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INTEGRAL HARDY INEQUALITIES, THEIR GENERALIZATIONS AND RELATED INEQUALITIES

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Abstract. Hardy inequalities have numerous applications in mathematical physics and spectral theory of unbounded operators. In this paper we describe direct generalizations of integral Hardy inequalities, their improvements and analogues. We systemize the relations between various interpretations of these inequalities and describe new one-dimensional integral inequalities. We show that these known and new inequalities are valid also for complex-valued functions.

We consider in details integral inequalities of Hardy, Rellich and Birman type for functions defined on bounded intervals. In particular, we provide the proofs for the generalizations and improvements of Birman integral inequalities for higher derivatives. We briefly discuss multidimensional analogues involving integrals of the powers of the modulus of the gradient of a function or of a polyharmonic operator.

Keywords: Hardy inequality, Rellich inequality, Birman inequality, Lamb constant, polyharmonic operator.

Mathematics Subject Classification: 26D10, 33C20

1. Introduction

As it is known, Hardy inequalities are employed in justifying embedding theorems in Sobolev spaces. Apparently, these applications played a key role in the popularization of one-dimensional Hardy integral inequalities. We note that in the monograph by S.L. Sobolev[1] involves a separate section "Hardy inequality". In this section, several versions of these inequalities and some generalizations are proved when the weight functions have the form $t^{-\lambda}|\ln t|^p$.

The appearance of different versions of Hardy inequality and related inequalities is due to a large number of various applications. In this article, Section 2 is devoted to the basic versions of Hardy inequality, where, in particular, we provide a justification for extending these inequalities to the case of complex-valued functions.

Our main Section 3 presents inequalities for higher derivatives, and Section 4 gives improvements of the inequalities in bounded intervals. We consistently describe the connections between different interpretations of Hardy integral inequality and inequalities of the Hardy, Rellich and Birman type for functions defined on infinite and bounded intervals. Thus, we systematize the connections between different interpretations of Hardy, Rellich and Birman integral inequalities. Among new results obtained in the article, we highlight Theorems 3.2 and 4.2 devoted to generalizations and improvements of Rellich and Birman integral inequalities involving the absolute values of a complex-valued function $f : X \to \mathbb{C}$ and of its derivative $f^{(k)}$ of order $k \geqslant 2$; the set X is $X = (0, \infty)$ in Theorem 3.2 and $X = (0, c)$, $c \in (0, \infty)$, is in Theorem 4.2.

In last Section 5 we briefly discuss the passage from one-dimensional integral inequalities to inequalities for complex-valued functions defined in domains of Euclidean space of dimension $n \geqslant 2$. In this case, we consider spatial analogues of inequalities for functions $u : \Omega \to \mathbb{C}$, when the integrals over the

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domain $\Omega \subset \mathbb{R}^n$ contain the modulus of this function and the modules of the gradient $\nabla u(x)$ or the polyharmonic operator $\Delta^{k/2}u(x)$.

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2. On basic versions of Hardy inequality

An original Hardy inequality can be formulated as follows, see [2, Thms. 327, 328, 330]:

Theorem 2.1. *Assume that $p \in [1,\infty)$, $s \in (-\infty,1)\cup(1,\infty)$ and we are given a function $f : (0,\infty) \to [0,\infty)$ obeying the condition $f/t^{s/p-1} \in L^p(0,\infty)$.*

We define a function $F : (0,\infty) \to [0,\infty)$ by the identities

$$F(t) = \int_0^t f(\tau)d\tau \quad \text{in the case} \quad s > 1, \qquad F(t) = \int_t^\infty f(\tau)d\tau \quad \text{in the case} \quad s < 1.$$

Then the following statements hold: if $p = 1$, then the identity

$$\int_0^\infty \frac{f(t)}{t^{s-1}}\,dt = |s-1| \int_0^\infty \frac{F(t)}{t^s}\,dt$$

is valid, while if $p > 1$, then

$$\int_0^\infty \frac{f^p(t)}{t^{s-p}}\,dt > \left(\frac{|s-1|}{p}\right)^p \int_0^\infty \frac{F^p(t)}{t^s}\,dt, \tag{2.1}$$

except for the case when $f \equiv 0$. The constant $(|s-1|/p)^p$ in this inequality is the best possible, that is, it is maximal among all possible ones.

The next theorem can be regarded as a version of Hardy theorem since it is both a corollary and a generalization of Theorem 2.1.

Theorem 2.2. *1) Let $1 \leqslant p < \infty$, $1 < s < \infty$. Assume that a function $g : [0,\infty) \to \mathbb{R}$ is absolutely continuous on each bounded segment $[0,a]$ and satisfies the conditions $g(0) = 0$, $g'/t^{s/p-1} \in L^p(0,\infty)$. Then*

$$\int_0^\infty \frac{|g'(t)|^p}{t^{s-p}}\,dt \geqslant \left(\frac{s-1}{p}\right)^p \int_0^\infty \frac{|g(t)|^p}{t^s}\,dt. \tag{2.2}$$

If $p > 1$ and $g \not\equiv 0$, then this inequality is strict and the constant $((s-1)/p)^p$ is sharp, that is, it is best possible.

2) Let $1 \leqslant p < \infty$, $-\infty < \sigma < 1$. Suppose that the function $g : (0,\infty] \to \mathbb{R}$ is absolutely continuous on each ray $[a,\infty]$, $a > 0$, and obeys the conditions

$$g(\infty) = 0 \quad \text{and} \quad g'/\tau^{\sigma/p-1} \in L^p(0,\infty).$$

Then

$$\int_0^\infty \frac{|g'(\tau)|^p}{\tau^{\sigma-p}}\,d\tau \geqslant \left(\frac{|\sigma-1|}{p}\right)^p \int_0^\infty \frac{|g(\tau)|^p}{\tau^\sigma}\,d\tau. \tag{2.3}$$

If $p > 1$ and $g \not\equiv 0$, then this inequality is strict but the constant $(|\sigma-1|/p)^p$ is sharp, that is, it is maximal among all possible ones.

It is clear that this theorem generalize and strengthens Theorem 2.1. We stress that Theorem 2.2 involves no conditions related with monotonicity of sign-definiteness of considered functions g and g'.

On the other hand, Theorem 2.2 is a corollary of Theorem 2.1 from the point of view of inequalities.

Indeed, let $1 < s < \infty$. We define functions f and F by the identities

$$f(t) = |g'(t)|, \qquad F(t) = \int_0^t |g'(\tau)| d\tau.$$

Since

$$\int_0^t |g'(\tau)| d\tau \geqslant |g(t)|,$$

then $F(t) \geqslant |g(t)|$ as $t \geqslant 0$ and inequality (2.2) follows from inequality (2.1) as $p > 1$, while for $p = 1$ it is implied by the identity corresponding to the case $p = 1$ in Theorem 2.1.

Inequality (2.3) is obtained from (2.2) by the change of the variable $\tau = 1/t$ and of the parameter $\sigma = 2 - s$. Thus, Theorem 2.2 is a corollary of Theorem 2.1 applied to the functions

$$F(t) = \int_0^t |g'(\tau)| d\tau \qquad s > 1, \qquad F(t) = \int_t^\infty |g'(\tau)|\, d\tau \qquad \sigma < 1,$$
$$f(t) = |g'(t)| \qquad s > 1, \qquad \sigma < 1.$$

It is obvious that such formulas for defining the functions $F : (0, \infty) \to [0, \infty)$ and $f : (0, \infty) \to [0, \infty)$ can be used also in the case, when the function g is complex-valued. This is why the following version of Hardy theorem holds true.

Theorem 2.3. *Suppose that $p \in [1, \infty)$, $s \in \mathbb{R} \setminus \{1\}$, a continuous function $g : (0, \infty) \to \mathbb{C}$ is differentiable almost everywhere, $|g'|/t^{s/p-1} \in L^p(0, \infty)$ and the following conditions are satisfied:*

1) if $s > 1$, then $g(0) := \lim_{t\to 0} g(t) = 0$ and the identity

$$g(t) = \int_0^t g'(\tau) d\tau$$

holds, where $0 \leqslant t < \infty$;

2) if $s < 1$, then

$$g(\infty) := \lim_{t\to\infty} g(t) = 0$$

and the identity

$$g(t) = \int_\infty^t g'(\tau) d\tau$$

holds, where $0 < t \leqslant \infty$.

Then the inequality holds:

$$\int_0^\infty \frac{|g'(t)|^p}{t^{s-p}}\, dt \geqslant \left(\frac{|s-1|}{p}\right)^p \int_0^\infty \frac{|g(t)|^p}{t^s}\, dt. \tag{2.4}$$

If $p > 1$ and $g \not\equiv 0$, then this inequality is strict but the constant $(|s-1|/p)^p$ is sharp, that is, it is maximal among all possible ones.

Let k be a natural number. As usually, by the symbol $C^k(\Omega)$ we denote the space of continuous and k times continuously differentiable functions $g : \Omega \to \mathbb{C}$, where Ω is a non-empty open set. By the symbol $C_0^k(\Omega)$ we denote a subfamily consisting of the functions $g \in C^k(\Omega)$, the compact supports of which are located in Ω.

Corollary 2.1. *For each $p \in [1, \infty)$ and each $s \in \mathbb{R} \setminus \{1\}$ the following inequality holds:*

$$\int_0^\infty \frac{|g'(t)|^p}{t^{s-p}}\, dt \geqslant \left(\frac{|s-1|}{p}\right)^p \int_0^\infty \frac{|g(t)|^p}{t^s}\, dt \tag{2.5}$$

for all $g \in C_0^1(0,\infty)$ *with a sharp constant* $(|s-1|/p)^p$. *For a function* $g \not\equiv 0$ *the inequality is strict for all* $p \in [1,\infty)$ *and* $s \in \mathbb{R} \setminus \{1\}$.

In many applications of Theorem 2.3 one needs its reduced versions related with usage of only one of boundary conditions $g(0) := \lim_{t\to 0} g(t) = 0$ and $g(\infty) := \lim_{t\to\infty} g(t) = 0$. These reduces versions are formally some generlizations of Theorem 2.3 in the case $s > 1$ or $s < 1$, but in fact they are its corollaries. Let us formulate two such corollaries.

Applying inequality (2.4) to the function defined by the identities $f(t) = g(t)$, $0 \leqslant t \leqslant t_0$ and $f(t) = g(t_0) = const$, $t_0 < t < \infty$, and taking into consideration the arguing by Hardy employed in the proof of the sharpness of the constants, we obtain the following statement.

Corollary 2.2. *Suppose that* $t_0 \in (0,\infty)$, $p \in [1,\infty)$, $s \in (1,\infty)$, *the function* $f : [0,t_0] \to \mathbb{C}$ *is absolutely continuous,* $f(0) = 0$ *and* $|f'|/t^{s/p-1} \in L^p(0,t_0)$. *Then the inequality holds:*

$$\int_0^{t_0} \frac{|f'(t)|^p}{t^{s-p}}\, dt \geqslant \left(\frac{s-1}{p}\right)^p \int_0^{t_0} \frac{|f(t)|^p}{t^s}\, dt. \tag{2.6}$$

If $p > 1$ *and* $f \not\equiv 0$, *then the inequality is strict and the constant* $((s-1)/p)^p$ *is sharp.*

The next statement can be proved in the same way as Corollary 2.2. The difference is that we apply inequality (2.4) to a function defined by the identities $f(t) = g(t)$, $t_0 \leqslant t \leqslant \infty$ and $f(t) = g(t_0) = const$, $0 < t < t_0$.

Corollary 2.3. *Assume that* $t_0 \in (0,\infty)$, $p \in [1,\infty)$, $s \in (-\infty,1)$, *the function* $f : [t_0,\infty] \to \mathbb{C}$ *is absolutely continuous,* $f(\infty) = 0$ *and* $|f'|/t^{s/p-1} \in L^p(t_0,\infty)$. *Then the inequality*

$$\int_{t_0}^{\infty} \frac{|f'(t)|^p}{t^{s-p}}\, dt \geqslant \left(\frac{|s-1|}{p}\right)^p \int_{t_0}^{\infty} \frac{|f(t)|^p}{t^s}\, dt \tag{2.7}$$

holds. If $p > 1$ *and* $f \not\equiv 0$, *then the inequality is strict and the constant* $((s-1)/p)^p$ *is sharp.*

The following corollary is true as well.

Corollary 2.4. *Assume that* $-\infty < a < b < \infty$. *For each* $p \in [1,\infty)$ *and each* $s \in \mathbb{R} \setminus \{1\}$ *the following inequality*

$$\int_a^b \frac{(b-\tau)^{p+s-2}}{(\tau-a)^{s-p}} |f'(\tau)|^p\, d\tau \geqslant (b-a)^p \left(\frac{|s-1|}{p}\right)^p \int_a^b \frac{(b-\tau)^{s-2}}{(\tau-a)^s} |f(\tau)|^p\, d\tau \quad \forall f \in C_0^1(a,b) \tag{2.8}$$

holds with a sharp constant $(b-a)^p\,(|s-1|/p)^p$. *For a function* $f \not\equiv 0$ *the inequality is strict for all* $p \in [1,\infty)$ *and* $s \in \mathbb{R} \setminus \{1\}$.

Inequality (2.8) is implied by inequality (2.5) under the change of the variable $t = (\tau-a)/(b-\tau)$ and the function $g(t) \equiv f(\tau)$.

The following corollary holds.

Corollary 2.5. *Assume that* $-\infty < a < b < \infty$, $\rho(\tau) := \min\{\tau-a, b-\tau\}$, *where* $\tau \in (a,b)$. *For each* $p \in [1,\infty)$ *and each* $s \in (1,\infty)$ *the inequality*

$$\int_a^b \frac{|f'(\tau)|^p}{\rho^{s-p}(\tau)}\, d\tau \geqslant \left(\frac{s-1}{p}\right)^p \int_a^b \frac{|f(\tau)|^p}{\rho^s(\tau)}\, d\tau \tag{2.9}$$

holds for all $f \in C_0^1(a,b)$ *with a sharp constant* $((s-1)/p)^p$. *For a function* $f \not\equiv 0$ *the inequality is strict for all* $p \in [1,\infty)$ *and* $s \in (1,\infty)$.

Proof. Applying inequality (2.6) for $t_0 = (b-a)/2$ and making linear changes of the variables of form $\tau = t + a$ and $\tau = b - t$, we obtain the inequalities

$$\int_a^{(a+b)/2} \frac{|f'(\tau)|^p}{(\tau-a)^{s-p}}\, d\tau \geqslant \left(\frac{s-1}{p}\right)^p \int_a^{(a+b)/2} \frac{|f(\tau)|^p}{(\tau-a)^s}\, d\tau \quad \forall f \in C_0^1(a,b),$$

$$\int_{(a+b)/2}^{b} \frac{|f'(\tau)|^p}{(b-\tau)^{s-p}}\, d\tau \geqslant \left(\frac{s-1}{p}\right)^p \int_{(a+b)/2}^{b} \frac{|f(\tau)|^p}{(b-\tau)^s}\, d\tau \quad \forall f \in C_0^1(a,b).$$

The sum of these inequalities gives desired inequality (2.9). The proof is complete. □

We note that the quantity $\rho(\tau) = \min\{\tau - a, b - \tau\}$ is equal to the distance from the point $\tau \in (a,b)$ to the boundary of the interval (a,b).

3. Inequalities for higher derivatives

Applying successively k times inequality (2.5) with the parameters $p = 2$ and $s = 2(k-j)$ to the functions $g = f^{(j)}$ with $s = k-1, \ldots, 0$, we obtain the following statement belonging to Hardy as $k = 1$, to Rellich [3] as $k = 2$ and to Birman [4] as $k \geqslant 3$.

Theorem 3.1. *Let k be a natural number. The following inequality with a sharp constant holds:*

$$\int_0^\infty |f^{(k)}(t)|^2 dt \geqslant \left(\frac{(2k-1)!!}{2^k}\right)^2 \int_0^\infty \frac{|f(t)|^2}{t^{2k}} dt \quad \forall f \in C_0^k(0,\infty). \tag{3.1}$$

If $f \not\equiv 0$, then the inequality is strict.

A detailed proof of inequality (3.1) was provided in a book by I.M. Glazman [5]. The proof of inequality (3.1) can be found in several papers, in particular, in paper by M.P. Owen [6] and in paper by F. Gesztesy, L.L. Littlejohn, I. Michael, R. Wellman [7]. In these works the sharpness of the constant $\big((2k-1)!!/2^k\big)^2$ was shown for each $k \geqslant 1$.

The Hardy theorem involves one special value of the parameter. Namely, for $s = \sigma = 1$ the inequality becomes meaningless since the corresponding constant vanishes. We denote $S_p^*(1) = \{1\}$. For the case $k \in \mathbb{N} \setminus \{1\}$ we shall need the set $S_p^*(k) := \bigcup_{j=1}^k \{1 + (j-1)p\}$ consisting of k special points.

The following direct analogue of the Hardy inequality holds true, which coincides with Theorem 2.3 as $k = 1$ and including Rellich and Birman inequalities (3.1) as a particular case.

Theorem 3.2. *Let $p \in [1,\infty)$, $k \in \mathbb{N}$ and $\sigma \in \mathbb{R} \setminus S_p^*(k)$, where $S_p^*(k) := \bigcup_{j=1}^k \{1 + (j-1)p\}$. Suppose that $f \in C^{k-1}(0,\infty)$ is a complex-valued function such that the derivative $f^{(k-1)}$ of order $k-1$ is differentiable almost everywhere and $t^{k-\sigma/p}|f^{(k)}| \in L^p(0,\infty)$.*

Let $j \in (\mathbb{N} \cup \{0\}) \cap [0, k-1]$. Suppose that $f^{(0)} := f$ and the following conditions are satisfied:

1) if $\sigma > 1 + (k-1)p$, then $f^{(j)}(0) := \lim_{t\to 0} f^{(j)}(t) = 0$ for all integer numbers $j \in [0, k-1]$ and the identity holds:

$$f^{(k-1)}(t) = \int_0^t f^{(k)}(\tau) d\tau, \quad 0 \leqslant t < \infty;$$

2) if $\sigma < 1$, then $f^{(j)}(\infty) := \lim_{t\to\infty} f^{(j)}(t) = 0$ for all integer numbers $j \in [0, k-1]$ and the identity holds:

$$f^{(k-1)}(t) = \int_\infty^t f^{(k)}(\tau) d\tau, \quad 0 < t \leqslant \infty;$$

3) if $1 + (m-1)p < \sigma < 1 + mp$, where $m \in [1, k-1]$ is a natural number, then $f^{(j)}(0) := \lim_{t\to 0} f^{(j)}(t) = 0$ for all integer numbers $j \in [0, m-1]$, and also $f^{(j)}(\infty) := \lim_{t\to\infty} f^{(j)}(t) = 0$ for all natural numbers

$j \in [m, k-1]$ and the identity

$$f^{(k-1)}(t) = \int_{\infty}^{t} f^{(k)}(\tau)d\tau, \quad 0 < t \leqslant \infty,$$

holds.
Then the inequality

$$\int_0^\infty \frac{\left|f^{(k)}(t)\right|^p}{t^{\sigma-kp}}dt \geqslant C_p(k,\sigma)\int_0^\infty \frac{|f(t)|^p}{t^\sigma}dt \tag{3.2}$$

holds, where

$$C_p(k,\sigma) := \prod_{j=1}^{k} |(\sigma-1)/p - j + 1|^p \,.$$

The constant $C_p(k,\sigma)$ is the best possible. If $p>1$ and $f \not\equiv 0$, then inequality (3.2) is sharp.

Proof. We suppose that $k \geqslant 2$ since as $k=1$, the theorem coincides with the Hardy theorem, more precisely, with it version formulated as Theorem 2.3. Applying Theorem 2.3 with $s = \sigma - jp$ to the function $g = f^{(j)}$, we obtain:

$$\int_0^\infty \frac{\left|f^{(j+1)}(t)\right|^p}{t^{\sigma-(j+1)p}}dt \geqslant \frac{|\sigma - pj - 1|^p}{p^p}\int_0^\infty \frac{\left|f^{(j)}(t)\right|^p}{t^{\sigma-jp}}dt, \qquad j = k-1, \dots, 1, 0. \tag{3.3}$$

We stress that in Hardy inequality (3.3) by Theorem 2.3 we need only one boundary condition: the function $g = f^{(j)}$ should vanish either at the point $t=0$ (if $s = \sigma - jp > 1$) or at the point $t = \infty$ (if $s = \sigma - jp < 1$). These conditions are satisfied due to Conditions 1), 2), 3) in the formulation of Theorem 3.2. Moreover, while justifying Hardy inequality (3.3), we need to confirm the condition $t^{j-\sigma/p}|f^{(j)}| \in L^p(0,\infty)$ for all natural numbers $j \in [1,k]$. The condition $t^{k-\sigma/p}|f^{(k)}| \in L^p(0,\infty)$ is contained in the assumptions of Theorem 3.2. This is why the inequality

$$\int_0^\infty \frac{\left|f^{(k)}(t)\right|^p}{t^{\sigma-kp}}dt \geqslant \frac{|\sigma-(k-1)p-1|^p}{p^p}\int_0^\infty \frac{\left|f^{(k-1)}(t)\right|^p}{t^{\sigma-(k-1)p}}dt$$

holds true. This implies $t^{k-1-\sigma/p}|f^{(k-1)}| \in L^p(0,\infty)$. But then the inequality

$$\int_0^\infty \frac{\left|f^{(k-1)}(t)\right|^p}{t^{\sigma-(k-1)p}}dt \geqslant \frac{|\sigma-(k-2)p-1|^p}{p^p}\int_0^\infty \frac{\left|f^{(k-2)}(t)\right|^p}{t^{\sigma-(k-2)p}}dt$$

holds and this implies that $t^{k-2-\sigma/p}|f^{(k-2)}| \in L^p(0,\infty)$.

Subsequently reducing the order of the derivative in this arguing, we see that $t^{j-\sigma/p}|f^{(j)}| \in L^p(0,\infty)$ for each natural number $j \in [1,k]$ and this is the desired statement.

Applying subsequently inequalities (3.3) to the cases $j=k-1$, $j=k-2$, $\dots$, $j=0$, we get

$$\int_0^\infty \frac{|f^{(k)}(t)|^p}{t^{\sigma-kp}}dt \geqslant \frac{|\sigma-p(k-1)-1|^p}{p^p}\int_0^\infty \frac{|f^{(k-1)}(t)|^p}{t^{\sigma-(k-1)p}}dt$$

$$\geqslant \frac{|\sigma-p(k-1)-1|^p}{p^p}\frac{|\sigma-p(k-2)-1|^p}{p^p}\int_0^\infty \frac{|f^{(k-2)}(t)|^p}{t^{\sigma-(k-2)p}}dt \geqslant \dots$$

$$\geqslant \left(p^{-k}\prod_{j=1}^{k}|\sigma-1-p(j-1)|\right)^p \int_0^\infty \frac{|f^{(0)}(t)|^p}{t^\sigma}dt.$$

As a result we obtain required inequality (3.2).

We note that as $p = 2$ and $\sigma = 2k$, in Theorem 3.2 we have the Hardy, Rellich and Birman constants since

$$C_2(k, 2k) = \left(2^{-k}\prod_{j=1}^{k}(2k+1-2j)\right)^2 = \left(\frac{(2k-1)!!}{2^k}\right)^2.$$

It is obvious that as $k \geqslant 2$, the proof of inequality (3.2) does not allow us to state the sharpness of the constant $C_p(k,\sigma)$. This is why it remains to prove the sharpness of the constant in the general case as $k \geqslant 2$.

Suppose that the constant $C_p(k,\sigma)$ in Theorem 3.2 is not best possible. Then for some set $\{p, k, \sigma\}$ of fixed parameters $p \in [1,\infty)$, $k \in \mathbb{N} \setminus \{1\}$, $\sigma \in \mathbb{R} \setminus S_p^*(k)$ there exists $\varepsilon_0 > 0$ such that for each function $f : (0,\infty) \to \mathbb{C}$ obeying the conditions of Theorem 3.2 the inequality

$$\int_0^\infty \frac{|f^{(k)}(t)|^p}{t^{\sigma-kp}}dt \geqslant (\varepsilon_0 + C_p(k,\sigma))\int_0^\infty \frac{|f(t)|^p}{t^\sigma}dt \tag{3.4}$$

holds true.

Let $\varepsilon \in (0,1)$. We consider a function $f_\varepsilon \in C(0,\infty) \cap C^\infty((0,\infty)\setminus\{1\})$ defined by the identities

$$f_\varepsilon(t) = t^{(\sigma-1+\varepsilon)/p}, \qquad 0 < t \leqslant 1, \qquad f_\varepsilon(t) = t^{(\sigma-1-\varepsilon)/p}, \qquad 1 < t < \infty.$$

We construct a function $g_\varepsilon \in C^k(0,\infty)$ by letting

$$g_\varepsilon(t) = f_\varepsilon(t), \qquad t \in (0,1/2) \cup (2,\infty); \qquad g_\varepsilon(t) = H(t), \qquad t \in [1/2, 2],$$

where $H(t)$ is the Hermite interpolation polynomial for the function f_ε constructed by two nodes $t_0 = 1/2$ and $t_1 = 2$ of multiplicity $k+1$ and therefore, $2k+2$ conditions hold:

$$H^{(j)}(t_\nu) = f_\varepsilon^{(j)}(t_\nu), \qquad j = 0, 1, \ldots, k; \quad \nu = 0, 1.$$

It is known, see, for instance, [8], that the degree of the polynomial $H(t)$ does not exceed $2k+1$ and

$$H(t) = \sum_{\nu=0}^{1}\sum_{j=0}^{k}\sum_{q=0}^{k-j} c_{kjq} f_\varepsilon^{(j)}(t_\nu)\frac{(t-t_\nu)^{j+k}(t-t_{1-\nu})^{k+1}}{(t_\nu - t_{1-\nu})^{k+q+1}},$$

where $c_{kjq} = (-1)^q(k+q)!/(j!k!q!)$. It is easy to see

$$\sup_{\varepsilon\in(0,1)}\max_{t\in[1/2,2]}|H(t)| = M_0 < \infty, \qquad \sup_{\varepsilon\in(0,1)}\max_{t\in[1/2,2]}|H^{(k)}(t)| = M_k < \infty,$$

since

$$\max_{j,\nu}\sup_{\varepsilon\in(0,1)}|f_\varepsilon^{(j)}(t_\nu)| < \infty.$$

By straightforward calculations we obtain

$$\int_0^\infty \frac{|g_\varepsilon(t)|^p}{t^\sigma}dt = \frac{2}{2^\varepsilon \varepsilon} + \int_{1/2}^2 \frac{|H(t)|^p}{t^\sigma}dt, \tag{3.5}$$

$$\int_0^\infty \frac{|g_\varepsilon^{(k)}(t)|^p}{t^{\sigma-kp}}dt = \frac{C_p(k,\sigma-\varepsilon) + C_p(k,\sigma+\varepsilon)}{2^\varepsilon \varepsilon} + \int_{1/2}^2 \frac{|H^{(k)}(t)|^p}{t^{\sigma-kp}}dt. \tag{3.6}$$

We note that

$$\lim_{\varepsilon\to0}\varepsilon\int_{1/2}^2 \frac{|H(t)|^p}{t^\sigma}dt = \lim_{\varepsilon\to0}\varepsilon\int_{1/2}^2 \frac{|H^{(k)}(t)|^p}{t^{\sigma-kp}}dt = 0. \tag{3.7}$$

The function $g_\varepsilon \in C^k(0,\infty)$ satisfies the boundary conditions described in Items 1, 2 and 3 of Theorem 3.2. Therefore, for a function $f = g_\varepsilon$ in accordance with (3.4) we have the inequality

$$\frac{\varepsilon}{2}\int_0^\infty \frac{|g_\varepsilon^{(k)}(t)|^p}{t^{\sigma-kp}}dt \geqslant (\varepsilon_0 + C_p(k,\sigma))\frac{\varepsilon}{2}\int_0^\infty \frac{|g_\varepsilon(t)|^p}{t^\sigma}dt \tag{3.8}$$

obtained from inequality (3.4) for the function $f = g_\varepsilon$ by multiplying both sides of the inequality by $\varepsilon/2$.

Passing to the limit as $\varepsilon \to 0$ in inequality (3.8) and taking into consideration formulas (3.5), (3.6) and (3.7), we arrive at the relations

$$C_p(k,\sigma) = \lim_{\varepsilon\to 0} \frac{C_p(k,\sigma-\varepsilon) + C_p(k,\sigma+\varepsilon)}{2^{1+\varepsilon}} \geqslant \lim_{\varepsilon\to 0} \frac{\varepsilon_0 + C_p(k,\sigma)}{2^{\varepsilon}} = \varepsilon_0 + C_p(k,\sigma)$$

and this contradicts the positivity of the number ε_0.

In view of the boundary conditions the property $f \not\equiv 0$ implies similar properties for the derivatives: $f' \not\equiv 0, \ldots, f^{(k-1)} \not\equiv 0$. This is why as $p > 1$ and $f \not\equiv 0$ inequalities (3.3), and hence, inequality (3.2), are strict. The proof is complete. □

In what follows in the corollaries we consider only the case $k \geqslant 2$. Similar statements for the case $k = 1$ are also valid and they are formulated in the Introduction as the corollaries of the Hardy inequality.

A generalization of the Rellich and Birman inequalities is the following corollary.

Corollary 3.1. *Assume that $k \in \mathbb{N} \setminus \{1\}$, $1 < p < \infty$, and $f \in C^{k-1}[0,\infty)$ is a complex-valued function. If $f^{(k-1)}$ is differentiable almost everywhere, $f^{(k)} \in L^p(0,\infty)$, $f^{(j)}(0) = 0$ for all $j = 0, \ldots, k-1$ and*

$$f^{(k-1)}(t) = \int_0^t f^{(k)}(\tau)d\tau, \qquad 0 \leqslant t < \infty,$$

then the inequality

$$\int_0^\infty \left|f^{(k)}(t)\right|^p dt \geqslant \prod_{j=1}^k (j-1/p)^p \int_0^\infty \frac{|f(t)|^p}{t^{kp}}\, dt$$

holds true with a sharp constant. In particular, the inequality

$$\int_0^\infty \left|f^{(k)}(t)\right|^p dt \geqslant \prod_{j=1}^k (j-1/p)^p \int_0^\infty \frac{|f(t)|^p}{t^{kp}}\, dt$$

is valid for all $f \in C_0^k(0,\infty)$.

As $p = 1$, the latter inequality is meaningless since $\sigma = k \in S_1^*(k)$ and the constant vanishes.

Letting $p = 1$ and $\sigma = 0$ or $\sigma = k+1$ in Theorem 3.2, we obtain the following statement.

Corollary 3.2. *Let $k \in \mathbb{N} \setminus \{1\}$. Then for each complex-valued function $f \in C_0^k(0,\infty)$ the inequalities*

$$\int_0^\infty t^k \left|f^{(k)}(t)\right| dt \geqslant k! \int_0^\infty |f(t)|\, dt, \qquad \int_0^\infty \frac{\left|f^{(k)}(t)\right|}{t}\, dt \geqslant k! \int_0^\infty \frac{|f(t)|}{t^{k+1}}\, dt$$

hold. The constant $k!$ is sharp in both inequalities.

Corollary 3.3. *Assume that $k \in \mathbb{N} \setminus \{1\}$, $1 \leqslant p < \infty$, a complex-valued function f satisfies $f \in C^{k-1}(0,\infty)$ and $f^{(k-1)}$ is differentiable almost everywhere, $t^k f^{(k)} \in L^p(0,\infty)$. If $f^{(j)}(\infty) := \lim\limits_{t\to\infty} f^{(j)}(t) = 0$ for all $j = 0, \ldots, k-1$, $f^{(k-1)}(t) = \int_\infty^t f^{(k)}(\tau)d\tau$, where $0 < t \leqslant \infty$, then the inequality*

$$\int_0^\infty t^{kp} \left|f^{(k)}(t)\right|^p dt \geqslant \prod_{j=0}^{k-1} (j+1/p)^p \int_0^\infty |f(t)|^p\, dt$$

holds true. The constant $\prod_{j=0}^{k-1} (j+1/p)^p$ is sharp.

The next two theorems provide generalizations of inequalities (2.6) and (2.7).

Theorem 3.3. *Assume that* $k \in \mathbb{N}$, $0 < c < \infty$, $1 \leqslant p < \infty$ *and* $1 + (k-1)p < \sigma < \infty$. *Let* $f \in C^{k-1}(0,c)$ *be a complex-valued function such that the derivative* $f^{(k-1)}$ *of order* $k-1$ *is differentiable almost everywhere and* $t^{k-\sigma/p}|f^{(k)}| \in L^p(0,c)$. *If* $f^{(j)}(0) := \lim_{t\to 0} f^{(j)}(t) = 0$ *for all integer numbers* $j \in [0, k-1]$ *and the identity*

$$f^{(k-1)}(t) = \int_0^t f^{(k)}(\tau)d\tau, \qquad 0 \leqslant t < c,$$

holds true, then the inequality

$$\int_0^c \frac{|f^{(k)}(t)|^p}{t^{\sigma-kp}} dt \geqslant C_p(k,\sigma) \int_0^c \frac{|f(t)|^p}{t^\sigma} dt \tag{3.9}$$

is valid, where

$$C_p(k,\sigma) := \prod_{j=1}^k |(\sigma-1)/p - j + 1|^p .$$

The constant $C_p(k,\sigma)$ *is best possible.*

Proof. Let $\varepsilon \in (0,c)$ and $f \in C^{k-1}(0,c)$ be one of the functions obeying the assumptions of the theorem. By these assumptions, this function and its derivatives up to the order of $k-1$ are extended by the continuity at the point $t = 0$. We can assume that $f \in C^{k-1}[0,c)$, $f^{(j)}(0) = 0$ for all integer numbers $j \in [0, k-1]$ and the derivaitve $f^{(k-1)}$ of order $k-1$ is absolutely continuous on the segment $[0, c-\varepsilon]$ for each $\varepsilon \in (0,c)$.

Applying inequality (2.6) to the function $f^{(j)}$ as $t_0 = c - \varepsilon$, $s = \sigma - jp$ and $j = k-1, k-2, \dots, 0$, we obtain

$$\int_0^{c-\varepsilon} \frac{|f^{(j+1)}(t)|^p}{t^{\sigma-(j+1)p}} dt \geqslant \frac{|\sigma - pj - 1|^p}{p^p} \int_0^{c-\varepsilon} \frac{|f^{(j)}(t)|^p}{t^{\sigma-jp}} dt.$$

Employing the iterations of these inequalities, namely, applying this inequality to the case $j = k-1$ and then successively to the case $j = k-2$, ..., $j = 0$, we get

$$\int_0^{c-\varepsilon} \frac{|f^{(k)}(t)|^p}{t^{\sigma-kp}} dt \geqslant C_p(k,\sigma) \int_0^{c-\varepsilon} \frac{|f(t)|^p}{t^\sigma} dt.$$

Letting ε tend to the zero, we obtain desired inequality (3.9).

It remains to prove the sharpness of the constant. Suppose that the constant $C_p(k,\sigma)$ in Theorem 3.3 is not best possible. Then for some set $\{p, k, \sigma\}$ of fixed parameters $p \in [1,\infty)$, $k \in \mathbb{N}$, $\sigma \in (1 + (k-1)p, \infty)$ there exists $\varepsilon_0 > 0$ such that for each function $f : (0,c) \to \mathbb{C}$ obeying the assumptions of Theorem 3.3 the inequality holds

$$\int_0^c \frac{|f^{(k)}(t)|^p}{t^{\sigma-kp}} dt \geqslant (\varepsilon_0 + C_p(k,\sigma)) \int_0^c \frac{|f(t)|^p}{t^\sigma} dt.$$

We apply this inequality to the function $f_\varepsilon(t) = t^{(\sigma-1+\varepsilon)/p}$, $0 \leqslant t \leqslant c$, satisfying the assumptions of Theorem 3.3 for each $\varepsilon \in (0,1)$. We then obtain:

$$C_p(k, \sigma+\varepsilon)\frac{c^\varepsilon}{\varepsilon} \geqslant (\varepsilon_0 + C_p(k,\sigma))\,\frac{c^\varepsilon}{\varepsilon}.$$

Multiplying both sides by ε and passing to the limit as $\varepsilon \to 0$, we get $C_p(k,\sigma) \geqslant \varepsilon_0 + C_p(k,\sigma)$. The obtain contradiction proves the sharpness of the constant $C_p(k,\sigma)$ in Theorem 3.3. The proof is complete. □

Theorem 3.4. *Assume that $k \in \mathbb{N}$, $0 < b < \infty$, $1 \leqslant p < \infty$ and $-\infty < s < 1$. Let $f \in C^{k-1}(b,\infty)$ be a complex-valued function such that the derivative $f^{(k-1)}$ of order $k-1$ is differentiable almost everywhere and $t^{k-s/p}|f^{(k)}| \in L^p(b,\infty)$. If $f^{(j)}(\infty) := \lim_{t\to\infty} f^{(j)}(t) = 0$ for all integer numbers $j \in [0, k-1]$ and the identity*

$$f^{(k-1)}(t) = \int_{\infty}^{t} f^{(k)}(\tau)d\tau$$

holds, where $b \leqslant t < \infty$, then the inequality

$$\int_b^\infty \frac{|f^{(k)}(t)|^p}{t^{s-kp}}dt \geqslant C_p(k,s)\int_b^\infty \frac{|f(t)|^p}{t^s}dt \tag{3.10}$$

is valid, where

$$C_p(k,s) := \prod_{j=1}^{k} |(s-1)/p - j + 1|^p .$$

The constant $C_p(k,s)$ is best possible.

Proof. The proof follows the lines of the previous proof. The difference is that in justifying the required inequality of Theorem 3.4 we employ inequality (2.7) for $t_0 = b + \varepsilon$, where $\varepsilon > 0$, while in justification of the sharpness of the constant we consider the function $f_\varepsilon(t) = t^{(s-1-\varepsilon)/p}$, $b < t < \infty$, satisfying the assumptions of Theorem 3.4 for each $\varepsilon \in (0,1)$. This completes the proof of Theorem 3.4. □

Remark 3.1. *As $k = 1$, the statement of Theorems 3.3 and 3.4 for real-valued functions f are well-known, see, for instance, a monograph by S.L. Sobolev* [1].

Now we provide a corollary of Theorem 3.3 generalizing and strengthening Rellich-Birman inequality as $k \geqslant 2$.

Corollary 3.4. *Assume that $k \in \mathbb{N}$, $0 < c < \infty$, $1 < p < \infty$. Let $f \in C^{k-1}(0,c)$ be a complex-valued function such that $f^{(k-1)}$ is differentiable almost everywhere and $t^{k-\sigma/p}|f^{(k)}| \in L^p(0,c)$.*

If $f^{(j)}(0) := \lim_{t\to 0} f^{(j)}(t) = 0$ for all integer numbers $j \in [0, k-1]$ and $f^{(k-1)}(t) = \int_0^t f^{(k)}(\tau)d\tau$, $0 \leqslant t < c$, then the inequality

$$\int_0^c \left|f^{(k)}(t)\right|^p dt \geqslant \prod_{j=1}^{k}(j-1/p)^p \int_0^c \frac{|f(t)|^p}{t^{kp}}\,dt$$

is valid. The constant $\prod_{j=1}^{k}(j-1/p)^p$ is best possible.

4. Improvements of Rellich and Birman inequalities in bounded intervals

According to Corollary 2.3, for each $c \in (0,\infty)$ and each absolutely continuous function $f : [0,c] \to \mathbb{R}$ such that $f(0) = 0$ and $f' \in L^2(0,c)$ the inequality

$$\int_0^c |f'(t)|^2 dt \geqslant \frac{1}{4}\int_0^c \frac{|f(t)|^2}{t^2}dt \tag{4.1}$$

holds true with a best possible constant 1/4. H. Brezis and M. Marcus [9] employed the absence of the extremal function, at which the identity in (4.1) is achieved. Namely, they proved that under the same conditions on the function f identity (4.1) can be strengthened, namely, the following inequality

$$\int_0^c |f'(t)|^2 dt \geqslant \frac{1}{4}\int_0^c \frac{|f(t)|^2}{t^2}dt + \frac{\lambda}{c^2}\int_0^c |f(t)|^2 dt \tag{4.2}$$

holds, where $\lambda = 1/4$. K.-J. Wirths and the author [10] found the best possible value for the constant λ in Brezis-Marcus inequality (4.2). It is turned out that the best possible value for the constant λ is

λ_0^2, where $z = \lambda_0 \approx 0.940$ is the first positive root of the equation $J_0(z) + 2zJ_0'(z) = 0$ for the zero order Bessel function.

In accordance with Corollary 2.3, for each $c \in (0, \infty)$, each $s \in (1, \infty)$ and each absolutely continuous function $f : [0, c] \to \mathbb{R}$ such that $f(0) = 0$ and $f'/t^{s/2-1} \in L^2(0, a)$ the inequality

$$\int_0^c \frac{|f'(t)|^2}{t^{s-2}} dt \geqslant \frac{(s-1)^2}{4} \int_0^c \frac{|f(t)|^2}{t^s} dt \tag{4.3}$$

holds with the best possible constant $(s-1)^2/4$. This gives rise to a natural problem: prove that under the same conditions for the function f inequality (4.3) can be strengthened, namely, the following inequality

$$\int_0^c \frac{|f'(t)|^2}{t^{s-2}} dt \geqslant \frac{(s-1)^2}{4} \int_0^c \frac{|f(t)|^2}{t^s} dt + \frac{\lambda}{c^s} \int_0^c |f(t)|^2 dt \tag{4.4}$$

is true with some positive constant $\lambda > 0$. This problem was solved by K.-J. Wirths and the author in paper [11]. For an exact formulation of the appropriate result involving inequality (4.4) as a particular case, we need some notation.

Let (p, q) be a pair of positive numbers. We shall need a function

$$y = F_{\nu,p,q}(t) = t^{p/2} J_\nu \left(\lambda_\nu(2p/q)\, t^{q/2} \right), \quad t \in [0, 1],$$

where

$$J_\nu(t) = \sum_{m=0}^{\infty} \frac{(-1)^m\, t^{2m+\nu}}{2^{2m+\nu}\, m!\, \Gamma(m+1+\nu)}$$

is the Bessel function of order $\nu \geqslant 0$, while Γ is the Euler Gamma function and $\lambda_\nu(2p/q)$ is a Lamb constant defined as the first positive root of the equation $(2p/q)J_\nu(z) + 2zJ_\nu'(z) = 0$ for fixed $\nu \geqslant 0$ and $x = 2p/q > 0$.

The zeroes of the function $xJ_\nu(z) + 2zJ_\nu'(z)$ for fixed $\nu > 0$, $x > 0$ were studied by H. Lamb, see [12] and [13], while for $\nu = 0$ they were studied in papers by K.-J. Wirths and the author [10], [11]. In particular, it was found that $\lambda_0(1) = \lambda_0 \approx 0.940$.

By $z = \lambda_\nu(x)$ we denote the first positive root of the equation $xJ_\nu(z) + 2zJ_\nu'(z) = 0$ for fixed $x > 0$ and $\nu \in [0, x/2]$. It was proved in papers [10] and [11] that the function

$$z = \lambda_\nu : (0, \infty) \to (0, \infty)$$

is monotonically increasing, the value $z = \lambda_\nu(x)$ for each $x \in (0, 1]$ or $x \in [1, \infty)$ can be found as a solution of the Cauchy problem for the equation

$$\frac{dz}{dx} = \frac{2z}{x^2 - 4\nu^2 + 4z^2}$$

with the initial condition $z(1) = \lambda_\nu(1)$.

While solving the problems related with the inequalities of Hardy type in convex domains $\Omega \subset \mathbb{R}^n$, in paper [11] by K.-J. Wirths and the author the following statement was proved, see Lemmas 1, 2 and Theorem 2 for $p = s - 1$ in [11].

Theorem 4.1. *Let $s \in (1, \infty)$, $q \in (0, \infty)$ and $\nu \in [0, (s-1)/q]$ and*

$$z = \lambda_{\nu,s,q} := \lambda_\nu(2(s-1)/q)$$

be the Lamb constant defined as the first positive root of the equation

$$(2(s-1)/q)J_\nu(z) + 2zJ_\nu'(z) = 0$$

for fixed $\nu \geqslant 0$ and $x = 2(s-1)/q$. If $f : [0, 1] \to \mathbb{R}$ is an absolutely continuous function such that $f(0) = 0$ and $f'/t^{s/2-1} \in L^2(0, 1)$, then

$$\int_0^1 \frac{|f'(t)|^2}{t^{s-2}} dt \geqslant \frac{(s-1)^2 - \nu^2 q^2}{4} \int_0^1 \frac{|f(t)|^2}{t^s} dt + \frac{q^2 \lambda_{\nu,s,q}^2}{4} \int_0^1 \frac{|f(t)|^2}{t^{s-q}} dt. \tag{4.5}$$

If $\nu > 0$, then the identity in (4.5) is achieved if and only if $f(t) = C\,F_{\nu,s-1,q}(t)$, where $C = const$. If $\nu = 0$ and $f \not\equiv 0$, then the strict inequality

$$\int_0^1 \frac{|f'(t)|^2}{t^{s-2}}dt > \frac{(s-1)^2}{4}\int_0^1 \frac{|f(t)|^2}{t^s}dt + \frac{q^2\lambda_{0,s,q}^2}{4}\int_0^1 \frac{|f(t)|^2}{t^{s-q}}dt \tag{4.6}$$

holds true, where both constants $(s-1)^2/4$ and $q^2\lambda_{0,s,q}^2/4$ in inequality (4.6) are sharp, that is, they are best possible in the following sense: for each $\varepsilon > 0$ there exist functions $f_{1\varepsilon}$, $f_{2\varepsilon}$, obeying the assumptions of the theorem and the inequalities

$$\int_0^1 \frac{|f_{1\varepsilon}'(t)|^2}{t^{s-2}}dt < \frac{(s-1)^2+\varepsilon}{4}\int_0^1 \frac{|f_{1\varepsilon}(t)|^2}{t^s}dt,$$

$$\int_0^1 \frac{|f_{2\varepsilon}'(t)|^2}{t^{s-2}}dt < \frac{(s-1)^2}{4}\int_0^1 \frac{|f_{2\varepsilon}(t)|^2}{t^s}dt + \frac{q^2\lambda_{0,s,q}^2+\varepsilon}{4}\int_0^1 \frac{|f_{2\varepsilon}(t)|^2}{t^{s-q}}dt.$$

Inequality (4.5) also holds for complex-valued functions. Namely, the following corollary is true.

Corollary 4.1. *Assume that the numbers s, q, ν and $\lambda_{\nu,s,q}$ are the same as in Theorem 3.1. If $g : [0,1] \to \mathbb{C}$ is an absolutely continuous function such that $g(0) = 0$ and $|g'|/t^{s/2-1} \in L^2(0,1)$, then*

$$\int_0^1 \frac{|g'(t)|^2}{t^{s-2}}dt \geqslant \frac{(s-1)^2-\nu^2q^2}{4}\int_0^1 \frac{|g(t)|^2}{t^s}dt + \frac{q^2\lambda_{\nu,s,q}^2}{4}\int_0^1 \frac{|g(t)|^2}{t^{s-q}}dt. \tag{4.7}$$

Proof. Let $g(t) = f_1(t) + if_2(t)$. It is easy to see that the functions $f_1(t) = \mathrm{Re}\, g(t)$ and $f_2(t) = \mathrm{Im}\, g(t)$ satisfy the assumptions of the theorem. This is why we can write inequality (4.5) for the function $f = f_1$ and for the function $f = f_2$. Summing up the obtained inequalities and taking into consideration the identities

$$|g(t)|^2 = f_1^2(t) + f_2^2(t), \qquad |g'(t)|^2 = (f_1'(t))^2 + (f_2'(t))^2,$$

we arrive at inequality (4.7). □

By straightforward calculations using the changes $s = 2-\sigma$, $t = 1/\tau$, $g(1/\tau) = f(\tau)$ in the integrals in inequality (4.7), we obtain the following corollary.

Corollary 4.2. *Let $\sigma \in (-\infty, 1)$, $q \in (0,\infty)$ and $\nu \in [0, (1-\sigma)/q]$. Let*

$$z = \lambda_{\nu,\sigma,q} := \lambda_\nu(2(1-\sigma)/q)$$

be the Lamb constant defined as the first positive root of the equation

$$(2(1-\sigma)/q)J_\nu(z) + 2zJ_\nu'(z) = 0$$

for fixed $\nu \geqslant 0$ and $x = 2(1-\sigma)/q$. If $f : [1,\infty] \to \mathbb{C}$ is an absolutely continuous function such that $f(\infty) = 0$ and $f'/\tau^{\sigma/2-1} \in L^2(1,\infty)$, then

$$\int_1^\infty \frac{|f'(\tau)|^2}{\tau^{\sigma-2}}d\tau \geqslant \frac{(1-\sigma)^2-\nu^2q^2}{4}\int_1^\infty \frac{|f(\tau)|^2}{\tau^\sigma}d\tau + \frac{q^2\lambda_{\nu,\sigma,q}^2}{4}\int_1^\infty \frac{|f(\tau)|^2}{\tau^{\sigma+q}}d\tau.$$

In particular, letting $\tau = \nu = 0$ and $q = 2$, we obtain the inequality

$$\int_1^\infty \tau^2|f'(\tau)|^2d\tau \geqslant \frac{1}{4}\int_1^\infty |f(\tau)|^2d\tau + \lambda_0^2\int_1^\infty \frac{|f(\tau)|^2}{\tau^2}d\tau,$$

where both constants$1/4$ and $\lambda_0 = \lambda_0(1) \approx 0.940$ are best possible.

Letting $\nu = 0$ and $q = s$, using the changes $t = \tau/c$, $g(t) = f(\tau/c)$ in the integrals in inequalities (4.7), by straightforward calculations we obtain inequality of form (4.4) with sharp constants.

Corollary 4.3. *Let $c \in (0,\infty)$, $s \in (1,\infty)$, $z = \lambda_0(2-2/s)$ be the Lamb constant defined as the first positive root of the equation*

$$(2-2/s)J_0(z) + 2zJ_0'(z) = 0$$

for fixed $s > 1$. If $f : [0,c] \to \mathbb{C}$ is an absolutely continuous function such that $|f'|/t^{s/2-1} \in L^2(0,c)$ and $f(0) = 0$, then

$$\int_0^c \frac{|f'(t)|^2}{t^{s-2}} dt \geqslant \frac{(s-1)^2}{4} \int_0^c \frac{|f(t)|^2}{t^s} d\tau + \frac{s^2\ (\lambda_0(2-2/s))^2}{4c^s} \int_0^c |f(t)|^2 dt. \tag{4.8}$$

The next theorem gives improvements and generalizations of Hardy-Rellich-Birman inequalities for bounded intervals.

Theorem 4.2. *Let $k \in \mathbb{N}$, $c \in (0,\infty)$ and $f \in C^{k-1}[0,c]$ be a complex-valued function such that the derivative $f^{(k-1)}$ of order $k-1$ is differentiable almost everywhere and $|f^{(k)}| \in L^2(0,c)$. If $f^{(j)}(0) := \lim\limits_{t\to 0} f^{(j)}(t) = 0$ for all integer numbers $j \in [0, k-1]$ and the identity*

$$f^{(k-1)}(t) = \int_0^t f^{(k)}(\tau) d\tau, \quad 0 \leqslant t \leqslant c$$

holds, then

$$\int_0^c \left|f^{(k)}(t)\right|^2 dt \geqslant \left(\frac{(2k-1)!!}{2^k}\right)^2 \int_0^c \frac{|f(t)|^2}{t^{2k}} dt + \lambda_0^2 \frac{k(k+1)(2k+1)}{6\, c^{2k}} \int_0^c |f(t)|^2 dt, \tag{4.9}$$

where $\lambda_0 \approx 0.940$ is the Lamb constant.

Proof. If $s = 2m$, $m \in \mathbb{N}$, then $2 - 2/s \geqslant 1$. Therefore,

$$\lambda_0(2-2/s) \geqslant \lambda_0(1) = \lambda_0 \approx 0.940$$

since the function $\lambda_\nu(x)$ is monotonically increasing.

Let f_0 be a function satisfying the assumptions of the theorem. The proof of inequality (3.10) for the function f_0 follows the lines of the proof of the main inequality in Theorem 3.2.

Choosing $s = 2m$, $m \in \mathbb{N}$ and applying inequality (4.8) to the derivative $f_0^{(k-m)}$ with $m = 1, \ldots, k$ in view of inequalities $\lambda_0(2-1/m) \geqslant \lambda_0$ we get

$$\int_0^c \frac{|f_0^{(k-m+1)}(t)|^2}{t^{2m-2}} dt \geqslant \frac{(2m-1)^2}{4} \int_0^c \frac{|f_0^{(k-m)}(t)|^2}{t^{2m}} dt + \frac{m^2\lambda_0^2}{c^{2m}} \int_0^c |f_0(t)|^2 dt.$$

Employing the iterations of these inequalities, namely, applying this inequality to the case $m = 1$, and then successively to the cases $m = 2, \ldots, m = k$, we obtain:

$$\int_0^c \left|f_0^{(k)}(t)\right|^2 dt \geqslant \left(\frac{(2k-1)!!}{2^k}\right)^2 \int_0^c \frac{|f_0(t)|^2}{t^{2k}} dt + \lambda_0^2 \sum_{j=1}^k \frac{j^2}{c^{2j}} \int_0^c |f_0(t)|^2 dt.$$

Applying this inequality to the function g satisfying the assumptions of the theorem as $c = 1$ and taking into consideration the known identity $1^2 + 2^2 + \ldots + k^2 = k(k+1)(2k+1)/6$, we find that

$$\int_0^1 \left|g^{(k)}(\tau)\right|^2 d\tau \geqslant \left(\frac{(2k-1)!!}{2^k}\right)^2 \int_0^1 \frac{|g(\tau)|^2}{\tau^{2k}} d\tau + \lambda_0^2 \frac{k(k+1)(2k+1)}{6} \int_0^1 |g(\tau)|^2 d\tau.$$

By the change of the variable $\tau = t/c$ and the function $g(\tau) = f(t)$ we then get inequality (3.10) and this completes the proof. □

Remark 4.1. *The constant $((2k-1)!!/2^k)^2$ in inequality (3.10) is best possible also in the case when the second term is absent. Namely, as a corollary of Theorem 3.3 we have the following statement: for each $\varepsilon > 0$ there exists a function f_ε satisfying the assumptions of Theorem 4.2 and the inequality*

$$\int_0^c \left|f_\varepsilon^{(k)}(t)\right|^2 dt < \left(\frac{(2k-1)!!}{2^k} + \varepsilon\right)^2 \int_0^c \frac{|f_\varepsilon(t)|^2}{t^{2k}} dt.$$

The constant $\lambda_0^2 k(k+1)(2k+1)/(6c^{2k})$ at the second integral in the right hand side of inequality (3.10) is best possible only as $k = 1$.

5. Multidimensional analogues

Let us briefly describe a relation between one-dimensional integral inequalities and their multidimensional analogues.

Let $n \in \mathbb{N}$, $n \geqslant 2$. By $|x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$ we denote the Euclidean norm of a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $dx = dx_1 dx_2 \cdots dx_n$ is a differential of the volume (area as $n = 2$). We consider a domain $\Omega \subset \mathbb{R}^n$ and functions $u : \Omega \to \mathbb{C}$. For $u \in C^1(\Omega)$ the norm $|\nabla u(x)|$ of the Euclidean gradient

$$\nabla u(x) := \left(\frac{\partial u(x)}{\partial x_1}, \frac{\partial u(x)}{\partial x_2}, \ldots, \frac{\partial u(x)}{\partial x_n}\right) \in \mathbb{C}^n, \quad x \in \Omega \subset \mathbb{R}^n,$$

is defined by the identity

$$|\nabla u(x)| = \sqrt{\sum_{j=1}^n \left|\frac{\partial u(x)}{\partial x_j}\right|^2} = \sqrt{\sum_{j=1}^n \left(\mathrm{Re}\,\frac{\partial u(x)}{\partial x_j}\right)^2 + \left(\mathrm{Im}\,\frac{\partial u(x)}{\partial x_j}\right)^2}.$$

In an arbitrary domain $\Omega \subset \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$, a direct analogue of the Hardy inequality is the following one:

$$\int_\Omega \frac{|\nabla u(x)|^p}{\mathrm{dist}^{s-p}(x, \partial\Omega)} dx \geqslant C_p(s, \Omega) \int_\Omega \frac{|u(x)|^p}{\mathrm{dist}^s(x, \partial\Omega)} dx \quad \forall u \in C_0^1(\Omega), \tag{5.1}$$

where the constant $C_p(s, \Omega) \in [0, \infty)$ is supposed to be maximal among all possible ones.

In the multidimensional case the role of the parameters $p \in [1, \infty)$, $s \in \mathbb{R}$ is still important, but the main appearing problems are

1) how to describe geometrically "nice" domains, that is, ones for which $C_p(s, \Omega) > 0$;

2) how to obtain lower and upper bounds for $C_p(s, \Omega) > 0$ depending on the geometric characteristics of the domain and of the parameters p, s.

A series of results on studying inequality (5.1) can be found in recent monographs [14]–[16]. We just briefly describe some results on special cases of inequality (5.1).

We can provide several domains, in which inequality (5.1) is equivalent to inequality (2.5), that is, to

$$\int_0^\infty \frac{|g'(t)|^p}{t^{s-p}} dt \geqslant \left(\frac{|s-1|}{p}\right)^p \int_0^\infty \frac{|g(t)|^p}{t^s} dt \quad \forall g \in C_0^1(0, \infty).$$

We note that following Theorems 5.1 and 5.2 belong to a folklore of the theory of multidimensional Hardy inequalities.

Theorem 5.1. *Let $n \in \mathbb{N}$, $n \geqslant 2$. For each $p \in [1, \infty)$ and each $\sigma \in \mathbb{R}$ the following inequality*

$$\int_{\mathbb{R}^n} \frac{|\nabla u(x)|^p}{|x|^{\sigma-p}} dx \geqslant \left(\frac{|\sigma-n|}{p}\right)^p \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^\sigma} dx \tag{5.2}$$

holds for all $u \in C_0^1(\mathbb{R}^n \setminus \{0\})$ with a sharp constant $(|\sigma-n|/p)^p$.

We are going to give a brief proof of the equivalence of inequalities (2.5) and (5.2) for a fixed $n \geqslant 2$. We take $s = \sigma - n + 1$ and employ spherical coordinates

$$x = r\omega \in \mathbb{R}^n, \qquad r = |x| > 0, \qquad \omega \in S := \{y \in \mathbb{R}^n : |y| = 1\},$$

the formula $dx = r^{n-1} dr d\omega$ and the inequality $|\nabla u(x)| \geqslant |\partial u(x)/\partial r|$. Applying inequality (2.5) to a function $u \in C_0^1(\mathbb{R}^n \setminus \{0\})$ with a fixed $\omega \in S$, we obtain the inequality

$$\int_0^\infty \left|\frac{\partial u(r\omega)}{\partial r}\right|^p \frac{dr}{r^{s-p}} \geqslant \left(\frac{|s-1|}{p}\right)^p \int_0^\infty \frac{|u(r\omega)|^p}{r^s} dr,$$

which is equivalent to

$$\int_0^\infty \left|\frac{\partial u(r\omega)}{\partial r}\right|^p \frac{r^{n-1}dr}{|x|^{\sigma-p}} \geqslant \left(\frac{|\sigma-n|}{p}\right)^p \int_0^\infty \frac{|u(r\omega)|^p}{|x|^\sigma} r^{n-1} dr,$$

where $|x| = r$ and $\sigma = s + n - 1$. Multiplying both sides of the latter inequality by $d\omega$ and integrating over the sphere S, we arrive at the inequality

$$\int_{\mathbb{R}^n} \left|\frac{\partial u(r\omega)}{\partial r}\right|^p \frac{dx}{|x|^{\sigma-p}} \geqslant \left(\frac{|\sigma-n|}{p}\right)^p \int_{\mathbb{R}^n} \frac{|u(r\omega)|^p}{|x|^\sigma} dx,$$

and this yields (5.2). And vice versa, applying (5.2) to radial function defined by the identity $u(x) \equiv u(|x|) =: g(|x|)$, we get inequality (2.5) with $s = \sigma - n + 1$ and $t = r = |x|$.

If $\sigma = s + n - 1 < n$, then $s < 1$. Inequality (5.2) is valid under the boundary property $u(\infty) = 0$, which is implied by the condition $u \in C_0^1(\mathbb{R}^n)$. In particular, letting $p = \sigma$, we obtain the following corollary.

Corollary 5.1. *For each $p \in [1, n)$ the following inequality holds:*

$$\int_{\mathbb{R}^n} |\nabla u(x)|^p dx \geqslant \left(\frac{n-p}{p}\right)^p \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} dx \quad \forall u \in C_0^1(\mathbb{R}^n) \tag{5.3}$$

with a sharp constant $((n-p)/p)^p$.

Inequality (5.3) is usually called Leray inequality. It was discovered by J. Leray in 1933 for the case $p = 2$, $n = 3$ in work [17] devoted to studying Navier-Stokes equation. Thus, Leray was first who considered Hardy type inequality in a spatial domain.

It is easy to show that inequality (2.5) is equivalent to a corresponding inequality in the half-space $\mathbb{H}_n^+ = \{x \in \mathbb{R}^n : x_1 > 0\}$.

Theorem 5.2. *For each $p \in [1, \infty)$ and each $s \in \mathbb{R}$ the following inequality holds:*

$$\int_{\mathbb{H}_n^+} \frac{|\nabla u(x)|^p}{x_1^{s-p}} dx \geqslant \left(\frac{|s-1|}{p}\right)^p \int_{\mathbb{H}_n^+} \frac{|u(x)|^p}{x_1^s} dx \tag{5.4}$$

for all $u \in C_0^1(\mathbb{H}_n^+)$ with a sharp constant $(|s-1|/p)^p$.

We are going to describe several nontrivial results on inequality (5.1) in domains $\Omega \subset \mathbb{R}^n$ not coinciding with $\mathbb{R}^n \setminus \{0\}$ and $\mathbb{H}_n^+$. More precisely, we consider inequality (5.1) in domains $\Omega \subset \mathbb{R}^n$, when there are no simple formulas for finding the distance $\operatorname{dist}(x, \partial\Omega)$, $x \in \Omega$.

We first of all mention that *given below Theorems 5.3–5.7 were originally formulated and proved for real-valued functions. But these Theorems 5.3–5.7 are also true for complex-valued functions since their proofs are based on applying inequalities of form (2.4), which are also valid for complex-valued functions.*

For the case $s > 1$ the most complete results on inequality (5.1) we obtained for convex domains $\Omega \subset \mathbb{R}^n$. Owing to efforts of a series of mathematicians, namely, by E.B. Davies, T. Matskewich, P.E. Sobolevskii, H. Brezis, M. Marcus, V.J. Mitzel, I.K. Shafigullin and the author, see paper [18] and the references therein, the following statement was proved.

Theorem 5.3. *Let $n \geqslant 2$. For each $p \in [1, \infty)$, each $s \in (1, \infty)$ and each convex domain $\Omega \subset \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$, the inequality holds*

$$\int\limits_{\Omega} \frac{|\nabla u(x)|^p}{\mathrm{dist}^{s-p}(x, \partial\Omega)} dx \geqslant \left(\frac{s-1}{p}\right)^p \int\limits_{\Omega} \frac{|u(x)|^p}{\mathrm{dist}^{s}(x, \partial\Omega)} dx$$

for all $u \in C_0^1(\Omega)$, where the constant is best possible, that is, $C_p(s, \Omega) = ((s-1)/p)^p$ for each convex domain $\Omega \subset \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$ for all $p \in [1, \infty)$ and $s \in (1, \infty)$.

Theorem 5.3 is also interesting and amazing owing to the fact that various convex domains have the same Hardy constant equalling to $((s-1)/p)^p$.

In [19] we proved the following theorem.

Theorem 5.4. *Let $n \geqslant 2$. For each $p \in [1, \infty)$, each $s > n$ and each domain $\Omega \subset \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$, the inequality*

$$\int\limits_{\Omega} \frac{|\nabla u(x)|^p}{\mathrm{dist}^{s-p}(x, \partial\Omega)} dx \geqslant \left(\frac{s-n}{p}\right)^p \int\limits_{\Omega} \frac{|u(x)|^p}{\mathrm{dist}^{s}(x, \partial\Omega)} dx$$

holds for all $u \in C_0^1(\Omega)$, where the constant is optimal in the sense that there exist domains, for which the constant $((s-n)/p)^p$ is sharp.

We observe that this theorem involves no additional geometric restrictions for the boundary of the domain. This is a quite rare situation in embedding theorems of such type.

As $s \in (-\infty, 1)$ "nice" domains are the exteriors of convex compacts. Namely, the following theorem was proved by R.V. Makarov and the author in paper [20].

Theorem 5.5. *Suppose that $n \geqslant 2$, $1 \leqslant p < \infty$, $-\infty < s < n$ and a domain $\Omega \subset \mathbb{R}^n$ is such that $\mathbb{R}^n \setminus \Omega$ is a non-empty compact set. Then*

$$c_p(s, \Omega) \geqslant c_{psn} := \frac{\min_{j=1,2,\ldots,n} |s-j|^p}{p^p},$$

that is, for each complex-valued domain $u \in C_0^1(\Omega)$ we have

$$\int\limits_{\Omega} \frac{|\nabla u(x)|^p}{\mathrm{dist}^{s-p}(x, \partial\Omega)} dx \geqslant c_{psn} \int\limits_{\Omega} \frac{|u(x)|^p}{\mathrm{dist}^{s}(x, \partial\Omega)} dx,$$

where the constant is optimal in the sense that there exist domains obeying the assumptions of the theorem and for these domains the constant c_{psn} is sharp.

Multidimensional analogues of Rellich-Birman inequalities are related with polyharmonic operators of order $k \geqslant 2$.

Let Δ be the Laplace operator. For smooth functions $u \in C^k(\Omega)$ we consider a polyharmonic operator defined by the identities, see [21]:

$$\Delta^{k/2} u(x) := \begin{cases} \Delta^j u(x) & \text{if } k = 2j \text{ is even,} \\ \nabla \Delta^j u(x) & \text{if } k = 2j+1 \text{ is odd,} \end{cases}$$

with a formal convention $\Delta^{1/2} u := \nabla u$. It is obvious that in the one-dimensional case $\Delta^{k/2} f(t) = f^{(k)}(t)$ for a function $f \in C^k(a, b)$ of a variable $t \in (a, b)$.

The following theorem holds.

Theorem 5.6. *Let $n \geqslant 2$, $k \geqslant 2$ and let $\Omega \subset \mathbb{R}^n$ be a convex domain $\Omega \neq \mathbb{R}^n$. Then*

$$\int\limits_{\Omega} |\Delta^{k/2} u(x)|^2 dx \geqslant \frac{((2k-1)!!)^2}{4^k} \int\limits_{\Omega} \frac{|u(x)|^2}{\mathrm{dist}^{2k}(x, \partial\Omega)} dx$$

for all $u \in C_0^k(\Omega)$. For all $n \geqslant 2$, $k \geqslant 2$ the constant is sharp for each convex domain $\Omega \subset \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$.

The inequality in Theorem 5.6 was proved by M.P. Owen in paper [6], in which it was pointed out that the constant $A_k(\Omega) := ((2k-1)!!)^2/4^k$ is optimal since it is sharp for the half-space $x_1 > 0$. The sharpness of the constant for each convex domain $\Omega \subset \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$ was proven in papers [22] and [23].

There are also several generalizations of this theorem for the case of non-convex domains. For instance, the following theorem was proved in paper [24].

Theorem 5.7. *Let $k \geqslant 2$ and $\Omega \subset \mathbb{R}^2$ be a domain $\Omega \neq \mathbb{R}^2$. Suppose that the constant $A_k(\Omega) \in [0, \infty)$ is sharp, that is, it is the maximal among all possible ones in the inequality*

$$\int\limits_{\Omega} |\Delta^{k/2} u(x)|^2 dx \geqslant A_k(\Omega) \int\limits_{\Omega} \frac{|u(x)|^2}{\operatorname{dist}^{2k}(x, \partial\Omega)} dx$$

for all $u \in C_0^k(\Omega)$. Then

$$A_k(\Omega) \geqslant ((k-1)!)^2 A_1(\Omega),$$

and the following statement holds: for each $k \geqslant 2$ the inequality $A_k(\Omega) > 0$ holds if and only if the domain $\Omega \subset \mathbb{R}^2$ has a uniformly perfect boundary.

We note that in the proofs of Theorem 5.6 and 5.7 an essential role is played by Theorems 5.3, 5.4 and by the following generalized identity by O.A. Ladyzhenskaya [25, Ch. 2, Form. (6.26)] for $m = 2$ and [21, Ch. 2, Form. (2.12)] for the general case:

$$\int\limits_{\Omega} \left|\Delta^{m/2} u(x)\right|^2 dx = \int\limits_{\Omega} \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \cdots \sum_{k_m=1}^{n} \left(\frac{\partial^m u(x)}{\partial x_{k_1} \partial x_{k_2} \cdots \partial x_{k_m}}\right)^2 dx$$

for each function $u \in C_0^m(\Omega)$.

We provide several corollaries of Theorem 5.7. The boundary of the circle with a punctured center is not perfect set. This is why the following corollary holds.

Corollary 5.2. *If $\Omega_1 \subset \mathbb{R}^2$ is the circle $|x| < 3$ with a punctured center, then $A_k(\Omega_1) = 0$.*

Removing from the circle a sufficiently "dense" closed set of point, we can construct a domain with a uniformly perfect boundary. In particular, the next corollary is true.

Corollary 5.3. *If $\Omega_2 \subset \mathbb{R}^2$ is the circle $|x| < 3$, from which a classical Cantor set lying on the segment $[0, 1]$ is removed, then $A_k(\Omega_2) > 0$.*

We can provide explicit lower bounds for the quantity $A_k(\Omega_2)$ as well as for the constant $A_k(\Omega)$ with using modulus characteristic with the domain Ω. The simplest partial case is given in the following statement.

Corollary 5.4. *If $\Omega \subset \mathbb{R}^2$ is a simply-connected domain, $\Omega \neq \mathbb{R}^2$, then $A_k(\Omega) \geqslant ((k-1)!/4)^2$.*

In conclusion we mention that in recent papers [26] and [27] there was formulated a series of unsolved problems on multidimensional inequalities of Hardy and Rellich type.

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