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DIRECT AND INVERSE THEOREMS OF APPROXIMATION THEORY IN LEBESGUE SPACES WITH MUCKENHOUPT WEIGHTS

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Abstract. In this work we establish direct and inverse theorems of approximation theory in Lebesgue spaces $L_{p,w}$ with Muckenhoupt weights w on the axis and on a period. The classical definition of the modulus of continuity can be meaningless in weighted spaces. Therefore, as the modules of continuity, including non-integer order, we use the norms of powers of deviation of Steklov means. The properties of these quantities are derived, some of which are similar to the properties of usual modules of continuity. In addition to the direct and inverse theorems, we obtain equivalence relations between the modules of continuity and the K- and R-functionals.

The proofs are based on estimates for the norms of convolution operators and they do not employ a maximal function. This allows us to establish the results for all $p \in [1, +\infty)$ not excluding the case $p = 1$. Previously used methods that employed the maximal function in one form or another are unsuitable for $p \to 1$. In addition, by the convolution-based approach we can obtain results simultaneously in the periodic and non-periodic case. With rare exceptions, constants are not specified explicitly, but their dependence on parameters is always tracked. All constants in the estimates depend on $[w]_p$ (Muckenhoupt characteristics of weight w), and there is no other dependence on w and p. The norms of convolution operators are estimated explicitly in terms of $[w]_p$. The methods of this work can be applied to prove direct and inverse theorems in more general functional spaces.

Keywords: best approximations, modules of continuity, Muckenhoupt weights, convolution.

Mathematics Subject Classification: 41A17, 42A10

1. Introduction

1.1. Survey of results. A lot of works were devoted to extending the classical theorems of approximation theory [1] from L_p spaces to more general function spaces. We mention a few of them [2]– [7], which are directly related to the topic of this article, i.e., the Lebesgue spaces $L_{p,w}$ with Muckenhoupt weights w. In a number of works, these questions were studied in more general weighted spaces, including $L_{p,w}$ as a special case (Orlicz and Lorentz spaces, Lebesgue spaces with a variable exponent), see, for instance, [8], [9] and the references in [7], [9].

In the mentioned works there were considered spaces of periodic functions (except for [3]) and the case $p > 1$. The constants in the estimates depended on $[w]_p$, which is the Muckenhoupt characteristic of the weight $w \in A_p$, and on p. Due to the dependence of the constants on p, the results in this form can not be extended to the case $p = 1$ even if $w \in A_1$. Thus, the theorems in the classical, i.e., in the weightless case are not consequences of these estimates.

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This situation is not due to the essence of the matter, but because of the methods of proof, such as the use of the maximal function, the boundedness of the Hilbert transform, approximations using Fourier sums and other methods that are unsuitable for $p \to 1$. Instead of this we show that the direct and inverse theorems can be obtained as consequences of bounds for the norms of convolution operators with symmetrically decreasing kernels.

An approach to estimating convolutions without using a maximal function was employed in original works of Rosenblum [10] and Muckenhoupt [11]. Then it was developed in a series of papers by Nakhman and Osilenker [12]– [15] in connection with linear summation methods for Fourier series of periodic functions. The results in [12]– [15] are true for all $p \in [1, +\infty)$, but the constants in the estimates (at least in the formulations) also depend on p. Cases of different exponents, two-weight, multidimensional and vector-valued generalizations, studied in these sources, are not considered in this work.

With rare exceptions, we do not specify constants explicitly, but we always control their dependence on parameters. All constants in the estimates depend on $[w]_p$, and there is no other dependence on w and p. Moreover, we use a unified approach and obtain results both for the periodic and non-periodic cases.

1.2. Notation. In what follows $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, $\mathbb{N}$ are the sets of complex, real, nonnegative real, integer and natural numbers, respectively; $\mathbb{T} = [-\pi, \pi]$. If otherwise is not implied by the context, the spaces of functions can be both real and complex. At a point of removable discontinuity a function is defined by continuity, otherwise we let $\frac{0}{0} = 0$. Equivalent functions are identified. The symbol $C(\alpha, \beta, \ldots)$ denotes quantities depending only on the specified parameters and which do not necessarily coincide even within the same formula. A non-negative measurable almost everywhere finite and almost everywhere positive function is called a weight. If $p \in [1, +\infty)$, w is a weight, then $L_{p,w}(\mathbb{R})$ is the space of functions summable on $\mathbb{R}$ with pth degree and weight w; this space is equipped with norm

$$\|f\|_{p,w} = \left(\int_{\mathbb{R}} |f|^p w \right)^{1/p}.$$

In the same way we define the space of 2π-periodic functions $L_{p,w}(\mathbb{T})$. The norm in this space is introduced in the same way and the weight w is supposed to be 2π-periodic. The symbol $L_{p,w}$ stands for $L_{p,w}(\mathbb{R})$ or $L_{p,w}(\mathbb{T})$. The notation $L_{p,d\mu}$ has a similar meaning. We omit the unit weight in the notation and write simply L_p, $\|f\|_p$, etc.

Next, $\mathbf{E}_\sigma$ and $\mathbf{E}_{\sigma-0}$ are the sets of entire functions of exponential type at most σ and less than σ respectively,

$$\mathcal{A}_\sigma(f)_{p,w} = \inf_{g \in \mathbf{E}_\sigma} \|f - g\|_{p,w}$$

is the best approximation of the function $f \in L_{p,w}$ by functions from $\mathbf{E}_\sigma$ in the space $L_{p,w}$; the value $\mathcal{A}_{\sigma-0}(f)_{p,w}$ is determined similarly. In the periodic case, $\mathcal{A}_\sigma(f)$ coincides with $E_{\lfloor \sigma \rfloor}(f)$ ($\lfloor \sigma \rfloor$ is the integer part of a number σ), which is the best approximation of f by trigonometric polynomials of degree at most σ.

The function $F\colon \mathbb{R} \to [0, +\infty]$ is said to decrease symmetrically if it is even and decreases on $\mathbb{R}_+$. If $K\colon \mathbb{R} \to \mathbb{C}$, then K^* denotes a bell-shaped majorant of the function K, that is, a symmetrically decreasing function such that $|K| \leqslant K^*$. By $\mathcal{R}$ we denote the set of symmetrically decreasing functions summable on $\mathbb{R}$, and the symbol $\mathcal{R}^*$ stands for the set of functions with a summable bell-shaped majorant.

Let χ_E be the characteristic function of a set E,

$$S_h f(x) = \frac{1}{h}\int\limits_{-h/2}^{h/2} f(x-t)\,dt, \qquad S_{h,\tau} f(x) = S_h f(x+\tau)$$

be the two-sided and shifted first order Steklov function for a function f with a step $h>0$; then $S_{h,\frac{h}{2}} f$ is the one-sided Steklov function. As usually, if U is an operator, then U^m is its mth power, $U^0 = I$ is the identity operator.

The convolution and the Fourier transform are normalized by the identities

$$f*g(x) = \int\limits_{\mathbb{R}} f(x-t)g(t)\,dt, \qquad \mathcal{F}f(y) = \int\limits_{\mathbb{R}} f(t)e^{-iyt}\,dt.$$

Under such normalization $\mathcal{F}(f*g) = \mathcal{F}f\cdot\mathcal{F}g$.

2. Convolutions in weighted spaces

2.1. Estimates for convolutions in terms of Muckenhoupt characteristics. For a given weight w on $\mathbb{R}$ we denote

$$[w]_p = \begin{cases} \sup\limits_Q \left\{ \left(\frac{1}{|Q|}\int\limits_Q w\right)\left(\frac{1}{|Q|}\int\limits_Q w^{-\frac{1}{p-1}}\right)^{p-1}\right\}, & p\in(1,+\infty), \\ \sup\limits_Q \left\{ \left(\frac{1}{|Q|}\int\limits_Q w\right)\left(\operatorname*{ess\,inf}\limits_{t\in Q} w(t)\right)^{-1}\right\}, & p=1, \end{cases}$$

where the suprema are taken over all possible segments in $\mathbb{R}$ and $|Q|$ is the length of a segment Q. If $[w]_p < +\infty$, then the weight w is said to satisfy the Muckenhoupt condition A_p or to belong to the Muckenhoupt class A_p. The properties of Muckenhoupt weights can be found in [16], [17]. According to the Hölder inequality, the function $p\mapsto [w]_p$ decreases, whence classes A_p expand as p increases.

Most of the applications of Muckenhoupt weights are related to maximal functions and singular integral operators, however these questions play no role in this work. The only property of Muckenhoupt weights that is important for us is the estimate

$$\|f*K\|_{p,w} \leqslant B\|K\|_1\|f\|_{p,w}, \qquad K\in\mathcal{R}, \qquad f\in L_{p,w}, \tag{2.1}$$

where a constant B depend only on $[w]_p$.

The following characteristic property of the Muckenhoupt weights is well-known [17, Sect. 5.2.1]. Let μ be a Borel measure on $\mathbb{R}$, $p\in[1,+\infty)$, $K\in\mathcal{R}$, $K_\varepsilon(t) = \frac{1}{\varepsilon}K(\frac{t}{\varepsilon})$. If for all $f\in L_{p,d\mu}$ the inequality

$$\|f*K_\varepsilon\|_{p,d\mu} \leqslant B\|K\|_1\|f\|_{p,d\mu} \tag{2.2}$$

holds with a constant B independent on ε, then $d\mu(x) = w(x)\,dx$, where $w\in A_p$. And vice versa, if $w\in A_p$ and $d\mu(x) = w(x)\,dx$, then inequality (2.2) holds with a constant B independent of K and ε.

We denote

$$B_p[w] = \sup_{f\in L_{p,w}, K\in\mathcal{R}} \frac{\|f*K\|_{p,w}}{\|K\|_1\|f\|_{p,w}}.$$

Then $B_p[w]$ is the smallest independent on K constant B in inequality (2.1). This definition immediately implies the inequality

$$\|f*K\|_{p,w} \leqslant B_p[w]\|K^*\|_1\|f\|_{p,w}, \qquad K\in\mathcal{R}^*, \qquad f\in L_{p,w}. \tag{2.3}$$

Many classical kernels, such as Steklov, Fejér, Rogosinski, Vallée Poussin, Poisson and other kernels, belong to $\mathcal{R}^*$.

In the weightless case $w \equiv 1$ there is a well-known estimate

$$\|f * K\|_p \leqslant \|K\|_1 \|f\|_p, \qquad K \in L_1(\mathbb{R}), \qquad f \in L_p,$$

which coincides with (2.3) for $K \in \mathcal{R}$. However if $K \in \mathcal{R}^* \setminus \mathcal{R}$, then $\|K^*\|_1$ can be essentially greater than $\|K\|_1$.

Let us give an example. Let $\tau \in \mathbb{R}$, $h > 0$, $K = \frac{1}{h}\chi_{(-\tau-h/2,-\tau+h/2)}$ be a shifted first order Steklov kernel. Then $K^* = \frac{1}{h}\chi_{(-|\tau|-h/2,|\tau|+h/2)}$. Hence,

$$\|K^*\|_1 = 1 + \frac{2|\tau|}{h},$$

which is large if the shift $|\tau|$ is large in comparison with the step h. In particular, for the one-sided Steklov kernel ($\tau = h/2$) the passage to the bell-shaped majorant increases the L_1-norm twice.

We note that for a fixed $\tau \neq 0$ the family of operators $\{S_{h,\tau}\}_{h>0}$ is not bounded in spaces $L_{p,w}$ that are not closed under shifts. Indeed, let us take a function $f \in L_{p,w}$ such that $f \geqslant 0$ and $f(\cdot + \tau) \notin L_{p,w}$. Then $S_{h,\tau} f \to f(\cdot + \tau)$ for $h \to 0+$ almost everywhere and by the Fatou theorem

$$\varliminf_{h \to 0+} \int_{\mathbb{R}} \big(S_{h,\tau} f(x)\big)^p w(x)\, dx \geqslant \int_{\mathbb{R}} f^p(\tau + x) w(x)\, dx = +\infty.$$

Hence, the entire family of operators $\{S_{h,\tau}\}_{\tau \in \mathbb{R}, h>0}$ is not bounded.

In the following lemma we estimate the constants $B_p[w]$ by the Muckenhoupt characteristics.

Lemma 2.1. *Let $p \in [1, +\infty)$, $\frac{1}{p} + \frac{1}{q} = 1$, $w \in A_p$. Then*

$$B_p[w] \leqslant 2^{\min\{\frac{1}{p}, \frac{1}{q}\}} [w]_p^{1/p}. \tag{2.4}$$

If, in addition, $w \in A_1$, then $B_p[w] \leqslant B_1^{1/p}[w]$ and

$$B_1[w] = \sup_{h>0} \operatorname*{ess\,sup}_{t \in \mathbb{R}} \frac{S_h w(t)}{w(t)} \leqslant [w]_1. \tag{2.5}$$

Proof. 1. We first prove the lemma for the spaces $L_{p,w}(\mathbb{R})$. We denote

$$Q_x = [x - 1/2, x + 1/2], \qquad Q = Q_0.$$

Without loss of generality we can suppose that $f \geqslant 0$. It is sufficient to show estimate (2.1) for the kernel $K = \chi_Q$. Indeed, since the value $[w]_p$ remains the same under scaling, the proven is true for Steklov kernels $K = \frac{1}{h}\chi_{[-h/2,h/2]}$, and then for linear combinations of Steklov kernels with positive coefficients. In the general case, we approximate the kernel $K \in \mathcal{R}$ by an increasing sequence of such linear combinations and pass to the limit by the Levy theorem.

1.1. For each weight λ by the Hölder inequality we have

$$(f * \chi_Q)^p(x) = \left(\int_{Q_x} f\right)^p \leqslant \left(\int_{Q_x} f^p w \lambda\right) \left(\int_{Q_x} w^{-q/p} \lambda^{-q/p}\right)^{p/q}.$$

Integrating in x and switching the integration order, we find

$$\int\limits_{\mathbb{R}} (f*\chi_Q)^p(x)w(x)\,dx \leqslant \int\limits_{\mathbb{R}} \left(\int\limits_{Q_x} f^p(t)w(t)\lambda(t)\,dt\right)\left(\int\limits_{Q_x} w^{-q/p}\lambda^{-q/p}\right)^{p/q} w(x)\,dx = \int\limits_{\mathbb{R}} f^p(t)w(t)\lambda(t)I(t)\,dt, \tag{2.6}$$

where

$$I(t) = \int\limits_{Q_t}\left(\int\limits_{Q_x} w^{-q/p}\lambda^{-q/p}\right)^{p/q} w(x)\,dx.$$

1.2. Let $p \in [2,+\infty)$. We let $\lambda = 1$. By the definition of the quantity $[w]_p$ we have

$$I(t) = \int\limits_{Q_t}\left(\int\limits_{Q_x} w^{-q/p}\right)^{p/q} w(x)\,dx \leqslant [w]_p \int\limits_{Q_t}\left(\int\limits_{Q_x} w\right)^{-1} w(x)\,dx.$$

We denote by Q_t^- the left and by Q_t^+ the right half of the segment Q_t. If $x \in Q_t^{\pm}$, then $Q_x \supset Q_t^{\pm}$. This is why

$$\int\limits_{Q_t}\left(\int\limits_{Q_x} w\right)^{-1} w(x)\,dx \leqslant \int\limits_{Q_t^-}\left(\int\limits_{Q_t^-} w\right)^{-1} w(x)\,dx + \int\limits_{Q_t^+}\left(\int\limits_{Q_t^+} w\right)^{-1} w(x)\,dx = 2,$$

and hence $I(t) \leqslant 2[w]_p$. Substituting this estimate into (2.6), we get the desired fact.

1.3. Let $p \in (1,2]$ and $\lambda(t) = \left(\int\limits_{Q_t} w^{-q/p}\right)^{p/q}$. As above, partitioning the segment Q_x into two halves, we find

$$\int\limits_{Q_x} w^{-q/p}\lambda^{-q/p} = \int\limits_{Q_x} w^{-q/p}(u)\left(\int\limits_{Q_u} w^{-q/p}\right)^{-1} du \leqslant 2.$$

Therefore,

$$\lambda(t)I(t) \leqslant 2^{p/q}\lambda(t)\int\limits_{Q_t} w(x)\,dx \leqslant 2^{p/q}[w]_p.$$

It remains to substitute this estimate into (2.6).

1.4. Let $w \in A_1$. We are going to prove inequality (2.5). We denote its middle part by $C[w]$. For all $h > 0$ we have

$$\|S_h f\|_{1,w} = \int\limits_{\mathbb{R}} \frac{1}{h}\int\limits_{x-h/2}^{x+h/2} f(t)\,dt\,w(x)\,dx = \int\limits_{\mathbb{R}} f(t)\left(\frac{1}{h}\int\limits_{t-h/2}^{t+h/2} w(x)\,dx\right) dt \leqslant C[w]\int\limits_{\mathbb{R}} f(t)w(t)\,dt.$$

This implies the inequality $B_1[w] \leqslant C[w]$ and its sharpness. The estimate $C[w] \leqslant [w]_1$ is obvious.

The estimate $B_p[w] \leqslant B_1^{1/p}[w]$ follows from the Hölder inequality

$$(f * K)^p \leqslant \|K\|_1^{p/q}(f^p * K)$$

or from the interpolation Riesz-Thorin theorem.

2. In order to prove the lemma in the periodic case, we need to make just some minor changes. Let the function f and the weight w be 2ℓ-periodic. We recall that we still have $K \in L_1(\mathbb{R})$ and the convolution is defined as an integral over the entire axis. Then in formula (2.6) the external integrals are taken over $[-\ell, \ell]$ and the identity

$$\int_{-\ell}^{\ell} \int_{Q_x} g(x,t)\,dt\,dx = \int_{-\ell}^{\ell} \int_{Q_t} g(x,t)\,dx\,dt$$

is used; this identity is true if a function g is $2l$-periodic with respect to both variables. In this way we obtain estimate (2.4) for the kernel $K = \chi_Q$. By scaling we see that if inequality (2.4) is true for the kernel K and the functions f and w with the period 2ℓ, then it is true for the kernel $K_h = \frac{1}{h}K(\frac{\cdot}{h})$ and the functions $f(h\cdot)$ and $w_{1/h} = hw(h\cdot)$ of the period $2\ell/h$. Since $[w_{1/h}]_p = [w]_p$ and ℓ is arbitrary, we obtain (2.4) for all Steklov kernels. The proof can be completed in the same way. □

Sharp constants in the weight inequalities (2.1) are unknown. In [10], for the first time, a criterion was obtained for the boundedness of some convolution operators in the spaces $L_{p,w}(\mathbb{T})$, including those with Steklov, Poisson and Fejér kernels. Muckenhoupt [11] showed that this criterion is equivalent to Condition A_p. In [11] inequality (2.1) was obtained for the Steklov means in $L_{p,w}(\mathbb{R})$ with the constant $\frac{3}{2}\cdot 3^{1/p}[w]_p^{1/p}$, and in $L_{p,w}(\mathbb{T})$ with the constant $3\cdot 6^{1/p}[w]_p^{1/p}$. Then inequality (2.1) for Poisson integrals was derived from it without specifying a constant. Inequality (2.1) in $L_{p,w}(\mathbb{T})$ for kernels of class $\mathcal{R}^*$ was obtained in [14] and [15] with a two-sided estimate $B_p[w] \asymp C(p)[w]_p^{1/p}$. The observation that inequality (2.1) for the Steklov means implies an inequality with the same constant for all kernels $K \in \mathcal{R}$ is implicitly present in [17]. Analysis of Stein's reasoning leads to the upper bound $B_p[w] \leqslant 2^{1+\frac{1}{p}}[w]_p^{1/p}$. The same book contains also the lower bound $B_p[w] \geqslant \frac{1}{2}[w]_p^{1/p}$. From the proof in [15, Thm. 1] the same lower bound follows. In [18, Lm. 2.18] inequality (2.1) was proved for the Steklov means in $L_{p,w}(\mathbb{R})$ with a constant of the form $C(p)[w]_p^{1/p}$ and it was noted that the exponent $1/p$, generally speaking, cannot be reduced. We also mention an obvious lower bound $B_p[w] \geqslant 1$. In [14], [15], [17], [18] there are also multidimensional and two-weight generalizations, which we do not discuss here.

Corollary 2.1. *Let $p \in [1, +\infty)$, $w \in A_p$, $K \in \mathcal{R}^*$, $f \in L_{p,w}$, $\sigma > 0$. Then*

$$\mathcal{A}_\sigma(f * K)_{p,w} \leqslant B_p[w] \inf_{K_\sigma \in \mathbf{E}_\sigma} \|(K - K_\sigma)^*\|_1 \mathcal{A}_\sigma(f)_{p,w} \leqslant B_p[w]\|K^*\|_1 \mathcal{A}_\sigma(f)_{p,w}.$$

In this inequality $\mathcal{A}_\sigma$ and $\mathbf{E}_\sigma$ can be replaced by $\mathcal{A}_{\sigma-0}$ and $\mathbf{E}_{\sigma-0}$.

Proof. The right inequality is trivial and we are going to prove only the left one. For all functions $f_\sigma \in \mathbf{E}_\sigma \cap L_{p,w}$ and $K_\sigma \in \mathbf{E}_\sigma \cap \mathcal{R}^*$ we have

$$\mathcal{A}_\sigma(f * K)_{p,w} = \mathcal{A}_\sigma\big((f - f_\sigma) * (K - K_\sigma)\big)_{p,w}.$$

It remains to use estimate (2.3) and pass to the infimum over f_σ and K_σ on the right hand side. The inequality for $\mathcal{A}_{\sigma-0}$ can be proved in the same way. The proof is complete. □

2.2. Estimates for bell-shaped majorant. In applications the kernel K is usually given in terms of the Fourier transform and can depend on the parameters. In order to apply Lemma 2.1 we need to know whether the kernel K possesses a summable bell-shaped majorant and we need to be able to estimate its L_1-norm. Some estimates are collected in the following lemma.

Lemma 2.2. *Let*

$$K(t) = 2\pi \mathcal{F}^{-1}\varphi(t) = \int\limits_{\mathbb{R}} \varphi(y) e^{ity}\, dy.$$

1. If $\varphi \in L_1(\mathbb{R})$, then $|K(t)| \leqslant \|\varphi\|_1$.

2. Assume that $s-1 \in \mathbb{N}$, there exists an absolutely continuous derivative $\varphi^{(s-2)}$ and the functions $\varphi, \dots, \varphi^{(s-1)}$ tend to zero at infinity. If the variation of $\varphi^{(s-1)}$ (we denote it by $\|d\varphi^{(s-1)}\|_1$) is finite, then $|K(t)| \leqslant |t|^{-s}\|d\varphi^{(s-1)}\|_1$. In particular, if $\varphi^{(s-1)}$ is absolutely continuous and $\varphi^{(s)} \in L_1(\mathbb{R})$, then $|K(t)| \leqslant |t|^{-s}\|\varphi^{(s)}\|_1$.

3. Let

$$\varphi(y) = c(iy)^{-\alpha} + \varphi_1(y) \tag{2.7}$$

or

$$\varphi(y) = c|y|^{-\alpha} + \varphi_1(y), \tag{2.8}$$

where $c \in \mathbb{C}$, $\alpha \in (0,1)$, $\varphi_1 \in L_1(\mathbb{R})$. Then respectively

$$|K(t)| \leqslant \frac{\pi|c|}{\Gamma(\alpha)\cos\frac{\alpha\pi}{2}}|t|^{\alpha-1} + \|\varphi_1\|_1$$

or

$$|K(t)| \leqslant \frac{2\pi|c|}{\Gamma(\alpha)}|t|^{\alpha-1} + \|\varphi_1\|_1.$$

Proof. The first statement is obvious, the second can be obtained by integrating by parts. The third statement is implied by the identities

$$\int\limits_{\mathbb{R}} |y|^{-\alpha} e^{ity}\, dy = \frac{\pi}{\Gamma(\alpha)\cos\frac{\alpha\pi}{2}}|t|^{\alpha-1},$$

$$\int\limits_{\mathbb{R}} (iy)^{-\alpha} e^{ity}\, dy = \begin{cases} \frac{2\pi}{\Gamma(\alpha)} t^{\alpha-1}, & t>0, \\ 0, & t<0. \end{cases}$$

□

It follows from Lemma 2.1 that all functions f from the classes $L_{p,w}$ are locally summable and

$$\int\limits_{\mathbb{R}} \frac{|f(t)|}{(1+t^2)}\, dt < +\infty.$$

This is why we can speak about their Fourier transforms in the space of tempered distributions $\mathcal{S}'$.

Let $\alpha > 0$, $\theta \in \mathbb{R}$. The Weyl-Nagy derivative of order (α, θ) of a function $f \in L_{p,w}$ is defined in terms of the Fourier images by the identity

$$\mathcal{F} f^{(\alpha,\theta)}(y) = e^{i\frac{\theta\pi}{2}\operatorname{sign} y}|y|^{\alpha}\mathcal{F}f(y). \tag{2.9}$$

It is easy to check that as $\alpha \in \mathbb{N}$, we have $f^{(\alpha,\alpha)} = f^{(\alpha)}$, $f^{(\alpha,\alpha-1)} = \widetilde{f}^{(\alpha)}$, where $\widetilde{f}$ is a trigonometrically conjugate with f function, see Section 4. For a non-integer $\alpha > 0$ the first identity is used as the definition of the derivative of order α.

Let us show that identity (2.9) makes sense in the space $\mathcal{S}'$. In the weightless case this was done in [19]. We write

$$e^{i\frac{\theta\pi}{2}\operatorname{sign} y}|y|^\alpha = (1+y^2)^\beta \cdot \frac{e^{i\frac{\theta\pi}{2}\operatorname{sign} y}|y|^\alpha}{(1+y^2)^\beta}.$$

For sufficiently large β the factor $\frac{e^{i\frac{\theta\pi}{2}\operatorname{sign} y}|y|^\alpha}{(1+y^2)^\beta}$ is the Fourier transform of the function from $\mathcal{R}^*$. This can be easily confirmed by integrating by parts and Lemma 2.2. This is why it defines a convolution operator from $L_{p,w}$ into $L_{p,w}$. A multiplication for an infinitely smooth tempered function $(1+y^2)^\beta$ is a continuous operation in $\mathcal{S}'$. This is why identity (2.9) determines a linear continuous operator from $L_{p,w}$ into $\mathcal{S}'$.

By the symbols $W_{p,w}^{(\alpha,\theta)}$ we denote the Weyl-Nagy classes, that is, the sets of functions f in $L_{p,w}$ such that $f^{(\alpha,\theta)} \in L_{p,w}$. In the periodic case, in defining them, we need not distributions since identity (2.9) can be treated as equality of Fourier coefficients. As $\theta = \alpha$ we obtain the Sobolev classes $W_{p,w}^{(\alpha)} = W_{p,w}^{(\alpha,\alpha)}$.

The set $W_{p,w}^{(\alpha,\theta)}$ with the norm

$$\|f\|_{p,w} + \|f^{(\alpha,\theta)}\|_{p,w}$$

is a Banach space. This can be proved in the standard way as for the Sobolev spaces.

2.3. Inequalities for convolution operators. Let us describe the scheme of applying Lemmas 2.1 and 2.2. Let the operators U and V with the values in the space $L_{p,w}$ be defined in terms of the Fourier transforms as multipliers:

$$\mathcal{F}Uf(y) = u(y)\mathcal{F}f(y), \quad \mathcal{F}Vf(y) = v(y)\mathcal{F}f(y). \tag{2.10}$$

We need to estimate $\|Uf\|_{p,w}$ via $\|Vf\|_{p,w}$. If the function

$$\varphi = \frac{u}{v} \quad \text{or} \quad \varphi = \frac{u}{v} - 1 \tag{2.11}$$

is the Fourier transform of a function $K \in \mathcal{R}^*$, then

$$Uf = Vf * K \qquad \text{or} \qquad Uf = Vf + Vf * K$$

and the estimate holds:

$$\|Uf\|_{p,w} \leqslant B_p[w]\|K^*\|_1\|Vf\|_{p,w} \tag{2.12}$$

or

$$\|Uf\|_{p,w} \leqslant \bigl(1 + B_p[w]\|K^*\|_1\bigr)\|Vf\|_{p,w} \leqslant B_p[w]\bigl(1 + \|K^*\|_1\bigr)\|Vf\|_{p,w}, \tag{2.13}$$

respectively. In the second case for the sake of brevity and uniformity we increase the term 1 to $B_p[w]$. By means of Lemma 2.2 we can confirm that $K^* \in L_1(\mathbb{R})$ and track the dependence of the norm $\|K^*\|_1$ on the parameters; the normalization in Lemma 2.2 differs by the factor 2π. Statements 1 and 3 of Lemma 2.2 are to be applied for small t, while Statement 2 is to be applied for large t.

Let us clarify in more details how to combine the inequalities for the convolutions. Let

$$Uf = Vf + Vf * K_1, \qquad Vf = Wf + Wf * K_2. \tag{2.14}$$

Then

$$\|Uf\|_{p,w} \leqslant \bigl(1 + B_p[w]\|K_1^*\|_1\bigr)\|Vf\|_{p,w}, \tag{2.15}$$

$$\|Vf\|_{p,w} \leqslant \bigl(1 + B_p[w]\|K_2^*\|_1\bigr)\|Wf\|_{p,w}. \tag{2.16}$$

The substitution of (2.16) into (2.15) gives

$$\|Uf\|_{p,w} \leqslant \bigl(1 + B_p[w]\|K_1^*\|_1\bigr)\bigl(1 + B_p[w]\|K_2^*\|_1\bigr)\|Wf\|_{p,w}. \tag{2.17}$$

However, if instead we combine convolution representations (2.14), we obtain

$$\begin{aligned} &Uf = Wf + Wf * (K_1 + K_2 + K_1 * K_2),\\ &\|Uf\|_{p,w} \leqslant \big(1 + B_p[w]\|(K_1 + K_2 + K_2 * K_1)^*\|_1\big)\|Wf\|_{p,w}\\ &\qquad \leqslant \big(1 + B_p[w](\|K_1^*\|_1 + \|K_2^*\|_1 + \|(K_2 * K_1)^*\|_1)\big)\|Wf\|_{p,w}. \end{aligned}$$

Since the convolution of symmetrically decreasing functions decreases symmetrically, the function $K_2^* * K_1^*$ is a bell-shaped majorant for $K_2 * K_1$. This yields

$$\|Uf\|_{p,w} \leqslant \big(1 + B_p[w](\|K_1^*\|_1 + \|K_2^*\|_1 + \|K_1^*\|_1\|K_2^*\|_1)\big)\|Wf\|_{p,w}. \tag{2.18}$$

The constant in inequality (2.18) is generally speaking less than in (2.17) since $B_p[w]$ is not squared. In the weightless case, when $B_p[w] = 1$, this difference disappears.

Similarly, inequalities (2.12) are combined with each other and with (2.13) without taking the square of $B_p[w]$.

3. Deviations of Steklov means as modules of continuity

3.1. Modification of modules of continuity. Generally speaking, the spaces $L_{p,w}$ are not closed with respect to shifts: the belonging $f \in L_{p,w}$ does not imply $f(\cdot + t) \in L_{p,w}$. This is why the classical definition of the modulus of continuity can make no sense for weighted spaces. Instead of the modules of continuity, in the literature there were used quantities constructed by averaging:

$$\begin{aligned} \Omega_{2\alpha}^{(1)}(f,h)_{p,w} &= \sup_{0\leqslant u_j,t\leqslant h}\left\|\left(\prod_{j=1}^{\lfloor\alpha\rfloor}(I - S_{u_j})\right)(I - S_t)^{\alpha-\lfloor\alpha\rfloor}f\right\|_{p,w},\\ \Omega_{2\alpha}^{(2)}(f,h)_{p,w} &= \sup_{0\leqslant u\leqslant h}\big\|(I - S_u)^\alpha f\big\|_{p,w},\\ \Omega_{\alpha}^{(3)}(f,h)_{p,w} &= \sup_{0\leqslant u\leqslant h}\left\|\frac{1}{u}\int_0^u |\Delta_t^\alpha f(\cdot)|\,dt\right\|_{p,w}, \end{aligned}$$

where $\Delta_t^\alpha f$ is the usual forward difference. The modulus $\Omega^{(1)}$ was employed, for instance, in [2], [4]– [7], the modulus $\Omega^{(2)}$ was used in [2], [7], [9], and the modulus $\Omega^{(3)}$ was used in [3], [8]. In these formulas $\alpha > 0$ is not necessary integer.

Let $p \in (1,+\infty)$, $L_{p,w} = L_{p,w}(\mathbb{T})$, $\alpha \in \mathbb{N}$. In [2], there was proved the equivalence of the modulus $\Omega_{2\alpha}^{(1)}$ to the corresponding K-functional, see Theorem 4.6 below. In [3] the same was done for the modulus $\Omega_{\alpha}^{(3)}$, as well as in the spaces $L_{p,w}$ on an arbitrary segment. In [7] this was done for the modulus $\Omega_{2\alpha}^{(2)}$ and this implied that $\Omega_{2\alpha}^{(1)}$, $\Omega_{2\alpha}^{(2)}$ and $\Omega_{2\alpha}^{(3)}$ are mutually equivalent. For non-integer α this result was extended for $\Omega_{2\alpha}^{(1)}$ in [6] and formulated for $\Omega_{2\alpha}^{(2)}$ in [7], see the comments in Section 5 in this work. The constants in the mentioned estimates depend on α, p and $[w]_p$.

As it is known, [20, Ch. 8], in the weightless case for all $r \in \mathbb{N}$, $p \in [1,+\infty)$ and $f \in L_p$ the relations hold:

$$\begin{aligned} \sup_{0\leqslant u\leqslant h}\big\|(I - S_u^r)f\big\|_p &\asymp \big\|(I - S_h^r)f\big\|_p \asymp \omega_2(f,h)_p,\\ \sup_{0\leqslant u\leqslant h}\big\|\big(I - S_{u,\frac{u}{2}}^r\big)f\big\|_p &\asymp \big\|\big(I - S_{h,\frac{h}{2}}^r\big)f\big\|_p \asymp \omega_1(f,h)_p, \end{aligned}$$

where ω_j are the classical modules of continuity and the constants depend only on r. We stress that for the equivalence to the first order modulus of continuity one needs the one-sided Steklov function.

Next we define a family of quantities of the type $\Omega_\alpha^{(1)}$ and $\Omega_\alpha^{(2)}$ based on deviations of Steklov means of any order, not just of first order and we shall establish their properties. We shall show that if we omit taking the supremum, we get an equivalent quantity. The estimates are valid for all $p \in [1, +\infty)$, and the constants depend on $[w]_p$ and other parameters, but not on p. The modulus $\Omega_\alpha^{(3)}$ is not discussed in this work.

For all $k \in \mathbb{N}$ and $h > 0$ by the symmetric decreasing of the Steklov kernels we have

$$\|S_h^k f\|_{p,w} \leqslant B_p[w]\|f\|_{p,w}. \tag{3.1}$$

In weightless spaces L_p the norms of the Steklov operators are equal to 1 and this allows us to define the αth power of the deviations $I - S_h^r$ and $I - S_{h,\frac{h}{2}}^r$ $(r \in \mathbb{N})$ for each $\alpha > 0$, not necessarily integer, by the identities

$$(I - S_h^r)^\alpha = \sum_{k=0}^{\infty} (-1)^k C_\alpha^k S_h^{rk}, \tag{3.2}$$

$$\big(I - S_{h,\frac{h}{2}}^r\big)^\alpha = \sum_{k=0}^{\infty} (-1)^k C_\alpha^k S_{h,\frac{h}{2}}^{rk}. \tag{3.3}$$

Estimate (3.1) allows to adopt identity (3.2) as the definition in the weighted spaces $L_{p,w}$. Indeed, the series on the right hand side converges absolutely in the operator norm and this implies the correctness of the definition and the estimate

$$\big\|(I - S_h^r)^\alpha\big\| \leqslant 1 + \sum_{k=1}^{\infty} |C_\alpha^k| B_p[w] \leqslant 1 + \big(2^{\lceil\alpha\rceil} - 1\big) B_p[w].$$

Here $\lceil\alpha\rceil = \min\{n \in \mathbb{Z} : n \geqslant \alpha\}$. The convergence of the operator series in (3.3) is not obvious. This is why we use another way and define the operators $(I - S_h^r)^\alpha$ and $(I - S_{h,\frac{h}{2}}^r)^\alpha$ in terms of the Fourier transforms.

We write the Fourier transforms of the Steklov kernels:

$$\mathcal{F} S_h f(y) = \frac{2}{hy} \sin\frac{hy}{2} \mathcal{F} f(y), \qquad \mathcal{F} S_{h,\frac{h}{2}} f(y) = \frac{e^{ihy} - 1}{ihy} \mathcal{F} f(y).$$

For the functions $f \in L_p$ this gives

$$(I - S_h^r)^\alpha f = f + f * K, \quad \mathcal{F}K(y) = \left(1 - \left(\frac{2}{hy}\sin\frac{hy}{2}\right)^r\right)^\alpha - 1, \tag{3.4}$$

$$\big(I - S_{h,\frac{h}{2}}^r\big)^\alpha f = f + f * K, \quad \mathcal{F}K(y) = \left(1 - \left(\frac{e^{ihy} - 1}{ihy}\right)^r\right)^\alpha - 1. \tag{3.5}$$

By Lemma 2.2 we have $K \in \mathcal{R}^*$. This is why identities (3.4) and (3.5) define linear continuous operators in the spaces $L_{p,w}$, that is,

$$\big\|(I - S_h^r)^\alpha f\big\|_{p,w}, \ \big\|\big(I - S_{h,\frac{h}{2}}^r\big)^\alpha f\big\|_{p,w} \leqslant C(r,\alpha) B_p[w] \|f\|_{p,w}. \tag{3.6}$$

At the same time, definitions (3.2) and (3.4) are equivalent.

3.2. Properties of modified modules of continuity. In the next theorem we collect the properties of the powers of the deviations of the Steklov means, some of which are similar to the properties of the usual modules of continuity. In the weightless case these properties follow from the comparison principle for linear operators [20, Ch. 8].

Theorem 3.1. *Let $p \in [1, +\infty)$, $w \in A_p$, $f \in L_{p,w}$, $r, m \in \mathbb{N}$, $h > 0$, $0 < \beta \leqslant \alpha$. Then*

$$\big\|(I - S_h^r)^\alpha f\big\|_{p,w} \leqslant C(r, m, \alpha) B_p[w] \big\|(I - S_h^m)^\alpha f\big\|_{p,w}, \tag{3.7}$$

$$\big\|\big(I - S_{h,\frac{h}{2}}^r\big)^\alpha f\big\|_{p,w} \leqslant C(r, m, \alpha) B_p[w] \big\|\big(I - S_{h,\frac{h}{2}}^m\big)^\alpha f\big\|_{p,w}, \tag{3.8}$$

$$\big\|(I - S_{\lambda h}^r)^\alpha f\big\|_{p,w} \leqslant C(r, \alpha, \lambda) B_p[w] \big\|(I - S_h^r)^\alpha f\big\|_{p,w}, \tag{3.9}$$

$$\big\|\big(I - S_{\lambda h,\frac{\lambda h}{2}}^r\big)^\alpha f\big\|_{p,w} \leqslant C(r, \alpha, \lambda) B_p[w] \big\|\big(I - S_{h,\frac{h}{2}}^r\big)^\alpha f\big\|_{p,w}; \tag{3.10}$$

in (3.9) *and* (3.10) *the constants are bounded in λ on each segment;*

$$\begin{aligned} \frac{1}{C(r,k) B_p[w]} \big\|(I - S_h^r)^k f\big\|_{p,w} &\leqslant \big\|\big(I - S_{h,\frac{h}{2}}^r\big)^{2k} f\big\|_{p,w} \\ &\leqslant C(r,k) B_p[w] \big\|(I - S_h^r)^k f\big\|_{p,w}, \quad k \in \mathbb{N}. \end{aligned} \tag{3.11}$$

Moreover, if $f \in W_{p,w}^{(2\beta,0)}$, then

$$\big\|(I - S_h^r)^\alpha f\big\|_{p,w} \leqslant C(r, \beta) B_p[w] h^{2\beta} \big\|(I - S_h^r)^{\alpha-\beta} f^{(2\beta,0)}\big\|_{p,w} \tag{3.12}$$

and, in particular, as $\beta = \alpha$,

$$\big\|(I - S_h^r)^\alpha f\big\|_{p,w} \leqslant C(r, \alpha) B_p[w] h^{2\alpha} \big\|f^{(2\alpha,0)}\big\|_{p,w}, \tag{3.13}$$

and if $f \in W_{p,w}^{(\beta)}$, then

$$\big\|\big(I - S_{h,\frac{h}{2}}^r\big)^\alpha f\big\|_{p,w} \leqslant C(r, \beta) B_p[w] h^{\beta} \big\|\big(I - S_{h,\frac{h}{2}}^r\big)^{\alpha-\beta} f^{(\beta)}\big\|_{p,w} \tag{3.14}$$

and, in particular, as $\beta = \alpha$,

$$\big\|\big(I - S_{h,\frac{h}{2}}^r\big)^\alpha f\big\|_{p,w} \leqslant C(r, \alpha) B_p[w] h^{\alpha} \big\|f^{(\alpha)}\big\|_{p,w}. \tag{3.15}$$

Proof. All relations in the theorem can be proved by the same way described in Section 2.3. In each case we write the functions u, v and φ from representations (2.10) and (2.11) and we clarify how Lemma 2.2 is employed. We recall that $K = \mathcal{F}^{-1}\varphi$.

First of all we note that the constants are independent of h since for different h the inequalities are obtained one from another by scaling. This is why we can suppose that $h = 1$.

1. Inequality (3.7). We have

$$u(y) = \left(1 - \left(\frac{2}{y}\sin\frac{y}{2}\right)^r\right)^\alpha, \qquad v(y) = \left(1 - \left(\frac{2}{y}\sin\frac{y}{2}\right)^m\right)^\alpha,$$

$\varphi = \frac{u}{v} - 1$. If $r > 1$ and $m > 1$, then the function φ is summable together with all its derivatives. If $m > r = 1$ or $r > m = 1$, then

$$\varphi(y) = \pm\alpha\frac{2}{y}\sin\frac{y}{2} + \varphi_1(y),$$

where the function φ_1 is summable together with all its derivatives.

2. Inequality (3.8). We have

$$u(y) = \left(1 - \left(\frac{e^{iy} - 1}{iy}\right)^r\right)^\alpha, \qquad v(y) = \left(1 - \left(\frac{e^{iy} - 1}{iy}\right)^m\right)^\alpha,$$

$\varphi = \frac{u}{v} - 1$. If $r > 1$ and $m > 1$, then the function φ is summable together with all its derivatives. If $m > r = 1$ or $r > m = 1$, then

$$\varphi(y) = \pm\alpha\frac{e^{iy} - 1}{iy} + \varphi_1(y),$$

where the function φ_1 is summable together with all its derivatives.

3. Inequality (3.9). We have

$$u(y)=\left(1-\left(\frac{2}{\lambda y}\sin\frac{\lambda y}{2}\right)^r\right)^\alpha,\qquad v(y)=\left(1-\left(\frac{2}{y}\sin\frac{y}{2}\right)^r\right)^\alpha,$$

$\varphi=\frac{u}{v}-1$. In view of inequality (3.7) it is sufficient to prove (3.9) for a single value of r, for instance, for $r=2$. Then the function φ is summable together with all its derivatives and L_1-norms of the derivatives depend continuously on λ.

4. Inequality (3.10). We have

$$u(y)=\left(1-\left(\frac{e^{i\lambda y}-1}{i\lambda y}\right)^r\right)^\alpha,\qquad v(y)=\left(1-\left(\frac{e^{iy}-1}{iy}\right)^r\right)^\alpha,$$

$\varphi=\frac{u}{v}-1$. In view of inequality (3.8) it is sufficient to show (3.10) for a single value of r, for instance, for $r=2$. Then the function φ is summable together with all its derivatives and L_1-norms of the derivatives depend continuously on λ.

5. Inequality (3.11). In view of inequalities (3.7) and (3.8) it is sufficient to prove (3.11) for $r=2$. We have

$$u(y)=\left(1-\left(\frac{e^{iy}-1}{iy}\right)^r\right)^{2k},\qquad v(y)=\left(1-\left(\frac{2}{y}\sin\frac{y}{2}\right)^r\right)^k.$$

The functions $\frac{u}{v}-1$ and $\frac{v}{u}-1$ taken as φ are summable together with all its derivatives.

6. Inequalities (3.12). We have

$$u(y)=\left(1-\left(\frac{2}{y}\sin\frac{y}{2}\right)^r\right)^\beta,\qquad v(y)=|y|^{2\beta},\qquad \varphi=\frac{u}{v}.$$

It is clear that it is sufficient to consider $\beta\in(0,1/2)$. The function φ is represented in form (2.8) and have summable derivatives. By Statements 2 and 3 of Lemma 2.2 we obtain $K^*\in L_1(\mathbb{R})$.

7. Inequality (3.14). We have

$$u(y)=\left(1-\left(\frac{e^{iy}-1}{iy}\right)^r\right)^\beta,\qquad v(y)=(iy)^\beta,\qquad \varphi=\frac{u}{v}.$$

It is clear that it is sufficient to consider $\beta\in(0,1)$. The function φ is represented in form (2.7) and have summable derivatives. By Statements 2 and 3 of Lemma 2.2 we obtain $K^*\in L_1(\mathbb{R})$. The proof is complete. □

Corollary 3.1. *Under the assumptions of Theorem 3.1 we have*

$$\sup_{0\leqslant u\leqslant h}\big\|(I-S_u^r)^\alpha f\big\|_{p,w}\leqslant C(r,\alpha)B_p[w]\big\|(I-S_h^r)^\alpha f\big\|_{p,w},$$

$$\sup_{0\leqslant u\leqslant h}\big\|\big(I-S^r_{u,\frac{u}{2}}\big)^\alpha f\big\|_{p,w}\leqslant C(r,\alpha)B_p[w]\big\|\big(I-S^r_{h,\frac{h}{2}}\big)^\alpha f\big\|_{p,w}.$$

We make some remarks on Theorem 3.1.

1. The expansion

$$I-S_h^m=(I+\ldots+S_h^{m-1})(I-S_h)$$

and the symmetric decreasing of the Steklov kernels imply the estimate

$$\|(I-S_h^m)f\|_{p,w}\leqslant\big(1+(m-1)B_p[w]\big)\|(I-S_h)f\|_{p,w}.$$

2. We stress that not only estimate (3.11), but also estimates (3.7)–(3.10) are two-sided since the opposite inequalities are obtained by switching the roles of the parameters.

3. The considered modules of continuity are equivalent:

$$\big\|(I-S_h)^\alpha f\big\|_{p,w}\leqslant\Omega^{(2)}_{2\alpha}(f,h)_{p,w}\leqslant\Omega^{(1)}_{2\alpha}(f,h)_{p,w}\leqslant C(\alpha)B_p[w]\big\|(I-S_h)^\alpha f\big\|_{p,w}.$$

Here the first two inequalities are obvious, and the third inequality is true by Corollary 3.1 and Theorem 3.1.

4. Direct and inverse theorems

4.1. Direct theorems. In order to prove direct theorems, as an approximating operator we can employ the means of Vallée Poussin type, which are defined as follows.

We take a function $\eta\colon \mathbb{R}\to\mathbb{R}$ with the properties: η is infinitely differentiable, even, $\eta(y)=1$ as $|y|\leqslant\frac{1}{2}$, $\eta(y)=0$ as $|y|\geqslant 1$. For $f\in L_{p,w}$ we define $V_\sigma f = f*\big(\mathcal{F}^{-1}(\eta(\frac{\cdot}{\sigma}))\big)$. It is clear that $V_\sigma f\in \mathbf{E}_\sigma$ and this is why

$$\mathcal{A}_\sigma(f)_{p,w}=\mathcal{A}_\sigma(f-V_\sigma f)_{p,w}.$$

Theorem 4.1. *Let $p\in[1,+\infty)$, $w\in A_p$, $f\in L_{p,w}$, $r\in\mathbb{N}$, $\alpha,\gamma,\sigma>0$. Then*

$$\mathcal{A}_\sigma(f)_{p,w}\leqslant C(r,\alpha,\gamma)B_p[w]\mathcal{A}_\sigma\left(\left(I-S^r_{\frac{\gamma\pi}{\sigma}}\right)^\alpha f\right)_{p,w}, \tag{4.1}$$

$$\mathcal{A}_\sigma(f)_{p,w}\leqslant C(r,\alpha,\gamma)B_p[w]\mathcal{A}_\sigma\left(\left(I-S^r_{\frac{\gamma\pi}{\sigma},\frac{\gamma\pi}{2\sigma}}\right)^\alpha f\right)_{p,w}. \tag{4.2}$$

In inequalities (4.1) *and* (4.2), $\mathcal{A}_\sigma$ *can be replaced by* $\mathcal{A}_{\sigma-0}$.

Proof. We let $u(y)=1-\eta\left(\frac{y}{\sigma}\right)$,

$$v_1(y)=\left(1-\left(\frac{2\sigma}{\gamma\pi y}\sin\frac{\gamma\pi y}{2\sigma}\right)^r\right)^\alpha,\quad v_2(y)=\left(1-\left(\frac{e^{\frac{i\gamma\pi y}{\sigma}}-1}{\frac{i\gamma\pi y}{\sigma}}\right)^r\right)^\alpha.$$

In view of inequalities (3.7) and (3.8) it is sufficient to prove the theorem for a single value of r, for instance, for $r=2$. Then the functions $\varphi_j=\frac{u}{v_j}-1$ are summable together with all their derivatives. By Lemma 2.2 we have $K_j=\mathcal{F}^{-1}\varphi_j\in\mathcal{R}^*$ and the norms $\|K_j^*\|_1$ are independent of σ. Applying Corollary 2.1, we obtain the inequalities for $\mathcal{A}_\sigma$. The inequalities for $\mathcal{A}_{\sigma-0}$ are obtained by the passage to a limit. The proof is complete. □

Corollary 4.1. *Let $p\in[1,+\infty)$, $w\in A_p$, $\gamma,\sigma>0$, $0<\beta\leqslant\alpha$. If $f\in W^{(2\beta,0)}_{p,w}$, then*

$$\mathcal{A}_\sigma(f)_{p,w}\leqslant C(r,\alpha,\beta,\gamma)B_p[w]\frac{1}{\sigma^{2\beta}}\mathcal{A}_\sigma\left(\left(I-S^r_{\frac{\gamma\pi}{\sigma}}\right)^{\alpha-\beta}f^{(2\beta,0)}\right)_{p,w}, \tag{4.3}$$

and, in particular, as $\beta=\alpha$,

$$\mathcal{A}_\sigma(f)_{p,w}\leqslant C(\alpha)B_p[w]\frac{1}{\sigma^{2\alpha}}\mathcal{A}_\sigma\big(f^{(2\alpha,0)}\big)_{p,w}, \tag{4.4}$$

and if $f\in W^{(\beta)}_{p,w}$, then

$$\mathcal{A}_\sigma(f)_{p,w}\leqslant C(r,\alpha,\beta,\gamma)B_p[w]\frac{1}{\sigma^{\beta}}\mathcal{A}_\sigma\left(\left(I-S^r_{\frac{\gamma\pi}{\sigma},\frac{\gamma\pi}{2\sigma}}\right)^{\alpha-\beta}f^{(\beta)}\right)_{p,w} \tag{4.5}$$

and, in particular, as $\beta=\alpha$,

$$\mathcal{A}_\sigma(f)_{p,w}\leqslant C(\alpha)B_p[w]\frac{1}{\sigma^{\alpha}}\mathcal{A}_\sigma\big(f^{(\alpha)}\big)_{p,w}. \tag{4.6}$$

In inequalities (4.3)–(4.6), $\mathcal{A}_\sigma$ *can be replaced by* $\mathcal{A}_{\sigma-0}$.

In order to prove this corollary, it is sufficient to compare Theorem 4.1 with inequalities (3.12)–(3.15).

The next lemma says that if we do not focus on the constants, then the inequalities of Bernstein and Riesz type are simple corollaries of the estimates for convolutions. We note that the Bernstein inequality is true for a wider class of weights, see [21].

Lemma 4.1. *Let $p \in [1, +\infty)$, $w \in A_p$, $\alpha, \gamma, \sigma > 0$, $\theta \in \mathbb{R}$, $T \in \mathbf{E}_\sigma \cap L_{p,w}$. Then*

$$\left\|T^{(\alpha,\theta)}\right\|_{p,w} \leqslant C(\alpha,\theta) B_p[w] \sigma^\alpha \|T\|_{p,w}, \tag{4.7}$$

$$\left\|T^{(2\alpha,0)}\right\|_{p,w} \leqslant C(r,\alpha,\gamma) B_p[w] \sigma^{2\alpha} \left\| \left(I - S^r_{\frac{\gamma\pi}{\sigma}}\right)^\alpha T \right\|_{p,w}, \tag{4.8}$$

$$\left\|T^{(\alpha)}\right\|_{p,w} \leqslant C(r,\alpha,\gamma) B_p[w] \sigma^{\alpha} \left\| \left(I - S^r_{\frac{\gamma\pi}{\sigma},\frac{\gamma\pi}{2\sigma}}\right)^\alpha T \right\|_{p,w}. \tag{4.9}$$

Proof. Let us prove inequality (4.7). We let $g_1(y) = e^{i\frac{\theta\pi}{2}\operatorname{sign} y}|y|^\alpha$ as $y \in [-1, 1]$. We continue the function g_1 on $\mathbb{R}$ to the Fourier transform of a function K_1 from $\mathcal{R}^*$. By Lemma 2.2, such continuation exists. Then

$$T^{(\alpha,\theta)}(x) = \sigma^\alpha \int\limits_{\mathbb{R}} T(x-t)\sigma K_1(\sigma t)\, dt,$$

and by Lemma 2.1 this implies (4.7) with the constant $\|K_1^*\|_1$.

Inequalities (4.8) and (4.9) can be proved in the same way with the help of the functions

$$g_2(y) = |y|^{2\alpha}\left(1 - \frac{2\sigma}{\gamma\pi y}\sin\frac{\gamma\pi y}{2\sigma}\right)^{-\alpha}, \qquad g_3(y) = (iy)^\alpha\left(1 - \frac{e^{i\frac{\gamma\pi y}{2\sigma}} - 1}{i\frac{\gamma\pi y}{2\sigma}}\right)^{-\alpha}.$$

The proof is complete. □

Corollary 4.2. *Let $p \in [1, +\infty)$, $w \in A_p$, $\alpha, \gamma, \sigma > 0$, $T \in \mathbf{E}_\sigma \cap L_{p,w}$. Then*

$$\left\| \left(I - S^r_{\frac{\gamma\pi}{\sigma}}\right)^\alpha T \right\|_{p,w} \leqslant C(r,\alpha) B_p[w] (\gamma\pi)^{2\alpha} \|T\|_{p,w}, \tag{4.10}$$

$$\left\| \left(I - S^r_{\frac{\gamma\pi}{\sigma},\frac{\gamma\pi}{2\sigma}}\right)^\alpha T \right\|_{p,w} \leqslant C(r,\alpha) B_p[w] (\gamma\pi)^{\alpha} \|T\|_{p,w}. \tag{4.11}$$

Proof. Inequality (4.10) follows from (3.13) and (4.7) as $\theta = 0$. Inequality (4.11) is implied by (3.15) and (4.7) as $\theta = \alpha$. The proof is complete. □

Direct theorem 4.1 with Lemma 4.1 allow us to specify the behavior of the constants in estimates (3.9) and (3.10): the growth order of the constants in λ is the same as in the classical case.

Theorem 4.2. *Let $p \in [1, +\infty)$, $w \in A_p$, $f \in L_{p,w}$, $r \in \mathbb{N}$, $\alpha, \lambda, h > 0$. Then*

$$\left\|(I - S^r_{\lambda h})^\alpha f\right\|_{p,w} \leqslant C(r,\alpha) B_p[w](1 + \lambda^{2\alpha}) \left\|(I - S^r_h)^\alpha f\right\|_{p,w}, \tag{4.12}$$

$$\left\|\left(I - S^r_{\lambda h,\frac{\lambda h}{2}}\right)^\alpha f\right\|_{p,w} \leqslant C(r,\alpha) B_p[w](1 + \lambda^{\alpha}) \left\|\left(I - S^r_{h,\frac{h}{2}}\right)^\alpha f\right\|_{p,w}. \tag{4.13}$$

Proof. Let us prove (4.12). We let $\sigma = \frac{\pi}{h}$. By (4.1), (3.13) and (4.8) (as $\gamma = 1$) we have

$$\begin{aligned}\left\|(I - S^r_{\lambda h})^\alpha f\right\|_{p,w} &\leqslant \left\|(I - S^r_{\lambda h})^\alpha (I - V_\sigma) f\right\|_{p,w} + \left\|(I - S^r_{\lambda h})^\alpha V_\sigma f\right\|_{p,w} \\ &\leqslant C(r,\alpha) B_p[w] \left\|(I - S^r_h)^\alpha f\right\|_{p,w} + C(r,\alpha) B_p[w] \lambda^{2\alpha} \left\|(I - S^r_h)^\alpha f\right\|_{p,w}.\end{aligned}$$

Inequality (4.13) can be proved in the same way, with the help of (4.2), (3.15) and (4.9). □

Now we can refine the information on the constants in Theorem 4.1.

Corollary 4.3. *The constants in inequalities* (4.1) *and* (4.2) *satisfy respectively the estimates*

$$C(r,\alpha,\gamma) \leqslant C_1(r,\alpha)(1 + \gamma^{-2\alpha}), \qquad C(r,\alpha,\gamma) \leqslant C_1(r,\alpha)(1 + \gamma^{-\alpha}).$$

G.I. Natanson and M.F. Timan [22] refined the Jackson inequality in spaces $L_p(\mathbb{T})$:

$$\sqrt[n]{\prod_{j=0}^{n-1} E_j(f)_p} \leqslant C(\alpha)\, \omega_\alpha\left(f, \frac{1}{n}\right)_p, \quad \alpha \in \mathbb{N}.$$

We write a similar refinement in the integral form.

Corollary 4.4. *Let $p \in [1, +\infty)$, $w \in A_p$, $f \in L_{p,w}$, $r \in \mathbb{N}$, $\alpha, h > 0$. Then*

$$\exp\left(\frac{1}{\sigma}\int_0^\sigma \ln \mathcal{A}_u(f)_{p,w}\, du\right) \leqslant C(r,\alpha)(1+\gamma^{-2\alpha})B_p[w] \left\|\left(I - S^r_{\frac{\gamma\pi}{\sigma}}\right)^\alpha f\right\|_{p,w}, \tag{4.14}$$

$$\exp\left(\frac{1}{\sigma}\int_0^\sigma \ln \mathcal{A}_u(f)_{p,w}\, du\right) \leqslant C(r,\alpha)(1+\gamma^{-\alpha})B_p[w] \left\|\left(I - S^r_{\frac{\gamma\pi}{\sigma},\frac{\gamma\pi}{2\sigma}}\right)^\alpha f\right\|_{p,w}. \tag{4.15}$$

Proof. Let us prove (4.14). Replacing σ by u and γ by $\frac{\gamma u}{\sigma}$ in (4.1), we find

$$\begin{aligned}\frac{1}{\sigma}\int_0^\sigma \ln \mathcal{A}_u(f)_{p,w}\, du &\leqslant \frac{1}{\sigma}\int_0^\sigma \ln\left\{C(r,\alpha)\left(1+\left(\frac{\gamma u}{\sigma}\right)^{-2\alpha}\right) B_p[w]\mathcal{A}_u\left(\left(I - S^r_{\frac{\gamma\pi}{\sigma}}\right)^\alpha f\right)_{p,w}\right\} du \\ &\leqslant \int_0^1 \ln\left(1+(\gamma t)^{-2\alpha}\right) dt + \ln\left\{C(r,\alpha)B_p[w]\left\|\left(I - S^r_{\frac{\gamma\pi}{\sigma}}\right)^\alpha f\right\|_{p,w}\right\}.\end{aligned}$$

We have

$$\int_0^1 \ln\left(1+(\gamma t)^{-2\alpha}\right) dt \leqslant \int_0^1 \ln\left((1+\gamma^{-2\alpha})t^{-2\alpha}\right) dt = 2\alpha + \ln\left(1+\gamma^{-2\alpha}\right).$$

Taking the exponentials, we get the needed inequality. Inequality (4.15) can be proved in the same way. The proof is complete. □

4.2. Inverse theorems. As usually, inequalities of Bernstein and Riesz type imply inverse theorems.

Theorem 4.3. *Let $p \in [1, +\infty)$, $w \in A_p$, $f \in L_{p,w}$, $r \in \mathbb{N}$, $\alpha, \sigma > 0$. Then*

$$\left\|\left(I - S^r_{\frac{\pi}{\sigma}}\right)^\alpha f\right\|_{p,w} \leqslant C(r,\alpha)B_p[w]\frac{1}{\sigma^{2\alpha}}\int_0^\sigma \mathcal{A}_u(f)_{p,w}\, du^{2\alpha},$$

$$\left\|\left(I - S^r_{\frac{\pi}{\sigma},\frac{\pi}{2\sigma}}\right)^\alpha f\right\|_{p,w} \leqslant C(r,\alpha)B_p[w]\frac{1}{\sigma^{\alpha}}\int_0^\sigma \mathcal{A}_u(f)_{p,w}\, du^{\alpha}.$$

Theorem 4.4. *Let $p \in [1, +\infty)$, $w \in A_p$, $f \in L_{p,w}$, $\alpha, \sigma > 0$, $\theta \in \mathbb{R}$ and*

$$\int_0^{+\infty} \mathcal{A}_u(f)_{p,w}\, du^\alpha < +\infty.$$

Then $f \in W_{p,w}^{(\alpha,\theta)}$ *and*

$$\left\|f^{(\alpha,\theta)}\right\|_{p,w} \leqslant C(\alpha,\theta)B_p[w]\left(\int_0^{+\infty} \mathcal{A}_u(f)_{p,w}\,du^\alpha\right),$$

$$\mathcal{A}_\sigma\left(f^{(\alpha,\theta)}\right)_{p,w} \leqslant C(\alpha,\theta)B_p[w]\left(\sigma^\alpha \mathcal{A}_\sigma(f)_{p,w} + \int_\sigma^{+\infty} \mathcal{A}_u(f)_{p,w}\,du^\alpha\right).$$

Corollary 4.5. *Under the assumptions of Theorem 4.4 as* $r \in \mathbb{N}$ *and* $\beta > 0$ *we have*

$$\left\|\left(I - S^r_{\frac{\pi}{\sigma}}\right)^\beta f^{(\alpha,\theta)}\right\|_{p,w} \leqslant C(r,\alpha,\beta,\theta)B_p[w] \cdot \left(\frac{1}{\sigma^{2\beta}}\int_0^\sigma \mathcal{A}_u(f)_{p,w}\,du^{\alpha+2\beta} + \int_\sigma^{+\infty} \mathcal{A}_u(f)_{p,w}\,du^\alpha\right),$$

$$\left\|\left(I - S^r_{\frac{\pi}{\sigma},\frac{\pi}{2\sigma}}\right)^\beta f^{(\alpha,\theta)}\right\|_{p,w} \leqslant C(r,\alpha,\beta,\theta)B_p[w] \cdot \left(\frac{1}{\sigma^{\beta}}\int_0^\sigma \mathcal{A}_u(f)_{p,w}\,du^{\alpha+\beta} + \int_\sigma^{+\infty} \mathcal{A}_u(f)_{p,w}\,du^\alpha\right).$$

The proofs of Theorems 4.3 and 4.4 are standard, see, for instance, [1], and this is why we omit them. Here we use inequalities (3.6) and (4.7)–(4.11). An abstract scheme for proving inverse theorems can be found in [23].

In the periodic case the integrals on the right hand sides of the inequalities become sums. For instance, as $\sigma \in \mathbb{N}$, we have

$$\frac{1}{\sigma^{2\alpha}}\int_0^\sigma \mathcal{A}_u(f)_{p,w}\,du^{2\alpha} = \frac{1}{\sigma^{2\alpha}}\sum_{k=0}^{\sigma-1}((k+1)^{2\alpha} - k^{2\alpha})\mathcal{A}_k(f)_{p,w}.$$

The usage of integrals is convenient for the uniform formulation of inverse theorems in the periodic and non-periodic case, see [23].

As it is known, for $p \in (1,+\infty)$ the direct and inverse theorems can be refined [4]. However, we can add no new information to these refinements since the dependence of the constants on p in these refinements is generated not by the way of proving but by the form of the inequality.

By $\widetilde{f}$ we denote the function trigonometrically conjugate with f; in other words, the Hilbert transform of f. If $p \in [1,+\infty)$, $f \in L_{p,w}$, then $\widetilde{f}$ can be defined as the principal value of the integral

$$\widetilde{f}(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)}{x-t}\,dt.$$

Then $\widetilde{f}(x)$ exists and is finite for almost all x. For $p \in (1,+\infty)$ we have $\widetilde{f} \in L_{p,w}$ and moreover, the operator of trigonometric conjugate is continuous in $L_{p,w}$ [24], see also [17, Sect. 5.4]. For sufficiently nice functions f, for instance, from the Schwartz class, we have $\mathcal{F}\widetilde{f}(y) = (-i\,\mathrm{sign}\,y)\mathcal{F}f(y)$.

The proof of next Theorem 4.5 is based on the known idea of passing to a primitive, see [1, Sect. 5.9].

Theorem 4.5. *Let $p \in [1,+\infty)$, $w \in A_p$, $f \in L_{p,w}$ and*

$$\int_1^{+\infty} \mathcal{A}_u(f)_{p,w} \frac{du}{u} < +\infty. \tag{4.16}$$

Then for each $\sigma > 0$ we have $(I - V_\sigma)\widetilde{f} \in L_{p,w}$ and

$$\mathcal{A}_\sigma\big((I - V_\sigma)\widetilde{f}\big)_{p,w} \leqslant CB_p[w] \left(\mathcal{A}_\sigma(f)_{p,w} + \int_\sigma^{+\infty} \mathcal{A}_u(f)_{p,w} \frac{du}{u} \right).$$

Proof. The function $y \mapsto \frac{1}{|y|}\left(1 - \eta\left(\frac{y}{\sigma}\right)\right)$ is the Fourier transform of some function $K \in \mathcal{R}^*$. We let $F = f * K$. Then $F \in L_{p,w}$ and $F^{(1,0)} = (I - V_\sigma)f \in L_{p,w}$. Hence, $F \in W^{(1,0)}_{p,w}$. By inequality (4.4),

$$A_u(F)_{p,w} \leqslant \frac{CB_p[w]}{u} A_u(f - V_\sigma f)_{p,w}. \tag{4.17}$$

We note that $A_u(f - V_\sigma f)_{p,w} = A_u(f)_{p,w}$ as $u \geqslant \sigma$. By condition (4.16) we obtain

$$\int_0^{\infty} A_u(F)_{p,w}\, du < +\infty.$$

By Theorem 4.4 (as $\alpha = \theta = 1$) we have $F \in W^{(1)}_{p,w}$ and hence $-F' = (I - V_\sigma)\widetilde{f} \in L_{p,w}$. By the same theorem

$$\mathcal{A}_\sigma\big((I - V_\sigma)\widetilde{f}\big)_{p,w} = A_\sigma(F')_{p,w} \leqslant CB_p[w] \left(\sigma\mathcal{A}_\sigma(F)_{p,w} + \int_\sigma^{+\infty} \mathcal{A}_u(F)_{p,w}\, du \right).$$

It remains to apply inequality (4.17); as it has been explained in Section 2.3, while combining the inequalities, the square of $B_p[w]$ is not taken. The proof is complete. □

The belonging $(I - V_\sigma)\widetilde{f} \in L_{p,w}$ is meaningful only if $p = 1$. However in the formulation and proof of Theorem 4.5 the existence of $\widetilde{f}$ was not used in any sense. The operator $f \mapsto (I - V_\sigma)\widetilde{f}$ treated as a single operator is well-defined and continuous from $L_{p,w}$ into $\mathcal{S}'$. In terms of the Fourier images this is the multiplication by $(-i \operatorname{sign} y)\left(1 - \eta\left(\frac{y}{\sigma}\right)\right)$.

In the periodic case if $f \in L_{p,w}(\mathbb{T})$, then Fourier series of the function $(I - V_\sigma)\widetilde{f}$ and the trigonometric series conjugate to the Fourier series of f coincide as $\sigma < 1$. This is why the statement simplifies: $\widetilde{f} \in L_{p,w}(\mathbb{T})$. In the non-periodic case as $p = 1$ we can not conclude this. The example

$$f(x) = \int_0^1 (1 - t)\cos xt\, dt = \frac{1 - \cos x}{x^2},$$

$$\widetilde{f}(x) = \int_0^1 (1 - t)\sin xt\, dt = \frac{x - \sin x}{x^2}$$

shows that condition (4.16) does not ensure the belonging $\widetilde{f} \in L_{1,w}(\mathbb{R})$ even in the weightless case and it is not sufficient for belonging to subtract from $\widetilde{f}$ an entire function of zero type, in particular, a constant.

4.3. K- and R-functionals and modules of continuity. We are going to establish the equivalence of K- and R-functionals to the modified modules of continuity. The results of such type in the weightless case can be found, for instance, in [25]. For the weighted spaces we refer to [6]. We define the family of K- and R-functionals by the identities

$$K_{\alpha,\theta}(f,h)_{p,w} = \inf_{g\in W_{p,w}^{(\alpha,\theta)}} \left\{ \|f-g\|_{p,w} + h^\alpha \left\|g^{(\alpha,\theta)}\right\|_{p,w} \right\},$$

$$R_{\alpha,\theta}(f,h)_{p,w} = \inf_{g\in \mathbf{E}_{1/h}\cap L_{p,w}} \left\{ \|f-g\|_{p,w} + h^\alpha \left\|g^{(\alpha,\theta)}\right\|_{p,w} \right\}.$$

Theorem 4.6. *Let $p\in[1,+\infty)$, $w\in A_p$, $f\in L_{p,w}$, $r\in\mathbb{N}$, $\alpha,h>0$. Then*

$$\left\|(I-S_h^r)^\alpha f\right\|_{p,w} \asymp K_{2\alpha,0}(f,h)_{p,w} \asymp R_{2\alpha,0}(f,h)_{p,w}, \tag{4.18}$$

$$\left\|\left(I-S_{h,\frac{h}{2}}^r\right)^\alpha f\right\|_{p,w} \asymp K_{\alpha,\alpha}(f,h)_{p,w} \asymp R_{\alpha,\alpha}(f,h)_{p,w}. \tag{4.19}$$

The constants in the estimates are of the form $C(r,\alpha)B_p[w]$.

Proof. We are going to prove (4.18). The inequality $K\leqslant R$ is trivial. Let us prove that

$$R_{2\alpha,0}(f,h)_{p,w} \leqslant C(r,\alpha)B_p[w]\left\|(I-S_h^r)^\alpha f\right\|_{p,w}.$$

We take $\sigma=\frac{1}{h}$ and write

$$\begin{aligned} R_{2\alpha,0}(f,h)_{p,w} &\leqslant \|f-V_\sigma f\|_{p,w} + h^{2\alpha}\left\|(V_\sigma f)^{(2\alpha,0)}\right\|_{p,w} \\ &\leqslant C(r,\alpha)B_p[w]\left\|(I-S_h^r)^\alpha f\right\|_{p,w} + C(r,\alpha)B_p[w]\left\|(I-S_h^r)^\alpha V_\sigma f\right\|_{p,w}. \end{aligned}$$

We have estimated the first term by inequality (4.1), and the latter by inequality (4.8). It remains to use the boundedness of the family $\{V_\sigma\}$:

$$\left\|(I-S_h^r)^\alpha V_\sigma f\right\|_{p,w} = \left\|V_\sigma (I-S_h^r)^\alpha f\right\|_{p,w} \leqslant CB_p[w]\left\|(I-S_h^r)^\alpha f\right\|_{p,w}.$$

Let us prove that

$$\left\|(I-S_h^r)^\alpha f\right\|_{p,w} \leqslant C(r,\alpha)B_p[w]K_{2\alpha,0}(f,h)_{p,w}.$$

For each function $g\in W_{p,w}^{(2\alpha,0)}$ by inequalities (3.6) and (3.13) we have

$$\begin{aligned} \left\|(I-S_h^r)^\alpha f\right\|_{p,w} &\leqslant \left\|(I-S_h^r)^\alpha (f-g)\right\|_{p,w} + \left\|(I-S_h^r)^\alpha g\right\|_{p,w} \\ &\leqslant C(r,\alpha)B_p[w]\|f-g\|_{p,w} + C(r,\alpha)B_p[w]h^{2\alpha}\left\|g^{(2\alpha,0)}\right\|_{p,w}. \end{aligned}$$

It remains to pass to the infimum over g.

Relation (4.19) can be proved in the same way. The proof is complete. □

5. Concluding remarks

In [7], [9], [26], [27] there was used a series of close statements called transference results. We note that paper [27] was devoted to approximation in the spaces $L_{p,w}(\mathbb{R})$ and involves the case $p=1$. Let us formulate one of such statements [9, Thm. 3.6] applied to the space $L_{p,w}(\mathbb{T})$. Let $p\in[1,+\infty)$, $w\in A_p$. For $f\in L_{p,w}(\mathbb{T})$ and a simple function G we let

$$F_{f,G}(u) = \int_{-\pi}^{\pi} f(u+x)|G(x)|\,dx.$$

If $f,g\in L_{p,w}(\mathbb{T})$ and the inequality

$$\sup_{u\in\mathbb{R}} |F_{f,G}(u)| \leqslant C_1 \sup_{u\in\mathbb{R}} |F_{g,G}(u)| \tag{5.1}$$

hods with some absolute constant C_1 for all simple functions G, then the inequality

$$\|f\|_{p,w} \leqslant C_2\|g\|_{p,w}$$

also holds with some constant C_2 independent of p and w.

This statement is wrong. Indeed, let us fix $\tau \in \mathbb{R}$ and take $g(t) = f(t+\tau)$. Then it is obvious that inequality (5.1) becomes the identity as $C_1 = 1$, while the quotient $\|f\|_{p,w}/\|g\|_{p,w}$ can be arbitrarily large if the space $L_{p,w}$ is not closed with respect to shift. This is why the statements proved with the help of this trick require a different proof. Those of them concerning the Lebesgue spaces with Muckenhoupt weights have been proved in the present work.

In conclusion we mention that the methods of the present work can be applied for proving direct and inverse theorems in more general functional spaces. If one succeeds to establish estimates for convolutions of type (2.1), then the results of Sections 3 and 4 are obtained immediately. For the sake of clarity, we have restricted ourselves to the Lebesgue spaces with Muckenhoupt weights.

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