ISSN 2304-0122 Ufa Mathematical Journal. Vol. 16. No 1 (2023). P. 100-111.

doi:10.13108/2023-16-1-100

HOMOGENIZATION OF MOTION EQUATIONS FOR MEDIUM CONSISTING OF ELASTIC MATERIAL AND INCOMPESSIBLE KELVIN-VOIGT FLUID

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Abstract. We consider an initial-boundary problem describing the motion of a two-phase medium with a periodic structure. The first phase of the medium is an isotropic elastic material and the second phase is an incompressible viscoelastic Kelvin-Voigt fluid. This problem consists of second and fourth order partial differential equations, conditions of continuity of displacements and stresses at the phase boundaries, and homogeneous initial and boundary conditions. Using the Laplace transform method, we derive a homogenized problem, which is an initial boundary value problem for the system of fourth order partial integro-differential equations with constant coefficients. The coefficients and convolution kernels of the homogenized equations are found by using solutions of auxiliary periodic problems on the unit cube. In the case of a layered medium, the solutions of the periodic problems are written explicitly, and this allows us to find analytic expressions for the homogenized coefficients and convolution kernels. In particular, we establish that the type and properties of the homogenized convolution kernels depend on the volume fraction of the fluid layers inside the periodicity cell.

Keywords: homogenization, equations of motion, two-phase medium, elastic material, Kelvin-Voigt fluid

Mathematics Subject Classification: 35B27

1. Introduction

The construction of rigorously justified homogenized models of micro-inhomogeneous media is among the main directions of the homogenization theory for partial differential equations. From the point of view of practical applications a large interest is attracted by homogenized models of two-phase media possessing ε-periodic structure and consisting of a solid material and a fluid. The dynamics of such solid-fluid media is described by initial boundary value problems for partial differential or integro-differential equations, the coefficients of which are ε-periodic functions of spatial variables. A straightforward numerical solving for media consisting of many thousands or millions of periodicity cells causes serious troubles. On the other hand, for some models of two-phase media one succeeds to derive corresponding homogenized models constructed as $\varepsilon \to 0$. As a rule, the homogenized models of periodic rigid-fluid media are initial boundary value problems for equations with constant coefficients. It should be recalled that according the main idea of the homogenization, the solutions of perturbed problems and of the corresponding homogenized ones should be close from small ε.

In works [1]–[9] homogenized motion models were constructed for periodic two-phase medium, one phase of which consisted of an elastic or viscoelastic material, while the other phase did of

A.S. Shamaev, V.V. Shumilova, Homogenization of motion equations for medium consisting of elastic material and incompessible Kelvin-Voigt fluid.

The work is made under the state task (state registration number 124012500443-0).

Submitted February 14, 2023.

a viscous Newtonian fluid. According to the results of these works, the homogenized equations are integro-differential even in the case, when the equations for each phase are differential.

In the present work we consider the homogenization for an initial boundary value problem describing the motion of a two-phase medium with an ε-periodic structure. As the phases, we choose an isotropic elastic material and an incompressible viscoelastic Kelvin-Voigt fluid. The description and properties of the non-Newtonian fluids of such kind can be found, for instance, in works [10], [11]. By means of the Laplace transform and the results of works [6]–[8] we write out the homogenized problem, which is an initial boundary value problem for the system of integro-differential equations with constant coefficients. The coefficients and the convolution kernels of the homogenized equations can be found by means of a series of auxiliary periodic problems on the unit cube. We show that in the case of layered medium the solutions of the periodic problems can be written explicitly and it is possible to obtain in this way explicit analytic expressions for the homogenized coefficients and the convolution kernels.

2. Original model of two-phase medium

We consider a bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$ filled by a two-phase medium with a periodic microstructure. The period of this medium is the cube $Y_\varepsilon = \varepsilon Y$, where $Y = (0,1)^3$ is the unit cube and the quantity ε is much less than the linear sizes of the domain Ω. We partition Y into two open disjoint subsets Y_1 and Y_2 with a common smooth boundary Γ: $Y = Y_1 \cup Y_2 \cup \Gamma$. We introduce the sets

$$E_s = \bigcup_{k\in\mathbb{Z}^3} \left(Y_s \cup (\partial Y_s \cap \partial Y) + k\right), \quad s = 1,2,$$

obtained by the Y-periodic continuation of the sets Y_s on the entire space $\mathbb{R}^3$. We denote $\Omega_{s\varepsilon} = \Omega \cap \varepsilon E_s$ and in what follows we suppose that the set $\Omega_{1\varepsilon}$ is filled by an isotropic elastic material, while the set $\Omega_{2\varepsilon}$ is filled by an incompressible viscoelastic Kelvin–Voigt fluid.

The constitutive relations for the components of the stress tensor and small deformations in the elastic phase $\Omega_{1\varepsilon}$ read as

$$\sigma^\varepsilon_{ij} = a_{ijkh} e_{kh}(u^\varepsilon), \tag{2.1}$$

where $u^\varepsilon(x,t)$ is the vector of displacements, σ^ε and $e(u^\varepsilon)$ are the stress tensor and tensor of small deformations, respectively, a is a positive definite tensor of elasticity modules,

$$e_{kh}(u^\varepsilon) = e^x_{kh}(u^\varepsilon) = \frac{1}{2}\left(\frac{\partial u^\varepsilon_k}{\partial x_h} + \frac{\partial u^\varepsilon_h}{\partial x_k}\right), \qquad a_{ijkh} = \lambda\delta_{ij}\delta_{kh} + \mu(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}).$$

Here λ and μ are the Lamé parameters, $\mu > 0$, $3\lambda + 2\mu > 0$, and δ_{ij} is the Kronecker delta. We note that in (2.1) and in what follows we suppose the summation over repeating indices and, unless else is stated, the indices i, j, k, h range from 1 to 3.

The constitutive relations in the fluid phase $\Omega_{2\varepsilon}$ read as

$$\sigma^\varepsilon_{ij} = -\delta_{ij} p^\varepsilon + 2\eta e_{ij}\left(\frac{\partial u^\varepsilon}{\partial t}\right) + 2\theta e_{ij}\left(\frac{\partial^2 u^\varepsilon}{\partial t^2}\right), \tag{2.2}$$

where $p^\varepsilon(x,t)$ is the pressure, η is the viscosity coefficient of the fluid, and $\tau = \theta/\eta$ is the retardation time [10].

The initial boundary value problem describing the motion of the two phase medium in the domain Ω is written as

$$\begin{gathered}\rho_s\frac{\partial^2 u_i^\varepsilon}{\partial t^2}=\frac{\partial\sigma_{ij}^\varepsilon}{\partial x_j}+f_i(x,t)\quad\text{in}\quad\Omega_{s\varepsilon}\times(0,T),\quad s=1,2,\\ \operatorname{div}\frac{\partial u^\varepsilon}{\partial t}=0\quad\text{in}\quad\Omega_{2\varepsilon}\times(0,T),\qquad [u^\varepsilon]|_{\Gamma_\varepsilon}=0,\quad[\sigma_{ij}^\varepsilon n_j]|_{\Gamma_\varepsilon}=0,\\ u^\varepsilon(x,t)|_{\partial\Omega}=0,\qquad u^\varepsilon(x,0)=\frac{\partial u^\varepsilon}{\partial t}(x,0)=0,\end{gathered}\tag{2.3}$$

where $\rho_s=\text{const}$ is the density of the medium in $\Omega_{s\varepsilon}$; $f(x,t)\in H^2(0,T;L^2(\Omega)^3)$ is the vector of a volume force; $[g]|_{\Gamma_\varepsilon}$ is the jump of a function g at the surface $\Gamma_\varepsilon=\partial\Omega_{1\varepsilon}\cap\partial\Omega_{2\varepsilon}$; $n=(n_1,n_2,n_3)$ is the unit normal vector to the surface Γ_ε directed from the rigid phase into the fluid one.

The variational formulation of problem (2.3) is as follows: find $u^\varepsilon(t)$ and $p^\varepsilon(t)$ with values in $H_0^1(\Omega)^3$ and $L^2(\Omega_{2\varepsilon})$, respectively, satisfying the integral identity

$$\begin{aligned}\sum_{s=1}^{2}\rho_s\int\limits_{\Omega_{s\varepsilon}}\frac{\partial^2u^\varepsilon}{\partial t^2}\cdot v\,dx+\int\limits_{\Omega_{1\varepsilon}}a_{ijkh}e_{kh}(u^\varepsilon)e_{ij}(v)\,dx\\ +2\eta\int\limits_{\Omega_{2\varepsilon}}e_{ij}\left(\frac{\partial u^\varepsilon}{\partial t}\right)e_{ij}(v)\,dx+2\theta\int\limits_{\Omega_{2\varepsilon}}e_{ij}\left(\frac{\partial^2u^\varepsilon}{\partial t^2}\right)e_{ij}(v)\,dx\\ -\int\limits_{\Omega_{2\varepsilon}}p^\varepsilon\operatorname{div}v\,dx=\int\limits_{\Omega}f\cdot v\,dx\qquad\forall v\in H_0^1(\Omega)^3\end{aligned}\tag{2.4}$$

for almost each $t\in(0,T)$, the incompressibility condition of the fluid phase

$$\operatorname{div}\frac{\partial u^\varepsilon}{\partial t}=0\quad\text{in}\quad\Omega_{2\varepsilon}\times(0,T)\tag{2.5}$$

and homogeneous initial conditions

$$u^\varepsilon(0)=0,\qquad\frac{\partial u^\varepsilon}{\partial t}(0)=0.\tag{2.6}$$

The unique solvability of problem (2.4)–(2.6) for each fixed $\varepsilon>0$ can be established as in works [1], [2], in which there was considered a case of two-phase medium consisting of an elastic material and a viscous Newtonian fluid.

From integral identity (2.4) we can derive a series of apriori estimates uniform in ε. First of all letting $v=\partial u^\varepsilon/\partial t$ in (2.4), we obtain

$$\frac{dz^\varepsilon}{dt}\leqslant 2\int\limits_{\Omega}f\cdot\frac{\partial u^\varepsilon}{\partial t}\,dx,\tag{2.7}$$

where

$$z^\varepsilon(t)=\sum_{s=1}^{2}\rho_s\int\limits_{\Omega_{s\varepsilon}}\left|\frac{\partial u^\varepsilon}{\partial t}\right|^2dx+\int\limits_{\Omega_{1\varepsilon}}a_{ijkh}e_{kh}(u^\varepsilon)e_{ij}(u^\varepsilon)\,dx+2\theta\int\limits_{\Omega_{2\varepsilon}}\left|e\left(\frac{\partial u^\varepsilon}{\partial t}\right)\right|^2dx.$$

By (2.7) we have

$$\frac{dz^\varepsilon}{dt}\leqslant\int\limits_{\Omega}|f|^2dx+\int\limits_{\Omega}\left|\frac{\partial u^\varepsilon}{\partial t}\right|^2dx\leqslant\int\limits_{\Omega}|f|^2dx+k_1z^\varepsilon,\qquad k_1=\frac{1}{\min\{\rho_1,\rho_2\}},$$

which by Grönwall inequality yield the estimates

$$\left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^\infty(0,T;L^2(\Omega)^3)} \leqslant C\|f\|_1, \qquad ||e(u^\varepsilon)||_{L^\infty(0,T;L^2(\Omega_{1\varepsilon})^3)} \leqslant C\|f\|_1,$$

$$\left\| e\left(\frac{\partial u^\varepsilon}{\partial t}\right) \right\|_{L^\infty(0,T;L^2(\Omega_{2\varepsilon})^3)} \leqslant C\|f\|_1, \qquad \|f\|_1 = \|f\|_{L^2(0,T;L^2(\Omega)^3)},$$

where C stand for various positive constants independent of ε. It also follows from these estimates and the Korn inequality that

$$\|u^\varepsilon\|_{L^\infty(0,T;H_0^1(\Omega)^3)} \leqslant C\|f\|_1.$$

In the same way we show that

$$\left\| \frac{\partial^3 u^\varepsilon}{\partial t^3} \right\|_{L^\infty(0,T;L^2(\Omega)^3)} \leqslant C\|f\|_{H^2(0,T;L^2(\Omega)^3)},$$

$$\left\| \frac{\partial^r u^\varepsilon}{\partial t^r} \right\|_{L^\infty(0,T;H_0^1(\Omega)^3)} \leqslant C\|f\|_{H^r(0,T;L^2(\Omega)^3)}, \quad r = 1, 2.$$

In particular, the obtained estimates imply that after a possible change on a set of zero Lebesgue measure we get $u^\varepsilon \in C^1([0,T]; H_0^1(\Omega)^3) \cap C^2([0,T]; L^2(\Omega)^3)$ [11].

It remains to obtain an estimate for the pressure $p^\varepsilon(x,t)$. In order to do this we consider the problem

$$\operatorname{div} q^\varepsilon = P^\varepsilon \quad \text{in} \quad \Omega \times (0,T), \quad q^\varepsilon(x,t)|_{\partial\Omega} = 0, \tag{2.8}$$

where

$$P^\varepsilon(x,t) = \begin{cases} -\dfrac{1}{|\Omega_{1\varepsilon}|} \displaystyle\int\limits_{\Omega_{2\varepsilon}} p^\varepsilon(x,t) & \text{in} \quad \Omega_{1\varepsilon} \times (0,T), \\ \qquad p^\varepsilon(x,t) & \text{in} \quad \Omega_{2\varepsilon} \times (0,T). \end{cases}$$

It is easy to confirm that

$$\int\limits_\Omega P^\varepsilon(x,t)\, dx = 0, \qquad \|P^\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leqslant C\|p^\varepsilon\|_{L^2(0,T;L^2(\Omega_{2\varepsilon}))}.$$

It is known [12] that there exists a solution to problem (2.8) and it satisfies the estimate

$$\|q^\varepsilon\|_{L^2(0,T;H_0^1(\Omega)^3)} \leqslant C\|P^\varepsilon\|_{L^2(0,T;L^2(\Omega))}.$$

In the integral identity we take $v = q^\varepsilon$, a solution of problem (2.8). Taking into consideration the above estimates for u^ε, it is easy to obtain

$$\|p^\varepsilon\|_{L^2(0,T;L^2(\Omega_{2\varepsilon}))} \leqslant C.$$

3. Homogenized model of two-phase medium

In order to write out a homogenized problem corresponding to (2.3) as $\varepsilon \to 0$, we continue $f(x,t)$ by zero for $t < 0$ and $t > T$. Then we apply the Laplace transform in the time t to (2.4)–(2.6) and we denote the image of the function $g(t)$ by g_λ or $g(\lambda)$, where λ is the parameter of the Laplace transform. Following the arguing from works [6]–[8] used in derivations of the homogenized models for two-phase medium, we can show that

$$\lim_{\varepsilon\to 0} \int\limits_\Omega |u_\lambda^\varepsilon(x) - u_\lambda(x)|^2\, dx = 0, \qquad \lim_{\varepsilon\to 0} \int\limits_{\Omega_{2\varepsilon}} |p_\lambda^\varepsilon(x) - p_\lambda(x, \varepsilon^{-1}x)|^2\, dx = 0,$$

$$\lim_{\varepsilon\to 0} \int\limits_\Omega |e(u_\lambda^\varepsilon(x)) - e(u_\lambda(x)) - w_\lambda(x, \varepsilon^{-1}x)|^2\, dx = 0, \qquad w_\lambda(x,y) = e_y(v_\lambda(x,y)),$$

$$v_\lambda(x,y) = Q_\lambda^{kh}(y)\frac{\partial u_{\lambda k}}{\partial x_h}, \qquad p_\lambda(x,y) = P_\lambda^{kh}(y)\frac{\partial u_{\lambda k}}{\partial x_h},$$

where the pairs $\{Q_\lambda^{kh}(y) \in H^1_{\mathrm{per}}(Y)^3/\mathbb{R}^3, P_\lambda^{kh}(y) \in L^2_{\mathrm{per}}(Y_2)\}$ are solutions of auxiliary problems on the cell Y ("per" stands for the Y-periodicity), while $u_\lambda(x) \in H^1_0(\Omega)^3$ is the solution of the homogenized problem in Laplace transforms

$$\rho_0\lambda^2 u_{\lambda i} = \frac{\partial \sigma^\lambda_{ij}}{\partial x_j} + f_{\lambda i}(x) \quad \text{in} \quad \Omega, \qquad u_\lambda|_{\partial\Omega} = 0. \tag{3.1}$$

Here $\rho_0 = |Y_1|\rho_1 + |Y_2|\rho_2$ is the density and σ^λ is the stress tensor of the homogenized medium written in the Laplace images. For the considered two-phase medium the components of the tensor σ^λ read as

$$\sigma^\lambda_{ij} = D_{ijkh}(\lambda)e_{kh}(u_\lambda) \tag{3.2}$$

with

$$D_{ijkh}(\lambda) = |Y_1|a_{ijkh} + |Y_2|\lambda(\eta+\theta\lambda)(\delta_{ik}\delta_{kh} + \delta_{ih}\delta_{jk}) + \int_{Y_1} a_{ijlm}e^y_{lm}(Q^{kh}_\lambda)\,dy$$
$$+ \int_{Y_2}\Big(2\lambda(\eta+\theta\lambda)e^y_{ij}(Q^{kh}_\lambda) - \delta_{ij}P^{kh}_\lambda\Big)\,dy,$$

and the pairs $\{Q^{kh}_\lambda(y), P^{kh}_\lambda(y)\}$ are solutions to Y-periodic problems

$$\begin{cases} \dfrac{\partial}{\partial y_j}\Big(\sigma^{(0)}_{ij}(Q^{kh}_\lambda, P^{kh}_\lambda)\Big) = 0 \quad \text{in} \quad Y, \qquad \operatorname{div}_y Q^{kh}_\lambda = -\delta_{kh} \quad \text{in} \quad Y_2, \\ \displaystyle\int_Y Q^{kh}_\lambda\,dy = 0, \qquad [Q^{kh}_\lambda]|_\Gamma = 0, \qquad [\sigma^{(0)}_{ij}(Q^{kh}_\lambda, P^{kh}_\lambda)\nu_j]|_\Gamma = 0, \end{cases} \tag{3.3}$$

where ν_j are the components of the unit normal vector to the surface Γ,

$$\sigma^{(0)}_{ij}(Q^{kh}_\lambda, P^{kh}_\lambda) = \begin{cases} a_{ijkh} + a_{ijlm}e^y_{lm}(Q^{kh}_\lambda) & \text{in} \quad Y_1, \\ \lambda(\eta+\theta\lambda)(2e^y_{ij}(Q^{kh}_\lambda) + \delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}) - \delta_{ij}P^{kh}_\lambda & \text{in} \quad Y_2. \end{cases}$$

In order to return back to the original variables x and t, we make the inverse Laplace transform in (3.2) and (3.3). It follows from (3.2) that the components of the homogenized tensor σ are of the form

$$\sigma_{ij} = D_{ijkh}(t) * e_{kh}(u),$$

where $D(t)$ is the tensor of relaxation kernels for the homogenized medium, while the symbol $*$ denotes the convolution in the variable t,

$$g_1(t) * g_2(t) = \int_0^t g_1(t-s)g_2(s)\,ds.$$

It is easy to see that both the originals of solutions to problems (3.3) and the components of the tensor $D(t)$ depend on the delta-function $\delta(t)$. Let us show that $Q^{kh}(y,t)$ and $P^{kh}(y,t)$ can be represented as

$$Q^{kh}(y,t) = \delta(t)Z^{kh}(y) + W^{kh}(y,t), \quad y \in Y,$$
$$P^{kh}(y,t) = \delta''(t)A^{kh}(y) + \delta'(t)B^{kh}(y) + \delta(t)C^{kh}(y) + S^{kh}(y,t), \quad y \in Y_2,$$

where $W^{kh}(y,t)$ and $S^{kh}(y,t)$ are independent of $\delta(t)$. Indeed, it follows immediately from (3.3) that the pairs $\{Z^{kh}(y), A^{kh}(y)\}$ and $\{W^{kh}(y,t), S^{kh}(y,t)\}$ are Y-perioidc solutions to problems

$$\begin{cases} \dfrac{\partial}{\partial y_j}\left(\sigma_{ij}^{(s)}(Z^{kh}, A^{kh})\right) = 0 \quad \text{in} \quad Y_s, \quad s=1,2, \qquad \operatorname{div}_y Z^{kh} = -\delta_{kh} \quad \text{in} \quad Y_2, \\ \displaystyle\int_Y Z^{kh} dy = 0, \qquad [Z^{kh}]|_\Gamma = 0, \qquad \sigma_{ij}^{(2)}(Z^{kh}, A^{kh})\nu_j|_\Gamma = 0, \end{cases} \tag{3.4}$$

and

$$\begin{cases} \dfrac{\partial}{\partial y_j}\left(\sigma_{ij}^{(3)}(W^{kh}, S^{kh})\right) = 0 \quad \text{in} \quad Y, \qquad \operatorname{div}_y W^{kh} = 0 \quad \text{in} \quad Y_2, \\ W^{kh}(y,0) = D^{kh}(y), \qquad \dfrac{\partial W^{kh}}{\partial t}(y,0) = N^{kh}(y) \quad \text{in} \quad Y_2, \\ \displaystyle\int_Y W^{kh} dy = 0, \qquad [W^{kh}]|_\Gamma = 0, \qquad [\sigma_{ij}^{(3)}(W^{kh}, S^{kh})\nu_j]|_\Gamma = 0, \end{cases} \tag{3.5}$$

respectively, where

$$\sigma_{ij}^{(1)}(Z^{kh}, A^{kh}) = a_{ijkh} + a_{ijlm} e_{lm}^y(Z^{kh}), \quad y \in Y_1,$$
$$\sigma_{ij}^{(2)}(Z^{kh}, A^{kh}) = 2\theta e_{ij}^y(Z^{kh}) + \theta(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}) - \delta_{ij}A^{kh}, \quad y \in Y_2,$$
$$\sigma_{ij}^{(3)}(W^{kh}, S^{kh}) = \begin{cases} a_{ijlm} e_{lm}^y(W^{kh}), \quad y \in Y_1, \\ 2\eta e_{ij}^y\left(\dfrac{\partial W^{kh}}{\partial t}\right) + 2\theta e_{ij}^y\left(\dfrac{\partial^2 W^{kh}}{\partial t^2}\right) - \delta_{ij} S^{kh}, \quad y \in Y_2. \end{cases}$$

At the same time, the pairs $\{D^{kh}(y), B^{kh}(y)\}$ and $\{N^{kh}(y), C^{kh}(y)\}$ are Y-periodic solutions of problems

$$\begin{cases} \dfrac{\partial}{\partial y_j}\left(\sigma_{ij}^{(4)}(D^{kh}, B^{kh})\right) = 0, \qquad \operatorname{div}_y D^{kh} = 0 \quad \text{in} \quad Y_2, \\ \displaystyle\int_{Y_2} D^{kh} dy = 0, \qquad \sigma_{ij}^{(4)}(D^{kh}, B^{kh})\nu_j|_\Gamma = 0, \end{cases} \tag{3.6}$$

and

$$\begin{cases} \dfrac{\partial}{\partial y_j}\left(\sigma_{ij}^{(5)}(N^{kh}, C^{kh})\right) = 0, \qquad \operatorname{div}_y N^{kh} = 0 \quad \text{in} \quad Y_2, \\ \displaystyle\int_{Y_2} N^{kh} dy = 0, \quad \sigma_{ij}^{(5)}(N^{kh}, C^{kh})\nu_j|_\Gamma = \sigma_{ij}^{(1)}(Z^{kh}, A^{kh})\nu_j|_\Gamma, \end{cases} \tag{3.7}$$

respectively, where

$$\sigma_{ij}^{(4)}(D^{kh}, B^{kh}) = 2\eta e_{ij}^y(Z^{kh}) + 2\theta e_{ij}^y(D^{kh}) + \eta(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}) - \delta_{ij}B^{kh},$$
$$\sigma_{ij}^{(5)}(N^{kh}, C^{kh}) = 2\eta e_{ij}^y(D^{kh}) + 2\theta e_{ij}^y(N^{kh}) - \delta_{ij}C^{kh}.$$

By means of solutions to problems (3.4)–(3.7), the tensor of the relaxation kernels $D(t)$ can be written as

$$D(t) = \delta(t)\alpha + \delta'(t)\beta + \delta''(t)\gamma - g(t), \tag{3.8}$$

where the components of the tensors α, β, γ, $g(t)$ are given by the formulas

$$\alpha_{ijkh} = |Y_1| a_{ijkh} + \int_{Y_1} a_{ijlm} e_{lm}^y(Z^{kh})\, dy + \int_{Y_2} \left(2\eta e_{ij}^y(D^{kh}) + 2\theta e_{ij}^y(N^{kh}) - \delta_{ij}C^{kh}\right) dy,$$

$$\beta_{ijkh} = \eta|Y_2|(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}) + \int\limits_{Y_2} \Big(2\eta e^y_{ij}(Z^{kh}) + 2\theta e^y_{ij}(D^{kh}) - \delta_{ij}B^{kh}\Big)\,dy,$$

$$\gamma_{ijkh} = \theta|Y_2|(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}) + \int\limits_{Y_2} \Big(2\theta e^y_{ij}(Z^{kh}) - \delta_{ij}A^{kh}\Big)\,dy,$$

$$g_{ijkh}(t) = -\int\limits_{Y_1} a_{ijlm}e^y_{lm}(W^{kh})\,dy - \int\limits_{Y_2} \left(2\eta e^y_{ij}\left(\frac{\partial W^{kh}}{\partial t}\right) + 2\theta e^y_{ij}\left(\frac{\partial^2 W^{kh}}{\partial t^2}\right) - \delta_{ij}S^{kh}\right)dy.$$

It is easy to confirm that $Z^{kh}(y) = Z^{hk}(y)$ and similarly for $D^{kh}(y)$, $N^{kh}(y)$, $W^{kh}(y,t)$, $A^{kh}(y)$, $B^{kh}(y)$, $C^{kh}(y)$ and $S^{kh}(y,t)$. Moreover, the components of the tensors α, β, γ, $g(t)$ obey classical symmetry conditions, that is,

$$\alpha_{ijkh} = \alpha_{jikh} = \alpha_{ijhk} = \alpha_{hkij}$$

and similarly for the tensors β, γ and $g(t)$.

Making the inverse Laplace transform in (3.1), we obtain the homogenized problem corresponding to problem (2.3):

$$\begin{gathered}\rho_0\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i(x,t) \quad \text{in} \quad \Omega\times(0,T),\\ u(x,t)|_{\partial\Omega} = 0, \quad u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0,\end{gathered} \tag{3.9}$$

where, in view of representation (3.8), the components of the homogenized stress tensor σ are of the form

$$\sigma_{ij} = \alpha_{ijkh}e_{kh}(u) + \beta_{ijkh}e_{kh}\left(\frac{\partial u}{\partial t}\right) + \gamma_{ijkh}e_{kh}\left(\frac{\partial^2 u}{\partial t^2}\right) - g_{ijkh}(t) * e_{kh}(u).$$

Thus, the homogenized motion model of the original two-phase medium is written as an initial boundary value problem for a system of integro-differential equations with constant coefficients.

The proof of the unique solvability of periodic problems (3.4)–(3.7) in the case, when the sets Y_2 and E_2 are connected respectively in Y and $\mathbb{R}^3$ is based on the arguing given in [3] in studying solutions of periodic problems of such kind. Moreover, under the mentioned connectedness conditions the tensors α and $D(\lambda)$ as $\lambda > 0$ are positive definite. In its turn, the positive definiteness of the latter tensor is a sufficient condition for the unique solvability of homogenized problem (3.9) [3], [14]. At the same time, as it will be shown in the next section on the example of a layered medium, the connectedness condition for the sets Y_2 and E_2 is not necessary for the unique solvability of homogenized problem (3.9).

The results obtained in the present section can be formulated as the following theorem.

Theorem 3.1. *Let $u^\varepsilon(x,t)$ be a solution of problem (2.3). Then for all $t \in [0,T]$*

$$\lim_{\varepsilon\to 0}\int\limits_{\Omega} |u^\varepsilon(x,t) - u(x,t)|^2\,dx = 0, \qquad \lim_{\varepsilon\to 0}\int\limits_{\Omega_{2\varepsilon}} |p^\varepsilon(x,t) - p(x,\varepsilon^{-1}x,t)|^2\,dx = 0,$$

$$\lim_{\varepsilon\to 0}\int\limits_{\Omega} |e(u^\varepsilon(x,t)) - e(u(x,t)) - w(x,\varepsilon^{-1}x,t)|^2\,dx = 0,$$

where $u(x,t)$ is the solution of the homogenized problem (3.9),

$$w(x,y,t) = e_y(v(x,y,t)), \qquad v(x,y,t) = Z^{kh}(y)\frac{\partial u_k}{\partial x_h} + W^{kh}(y,t) * \frac{\partial u_k}{\partial x_h},$$

$$W^{kh}(y,0) = D^{kh}(y), \qquad \frac{\partial W^{kh}}{\partial t}(y,0) = N^{kh}(y) \quad in \quad Y_2,$$

$$p(x,y,t) = A^{kh}(y)\frac{\partial^3 u_k}{\partial x_h \partial t^2} + B^{kh}(y)\frac{\partial^2 u_k}{\partial x_h \partial t} + C^{kh}(y)\frac{\partial u_k}{\partial x_h} + S^{kh}(y,t) * \frac{\partial u_k}{\partial x_h},$$

while the pairs $\{Z^{kh}(y), A^{kh}(y)\}$, $\{W^{kh}(y,t), S^{kh}(y,t)\}$, $\{D^{kh}(y), B^{kh}(y)\}$ *and* $\{N^{kh}(y), C^{kh}(y)\}$ *are* Y*-periodic solutions to problems* (3.4)–(3.7).

4. Case of layered medium

In the case, when the medium consists of alternating elastic and fluid layers, we can obtain explicit formulas for the components of the tensors α, β, γ and $g(t)$. In order to do this, we adopt that the layers are parallel to the plane Ox_2x_3, and the sets Y_1 and Y_2 are given by

$$Y_1 = \bigcup_{m=0}^{M} (q_{2m}, q_{2m+1}) \times (0,1)^2, \qquad Y_2 = \bigcup_{m=1}^{M} (q_{2m-1}, q_{2m}) \times (0,1)^2, \quad M \geqslant 1,$$

where $0 = q_0 < q_1 < q_2 < \ldots < q_{2M} < q_{2M+1} = 1$. We note that under such assumption the period Y_ε consists M fluid layers and $M+1$ elastic layers, while in the boundary conditions of the periodic problems we have $\nu_1 = 1$, $\nu_2 = \nu_3 = 0$. Moreover, if by q we denote the total volume fraction of the fluid inside the period Y_ε, then

$$q = \frac{|\varepsilon Y_2|}{|Y_\varepsilon|} = \frac{|Y_2|}{|Y|} = \sum_{m=1}^{M} (q_{2m} - q_{2m-1}), \quad 0 < q < 1.$$

We write out solutions for periodic problems (3.4)–(3.7) as $k \leqslant h$. It is easy to confirm that as $k = 2$ and $h = 3$

$$Z^{23}(y) = D^{23}(y) = N^{23}(y) = W^{23}(y,t) = (0,0,0),$$
$$A^{23}(y) = B^{23}(y) = C^{23}(y) = S^{23}(y,t) = 0.$$

In order to write out the solutions to other periodic problems, we introduce a piecewise-linear function

$$z(y_1) = \begin{cases} -y_1 + C_{2m}, & y_1 \in (q_{2m-1}, q_{2m}), \quad m = 1, \ldots, M, \\ \dfrac{q y_1}{1-q} + C_{2m+1}, & y_1 \in (q_{2m}, q_{2m+1}), \quad m = 0, \ldots, M, \end{cases}$$

where the constants C_{2m} and C_{2m+1} are given by the expressions

$$C_1 = \frac{1}{2(1-q)} \left(q - \sum_{m=1}^{2M} (-1)^m q_m^2 \right),$$

$$C_m = \frac{1}{2(1-q)} \left(-q - \sum_{k=1}^{2M} (-1)^k q_k^2 + 2\sum_{k=m}^{2M} (-1)^k q_k \right), \quad m = 2, \ldots, M,$$

$$C_{2M+1} = -\frac{1}{2(1-q)} \left(q + \sum_{m=1}^{2M} (-1)^m q_m^2 \right).$$

As it is known [15], the function $z(y_1)$ satisfies the following conditions:

$$z(+0) = z(1-0), \qquad \int_0^1 z(y_1)\, dy_1 = 0, \qquad [z]|_{y_1 = q_m} = 0, \quad m = 1, \ldots, 2M.$$

We continue the function $z(y_1)$ periodically with the period 1 to the entire real line and keep the same notation for the continuation. It is easy to confirm that the solutions of stationary periodic problems are written as

$$\begin{aligned}
&Z^{ii}(y) = (z(y_1), 0, 0), \qquad Z^{12}(y) = (0, z(y_1), 0), \qquad Z^{13}(y) = (0, 0, z(y_1)),\\
&A^{1i}(y) = B^{1i}(y) = 0, \qquad A^{jj}(y) = -2\theta, \qquad B^{jj}(y) = -2\eta,\\
&D^{ii}(y) = D^{1j}(y) = N^{ii}(y) = (0,0,0), \qquad N^{12}(y) = (0, c_1 z(y_1), 0),\\
&N^{13}(y) = (0, 0, c_1 z(y_1)), \qquad C^{11}(y) = -\frac{\lambda + 2\mu}{1-q}, \qquad C^{jj}(y) = -\frac{\lambda + 2\mu q}{1-q},\\
&C^{1j}(y) = 0, \qquad c_1 = -\frac{\mu}{\theta(1-q)}, \quad i = 1,2,3, \quad j = 2,3.
\end{aligned}$$

The solutions of evolution periodic problems read as

$$\begin{aligned}
&W^{ii}(y,t) = (0,0,0), \qquad\qquad W^{12}(y,t) = (0, z(y_1)w(t), 0),\\
&W^{13}(y,t) = (0, 0, z(y_1)w(t)), \qquad S^{ii}(y,t) = S^{1j}(y,t) = 0, \quad i = 1,2,3, \quad j = 2,3,
\end{aligned}$$

where $w(t)$ is the solution to the differential equation

$$\theta(1-q)\frac{d^2 w}{dt^2} + \eta(1-q)\frac{dw}{dt} + \mu q w(t) = 0$$

satisfying the initial conditions

$$w(0) = 0, \qquad \frac{dw}{dt}(0) = c_1.$$

It is easy to confirm that $w(t) = c_1 w_0(t)$, where

$$w_0(t) = t \exp\left(-\frac{\eta}{2\theta}t\right)$$

if $\eta^2(1-q) = 4q\mu\theta$,

$$w_0(t) = \frac{\theta}{\sqrt{D}}\left(\exp\left(-\frac{\eta - \sqrt{D}}{2\theta}t\right) - \exp\left(-\frac{\eta + \sqrt{D}}{2\theta}t\right)\right)$$

if $\eta^2(1-q) > 4q\mu\theta$, and

$$w_0(t) = \frac{\theta}{\sqrt{-D}} \exp\left(-\frac{\eta}{2\theta}t\right) \sin\left(\frac{\sqrt{-D}}{2\theta}t\right)$$

if $\eta^2(1-q) < 4q\mu\theta$. Here by D we denote the discriminant of the square equation

$$\theta\lambda^2 + \eta\lambda + \frac{q\mu}{1-q} = 0, \tag{4.1}$$

that is,

$$D = \eta^2 - \frac{4q\mu\theta}{1-q}.$$

Once we know explicit solutions of all periodic problems, we can proceed to calculating of the components of the tensors α, β, γ and $g(t)$. But before we should have in mind that $D_{ijkh}(t) = 0$ as soon as $\delta_{ij}\delta_{kh} + \delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk} = 0$. Moreover, the homogenized medium is transversally isotropic, that is, the relations

$$\begin{aligned}
&D_{2222}(t) = D_{3333}(t), \qquad D_{1122}(t) = D_{1133}(t),\\
&D_{1212}(t) = D_{1313}(t), \qquad D_{2222}(t) - D_{2233}(t) = 2D_{2323}(t)
\end{aligned}$$

hold true.

Now we substitute the solutions of problems (3.4)–(3.7) into the formulas for the components of the tensors α, β, γ and $g(t)$ given in the previous section. After simple transformations we arrive at the following result.

Theorem 4.1. *For a layered medium of the above described type the homogenized tensors α, β, γ and $g(t)$ have the following non-zero components:*

$$\alpha_{1111} = \frac{\lambda + 2\mu}{1-q}, \qquad \alpha_{iiii} = \frac{\lambda + 2\mu(1-2q+2q^2)}{1-q}, \qquad \alpha_{iijj} = \frac{\lambda + 2\mu q^2}{1-q},$$
$$\alpha_{11ii} = \alpha_{ii11} = \frac{\lambda + 2\mu q}{1-q}, \qquad \alpha_{1i1i} = \alpha_{1ii1} = \alpha_{i1i1} = \alpha_{i11i} = \frac{\mu}{1-q},$$
$$\alpha_{ijij} = \alpha_{ijji} = \mu(1-q), \qquad \beta_{iiii} = 4\eta q, \qquad \beta_{iijj} = 2\eta q,$$
$$\beta_{ijij} = \beta_{ijji} = \eta q, \qquad \gamma_{iiii} = 4\theta q, \qquad \gamma_{iijj} = 2\theta q, \qquad \gamma_{ijij} = \gamma_{ijji} = \theta q,$$
$$g_{1i1i}(t) = g_{1ii1}(t) = g_{i1i1}(t) = g_{i11i}(t) = \frac{\mu^2 q w_0(t)}{\theta(1-q)^2}, \quad i = 2,3, \quad j = 5-i.$$

It is easy to confirm that α and $D(\lambda) = \alpha + \lambda\beta + \lambda^2\gamma - g(\lambda)$ as $\lambda > 0$ are positive definite tensors, while β, γ and $g(t)$ are degenerate tensors.

To compare, we briefly describe the homogenized tensors for a layered medium consisting of an isotropic elastic material and an incompressible Newtonian fluid, for which we should let $\theta = 0$ in constitutive relations (2.2). As it was shown in [16], for such medium the components of the tensors α and β are calculated exactly by the same formulas as above, while the tensor γ is zero. Concerning the tensor $g(t)$, all its components are zero except for

$$g_{1i1i}(t) = g_{1ii1}(t) = g_{i1i1}(t) = g_{i11i}(t) = \frac{\mu^2 q w_1(t)}{\eta(1-q)^2}, \quad i = 2,3,$$

where we have denoted

$$w_1(t) = \exp\left(-\frac{\mu q}{\eta(1-q)} t\right).$$

We see that the comparing of the properties of the tensors $g(t)$ for two layered medium is reduced to comparing the properties of the functions $w_0(t)$ and $w_1(t)$. We first of all observe a common property of these functions, which is $w_0(t) \to 0$ and $w_1(t) \to 0$ as $t \to +\infty$. While in other aspects, these functions differ essentially. Namely, $w_1(t)$ is a positive function strictly decreasing as $t > 0$, while the behavior of the function $w_0(t)$ depends on the fraction of the fluid q within the period. If $q < \eta^2/(\eta^2 + 4\mu\theta)$, then $w_0(t)$ is a positive function, which first strictly increases achieving the maximum at

$$t = \frac{\theta}{\sqrt{D}} \ln \frac{\eta + \sqrt{D}}{\eta - \sqrt{D}},$$

and then it strictly decreases. If $q = \eta^2/(\eta^2 + 4\mu\theta)$, then the function $w_0(t)$ is also positive and first it strictly increases achieving the maximum at $t = 2\theta/\eta$, and then it strictly decreases. If $q > \eta^2/(\eta^2 + 4\mu\theta)$, then $w_0(t) > 0$ as

$$t \in \left(\frac{4\theta\pi k}{\sqrt{-D}}, \ \frac{2\theta\pi}{\sqrt{-D}} + \frac{4\theta\pi k}{\sqrt{-D}}\right), \quad k = 0, 1, 2, \ldots$$

and $w_0(t) < 0$ for other values $t > 0$. Moreover, the function $w_0(t)$ strictly decreases on the intervals

$$\frac{2\theta}{\sqrt{-D}}\left(2\pi k + \arccos\frac{\eta}{\sqrt{\eta^2 - D}}\right) < t < \frac{2\theta}{\sqrt{-D}}\left(\pi + 2\pi k + \arccos\frac{\eta}{\sqrt{\eta^2 - D}}\right)$$

and strictly increases on other intervals in the semi-axis $t > 0$. Thus, it possesses infinitely many maximum and minimum points, and the absolute values of its maximal and minimal values decay exponentially as $t \to +\infty$.

In conclusion we mention that we have first found the presence of the maxima for the convolution kernels for the integro-differential equations appearing in homogenization of layered two-phase medium with a periodic structure. Earlier we showed that if the first phase is an elastic material or a viscoelastic Kelvin–Voigt material, while the second phase is a viscoelastic Kelvin–Voigt material or a viscous Newtonian fluid, then the convolution kernels of the homogenized equations are decaying exponents [16]–[19]. Thus, for all these media the convolution kernels are positive functions strictly decaying as $t > 0$. In contrast to them, for the media consisting of an elastic material and a Kelvin–Voight fluid, the numbers of monotonicity intervals and of the maximum points of the convolution kernels depend on the fraction of the fluid q within the periodicity cell. Namely, we can find a number M_0 such that as $0 < q \leqslant M_0$ the convolution kernels possess two monotonicity intervals and one maximum point, while as $M_0 < q < 1$ they have infinitely many alternating intervals of increasing and decreasing as well as infinitely many maximum and minimum points.

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