ISSN 2304-0122 Ufa Mathematical Journal. Vol. 15. No 4 (2023). P. 31-41.

doi:10.13108/2023-15-4-31

SYLVESTER PROBLEM, COVERINGS BY SHIFTS, AND UNIQUENESS THEOREMS FOR ENTIRE FUNCTIONS

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Abstract. The idea to write this note arose during the discussion that followed the report of the first author at the International Scientific Conference "Ufa Autumn Mathematical School-2022". We propose three general methods for constructing uniqueness sets in classes of entire functions with growth restrictions. In all three cases, the sequence of zeros of an entire function with special properties is chosen as such set. The first method is related to Sylvester famous problem on the smallest circle containing a given set of points on a plane, and theorems of convex geometry. The second method initially relies on Helly theorem on the intersection of convex sets and its application to the possibility of covering one set by shifting another. The third method is based on the classical Jensen formula, which allows one to estimate the type of an entire function via the averaged upper density of the sequence of its zeros. We present only basic results now. The development of our approaches is expected to be presented in subsequent works.

Keywords: Sylvester problem, Jung theorem, Helly theorem, uniqueness set, type of entire function, sequence of zeros, indicator of an entire function, averaged upper density, Jensen formula, indicator diagram, smallest circle.

Mathematics Subject Classification: 30D15, 52A10

1. Preliminaries. Formulation and discussion of results

1.1. Uniqueness sets for classes of entire functions of exponential type with restriction for type. In 1857 English mathematician James Joseph Sylvester posed the following problem in note [1]. Given a finite set of the points in the plane, find a circle of the minimal radius containing these points. The Sylvester problem then was generalized many times up to an abstract formulation for an arbitrary set in a metric space. The exposition of the current state-of-art in the framework of the approximation theory, including applications, was presented, for instance, in survey [2]. An important role in the developing of these problems was played by a Heinrich Jung theorem [3] proved in 1901; according to this theorem, each compact set K of a diameter $d := \max\{|z_1 - z_2| \colon z_1, z_2 \in K\}$ in the *complex plane* $\mathbb{C}$ can be put in a closed circle of a radius $d/\sqrt{3}$. For each flat compact set K a corresponding closed circle of the smallest radius r is found uniquely, but there is a problem on exact calculating of the quantity r in terms of appropriate geometric characteristics of the compact set K. For a single-point compact set such circle obviously generates into a point, that is, $r = 0$. Except for this trivial case, we always have $r > 0$. Simple examples like a segment and a regular triangle confirms the shaprness of the two-sided estimate $d/2 \leqslant r \leqslant d/\sqrt{3}$ for the extremal radius r. We also note that for an acute triangle and a right triangle the extremal quantity r is equal to the radius of the circumscribed circle of a triangle, while for the obtuse triangle r is the half of the greatest

G.G. Braichev, B.N. Khabibullin, V.B. Sherstyukov, Sylvester problem, coverings by shifts, and uniqueness theorems for entire functions.

The work by the second author is made in the framework of the state task of the Ministry of Science and Higher Education of Russian Federation (code of scientific theme FMRS-2022-0124).

Submitted March 6, 2023.

side, which is less then the radius of the circumscribed circle. A nice elementary introduction in the corresponding part of the convex geometry was given in [4], [5].

Let us describe a connection of the Sylvester problem with the uniqueness issues in the traditional classes of entire functions of one variable. By the symbol $\mathbb{N}$ we denote the set of all natural numbers. We mostly use standard for the theory of entire functions terminology and notation [6]. For an entire function f on $\mathbb{C}$ we let

$$M(f;r) := \max_{|\lambda|\leqslant r} |f(\lambda)| = \max_{|\lambda|=r} |f(\lambda)|, \qquad r \geqslant 0.$$

We recall that an entire function f has an exponential type if the upper limit

$$\sigma(f) := \overline{\lim_{r\to+\infty}} \frac{\ln M(f;r)}{r}$$

is finite. The quantity $\sigma(f)$ is called an (exponential) *type* of entire function f.

An entire function f *vanishes on a sequence* $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ of complex numbers λ_n if the multiplicity of the zero of the function f at each point $\lambda \in \mathbb{C}$ does not less than the number of points λ_n in Λ, which are equal to λ. If the multiplicity of the zero of the function f at each point $\lambda \in \mathbb{C}$ coincides with the number of points λ_n in Λ equalling to λ, then Λ is a *sequence of all zeroes of the function* f, which is denoted in what follows by $\Lambda(f)$. A sequence Λ is called a *uniqueness set* for some class of entire function if each function from this class vanishing on Λ is zero. Otherwise Λ is a *non-uniqueness set* for this class.

An *indicator* of an entire function f of an exponential type is the characteristics

$$h(f;\,\theta) := \overline{\lim_{r\to+\infty}} \frac{\ln|f(re^{i\theta})|}{r}, \qquad 0 \leqslant \theta \leqslant 2\pi. \tag{1.1}$$

If in this definition for a fixed θ a limit is well-defined as $r \to +\infty$ outside some set of zero relative linear Lebesgue measure, then f is a function of a completely regular growth on the ray $\arg\lambda = \theta$. A function has a *completely regular growth* (in the sense of Levin-Pfluger) if such special limit is well-defined for each θ in the segment $[0, 2\pi]$.

A geometric counterpart of the function f is its *indicator diagram*, which is a convex set defined as

$$D(f) := \bigcap_{0\leqslant\theta\leqslant 2\pi} \left\{\lambda \in \mathbb{C} :\ \mathrm{Re}\,(\lambda e^{-i\theta}) \leqslant h(f;\,\theta)\right\}.$$

We also note an universal identity for all entire functions of exponential type

$$\max_{0\leqslant\theta\leqslant 2\pi} h(f;\theta) = \sigma(f). \tag{1.2}$$

For a number $\sigma > 0$ by $\mathrm{Ent}[1,\sigma)$ we denote the class of entire functions f of exponential type $\sigma(f) < \sigma$. It is obvious that if Λ is a uniqueness set for a class $\mathrm{Ent}[1,\sigma)$, then it is the same for each class $\mathrm{Ent}[1,\sigma')$ as $0 < \sigma' < \sigma$. It is also clear that if Λ contains only finitely many elements, then Λ can not serve as an uniqueness set for any class $\mathrm{Ent}[1,\sigma)$. The condition $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ excludes such situation from the definition of the uniqueness set. This is in formulation of our results we suppose, not saying explicitly, that the generating function f has infinitely many zeroes forming the sequence $\Lambda(f)$. For the sake of definiteness, we arrange the elements of this sequence in the non-ascending order of their absolute values and with the multiplicities taken into account.

Our first result is as follows.

Theorem 1.1. *Let $f \not\equiv 0$ be an entire function of exponential type of a completely regular growth, while $r(f)$ be the radius of the smallest circle containing the indicator diagram $D(f)$. The sequence $\Lambda(f)$ is a uniqueness set for* $\mathrm{Ent}[1,\sigma)$ *if and only if $\sigma \leqslant r(f)$.*

We stress that earlier in similar statements instead of a geometric characteristics $r(f)$ usually various densities of the sequence $\Lambda(f)$ were employed. We also mention a possibility of reformulating the theorem in terms of the completeness radius of a corresponding exponential system. A theoretical base for such reformulation and main results in this direction by 2012 were presented in details in monograph-survey [7].

We consider two examples, in which Theorem 1.1 is characterized by simplicity and clarity. More complicated constructions covered by this theorem can be found in [8], [9].

Example 1. *We take an entire function of an exponential type of a completely regular growth* $f(\lambda) \underset{\lambda\in\mathbb{C}}{=} \sin\pi\lambda$. *Then* $\Lambda(f)=\mathbb{Z}$ *is the set of all integer numbers,* $D(f)$ *is the segment on the imaginary axis with the end-points at* $\pm\pi i$. *Here* $r(f)=\pi$. *Applying Theorem 1.1, we obtain a well-known result by F. Carlson* [10] *stating that* $\mathbb{Z}$ *is the uniqueness set in the class* $\operatorname{Ent}[1,\pi)$.

Example 2. *Let a canonical product* f *be constructed by the set* Λ *formed by* $\pm\Omega$ *and* $\pm iM$, *where* $\Omega=\{\omega_n\}_{n\in\mathbb{N}}$ *and* $M=\{\mu_n\}_{n\in\mathbb{N}}$ *are increasing sequences of positive numbers with densities* ω *and* μ, *respectively, that is, there exist limits*

$$\lim_{n\to\infty}\frac{n}{\omega_n}=\omega>0,\qquad \lim_{n\to\infty}\frac{n}{\mu_n}=\mu>0.$$

Then the indicator diagram $D(f)$ *is the rectangle* $|\operatorname{Re}\lambda|\leqslant\pi\mu$, $|\operatorname{Im}\lambda|\leqslant\pi\omega$ *and on the base of Theorem 1.1 we conclude that* $\Lambda(f)=\Lambda$ *is the uniqueness set in the classes* $\operatorname{Ent}[1,\sigma)$, *where* $\sigma\leqslant r(f)=\pi\sqrt{\omega^2+\mu^2}$.

Theorem 1.1 and Jung theorem yield the following statement.

Corollary 1.1. *Let an entire function* f *be the same as in Theorem 1.1 and* d *be the diameter of the indicator diagram* $D(f)$. *If* $0<\sigma\leqslant d/2$, *then* $\Lambda(f)$ *is a uniqueness set for* $\operatorname{Ent}[1,\sigma)$, *and if* $\sigma>d/\sqrt{3}$, *then* $\Lambda(f)$ *is a non-uniqueness set for* $\operatorname{Ent}[1,\sigma)$.

Indeed, if $0<\sigma\leqslant d/2$, then the Jung theorem gives $r(f)\geqslant d/2\geqslant\sigma$ and $\Lambda(f)$ is a uniqueness set for $\operatorname{Ent}[1,\sigma)$ by Theorem 1.1. If $d/\sqrt{3}<\sigma$, then by the Jung theorem $r(f)\leqslant d/\sqrt{3}<\sigma$. In according with Theorem 1.1 this means that $\Lambda(f)$ is a non-uniqueness set for $\operatorname{Ent}[1,\sigma)$.

1.2. Uniqueness sets for classes of entire functions of exponential type with restrictions for indicator. We proceed to inqueness theorems for the classes of entire functions f of exponential type with restrictions for their growth not only for the type but also for a more gentle characteristics, indicator (1.1), which is convenient to continue 2π-periodically on the entire *real line* $\mathbb{R}$. This continuation will be denoted by $h_f(\theta)\overset{(1.1)}{:=}h(f;\theta)$, where $\theta\in\mathbb{R}$.

An order completing of the set $\mathbb{R}$ by the supremum and infimum

$$+\infty:=\sup\mathbb{R}=\inf\emptyset\notin\mathbb{R}\quad\text{and}\quad-\infty:=\inf\mathbb{R}=\sup\emptyset\notin\mathbb{R},$$

where $\emptyset$ is the *empty set*, defines an *extended* real line $\overline{\mathbb{R}}:=\mathbb{R}\bigcup\{\pm\infty\}$, where, in addition to standard operations, we let $0\cdot(\pm\infty)=(\pm\infty)\cdot 0:=0$.

For a *support function* of an arbitrary set $S\subseteq\mathbb{C}$ we use the notation

$$h_S(\theta):=\sup_{s\in S}\operatorname{Re}\bigl(se^{-i\theta}\bigr)\in\overline{\mathbb{R}},\quad\theta\in\mathbb{R},\tag{1.3}$$

which cause no discrepancies with the notation h_f for the indicator, since in the latter case subscript $_f$ is a function, while in (1.3) the subscript $_S$ is a set.

Let a 2π-periodic function $H\colon\mathbb{R}\to\overline{\mathbb{R}}$ be *trigonometrically convex* [6], [11], that is,

$$H(\theta)\leqslant\frac{\sin(\theta_2-\theta)}{\sin(\theta_2-\theta_1)}H(\theta_1)+\frac{\sin(\theta-\theta_1)}{\sin(\theta_2-\theta_1)}H(\theta_2)\quad\text{for all}\quad\theta\in(\theta_1,\theta_2)\subset\mathbb{R},\quad 0<\theta_2-\theta_1<\pi.$$

Each such function H is a support function of a *convex closed* in $\mathbb{C}$ set

$$D_H := \{z \in \mathbb{C}\colon \operatorname{Re}(ze^{-i\theta}) \leqslant H(\theta)\}, \tag{1.4}$$

which becomes a non-empty *compact set*, when the function H is *finite*, that is, $H(\mathbb{R}) \subset \mathbb{R}$. We note that for each $S \subseteq \mathbb{C}$ the support function h_S in (1.3) is trigonometrically convex and 2π-periodic and if the set $S \neq \emptyset$ is bounded in $\mathbb{C}$, then the support function h_S is finite.

A closed (open) *triangle* in $\mathbb{C}$ is a non-empty intersection of three closed (open) half-planes; if it is unbounded in $\mathbb{C}$, we postulate that the boundaries of two half-planes are parallel or coincide.

By $\operatorname{Ent}[1, H]$ we denote the class of all entire functions f of exponential type with the indicator $h_f(\theta) \leqslant H(\theta)$ for each $\theta \in \mathbb{R}$. (Non-)uniqueness sets for such classes of functions determined by restrictions for the indicator were considered in details in survey [7, Ch. 3]; in terms of limiting sets for entire and subharmonic functions they were studied in a monograph by V.S. Azarin [12, Ch. 6] in 2009 and in more details and in more general setting and often in subharmonic versions, they were studied in works [13, Sect. 3], [14, Thms. 2, 4, 5], [15, Thm. 3].

In the next criterion a key statement on the uniqueness set is formulated in the end since this is more convenient for a more coherent structuring of the proof.

Theorem 1.2. *Let an entire function f be the same as in Theorem 1.1 and $H\colon \mathbb{R} \to \mathbb{R}$ be a trigonometrically convex 2π-periodic function. Then the following statements are equivalent:*

I. *There is no shift of the indicator diagram $D(f)$ contained in D_H from* (1.4).

II. *There exist two closed triangles, one contains D_H, the other is contained in $D(f)$, such that each shift of the first triangle does not contain the second triangle.*

III. *There exists a triple of numbers $z_1, z_2, z_3 \in \mathbb{C}$ with the property $z_1 + z_2 + z_3 = 0$, for which*

$$\sum_{j=1}^{3} |z_j|\, h_{D(f)}(\arg z_j) > \sum_{j=1}^{3} |z_j|\, H(\arg z_j). \tag{1.5}$$

IV. *There exists $\theta \in \mathbb{R}$, for which $h_f(\theta) + h_f(\theta + \pi) > H(\theta) + H(\theta + \pi)$, or there exists a triple $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$, for which the difference $\theta_2 - \theta_1$ is not a multiple of π and the inequality holds:*

$$\begin{aligned} h_f(\theta_1)\,\frac{\sin(\theta_3-\theta_2)}{\sin(\theta_2-\theta_1)} + h_f(\theta_3) + h_f(\theta_2)\,\frac{\sin(\theta_1-\theta_3)}{\sin(\theta_2-\theta_1)} \\ > H(\theta_1)\,\frac{\sin(\theta_3-\theta_2)}{\sin(\theta_2-\theta_1)} + H(\theta_3) + H(\theta_2)\,\frac{\sin(\theta_1-\theta_3)}{\sin(\theta_2-\theta_1)}. \end{aligned}$$

V. *The sequence $\Lambda(f)$ is the uniqueness set for the class* $\operatorname{Ent}[1, H]$.

By $\operatorname{Ent}[1, H)$ we denote the class of all entire functions f of exponential type, the indicator of which satisfies $h_f(\theta) < H(\theta)$ for each $\theta \in \mathbb{R}$. For instance, as $H(\theta) \equiv \sigma > 0$, the class $\operatorname{Ent}[1, H)$ is the previous class $\operatorname{Ent}[1, \sigma)$ discussed in Subsection 1.1. If an addition condition

$$\inf_{\theta \in \mathbb{R}} \bigl(H(\theta) + H(\theta + \pi)\bigr) > 0 \tag{1.6}$$

is satisfied, then H is a support function of non-empty *convex domain*

$$O_H := \{z \in \mathbb{C}\colon \operatorname{Re}(ze^{-i\theta}) < H(\theta)\}, \tag{1.7}$$

and the class $\operatorname{Ent}[1, H)$ contains non-constant functions. Theorem 1.2 implies the following fact.

Corollary 1.2. *Let an entire function f be the same as in Theorem 1.1, and $H\colon \mathbb{R} \to \mathbb{R}$ be a trigonometrically convex 2π-periodic function obeying condition* (1.6). *Then the following six statements are equivalent.*

I. *There is no shift of the indicator diagram $D(f)$ contained in O_H from* (1.7).

II. *For each number* $c \in (0,1) \subset \mathbb{R}$ *there exists an open triangle circumscribed around* cO_H *and a closed triangle contained in* $D(f)$*, for which each shift of the open triangle does not contain the closed triangle.*

III. *For each* $c \in (0,1)$ *there exist* $z_1, z_2, z_3 \in \mathbb{C}$ *with* $z_1 + z_2 + z_3 = 0$, $|z_1| + |z_2| + |z_3| \neq 0$ *and*

$$\sum_{j=1}^{3} |z_j| \, h_{D(f)}(\arg z_j) \geqslant c \sum_{j=1}^{3} |z_j| \, H(\arg z_j). \tag{1.8}$$

IV. *For each* $c \in (0,1)$ *there exists* $\theta \in \mathbb{R}$*, for which*

$$h_f(\theta) + h_f(\theta + \pi) > cH(\theta) + cH(\theta + \pi),$$

or there exists a triple $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$*, for which the difference* $\theta_2 - \theta_1$ *is not a multiple of* π *and*

$$h_f(\theta_1) \frac{\sin(\theta_3 - \theta_2)}{\sin(\theta_2 - \theta_1)} + h_f(\theta_3) + h_f(\theta_2) \frac{\sin(\theta_1 - \theta_3)}{\sin(\theta_2 - \theta_1)}$$
$$\geqslant cH(\theta_1) \frac{\sin(\theta_3 - \theta_2)}{\sin(\theta_2 - \theta_1)} + cH(\theta_3) + cH(\theta_2) \frac{\sin(\theta_1 - \theta_3)}{\sin(\theta_2 - \theta_1)}.$$

V. *The sequence* $\Lambda(f)$ *is the uniqueness set in* $\mathrm{Ent}[1, cH]$ *for each* $c \in (0,1)$.

VI. *The sequence* $\Lambda(f)$ *is the uniqueness set for the class* $\mathrm{Ent}[1, H)$.

Corollary 1.2 will be derived from Theorem 1.2 after its proof in Section 2.

1.3. Uniqueness sets for classes of entire functions of arbitrary growth order. Here we provide a result of a different nature admitting a rather arbitrary growth of entire functions. We recall that the *order* of an entire function f is the quantity

$$\rho(f) := \varlimsup_{r \to +\infty} \frac{\ln \ln M(f; r)}{\ln r}.$$

As Valiron already showed, for each entire function f of a finite positive order ρ, there exists an unboundedly growing differential on some ray of the positive semi-axis function $\nu(r)$, in what follows called weight function, for which the limit

$$\lim_{r \to +\infty} \frac{r\nu'(r)}{\nu(r)} = \rho \tag{1.9}$$

is well-defined and the quantity

$$\varlimsup_{r \to +\infty} \frac{\ln M(f;\, r)}{\nu(r)}$$

is finite and positive. The original statement by Valiron was extended for entire functions of zero and infinite orders, see [16]. Thus, for each entire function f of order ρ, where $0 \leqslant \rho \leqslant +\infty$, there exists an entire function ν obeying condition (1.9) such that

$$0 < \sigma_\nu(f) := \varlimsup_{r \to +\infty} \frac{\ln M(f; r)}{\nu(r)} < +\infty. \tag{1.10}$$

The quantity $\sigma_\nu(f)$ is called ν*-type* of an entire function f. Chapter II of thesis [17] was devoted to various approaches to describing the growth of entire functions and many references were provided.

In what follows we always assume that the function ν increases to $+\infty$, satisfies (1.9) with some $0 \leqslant \rho \leqslant +\infty$ and is such that

$$\ln r = o(\nu(r)), \quad r \to +\infty. \tag{1.11}$$

Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of complex numbers with no finite accumulation points arranged in the ascending order of their absolute values. By $n_\Lambda(r) = \max\{n\colon |\lambda_n| \leqslant r\}$, $r \geqslant 0$, we denote its *counting function*, while an *averaged counting function* for Λ is defined as

$$N_\Lambda(r) = \int_0^r \frac{n_\Lambda(t) - n_\Lambda(0)}{t}\,dt, \quad r > 0.$$

If a number $\varepsilon > 0$ is less than the absolute value of the first non-zero term in the sequence Λ, then

$$N_\Lambda(r) = \int_\varepsilon^r \frac{n_\Lambda(t)}{t}\,dt \,-\, n_\Lambda(0)\ln\frac{r}{\varepsilon}, \quad r > \varepsilon.$$

As an asymptotic characteristics of the quantity $N_\Lambda(r)$ we consider an *averaged upper ν-density*

$$\overline{\Delta}_\nu^*(\Lambda) := \varlimsup_{r\to+\infty} \frac{N_\Lambda(r)}{\nu(r)} = \varlimsup_{r\to+\infty} \frac{1}{\nu(r)} \int_\varepsilon^r \frac{n_\Lambda(t)}{t}\,dt \tag{1.12}$$

for mentioned choice of ε. The coincidence of the upper limits in (1.12) is based on (1.11).

For $\sigma > 0$ by $\mathrm{Ent}[\nu,\sigma)$ we denote the class of all entire functions, the ν-type of which is less than σ. Owing to (1.11), the class $\mathrm{Ent}[\nu,\sigma)$ involves functions not being polynomials. A classical Jensen formula [6, Ch. I, Sect. 5] allows us to write the inequality

$$\sigma_\nu(f) \geqslant \overline{\Delta}_\nu^*(\Lambda) \tag{1.13}$$

relating ν-type (1.10) of an entire function f with an averaged upper ν-density (1.12) of the sequence of its zeroes $\Lambda = \Lambda(f)$. The following simple and at the same time rather general fact is immediately implied by inequality (1.13) but we did not see it in the literature.

Theorem 1.3. *If the ν-type of an entire function f with the sequence of zeroes $\Lambda := \Lambda(f)$ and an averaged upper ν-density of the sequence Λ coincide and are non-zero, that is,*

$$0 < \sigma := \sigma_\nu(f) = \overline{\Delta}_\nu^*(\Lambda), \tag{1.14}$$

then Λ is the uniqueness set for $\mathrm{Ent}[\nu,\sigma)$, *but it is not for* $\mathrm{Ent}[\nu,\sigma')$ *for each $\sigma' > \sigma$.*

Let us discuss how essential is the condition in Theorem 1.3 that the sequence Λ is a zero set of some entire function. We first consider the classes of entire functions of zero order, when the value ρ in condition (1.9) is zero. Appropriate examples of weight functions satisfying also (1.11) are $\nu(r) = \exp(\ln^\alpha r)$ with the parameter $\alpha \in (0,1)$ and $\nu(r) = \ln^\beta r$ with the parameter $\beta > 1$. In such cases for each sequence Λ with a finite averaged upper ν-density $\overline{\Delta}_\nu^*(\Lambda) > 0$ there exists an entire function f with the zero set $\Lambda(f) = \Lambda$. Such function is defined by a canonical Weierstrass-Hadamard product. At the same time, as it was shown in thesis [17, Sect. 2.2], equality (1.14) is true. In view of the made remarks, by Theorem 1.3 we get the following statement.

Corollary 1.3. *Let an increasing to $+\infty$ function ν satisfies condition* (1.9) *with the value $\rho = 0$ and condition* (1.11). *Then each sequence of complex numbers with an averaged upper ν-density $\overline{\Delta}_\nu^*(\Lambda) =: \sigma > 0$ is the uniqueness set for the class* $\mathrm{Ent}[\nu,\sigma)$.

For the functions of finite positive order the situation is different. If the weight function ν satisfies condition (1.9) with a finite $\rho > 0$ (then (1.11) holds as well), then not each sequence of complex numbers Λ with a finite averaged upper ν-density can serve as a set of zeroes of an entire function of a finite ν-type. The Weierstrass-Hadamard product constructed by Λ defines an entire function of a finite ν-type if ρ is a non-integer number. If the number ρ is integer, then for constructing an entire function the sequence Λ should possess a certain balance in the spirit of Lindelöf condition [18].

It is important to note that as $\rho > 0$, condition (1.14) in Theorem 1.3 is rather strict and it makes the entire function f to have a constant ν-indicator

$$h_\nu(f;\theta) := \varlimsup_{r\to+\infty} \frac{\ln|f(re^{i\theta})|}{\nu(r)} \equiv \sigma, \qquad 0 \leqslant \theta \leqslant 2\pi. \tag{1.15}$$

In fact, a simple idea to employ property (1.14) in the uniqueness issues was founded in survey [19, Sect. 2], which involves also an additional information related with restriction (1.15).

For a fixed $\sigma > 0$ condition (1.14) selects in the class of entire functions with ν-type $\sigma_\nu(f) = \sigma$ those, the sequence of zeroes $\Lambda(f)$ has a maximal possible averaged upper ν-density. Let us provide one more observation demonstrating a connection between Theorems 1.1 and 1.3. Let f be an entire function of exponential type with $\sigma(f) = \sigma > 0$. Suppose that the sequence of its zeroes $\Lambda := \Lambda(f)$ is *measurable*, that is, there exists a limit

$$\Delta^*(\Lambda) := \lim_{r\to+\infty} \frac{N_\Lambda(r)}{r},$$

and the identity $\Delta^*(\Lambda) = \sigma$ holds true. Then, see [19, Sect. 2], [20], the function f has a completely regular growth and the indicator diagram $D(f)$ exactly coincides with the smallest containing it circle $\{\lambda \in \mathbb{C}\colon |\lambda| \leqslant \sigma = r(f)\}$. By Theorem 1.1 such sequence Λ forms a uniqueness set for the class $[1,\sigma)$.

Following [17, Ch. 2] and [20], one can check identity (1.14) straightforwardly expressing it in terms of the Taylor coefficients and zeroes of an entire function f.

Corollary 1.4. *Let a strictly increasing to $+\infty$ function ν satisfy condition* (1.9) *with a finite value $\rho > 0$ and φ be the inverse function for ν, while $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ be the sequence of all zeroes of the entire function $f(\lambda) = \sum\limits_{n=0}^{\infty} a_n\lambda^n$, $\lambda \in \mathbb{C}$. If the condition*

$$\varlimsup_{n\to\infty} \frac{\varphi(n)}{\sqrt[n]{|\lambda_1\,\lambda_2\,\ldots\,\lambda_n|}} = \varlimsup_{n\to\infty} \varphi(n)\,\sqrt[n]{|a_n|} = (\sigma e\rho)^{1/\rho}$$

is satisfied, then Λ is the uniqueness set for the class $\mathrm{Ent}[\nu,\sigma)$.

2. Proof of main results

Proof of Theorem 1.1. In the case $r(f) = 0$ the indicator diagram $D(f) = \{a\}$ is a single-point set and $\sigma > r(f) = 0$. At the same time $\Lambda(f)$ is not the uniqueness set for $\mathrm{Ent}[1,\sigma)$ since a non-zero entire function $\lambda \underset{\lambda\in\mathbb{C}}{\longmapsto} f(\lambda)\,e^{-\bar{a}\lambda}$ (the bar stands for the complex conjugation) possesses a zero type, still vanishes on $\Lambda(f)$ and belongs to the class $\mathrm{Ent}[1,\sigma)$ for each $\sigma > 0$. We see that in this case Theorem 1.1 is true and in what follows we consider only the case $r(f) > 0$.

Let us show that under the assumptions of the theorem the sequence $\Lambda(f)$ is the uniqueness set for the class $\mathrm{Ent}\big[1, r(f)\big)$ and at the same time it is not the uniqueness set for any class $\mathrm{Ent}[1,\sigma)$ as $\sigma > r(f)$. Let $a \in \mathbb{C}$ be the center of the smallest circle of the radius $r(f)$ containing the indicator diagram $D(f)$ of the function f. We consider a function $f_{-\bar{a}}(\lambda) \underset{\lambda\in\mathbb{C}}{\equiv} f(\lambda)\,e^{-\bar{a}\lambda}$. This entire function also possesses an exponential type and its indicator diagram $D(f_{-\bar{a}})$ is located in the circle

$$\big\{\lambda \in \mathbb{C}\colon |\lambda| \leqslant r(f)\big\}. \tag{2.1}$$

Taking into consideration the extremal nature of such circle and property (1.2), we conclude that the exponential type $\sigma(f_{-\bar{a}})$ of the auxiliary entire function $f_{-\bar{a}}$ is equal to $r(f)$. Since $f_{-\bar{a}} \in \mathrm{Ent}[1,\sigma)$ for each $\sigma > r(f)$, the sequence $\Lambda(f_{-\bar{a}}) = \Lambda(f)$ is not a uniqueness set for $\mathrm{Ent}[1,\sigma)$ for each $\sigma > r(f)$. We observe that the necessary part of the theorem holds with no additional requirement on the regularity of growth of the function f. This requirement will be needed in the proof of sufficiency.

So, let the generating entire function f of exponential type have a completely regular growth. Let us show that the sequence of its zeros $\Lambda(f)$ forms a uniqueness set for the class $\mathrm{Ent}\big[1, r(f)\big)$, where $r(f)$ is the radius of the smallest circle containing the indicator diagram $D(f)$. As above, we work with the auxiliary function $f_{-\overline{a}}$, which has the same sequence of zeros as f. The indicator diagram $D(f_{-\overline{a}})$ is contained in circle (2.1), but cannot be placed into any circle of a smaller radius. Let $F \in \mathrm{Ent}\big[1, r(f)\big)$ and let F vanish on $\Lambda(f)$. Then the quotient $g(\lambda) \underset{\lambda\in\mathbb{C}}{\equiv} F(\lambda)/f_{-\overline{a}}(\lambda)$ defines an entire function of exponential type. Since $f_{-\overline{a}}$ has a completely regular growth, the rule for adding indicators applies:

$$h(gf_{-\overline{a}};\,\theta) = h(g;\,\theta) + h(f_{-\overline{a}};\,\theta) = h(F;\,\theta), \qquad 0 \leqslant \theta \leqslant 2\pi. \tag{2.2}$$

An extremal with respect to a compact set $D(f_{-\overline{a}})$ nature of circle (2.1) indicates that at least one of the following situations is realized.

1. There exist a direction $\theta_0 \in [0, \pi]$, for which

$$h(f_{-\overline{a}};\theta_0) = h(f_{-\overline{a}};\theta_0+\pi) = \sigma(f_{-\overline{a}}) = r(f).$$

Due to (2.2) and view of (1.2) and the choice of F, in such case we have

$$h(g;\,\theta_0) = h(F;\theta_0) - h(f_{-\overline{a}};\theta_0) \leqslant \sigma(F) - r(f) < 0,$$
$$h(g;\,\theta_0+\pi) = h(F;\theta_0+\pi) - h(f_{-\overline{a}};\theta_0+\pi) \leqslant \sigma(F) - r(f) < 0.$$

Here $h(g;\,\theta_0) + h(g;\,\theta_0+\pi) < 0$ and by a known property of indicator this can not hold for a non-zero function g.

2. There exist three directions $\theta_1 < \theta_2 < \theta_3$ on $[0, 2\pi]$ such that

$$\theta_2 - \theta_1 < \pi, \qquad \theta_3 - \theta_2 < \pi, \qquad \theta_3 - \theta_1 > \pi,$$

for which

$$h(f_{-\overline{a}};\theta_1) = h(f_{-\overline{a}};\theta_2) = h(f_{-\overline{a}};\theta_3) = \sigma(f_{-\overline{a}}) = r(f).$$

In this case, using again (2.2), we obtain that the values $h(g;\,\theta)$ are negative as $\theta = \theta_j$, where $j = 1, 2, 3$. By general properties of the indicator, the location of the mentioned points allows us to conclude that in this case $g(\lambda) \equiv 0$. Thus, in each case $g(\lambda) \equiv 0$ and hence $F(\lambda) \equiv 0$. Therefore, $\Lambda(f)$ is the uniqueness set for the class $\mathrm{Ent}\big[1, r(f)\big)$. The proof is complete. □

Proof of Theorem 1.2. First we are going to the prove the equivalence of the negations of Statements V and I. Let for some $a \in \mathbb{C}$ the shift $D(f) + a$ be contained in D_H. This means that the indicator diagram of an entire function $f_{\overline{a}}(\lambda) \underset{\lambda\in\mathbb{C}}{\equiv} f(\lambda)\, e^{\overline{a}\lambda}$ is located in the compact set D_H, that is, the convex compact $D(f_{\overline{a}})$ is contained in the convex compact set D_H. By the definition of the indicator diagram this implies the inequality $h_{f_{\overline{a}}}(\theta) \leqslant H(\theta)$ for each $\theta \in \mathbb{R}$. Therefore, $\Lambda(f_{\overline{a}}) = \Lambda(f)$ is the non-uniqueness set in the class $\mathrm{Ent}[1, H]$.

And vice versa, let $\Lambda(f)$ be a non-uniqueness set for $\mathrm{Ent}[1, H]$. Then there exists a non-zero entire function F of exponential type vanishing on the sequence $\Lambda(f)$ such that $h_F(\theta) \leqslant H(\theta)$ for each $\theta \in \mathbb{R}$. This means that the indicator diagram $D(F)$ is contained in D_H and F is divisible by the function f, that is, $F = gf$ for some non-zero entire function g. At the same time, g is an entire function of exponential type as the quotient of such functions. Since f is a completely regular growth, by the well-known theorem an addition indicators as in (2.2) we obtain

$$h_f(\theta) + h_g(\theta) \equiv h_{gf}(\theta) \equiv h_F(\theta) \leqslant H(\theta)$$

for all $\theta \in \mathbb{R}$. In terms of the indicator diagram and the convex compact set D_H with the support function H this means that $D(f) + D(g) \subseteq D_H$. In particular, for each point $z \in D(g)$

the inclusion $D(f)+z \subseteq D_H$ holds. Thus, some shift of $D(f)$ is located in D_H. The equivalence of Statements V and I is proven.

In proving the equivalence of Statement I to remaining three Statements II–IV we again employ their negations. A main role is played by the following result from [21].

Theorem 2.1 ([21, Thm. 2]). *Let C be a convex bounded set in $\mathbb{C}$, $\mathcal{S}$ be a family of sets from $\mathbb{C}$, and S be the union of all sets from $\mathcal{S}$. Suppose that C is a closed set or S is an open set. Then the following four statements are pairwise equivalent:*

(i) *some shift of the set S is contained in C;*

(ii) *for each triple of sets S_1, S_2, S_3 from $\mathcal{S}$ and each closed non-empty triangle circumscribed around C there exists a point $z \in \mathbb{C}$, for which all three shifts S_1+z, S_2+z, S_3+z are contained in this triangle;*

(iii) *for each triple of sets $S_1, S_2, S_3 \in \mathcal{S}$ and all triples of real numbers $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ and numbers $q_1, q_2, q_3 \geqslant 0$ under the condition $q_1e^{i\theta_1}+q_2e^{i\theta_2}+q_3e^{i\theta_3}=0$ the inequality holds:*

$$q_1h_{S_1}(\theta_1)+q_2h_{S_2}(\theta_2)+q_3h_{S_3}(\theta_3) \leqslant q_1h_C(\theta_1)+q_2h_C(\theta_2)+q_3h_C(\theta_3);$$

(iv) *for each triple of sets S_1, S_2, $S_3 \in \mathcal{S}$ and each set of numbers θ_1, θ_2, $\theta_3 \in \mathbb{R}$ such that each mutual difference is a multiple of π, as $\theta_j-\theta_k$ is not a multiple of 2π, the inequality*

$$h_{S_1}(\theta_k)+h_{S_2}(\theta_j) \leqslant h_C(\theta_k)+h_C(\theta_j)$$

holds, while if the difference $\theta_2-\theta_1$ is not a multiple of π, then the inequality

$$h_{S_1}(\theta_1)\,\frac{\sin(\theta_3-\theta_2)}{\sin(\theta_2-\theta_1)}+h_{S_3}(\theta_3)+h_{S_2}(\theta_2)\,\frac{\sin(\theta_1-\theta_3)}{\sin(\theta_2-\theta_1)}$$
$$\leqslant h_C(\theta_1)\,\frac{\sin(\theta_3-\theta_2)}{\sin(\theta_2-\theta_1)}+h_C(\theta_3)+h_C(\theta_2)\,\frac{\sin(\theta_1-\theta_3)}{\sin(\theta_2-\theta_1)}$$

holds true.

We let $S:=D(f)$ and $C:=D_H$. Under such choice both sets S and C are convex compact sets in $\mathbb{C}$. At the same time we immediately see that Statement (i) of Theorem 2.1 is the negation of Statement I in Theorem 1.2.

We proceed to Statement (ii). In Theorem 2.1 one can treat S as the union of all sets from the family $\mathcal{S}=\big\{\{s\}\colon s\in S\big\}$ of all single-point sets contained in S. Then Statement (ii) means that for each triple of points $s_1, s_2, s_3 \in S$ and each closed non-empty triangle circumscribed around C, there exists a point $z \in \mathbb{C}$, for which all three shifts s_1+z, s_2+z, s_3+z are contained in this triangle. Due to the convexity of S, this means that each closed triangle with arbitrary vertices $s_1, s_2, s_3 \in S$ can be placed in the mentioned closed triangle circumscribed around C. This easily shows that Statement (ii) is the negation of Statement II in Theorem 1.2 for the specified choice of the family $\mathcal{S}$.

We proceed to (iii). In Theorem 2.1 we regard S as the union of all sets from the family $\mathcal{S}=\{S\}$ consisting of one set S. Then Statement (iii) means that for each triple of complex numbers in polar form $z_1=q_1e^{i\theta_1}$, $z_2=q_2e^{i\theta_2}$, $z_3=q_3e^{i\theta_3}$, provided $z_1+z_2+z_3=0$, a non-strict inequality opposite to strict inequality (1.5) from Statement III of Theorem 1.2 holds. Thus, (iii) is the negation of III in Theorem 1.2.

We proceed to (iv). In Theorem 2.1 we again treat S as the union of all sets from the family $\mathcal{S}=\{S\}$ consisting of one set S, that is, $S_1=S_2=S_3=S$ in two non-strict inequalities of Statement (iv), which are opposite to the strict inequalities in Statement IV of Theorem 1.2. Thus, Statement (iv) is a negation of Statement IV of Theorem 1.2.

By Theorem 2.1 Statements (i)–(iv) in the above described versions are equivalent and this means that their negations I–IV from Theorem 1.2 are equivalent. This completes the proof. □

Derivation of Corollary 1.2 from Theorem 1.2. First we are going to justify the equivalence of Statements V and VI in Corollary 1.2 through the equivalence of their negations. The negation of Statement VI of Corollary 1.2 is that $\Lambda := \Lambda(f)$ is a non-uniqueness set for the class $\operatorname{Ent}[1, H)$. In other words, there is an entire function $F \not\equiv 0$ of exponential type, vanishing on Λ, with a compact indicator diagram $D(F)$ lying inside the convex region O_H. In particular, Λ is a non-uniqueness set for $\operatorname{Ent}[1, h_F]$. Therefore, there exists $c \in (0,1)$, for which some shift of the convex compact set D_{cH} with the support function cH includes $D(F)$ and is contained in O_H. Applying a shift to D_{cH} means that for some a_c an entire function of exponential type $F(\lambda)\, e^{a_c\lambda} \not\equiv 0$ for $\lambda \in \mathbb{C}$ still vanishes on Λ and belongs to the class $\operatorname{Ent}[1, cH]$. In this way we obtain that Λ is a non-uniqueness set for $\operatorname{Ent}[1, cH]$, and the negation of Statement VI of Corollary 1.2 implies the negation of Statement V of Corollary 1.2. Reversing this arguing, we see that the negation of Statement V of Corollary 1.2 yields the negation of Statement VI of Corollary 1.2. Thus, Statements V and VI of Corollary 1.2 are equivalent.

Equivalence of Statement V of Corollary 1.2 to all previous statements I–IV of Corollary 1.2 is exactly the equivalence of Statements V of Theorem 1.2 to preceding Statements I–IV of Theorem 1.2 for cH replaced by H. In this case, the replacement of strict inequalities $>$ from (1.5) and inequalities in Statement IV of Theorems 1.2 for non-strict $\geqslant$ in (1.8) and Statement IV in Corollary 1.2 is possible owing to the fact that the number $c \in (0,1)$ admits certain variations within $(0,1)$. The same applies to the open triangle described around cO_H in Statement II of Corollary 1.2 instead of the seemingly claimed closed triangle according to Statement II of Theorem 1.2 described around D_{cH}. The proof is complete. □

Proof of Theorem 1.3. Under the assumptions of the theorem let F be an entire function of ν-type $\sigma_\nu(F) < \sigma$ vanishing on Λ. Suppose that F is not identically zero. Then F possesses infinitely many zero forming a sequence $\Lambda(F)$, for which $\Lambda = \Lambda(f) \subseteq \Lambda(F)$. Here the embedding means that the number of appearance of each value $\lambda \in \mathbb{C}$ into $\Lambda(F)$ is at least the number of appearance in Λ. Therefore, for the counting functions of these sequence we have the inequality $n_{\Lambda(F)}(r) \geqslant n_\Lambda(r)$ for all $r \geqslant 0$. Using (1.12)–(1.14), we have

$$\sigma_\nu(F) \overset{(1.13)}{\geqslant} \overline{\Delta}_\nu^*\bigl(\Lambda(F)\bigr) \overset{(1.12)}{=} \varlimsup_{r\to+\infty} \frac{N_{\Lambda(F)}(r)}{\nu(r)} \geqslant \varlimsup_{r\to+\infty} \frac{N_\Lambda(r)}{\nu(r)} \overset{(1.12)}{=} \overline{\Delta}_\nu^*(\Lambda) \overset{(1.14)}{=} \sigma_\nu(f) \overset{(1.14)}{=} \sigma,$$

and this contradicts the condition $\sigma_\nu(F) < \sigma$. Therefore, $F(\lambda) \equiv 0$ on $\mathbb{C}$ and Λ is the uniqueness set for the class $\operatorname{Ent}[1,\sigma)$. If we take $\sigma' > \sigma$, then the class $\operatorname{Ent}[1,\sigma')$ contains the function f and this does not allow the sequence of its zeroes $\Lambda = \Lambda(f)$ to be the uniqueness set for such class $\operatorname{Ent}[1,\sigma')$ with $\sigma' > \sigma$. The proof is complete. □

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