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ON SYMMETRY CLASSIFICATION OF INTEGRABLE EVOLUTION EQUATIONS OF THIRD ORDER

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Abstract. We present new results in the framework of symmetry classification of third order integrable evolution vector equations. A technique proposed by G.A. Meshkov and V.V. Sokolov allowed us to find 12 equations satisfying necessary integrability conditions. We provide a short review of all known equations of the considered type and also clarify all computational difficulties not allowing us to complete the classification problem in the general form.

By imposing reasonable additional restrictions for the form of equations while classifying them we succeed to complete the calculations. The found equations possess several nontrivial conserved densities and they are likely exactly integrable. As the proof of their integrability, the Lax representation or Bäcklund autotransform could serve but finding them is a rather complicated problem requiring a sufficient motivation, for instance, an application value of some of these equations.

Keywords:integrable vector equations, canonical densities, conservation laws.

Mathematics Subject Classification: 37K10, 35Q53

1. Introduction

As classical examples of nonlinear third order evolution equations, the generalizations of modified KdV equations presented in [6] can serve:

$$\boldsymbol{U}_t = \boldsymbol{U}_{xxx} - 6(\boldsymbol{U},\boldsymbol{U})\boldsymbol{U}_x,$$
$$\boldsymbol{U}_t = \boldsymbol{U}_{xxx} - 3(\boldsymbol{U},\boldsymbol{U})\boldsymbol{U}_x - 3(\boldsymbol{U},\boldsymbol{U}_x)\boldsymbol{U}.$$

An interest to searching integrable vector cases increased after papers [12] and [11], in which the authors proposed an effective method for classification of equations of form

$$\boldsymbol{U}_t = \boldsymbol{U}_3 + \boldsymbol{U}_2 f_2 + \boldsymbol{U}_1 f_1 + \boldsymbol{U} f_0, \qquad \boldsymbol{U}_t = \frac{\partial \boldsymbol{U}}{\partial t}, \qquad \boldsymbol{U}_n = \frac{\partial_n \boldsymbol{U}}{\partial x^n}, \tag{1.1}$$

where $\boldsymbol{U} = \boldsymbol{U}(t,x)$ is a vector in the Euclidean space $\mathbb{R}^n$, while unknown functions f_i depend on the scalar products $(\boldsymbol{U}_i, \boldsymbol{U}_j) = u_{[i,j]}$, $0 \leqslant i \leqslant j \leqslant 2$. The variables $u_{[i,j]}$ are regarded as independent due to an arbitrary dimension of the space $\mathbb{R}^n$ and, according to a usual practice, an order $(\operatorname{ord} f)$ of a function $f = f(u_{[0,0]}, \dots, u_{[i,j]})$ is the order of the higher derivative of the variables $u_{[i,j]}$ involved in the variables of this function.

By now, in [1], [4], [5] and [7] there were obtained the lists of integrable equations (1.1) of following types:

$$\boldsymbol{U}_t = (\boldsymbol{U}_2 + \boldsymbol{U}_1 f_1 + \boldsymbol{U}_0 f_0)_x, \quad \operatorname{ord} f_i \leqslant 1;$$
$$\boldsymbol{U}_t = \boldsymbol{U}_3 + \boldsymbol{U}_1 f_1 + \boldsymbol{U}_0 f_0, \quad \operatorname{ord} f_i \leqslant 2;$$
$$\boldsymbol{U}_t = \boldsymbol{U}_3 + \boldsymbol{U}_2 f_2 + \boldsymbol{U}_1 f_1 + \boldsymbol{U} f_0, \quad \operatorname{ord} f_i \leqslant 2, \quad \operatorname{ord} f_0 \leqslant 1;$$

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$$\boldsymbol{U}_t = \boldsymbol{U}_3 - 3\boldsymbol{U}_2 \frac{u_{[0,1]}}{u_{[0,0]}} + \boldsymbol{U}_1 f_1 + \boldsymbol{U} f_0, \quad \operatorname{ord} f_i \leqslant 2.$$

In addition, extra three classification problems were solved in [2], [3] and [10], where apriori restrictions were not so much for unknown functions f_i, but rather for the presence of certain properties of (1.1).

The aim of the present work is to advance in classifying equations (1.1) and to find new integrable problems.

As in the above cited works, we apply a symmetry approach based on constructing *canonical densities*, which are specific local densities of conservation laws obtained by means of formal operator series. This method was proposed in [8] and generalized in [9], while its vector analogue was presented in [11]. The matter of the technique is that as a time Lax equation for (1.1) one takes $(-D_t + D_x^3 + f_2 D_x^2 + f_1 D_x + f_0)\psi = 0$ and make a standard substitution

$$\psi = \exp\left(\int R\, dx\right).$$

As a result one gets a Ricatti type equation

$$(D_x + R)^2 R + f_2 (D_x + R) R + f_1 R + f_0 = F, \quad D_x F = D_t R, \tag{1.2}$$

which possess formal solutions of form

$$R = \lambda^{-1} + \sum_{n=0}^{\infty} \rho_n \lambda^n, \qquad F = \lambda^{-3} + \sum_{n=0}^{\infty} \theta_n \lambda^n. \tag{1.3}$$

Substituting (1.3) into the first equation in (1.2), we arrive at a recurrent formula

$$\begin{aligned}\rho_{n+2} =& \frac{1}{3}\Big[\theta_n - f_0\,\delta_{n,0} - 2\, f_2\, \rho_{n+1} - f_2\, D_x \rho_n - f_1\, \rho_n\Big] \\ & - \frac{1}{3}\Big[f_2 \sum_{s=0}^{n} \rho_s\, \rho_{n-s} + \sum_{0 \leqslant s+k \leqslant n} \rho_s\, \rho_k\, \rho_{n-s-k} + 3 \sum_{s=0}^{n+1} \rho_s\, \rho_{n-s+1}\Big] \\ & - D_x \Big[\rho_{n+1} + \frac{1}{2} \sum_{s=0}^{n} \rho_s\, \rho_{n-s} + \frac{1}{3} D_x \rho_n\Big], \quad n \geqslant 0.\end{aligned}$$

Here $\delta_{i,j}$ is the Kronecker delta and

$$\rho_0 = -\frac{1}{3}\, f_2, \qquad \rho_1 = \frac{1}{9}\, f_2^2 - \frac{1}{3}\, f_1 + \frac{1}{3}\, D_x\, f_2. \tag{1.4}$$

Using (1.3), by the second equation in (1.2) we obtain an infinite series of conservation laws

$$D_t \rho_n = D_x \theta_n, \qquad n = 0, 1, 2, \ldots, \tag{1.5}$$

where ρ_n and θ_n are the functions of the variables $u_{[i,j]}$. At the same time, the differentiation operators are defined as

$$D_x = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} \boldsymbol{U}_{i+1} \frac{\partial}{\partial \boldsymbol{U}_i}, \qquad D_t = \frac{\partial}{\partial t} + \sum_{i=0}^{\infty} D_x^i (\boldsymbol{U}_3 + f_2 \boldsymbol{U}_2 + f_1 \boldsymbol{U}_1 + f_0 \boldsymbol{U}) \frac{\partial}{\partial \boldsymbol{U}_i}.$$

The differentiation rules for the scalar products $u_{[i,j]}$ are implied by the identity $D_x \boldsymbol{U}_i = \boldsymbol{U}_{i+1}$ and the bilinearity of the scalar product; the evolution derivative D_t is calculated by the chain rule.

The recursion formula allows us to find the functions θ_n straightforwardly from (1.5) since the expressions for ρ_n involve θ_k, $k \leqslant n-2$. For instance,

$$\rho_2 = -\frac{1}{3}\, f_0 + \frac{1}{3}\, \theta_0 - \frac{2}{81}\, f_2^3 + \frac{1}{9}\, f_1\, f_2 - D_x \left(\frac{1}{9}\, f_2^2 + \frac{2}{9}\, D_x\, f_2 - \frac{1}{3}\, f_1\right).$$

The functions ρ_n are called canonical densities of equation (1.1) and they are expressed via its coefficients. Thus, identities (1.5) are in fact conditions for determining f_i and this is why (1.5) are called *ρ_n-integrability conditions*.

A convenient tool allowing one to simplify the form of a studied equation in classification are point transforms $\boldsymbol{U} \to \boldsymbol{V}$:

$$\boldsymbol{U} = \left(\frac{f(v_{[0,0]})}{v_{[0,0]}}\right)^{1/2} \boldsymbol{V}, \tag{1.6}$$

where f is an arbitrary function since $f' \neq 0$. The square of (1.6) looks rather simple: $u_{[0,0]} = f(v_{[0,0]})$ and this indicates the non-degeneracy of this transform. Let us provide other transforms admitted by (1.1).

Scale transformations:

$$x \to \varepsilon x, \qquad t \to \varepsilon^3 t. \tag{1.7}$$

$$\boldsymbol{U} \to \lambda \boldsymbol{U}, \qquad u_{[i,k]} \to \lambda^2 u_{[i,k]}. \tag{1.8}$$

Galileo transformations:

$$\tilde{t} = t, \qquad \tilde{x} = x + ct. \tag{1.9}$$

Exponential transform:

$$\boldsymbol{U} = e^{pt+kx}\boldsymbol{V}, \quad \boldsymbol{U}_1 = e^{pt+kx}(\boldsymbol{V}_1 + k\boldsymbol{V}), \quad \boldsymbol{U}_t = e^{pt+kx}(\boldsymbol{V}_t + p\boldsymbol{V}), \quad \dots, \tag{1.10}$$

where p and k are parameters. It is obvious that this transforms is possible only for equations homogeneous in the sense of transform (1.8).

Together with the considered transforms, equation (1.1) is invariant with respect to the rotations in $\mathbb{R}^n$: $\boldsymbol{U}' = O\boldsymbol{U}$, $OO^T = E$, and this is why in classification it is sometimes convenient to pass to the spherical coordinate system. The passage from the Cartesian coordinates to the spherical ones is made by the following formulas:

$$\begin{array}{llll} \boldsymbol{U} = R\boldsymbol{V}, & v_{[0,0]} = 1; & \boldsymbol{U}_x = R_x\boldsymbol{V} + R\boldsymbol{V}_x, & \dots, \\ u_{[0,0]} = R^2, & u_{[0,1]} = R\,R_x, & u_{[1,1]} = R^2 v_{[1,1]} + R_x^2, & \dots, \end{array} \tag{1.11}$$

where R is the spherical radius, while the components of the vector $\boldsymbol{V}$ serve as angular variables. We observe that the differentiation of the identity $v_{[0,0]} = 1$ gives $v_{[0,1]} = 0$, $v_{[0,2]} = -v_{[1,1]}$ and so forth. As a result, all variables $v_{[0,k]}$, $k > 1$, are expressed via $v_{[i,j]}$, $1 \leqslant i \leqslant j \leqslant k$. The formulas for the inverse transform can be easily obtained directly from (1.11).

Definition 1.1. *If in variables (1.11) equation (1.1) is reduced to a system of form*

$$\begin{aligned} &\boldsymbol{V}_t = \boldsymbol{V}_3 + f_2\boldsymbol{V}_2 + f_1\boldsymbol{V}_1 + f_0\boldsymbol{V}, \quad f_i = f_i(v_{[1,1]}, v_{[1,2]}, v_{[2,2]}), \\ &R_t = R_3 + \Phi(R_2, R_1, R, v_{[1,1]}, v_{[1,2]}, v_{[2,2]}), \end{aligned}$$

then such system is called triangular.

In the present paper we do not consider equations equations, which pass to triangular systems since a complete list of vector equations (1.1) integrable on the sphere $\mathbb{S}^n$ was obtained in [11], see also [10].

2. Results of analysis of integrability conditions

It was established in work [11] that the even densities ρ_n are trivial, that is, $\rho_{2n} = D_x\chi_n$. Taking into consideration (1.4), without loss of generality we can let $f_2 = \frac{3}{2}D_x(\ln f)$, $\operatorname{ord} f = 1$.

The analysis of the first of conditions (1.5) allows us to determine the dependence of f_1 on the variables of the second order and (1.1) becomes

$$\begin{aligned} \boldsymbol{U}_t =& \boldsymbol{U}_3 + \frac{3}{2} D_x(\ln f)\, \boldsymbol{U}_2 + \\ &+ \left(cf u_{[2,2]} + a_1 u_{[1,2]}^2 + a_2 u_{[1,2]} u_{[0,2]} + a_3 u_{[0,2]}^2 + a_4 u_{[1,2]} + a_5 u_{[0,2]} + a_6\right) \boldsymbol{U}_1 + f_0 \boldsymbol{U}, \end{aligned} \tag{2.1}$$

where $\operatorname{ord} a_i \leqslant 1$, $c = \mathrm{const}$.

The function ρ_2 has a forth order (due to the term θ_0), but after extracting and neglecting trivial terms being total derivatives in the variable x, we obtain a function of second order, that is, $\rho_2 \sim F(u_{[2,2]}, u_{[1,2]}, u_{[0,2]}, \dots)$. Since $u_{[2,2]} \notin \operatorname{Im} D$, the condition $\rho_2 \in \operatorname{Im} D$ involves the restriction $\partial F / \partial u_{[2,2]} = 0$ and its simplest implication are written as

$$\begin{aligned} \frac{\partial^2 f_0}{\partial u_{[2,2]}^2} \frac{\partial}{\partial u_{[1,1]}} \frac{1}{f} \left(2u_{[0,1]} \frac{\partial f}{\partial u_{[1,1]}} + u_{[0,0]} \frac{\partial f}{\partial u_{[0,1]}} \right) = 0, \\ \frac{\partial^2 f_0}{\partial u_{[2,2]}^2} \frac{\partial}{\partial u_{[0,1]}} \frac{1}{f} \left(2u_{[0,1]} \frac{\partial f}{\partial u_{[1,1]}} + u_{[0,0]} \frac{\partial f}{\partial u_{[0,1]}} \right) = 0. \end{aligned}$$

Thus, we come to the first fork.

The option $\partial^2 f_0 / \partial u_{[2,2]}^2 \neq 0$ is completely calculated. In this case, all equations in coordinates (1.11) pass to the known integrable equations and at the same time the function f_0 is not determined by the integrability condition and remains arbitrary.

Under the condition $\partial^2 f_0 / \partial u_{[2,2]}^2 = 0$ we obtain that $f_0 = g_1\, u_{[2,2]} + g_2$, where the functions g_1 and g_2 are independent of $u_{[2,2]}$ and their order does not increase 2. The case $\operatorname{ord} f_0 = 1$ was completely studied in [7] and this is why in what follows we let $\operatorname{ord} f_0 = 2$.

Compact equations can be obtained from the first and forth integrability conditions:

$$\left(2\, u_{[0,1]} \frac{\partial f}{\partial u_{[1,1]}} + u_{[0,0]} \frac{\partial f}{\partial u_{[0,1]}} \right) \left\{ \frac{\partial g_1}{\partial u_{[i,2]}},\ \frac{\partial^3 g_2}{\partial u_{[i,2]} \partial u_{[j,2]} \partial u_{[k,2]}} \right\} = 0, \tag{2.2}$$

$$\left\{ a_2\, u_{[0,0]} + 2\, a_1\, u_{[0,1]},\ a_2\, u_{[0,1]} + 2\, a_3\, u_{[0,0]} + 2\, c\, f \right\} \frac{\partial g_1}{\partial u_{[i,2]}} = 0, \qquad i, j, k = 0, 1. \tag{2.3}$$

We study the forks appearing (2.2) and (2.3) in the following order:

$$(\mathbf{a}) \quad 2\, u_{[0,1]} \frac{\partial f}{\partial u_{[1,1]}} + u_{[0,0]} \frac{\partial f}{\partial u_{[0,1]}} \equiv \psi \neq 0; \qquad (\mathbf{b}) \quad \psi = 0.$$

Option (a). It follows from (2.2) that $\operatorname{ord} g_1 < 2$ and the function g_2 is quadratic in the variables $u_{[0,2]}$, $u_{[1,2]}$, that is,

$$\begin{aligned} f_0 =& g_1\, u_{[2,2]} + b_1\, u_{[1,2]}^2 + b_2\, u_{[0,2]}^2 \\ &+ b_3\, u_{[1,2]}\, u_{[0,2]} + b_4\, u_{[1,2]} + b_5\, u_{[0,2]} + b_6 \end{aligned}$$

and the order of the functions b_i does not exceed one.

The analysis of the first six integrability conditions mostly consists in considering various conditions for unknown functions. For instance, assuming that $g_1 \neq 0$, we always obtain equations, in which point transformations allow to remove the term with $u_{[2,2]}$. As a result, in this option only five equations satisfy seven ρ_n-integrability conditions ($n = 0, \dots, 6$). We also establish that each of them possesses a higher symmetry of fifth order and, up to the point transforms, it is reduced to one of the equations in the following list:

$$
\begin{aligned}
\boldsymbol{U}_t =& \boldsymbol{U}_3 + f_x \boldsymbol{U}_2 + \frac{3}{2}\left(\frac{u_{[0,0]}u_{[2,2]}}{\eta} + \frac{m(g^2 - k^2 u_{[0,1]}^4)}{\eta u_{[0,0]}^2}\right)\boldsymbol{U}_1 \\
&- m\left(\frac{u_{[0,1]}^2 f_x}{u_{[0,0]}^2} + \frac{3u_{[0,1]}g - u_{[0,1]}^3}{u_{[0,0]}^3}\right)\boldsymbol{U},
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
\boldsymbol{U}_t =& \boldsymbol{U}_3 + f_x \boldsymbol{U}_2 \\
&+ \frac{3}{2}\left(\frac{u_{[0,0]}u_{[2,2]}}{\eta} - \frac{(u_{[0,0]}f_x + 3u_{[0,1]})^2}{9(\eta - u_{[0,0]}^2)} + \frac{m(g^2 - k^2 u_{[0,1]}^4) + k^2 u_{[0,0]}^2 u_{[0,1]}^2}{\eta u_{[0,0]}^2}\right)\boldsymbol{U}_1 \\
&- m\left(\frac{u_{[0,1]}^2 f_x}{u_{[0,0]}^2} + \frac{3u_{[0,1]}g - u_{[0,1]}^3}{u_{[0,0]}^3}\right)\boldsymbol{U},
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
\boldsymbol{U}_t =& \boldsymbol{U}_3 + f_x \boldsymbol{U}_2 \\
&+ \frac{3}{2}\left(\frac{u_{[0,0]}u_{[2,2]}}{\eta} + \frac{(u_{[0,0]}f_x + 3(k+1)u_{[0,1]})^2}{9u_{[0,0]}^2} + \frac{m(g^2 - k^2 u_{[0,1]}^4)}{\eta u_{[0,0]}^2}\right)\boldsymbol{U}_1 \\
&- m\left(\frac{u_{[0,1]}^2 f_x}{u_{[0,0]}^2} + \frac{3u_{[0,1]}g - u_{[0,1]}^3}{u_{[0,0]}^3}\right)\boldsymbol{U},
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
\boldsymbol{U}_t =& \boldsymbol{U}_3 + f_x \boldsymbol{U}_2 + 3\left(\frac{u_{[0,0]}u_{[2,2]}}{\eta} + \frac{(k+1)(2u_{[0,1]}u_{[0,0]}f_x + 3g + 3k^2 u_{[0,1]}^2)}{3u_{[0,0]}^2}\right. \\
&\left.+\frac{mg^2}{\eta u_{[0,0]}^2} - \frac{mku_{[0,1]}^2(\eta + ku_{[0,1]}^2)}{\eta u_{[0,0]}}\right)\boldsymbol{U}_1 - m\left(\frac{u_{[0,1]}^2 f_x}{u_{[0,0]}^2} + \frac{3u_{[0,1]}g - u_{[0,1]}^3}{u_{[0,0]}^3}\right)\boldsymbol{U},
\end{aligned}
\tag{2.7}
$$

where

$$
\begin{aligned}
f =& \frac{3}{2}\ln\frac{u_{[0,0]}}{\eta}, \qquad g = u_{[0,0]}(u_{[0,2]} + u_{[1,1]}) - u_{[0,1]}^2, \\
\eta =& u_{[0,0]}u_{[1,1]} + mu_{[0,1]}^2, \qquad k^2 = m + 1,\ m\,k \neq 0; \\
\boldsymbol{U}_t =& \boldsymbol{U}_3 + \frac{3}{2}(\ln f)_x \boldsymbol{U}_2 + 3\left(\frac{b\,u_{[0,1]}\,f_x^2}{f\,g^2} - \frac{(a + fu_{[0,2]})\,f_x}{f\,g}\right. \\
&+ \frac{4u_{[0,1]}^3(h^2 - (b^2+1)g^2 u_{[0,0]}^2)}{3g^4 u_{[0,0]}^3} \\
&\left.- \frac{u_{[0,1]}\,h}{u_{[0,0]}^2\,f\,g^2}\left(a + fu_{[0,2]} - \frac{u_{[0,1]}(3\,g - 4\,b)\,f_x}{2g} - \frac{(g^2+1)\,f\,u_{[0,1]}^2}{3\,g^2\,u_{[0,0]}}\right)\right)\boldsymbol{U} \\
&+ \frac{3}{2a}\left(u_{[2,2]}f - \frac{(a + fu_{[0,2]})^2}{u_{[0,0]}f}\right. \\
&\left.- \frac{(u_{[0,0]}g(ff_x u_{[0,0]}u_{[0,1]} - (a + fu_{[0,2]})g) + u_{[0,1]}^2 fh)^2}{u_{[0,0]}^3 fg^6}\right)\boldsymbol{U}_1.
\end{aligned}
\tag{2.8}
$$

Here a, b are constants,

$$
g = b + fu_{[0,0]}, \qquad h = g^2 + 2\,b\,g - 1,
$$

and f satisfies the equation

$$u_{[0,0]}u_{[1,1]} - u_{[0,1]}^2 = \frac{a\,u_{[0,0]}}{f} + \frac{u_{[0,1]}^2}{(b + fu_{[0,0]})^2}.$$

Option (b). We have not succeeded to calculate completely this option. The difficulty is due to the fact that among the results of [1], [3] and [4], this option involves many versions of the equations of the case $\partial^2 f_0/\partial u_{[2,2]}^2 \neq 0$. At the same time, in [2] and [7] there are examples of integrable equations, in which the condition $\psi = 0$ is satisfied and at the same time in (2.1) we have

$$f = \frac{u_{[0,0]}}{u_{[0,0]}u_{[1,1]} - u_{[0,1]}^2}. \tag{2.9}$$

This function f is invariant with respect to transform (1.6) and this is why, in order to obtain new integrable equations, we chose exactly (2.9) as an additional restriction. An analysis of integrability conditions allowed us to establish that under condition (2.9) in this options, up to point transforms, there are only the following seven equations obeying seven ρ_n-integrability conditions:

$$\begin{aligned}\boldsymbol{U}_t =&\boldsymbol{U}_3 + \frac{3}{2}(\ln f)_x\boldsymbol{U}_2\\
&+\frac{3}{2}\left(fu_{[2,2]} + \frac{g^2}{u_{[0,0]}^2} - \frac{(1+fu_{[0,2]})^2}{fu_{[0,0]}} - \frac{(u_{[0,1]} - 2k_1u_{[0,0]})u_{[0,1]}}{u_{[0,0]}^2}\right)\boldsymbol{U}_1\\
&-\frac{3}{2}\left(k_1fu_{[2,2]} - \frac{(k_1u_{[0,0]} - 2u_{[0,1]})g^2}{u_{[0,0]}^3}\right.\\
&-\frac{2(u_{[0,0]}u_{[0,2]} + 2(k_1u_{[0,0]} - u_{[0,1]})u_{[0,1]})g}{u_{[0,0]}^3}\\
&\left.-\frac{2u_{[1,2]}}{u_{[0,0]}} - \frac{k_1fu_{[0,2]}^2}{u_{[0,0]}} - \frac{u_{[0,1]}^2(3k_1u_{[0,0]} - 2u_{[0,1]})}{u_{[0,0]}^3} + \frac{k_1u_{[0,0]} + 2u_{[0,1]}}{fu_{[0,0]}^2}\right)\boldsymbol{U},\end{aligned} \tag{2.10}$$

$$\begin{aligned}\boldsymbol{U}_t =&\boldsymbol{U}_3 + \frac{3}{2}(\ln f)_x\boldsymbol{U}_2 + \frac{3}{2}\left(fu_{[2,2]} + \frac{g^2}{u_{[0,0]}^2} - \frac{(1+fu_{[0,2]})^2}{fu_{[0,0]}} + \frac{2u_{[0,1]}^2f}{3(a\xi+1)^2}\right)\boldsymbol{U}_1\\
&-3\left(\frac{2au_{[0,1]}^3f^2}{9\xi(a\xi+1)^3} - \frac{u_{[0,1]}g^2}{u_{[0,0]}^3(a\xi+1)^2} + \frac{(1+fu_{[0,2]})g}{fu_{[0,0]}^2(a\xi+1)}\right.\\
&\left.-\frac{(g+2u_{[0,1]})(u_{[0,0]}(1+fu_{[0,2]}) - 2u_{[0,1]}f) + 2fu_{[0,1]}^3}{fu_{[0,0]}^3} + \frac{u_{[0,1]}g^2}{u_{[0,0]}^3}\right)\boldsymbol{U},\end{aligned} \tag{2.11}$$

$$\begin{aligned}\boldsymbol{U}_t =&\boldsymbol{U}_3 + \frac{3}{2}(\ln f)_x\boldsymbol{U}_2\\
&+\frac{3}{2}\left(fu_{[2,2]} + \frac{g^2}{u_{[0,0]}^2} - \frac{(1+fu_{[0,2]})^2}{fu_{[0,0]}} + \frac{k_1(2u_{[0,0]} + 3u_{[0,1]})^2f}{u_{[0,0]}^2}\right)\boldsymbol{U}_1\\
&+\left(fu_{[2,2]} + \frac{3(1+fu_{[0,2]})g}{fu_{[0,0]}^2} - \frac{(1+fu_{[0,2]})^2}{fu_{[0,0]}} - \frac{(9k_1+4)(u_{[0,0]} + u_{[0,1]})u_{[0,1]}^2}{u_{[0,0]}^3}\right.\\
&\left.-\frac{(2u_{[0,0]} + 3u_{[0,1]})(2u_{[0,1]}fg - (1+fu_{[0,2]})u_{[0,0]})}{fu_{[0,0]}^3} - \frac{(u_{[0,0]} + 3u_{[0,1]})g^2}{u_{[0,0]}^3}\right)\boldsymbol{U},\end{aligned} \tag{2.12}$$

$$
\begin{aligned}
\boldsymbol{U}_t =& \boldsymbol{U}_3 + \frac{3}{2}(\ln f)_x \boldsymbol{U}_2 \\
&+ \frac{3}{2}\left(f u_{[2,2]} + \frac{g^2}{u_{[0,0]}^2} - \frac{(1+fu_{[0,2]})^2}{fu_{[0,0]}} - \frac{3(2k_1u_{[0,0]}+u_{[0,1]})^2 f}{4u_{[0,0]}(a\xi+1)^2} \right) \boldsymbol{U}_1 \\
&+ 3\left(k_1 f u_{[2,2]} + \frac{(2k_1u_{[0,0]}+u_{[0,1]})g^2}{u_{[0,0]}^3(a\xi+1)^2} - \frac{(k_1u_{[0,0]}+u_{[0,1]})(g+2u_{[0,1]})^2}{u_{[0,0]}^3} \right. \\
&+ \frac{k_1(1-f^2u_{[0,2]}^2)}{fu_{[0,0]}} + \frac{(u_{[0,1]}+g)(1+fu_{[0,2]})}{fu_{[0,0]}^2} + \frac{2u_{[0,1]}^2(4u_{[0,1]}+3g)}{3u_{[0,0]}^3} \\
&+ \frac{(fu_{[0,1]}^2-(1+fu_{[0,2]})u_{[0,0]})g}{fu_{[0,0]}^3(a\xi+1)} + \frac{af(3cu_{[0,0]}+u_{[0,1]})\xi u_{[0,1]}^2}{4u_{[0,0]}^2(a\xi+1)^3} \\
&\left. - \frac{3k_1f(4k_1u_{[0,0]}+u_{[0,1]})u_{[0,1]}}{4u_{[0,0]}(a\xi+1)^3} + \frac{c^3(3a\xi+1)}{a^2(a\xi+1)^3} \right) \boldsymbol{U},
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
\boldsymbol{U}_t =& \boldsymbol{U}_3 + \frac{3}{2}(\ln f)_x \boldsymbol{U}_2 \\
&+ \frac{3}{2}\left(f u_{[2,2]} + \frac{(g+u_{[0,1]})^2}{(1-f^{-1})u_{[0,0]}^2} - \frac{(1+fu_{[0,2]})^2}{fu_{[0,0]}} + \frac{u_{[0,1]}^2 f}{u_{[0,0]}(u_{[0,0]}+a)} \right) \boldsymbol{U}_1 \\
&+ 3\left(\frac{u_{[0,1]}^2 fg - (1+fu_{[0,2]})u_{[0,0]}u_{[0,1]}}{fu_{[0,0]}^2(u_{[0,0]}+a)} - \frac{(a-2)u_{[0,1]}^3}{u_{[0,0]}^2(u_{[0,0]}+a)^2} \right. \\
&\left. + b\sqrt{(u_{[0,0]}+a)(1-f^{-1})} \right) \boldsymbol{U},
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
\boldsymbol{U}_t =& \boldsymbol{U}_3 + \frac{3}{2}(\ln f)_x \boldsymbol{U}_2 \\
&+ \frac{3}{2}\left(f u_{[2,2]} + \left(\frac{g}{u_{0,0]}} - \frac{k_1\varphi}{u_{[0,0]}\xi} \right)^2 - \frac{(1+fu_{[0,2]})^2}{fu_{[0,0]}} \right. \\
&\left. - \frac{k_2(u_{[0,1]}-2u_{[0,0]})u_{[0,1]}}{u_{[0,0]}^2} \right) \boldsymbol{U}_1 \\
&- \frac{3}{2}\left(f u_{[2,2]} + \left(\frac{g}{u_{0,0]}} - \frac{k_1\varphi}{6u_{[0,0]}\xi} \right)^2 \right. \\
&+ \frac{2(1-fu_{[1,2]})(1+fu_{[0,2]})}{fu_{[0,0]}} - \frac{2\varphi(g+u_{[0,1]})^2}{u_{[0,0]}^3} \\
&- \frac{(u_{[0,0]}-2u_{[0,1]})(1+fu_{[0,2]})^2}{fu_{[0,0]}^2} \\
&+ \frac{k_2u_{[0,1]}^2(3u_{[0,0]}-2u_{[0,1]})}{3u_{[0,0]}^3} + \frac{2k_1\varphi^2 g}{u_{[0,0]}^3\xi} + \frac{6k_1\varphi u_{[0,1]}^2}{u_{[0,0]}^3\xi} \\
&\left. - \frac{6k_1\varphi(1+fu_{[0,2]})}{fu_{[0,0]}^2\xi} - \frac{2(3u_{[0,0]}-u_{[0,1]})u_{[0,1]}^2}{3u_{[0,0]}^3} \right) \boldsymbol{U},
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
\boldsymbol{U}_t =& \boldsymbol{U}_3 + \frac{3}{2}(\ln f)_x \boldsymbol{U}_2 + \frac{3}{2}\Bigg(f u_{[2,2]} + \left(\frac{g}{u_{[0,0]}} + \frac{2k_1\varphi}{u_{[0,0]}(a\xi+1)} \right)^2 - \frac{(1+fu_{[0,2]})^2}{fu_{[0,0]}} \\
& - \frac{k_2(a\xi u_{[0,1]} + u_{[0,0]})(a\xi(2u_{[0,0]} - u_{[0,1]}) + u_{[0,0]})}{u_{[0,0]}^2(a\xi+1)^2} \Bigg) \boldsymbol{U}_1 - \frac{3}{2}\Bigg(f u_{[2,2]} + \frac{2a\xi u_{[0,1]} g^2}{u_{[0,0]}^3(a\xi+1)} \\
& + \frac{2(a\xi u_{[0,1]} + u_{[0,0]})g^2}{u_{[0,0]}^3(a\xi+1)^2} - \frac{g^2}{u_{[0,0]}^2} - \frac{2a\xi(1+fu_{[0,2]})g}{fu_{[0,0]}^2(a\xi+1)} - \frac{4u_{[0,1]}\varphi g}{u_{[0,0]}^3} \\
& - \frac{2u_{[0,1]}^2 g}{u_{[0,0]}^3(a\xi+1)} + \frac{4k_1\varphi^2(2g + k_1 u_{[0,0]})}{u_{[0,0]}^3(a\xi+1)^2} + \frac{1 - f^2u_{[0,2]}^2}{fu_{[0,0]}} - \frac{2u_{[0,1]}(1+fu_{[0,2]})}{fu_{[0,0]}^2} \\
& + \frac{8k_1^2\varphi^3}{3u_{[0,0]}^3(a\xi+1)^3} - \frac{2a\xi k_2(afu_{[0,1]} + \xi)^2\varphi}{3fu_{[0,0]}^2(a\xi+1)^3} - \frac{k_2(afu_{[0,1]} + \xi)^2}{3fu_{[0,0]}(a\xi+1)^2} \\
& + \frac{4k_1\varphi(fu_{[0,1]}(g - u_{[0,1]}) + (1+fu_{[0,2]})u_{[0,0]})}{fu_{[0,0]}^3(a\xi+1)} - \frac{4(3u_{[0,0]} - 2u_{[0,1]})u_{[0,1]}^2}{3u_{[0,0]}^3} \Bigg) \boldsymbol{U},
\end{aligned}
\tag{2.16}
$$

where a, b, k_1, k_2 are constants and

$$
g = f(u_{[0,0]}u_{[1,2]} - u_{[0,1]}u_{[0,2]}) - 2u_{[0,1]}, \qquad \varphi = u_{[0,0]} - u_{[0,1]},
$$
$$
\xi^2 = fu_{[0,0]}, \qquad f = \frac{u_{[0,0]}}{u_{[0,0]}u_{[1,1]} - u_{[0,1]}^2}.
$$

3. Concluding remarks

The exact integrability of (2.4)–(2.8) and (2.10)–(2.16) could be proved by Bäcklund auto-transforms or differential substitutions relating their solutions one to another or with solutions of already known integrable cases, see [2] and [11]. Equation (2.10) was first obtained in [2] by means of first order differential substitution into an exactly integrable equation. We found no other differential substitutions, while the construction of Backlünd auto-transformations for such cumbersome equations is a rather labour-consuming problem requiring a convincing motivation. At the same time, all equations found in this work possess several non-trivial conservation laws and satisfy seven ρ_n-integrability conditions (1.5) and this is why they are likely integrable.

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