} + \mathfrak{a}^{\varepsilon j}).$$

We complete the system of differential equations

$$L(\nabla_x)\mathfrak{u}^\varepsilon(x) = \mathfrak{l}^\varepsilon \mathfrak{u}^\varepsilon(x), \quad x \in \Omega, \quad (4.18)$$

with boundary conditions

$$N(x, \nabla_x)\mathfrak{u}^\varepsilon(x) = 0, \quad x \in \partial\Omega \setminus \mathcal{P}, \quad (4.19)$$

and asymptotic conditions at the points $P^1, \dots, P^J$

$$\mathfrak{b}^{3j} = \mathfrak{l}^\varepsilon \rho_0 \varepsilon^2 |\varpi_j| n(P^j) n(P^j)^\top \mathfrak{a}^{\varepsilon j} \in \mathbb{R}^3 \quad (4.20)$$

coming from formulas (4.11) and (4.12).

The operator of problem (4.18)–(4.20) is realized as the mapping

$$\mathfrak{D} \ni \mathfrak{u}^\varepsilon \mapsto (L\mathfrak{u}^\varepsilon, N\mathfrak{u}^\varepsilon, \mathfrak{b}^\varepsilon) - \mathfrak{l}^\varepsilon(\mathfrak{u}^\varepsilon, 0, \varepsilon^2 T\mathfrak{a}^\varepsilon) \in \mathfrak{R} := L^2(\Omega)^3 \times \{0|_{\partial\Omega}\}^3 \times \mathbb{R}^{3\times J} \times \mathbb{R}^{3\times J}, \quad (4.21)$$

where

$$T = \rho_0 \operatorname{diag}\left\{|\varpi_1| n(P^1)n(P^1)^\top, \dots, |\varpi_J| n(P^J)n(P^J)^\top\right\}.$$

Theorem 4.2. *The spectrum of problem* (4.18)–(4.20) *is purely discrete. The terms of the corresponding sequence of eigenvalues*

$$0 = \mathfrak{l}_1^\varepsilon = \cdots = \mathfrak{l}_6^\varepsilon < \mathfrak{l}_7^\varepsilon \leqslant \mathfrak{l}_8^\varepsilon \leqslant \ldots \leqslant \mathfrak{l}_m^\varepsilon \leqslant \cdots \to +\infty$$

and the terms of sequence (1.16) *of eigenvalues of problem* (1.2)–(1.4) *satisfy the relation*

$$|\lambda_m^\varepsilon - \mathfrak{l}_m^\varepsilon| \leqslant \mathfrak{c}_m \varepsilon^3 \quad as \quad \varepsilon \in (0, \mathfrak{e}_m], \tag{4.22}$$

where $m \geqslant 7$, *while* $\mathfrak{c}_m$ *and* $\mathfrak{e}_m > 0$ *are some numbers.*

Proof. In view of the compactness of the embedding $\mathfrak{D} \subset \mathfrak{R}$ (we do not take into consideration the zero component, which is the third left one in (4.21)), the statement on the discreteness is obvious. Relation (4.22) is implied by Theorem 2.3 and estimate (2.24) as $\alpha = 0$. □

A mechanical interpretation of asymptotic conditions (4.20) is simple: a body Ω is connected by stiff springs to rigid profiles (see monograph [4] and cf. paper [20]). The question on creating numerical schemes for solving problem (1.2)–(1.4) remained completely open.

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Sergei Alexandrovich Nazarov,
Institute of Problems of Mechanical Engineering,
Russian Academy of Sciences,
Bolshoy av. of Vassilievsky Island 61,
199178, Saint–Petersburg, Russia
E-mail: `srgnazarov@yahoo.co.uk`