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QUADRATURE FORMULA FOR NORMAL DERIVATIVE OF DOUBLE LAYER POTENTIAL

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Abstract. Looking for a solution to the Dirichlet and Neumann boundary value problems for the Helmholtz equation in the form of a combination of simple and double layer potentials, the considered boundary value problems are reduced to a curvilinear integral equation depending on the operators generated by the simple and double layer potentials and by their normal derivative. It is known that the latter operators are weakly singular integral ones. However, a counterexample constructed by Lyapunov shows that for the double layer potential with continuous density, the derivative, generally speaking, does not exist, that is, the operator generated by the normal derivative of the double layer potential is a singular integral operator.

Since in many cases it is impossible to find exact solutions to integral equations, it is of interest to study an approximate solution of the obtained integral equations. In its turn, in order to find an approximate solution, it is necessary, first of all, to construct quadrature formulas for the simple and double layer potentials of the and for their normal derivatives. In this work we prove the existence theorem for the normal derivative of the double layer potential and we provide a formula for its calculation. In addition, we develop a new method for constructing a quadrature formula for a singular curvilinear integral and on the base of this we construct a quadrature formula for the normal derivative of the double layer potential and we estimate the error.

Keywords: quadrature formulas, singular integral, double layer potential, Hankel function, Lyapunov curve.

Mathematics Subject Classification: 45E05, 31B10

1. Introduction and formulation of problem

It is known that in a two-dimensional space, searching for a solution to the Dirichlet and Neumann boundary value problems for the Helmholtz equation $\Delta u + k^2 u = 0$ in the form of a combination of potentials of a simple and double layer, the considered boundary value problems are reduced to an integral equation (see [1, Ch. III]) depending on the operator generated by the normal derivative of the double layer potential:

$$(T\rho)(x) = 2\frac{\partial}{\partial\nu(x)}\int\limits_L \frac{\partial\Phi(x,y)}{\partial\nu(y)}\rho(y)\,dL_y, \qquad x=(x_1,x_2)\in L. \tag{1.1}$$

Here Δ is the Laplace operator, k is a wave number and $\operatorname{Im} k \geqslant 0$, $L\subset\mathbb{R}^2$ is a simple closed Lyapunov curve, $\rho(y)$ is a continuous function on the curve L, $\nu(y)$ is the outward unit normal

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at the point $y \in L$, and $\Phi(x, y)$ is the fundamental solution of the Helmholtz equation, that is,

$$\Phi(x,\, y) = \begin{cases} \dfrac{1}{2\pi} \ln \dfrac{1}{|x-y|} & \text{as} \quad k = 0, \\ \dfrac{i}{4} H_0^{(1)}\left(k\,|x-y|\right) & \text{as} \quad k \neq 0, \end{cases}$$

where by $|x-y|$ we denote the Euclidean distance between the points x and y; $H_0^{(1)}$ is the Hankel function of zero order and first kind defined by the formula $H_0^{(1)}(z) = J_0(z) + iN_0(z)$,

$$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{z}{2}\right)^{2m}$$

is the Bessel function of zero order,

$$N_0(z) = \frac{2}{\pi} \left(\ln \frac{z}{2} + C\right) J_0(z) + \sum_{m=1}^{\infty} \left(\sum_{l=1}^{m} \frac{1}{l}\right) \frac{(-1)^{m+1}}{(m!)^2} \left(\frac{z}{2}\right)^{2m}$$

is the Neumann function of zero order, see [2, Ch. XIII], while $C = 0.57721\ldots$ is the Euler constant. It should be mentioned that a counterexample constructed by Lyapunov, see [3, Ch. II] shows that, generally speaking, the double layer potential does not have a derivative.

Since in many cases it is impossible to find exact solutions of integral equations, it is interesting to study approximate solutions of the obtained integral equations. In order to find an approximate solution, it is necessary, first of all, to construct quadrature formulas for the simple layer potential and double layer potential and for their normal derivatives. We note that in work [4], by using the asymptotic formula for the Hankel functions of the first kind and zero order, a quadrature formula was constructed for the simple layer potential and double layer potential, which does not allow to determine the rate of convergence of these quadrature formulas. However, in work [5], in a more practical way, quadrature formulas for the simple layer potential and double layer potential were constructed, and in work [6] a quadrature formula was constructed for the normal derivative of the simple layer potential and estimates for the errors of the constructed quadrature formulas were given. In addition, in works [7], [8] quadrature formulas were constructed for the normal derivative of the logarithmic simple layer potential and double layer potential and there were studied approximate solutions of the integral equations of the external Dirichlet boundary value problem and the mixed boundary value problem for the Laplace equation in two-dimensional space. In works [9], [10] a new method was proposed for constructing a cubature formula for the normal derivative of the acoustic double layer potential and the collocation method was justified for integral equations of external boundary value problems of Dirichlet and Neumann for the Helmholtz equation in three-dimensional space. However, it is known that in the three-dimensional space the fundamental solution of the Helmholtz equation reads as

$$\Phi_k(x,\, y) = \frac{\exp\left(ik\,|x-y|\right)}{4\pi\,|x-y|}, \qquad x, y \in \mathbb{R}^3, \qquad x \neq y,$$

which strictly differs from the fundamental solution of the Helmholtz equation in the two-dimensional space. It also should be mentioned that in work [11], the normal derivative of the double layer potential was considered as a hypersingular integral treated as the principal value in the Hadamard sense and a quadrature formula for the normal derivative of the double layer potential was constructed by the methods of subdomains under an additional condition for the density ρ, see [11, Ch. XIII]. It is known that under this condition, the expression for the normal derivative of the double layer potential can be represented in the form of a singular integral, see [1, Ch. II], [11, Ch. IV], i.e. integral (1.1) exists in the sense of the Cauchy principal value. In addition, it should be noted that the quadrature formula constructed in [11] is not practical in the sense that its coefficients are singular integrals. Therefore, considering the normal derivative of the double layer potential as an integral in the sense of the Cauchy principal value, a more practical way of constructing a quadrature formula for integral (1.1) is important; our note is devoted to this issue.

2. Existence and formula for calculating normal derivative for double layer potential

By $C(L)$ we denote the space of all continuous functions on L with the norm $\|\rho\|_\infty = \max\limits_{x\in L} |\rho(x)|$, and for a function $\varphi(x) \in C(L)$ we introduce a continuity modulus of form

$$\omega(\varphi,\delta) = \delta \sup_{\tau \geqslant \delta} \frac{\bar{\omega}(\varphi,\tau)}{\tau}, \qquad \delta > 0,$$

where $\bar{\omega}(\varphi,\tau) = \max\limits_{\substack{|x-y|\leqslant\tau\\ x,y\in L}} |\varphi(x)-\varphi(y)|$. We note in the same way the continuity modulus for a continuous vector function $\varphi(x) = (\varphi_1(x), \varphi_2(x))$ is introduced with

$$|\varphi(x)-\varphi(y)| = \sqrt{(\varphi_1(x)-\varphi_1(y))^2 + (\varphi_2(x)-\varphi_2(y))^2}.$$

Theorem 2.1. *Let $L \subset \mathbb{R}^2$ be a simple closed Lyapunov curve, $\rho(x)$ be a continuously differentiable function on L and*

$$\int\limits_0^{\operatorname{diam} L} \frac{\omega(\operatorname{grad}\rho,\, t)}{t} dt < +\infty.$$

Then the double layer potential

$$W(x) = \int\limits_L \frac{\partial \Phi(x,\, y)}{\partial \nu(y)} \rho(y)\, dL_y, \qquad x \in L,$$

possesses on L the normal derivative and

$$\begin{aligned}\frac{\partial W(x)}{\partial \nu(x)} = &\int\limits_L \frac{\partial V(x,y)}{\partial \nu(x)} \rho(y)\, dL_y \\ &- \frac{1}{\pi}\int\limits_L \frac{(x-y,\nu(y))\,(x-y,\nu(x))}{|x-y|^4} (\rho(y)-\rho(x))\, dL_y \\ &+ \frac{1}{2\pi}\int\limits_L \frac{(\nu(y),\, \nu(x))}{|x-y|^2} (\rho(y)-\rho(x))\, dL_y, \qquad x\in L\end{aligned} \tag{2.1}$$

and

$$\left|\frac{\partial W(x)}{\partial \nu(x)}\right| \leqslant M^1 \left(\|\rho\|_\infty + \|\operatorname{grad}\rho\|_\infty + \int\limits_0^{\operatorname{diam} L} \frac{\omega(\operatorname{grad}\rho, t)}{t} dt \right) \quad \textit{for all} \quad x \in L.$$

Here by (a,b) we denote the scalar product in a and b and $\operatorname{diam} L = \sup\limits_{x,y\in L} |x-y|$ *and*

$$V(x,y) = \left(\frac{i}{4} - \frac{C}{2\pi} - \frac{1}{2\pi} \ln \frac{k\,|x-y|}{2} \right) (y-x, \nu(y)) \sum_{m=1}^{\infty} \frac{(-1)^m k^{2m} |x-y|^{2m-2}}{2^{2m-1}(m-1)!\, m!}$$

[1]Hereinafter by M we denote positive constants, which are different in various inequalities.

$$-\left(y-x,\nu\left(y\right)\right)\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{\left(-1\right)^{m+1}k^{2m}\left|x-y\right|^{2m-2}}{2^{2m+1}\left(m-1\right)!\,m!}$$
$$-\frac{1}{2\pi}\left(y-x,\nu\left(y\right)\right)\sum_{m=1}^{\infty}\frac{\left(-1\right)^{m}k^{2m}\left|x-y\right|^{2m-2}}{2^{2m}\left(m!\right)^{2}}.$$

Moreover, the first and second terms, the integrals in (2.1) are weakly-singular, while the last integral exists in the sense of the Cauchy principal value, that is,

$$\int_{L}\frac{\left(\nu\left(y\right),\nu\left(x\right)\right)}{\left|x-y\right|^{2}}\left(\rho\left(y\right)-\rho\left(x\right)\right)dL_{y}=\lim_{\epsilon\to+0}\int_{L\setminus L_{\epsilon}(x)}\frac{\left(\nu\left(y\right),\nu\left(x\right)\right)}{\left|x-y\right|^{2}}\left(\rho\left(y\right)-\rho\left(x\right)\right)dL_{y},$$

where $L_{\epsilon}\left(x\right)$ is the part of the curve L located inside the circle of the radius ϵ centered at the point $x\in L$.

Proof. By straightforward calculations we find that

$$\frac{\partial\Phi\left(x,y\right)}{\partial\nu\left(y\right)}=\frac{i}{4}\left(\frac{\partial J_{0}\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)}+i\,\frac{\partial N_{0}\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)}\right),$$

where

$$\frac{\partial J_{0}\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)}=\left(y-x,\nu\left(y\right)\right)\sum_{m=1}^{\infty}\frac{\left(-1\right)^{m}k^{2m}\left|x-y\right|^{2m-2}}{2^{2m-1}\left(m-1\right)!\,m!}$$

and

$$\frac{\partial N_{0}\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)}=\frac{2}{\pi}\left(\ln\frac{k\left|x-y\right|}{2}+C\right)\frac{\partial J_{0}\left(k\left|x-y\right|\right)}{\partial\nu\left(y\right)}+\frac{2\left(y-x,\nu\left(y\right)\right)}{\pi\left|x-y\right|^{2}}J_{0}\left(k\left|x-y\right|\right)$$
$$+\left(y-x,\nu\left(y\right)\right)\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{\left(-1\right)^{m+1}k^{2m}\left|x-y\right|^{2m-2}}{2^{2m-1}\left(m-1\right)!\,m!}.$$

Then the expression $W\left(x\right)$ can be represented as

$$W\left(x\right)=\int_{L}\left(\frac{\left(x-y,\nu\left(y\right)\right)}{2\pi\left|x-y\right|^{2}}+V\left(x,y\right)\right)\rho\left(y\right)dL_{y},\qquad x\in L.$$

It was shown in work [12] that if the function $\rho\left(x\right)$ is continuously differentiable on L and

$$\int_{0}^{\operatorname{diam}L}\frac{\omega\left(\operatorname{grad}\rho,\,t\right)}{t}dt<+\infty,$$

then the function

$$W_{0}\left(x\right)=\frac{1}{2\pi}\int_{L}\frac{\left(x-y,\nu\left(y\right)\right)}{\left|x-y\right|^{2}}\rho\left(y\right)dL_{y},\quad x\in L,$$

possesses on L the normal derivative and

$$\frac{\partial W_{0}\left(x\right)}{\partial\nu\left(x\right)}=-\frac{1}{\pi}\int_{L}\frac{\left(x-y,\nu\left(y\right)\right)\left(x-y,\nu\left(x\right)\right)}{\left|x-y\right|^{4}}\left(\rho\left(y\right)-\rho\left(x\right)\right)dL_{y}$$
$$+\frac{1}{2\pi}\int_{L}\frac{\left(\nu\left(y\right),\nu\left(x\right)\right)}{\left|x-y\right|^{2}}\left(\rho\left(y\right)-\rho\left(x\right)\right)dL_{y},\quad x\in L \tag{2.2}$$

and

$$\left|\frac{\partial W_0(x)}{\partial \nu(x)}\right| \leqslant M\left(\|\rho\|_\infty + \|\mathrm{grad}\,\rho\|_\infty + \int\limits_0^{\mathrm{diam}\,L} \frac{\omega(\mathrm{grad}\,\rho, t)}{t}dt\right), \quad \forall x \in L,$$

when the last integral in identity (2.2) exists in the sense of the Cauchy principal value.

Since [13, Ch. V]

$$|(x-y, \nu(x))| \leqslant M\,|x-y|^{1+\alpha}, \quad \forall x, y \in L, \tag{2.3}$$

then taking into consideration the inequalities

$$|J_0(k\,|x-y|)| = \left|\sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{k\,|x-y|}{2}\right)^{2m}\right| \leqslant \sum_{m=0}^{\infty} \frac{(|k|\ diamL)^{2m}}{4^m\,(m!)^2}, \quad \forall x, y \in L, \tag{2.4}$$

and

$$\begin{aligned} &\left|\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{(-1)^{m+1}\,k^{2m}\,|x-y|^{2m-2}}{2^{2m-1}\,(m-1)!\,m!}\right| \\ &\leqslant \sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{|k|^{2m}\,(diamL)^{2m-2}}{2^{2m-1}\,(m-1)!\,m!}, \qquad \forall x, y \in L, \end{aligned} \tag{2.5}$$

we obtain that

$$|V(x,y)| \leqslant M\ |x-y|, \quad \forall x, y \in L.$$

Therefore, the function

$$W_1(x) = \int\limits_L V(x,y)\ \rho(y)\,dL_y, \quad x \in L,$$

possesses the normal derivative on L and

$$\begin{aligned} \frac{\partial W_1(x)}{\partial \nu(x)} &= \int\limits_L \frac{\partial V(x,y)}{\partial \nu(x)}\,\rho(y)\,dL_y \\ &= \frac{1}{2\pi}\int\limits_L \frac{(y-x,\nu(x))\,(y-x,\nu(y))}{|x-y|^2}\sum_{m=1}^{\infty}\frac{(-1)^m\,k^{2m}\,|x-y|^{2m-2}}{2^{2m-1}\,(m-1)!\,m!}\rho(y)\,dL_y \\ &\quad - \int\limits_L \left(\frac{i}{4} - \frac{C}{2\pi} - \frac{1}{2\pi}\ln\frac{k\ |x-y|}{2}\right)(\nu(y), \nu(x)) \\ &\quad \cdot \sum_{m=1}^{\infty}\frac{(-1)^m\,k^{2m}\,|x-y|^{2m-2}}{2^{2m-1}\,(m-1)!\,m!}\rho(y)\,dL_y \\ &\quad + \int\limits_L \left(\frac{i}{4} - \frac{C}{2\pi} - \frac{1}{2\pi}\ln\frac{k\ |x-y|}{2}\right)(y-x, \nu(y)) \\ &\quad \cdot (x-y,\nu(x))\sum_{m=2}^{\infty}\frac{(-1)^m\,k^{2m}\,|x-y|^{2m-4}}{2^{2m-2}\,(m-2)!\,m!}\rho(y)\,dL_y \\ &\quad + \int\limits_L (\nu(y),\nu(x))\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{(-1)^{m+1}\,k^{2m}\,|x-y|^{2m-2}}{2^{2m+1}\,(m-1)!\,m!}\rho(y)\,dL_y \\ &\quad - \int\limits_L (x-y,\nu(x))\ (y-x,\nu(y))\sum_{m=2}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{(-1)^{m+1}\,k^{2m}\,|x-y|^{2m-4}}{2^{2m}\,(m-2)!\,m!}\rho(y)\,dL_y \end{aligned}$$

$$+\frac{1}{2\pi}\int\limits_{L}(\nu(y),\nu(x))\,(y-x,\nu(y))\sum_{m=1}^{\infty}\frac{(-1)^{m}k^{2m}|x-y|^{2m-2}}{2^{2m}(m!)^{2}}\rho(y)\,dL_y$$
$$-\frac{1}{2\pi}\int\limits_{L}(x-y,\nu(x))\,(y-x,\nu(y))\sum_{m=2}^{\infty}\frac{(-1)^{m}(m-1)k^{2m}|x-y|^{2m-4}}{2^{2m-1}(m!)^{2}}\rho(y)\,dL_y$$

and

$$\left|\frac{\partial V(x,y)}{\partial\nu(x)}\right|\leqslant M\,|\ln|x-y||\,,\quad\forall x,y\in L. \tag{2.6}$$

This yields

$$\left|\frac{\partial W_1(x)}{\partial\nu(x)}\right|\leqslant M\,\|\rho\|_{\infty}\,,\quad\forall x\in L.$$

The proof is complete. □

3. Quadrature formula for normal derivative of double layer potential

Suppose that the curve L is given by the parametric equation $x(t)=(x_1(t),x_2(t))\,,\,t\in[a,b]$. We partition the segment $[a,b]$ into $n>2M_0(b-a)/d$ equal parts:

$$t_p=a+\frac{(b-a)\,p}{n},\quad p=\overline{0,n},$$

where $M_0=\max\limits_{t\in[a,b]}\sqrt{(x_1'(t))^2+(x_2'(t))^2}<+\infty$, see [14, Ch. VI]) and d is the standard basis, see [13, Ch. V]). As nodes we take $x(\tau_p)$, $p=\overline{1,n}$, where $\tau_p=a+\frac{(b-a)(2p-1)}{2n}$. Then the curve L is partitioned into elementary parts:

$$L=\bigcup_{p=1}^{n}L_p,\quad\text{where}\quad L_p=\{\,x(t):\ t_{p-1}\leqslant t\leqslant t_p\}\,.$$

It is known that [15]

(1) $\forall p\in\{1,2,\ldots,n\}$: $r_p(n)\sim R_p(n)$, where

$$r_p(n)=\min\{|x(\tau_p)-x(t_{p-1})|\,,|x(t_p)-x(\tau_p)|\}\,,$$
$$R_p(n)=\max\{|x(\tau_p)-x(t_{p-1})|\,,\,|x(t_p)-x(\tau_p)|\}\,,$$

and the writing $a(n)\sim b(n)$ means that $C_1\leqslant\frac{a(n)}{b(n)}\leqslant C_2$, where C_1 and C_2 are positive constants independent of n.

(2) $\forall p\in\{1,2,\ldots,n\}:R_p(n)\leqslant d/2$;

(3) $\forall p,j\in\{1,2,\ldots,n\}:r_j(n)\sim r_p(n)$;

(4) $r(n)\sim R(n)\sim\frac{1}{n}$, where $R(n)=\max\limits_{p=\overline{1,n}}R_p(n)$, $r(n)=\min\limits_{p=\overline{1,n}}r_p(n)$.

In what follows such partition is called a partition of the curve L into regular elementary parts. The following lemma holds.

Lemma 3.1 ([15]). *There exist constants $C_0'>0$ and $C_1'>0$ independent of n, such that for all $p,j\in\{1,2,\ldots,n\}$, $j\neq p$, and for all $y\in L_j$ the inequalities*

$$C_0'\,|y-x(\tau_p)|\leqslant|x(\tau_j)-x(\tau_p)|\leqslant C_1'\,|y-x(\tau_p)|$$

hold true.

It is obvious that there exists a natural number n_0 such that

$$(R(n))^{\frac{1}{1+\alpha}}\leqslant\min\{1,d/2\}\,,\qquad\forall n>n_0.$$

Let

$$Q_l = \left\{ j \mid 1 \leqslant j \leqslant n,\ |x(\tau_l) - x(\tau_j)| > (R(n))^{\frac{1}{1+\alpha}} \right\}$$

and

$$\begin{aligned} V_n(x,y) = & \left(\frac{i}{4} - \frac{C}{2\pi} - \frac{1}{2\pi} \ln \frac{k\ |x-y|}{2} \right) (y-x, \nu(y)) \sum_{m=1}^{n} \frac{(-1)^m k^{2m} |x-y|^{2m-2}}{2^{2m-1}(m-1)!\, m!} \\ & - (y-x, \nu(y)) \sum_{m=1}^{n} \left(\sum_{l=1}^{m} \frac{1}{l} \right) \frac{(-1)^{m+1} k^{2m} |x-y|^{2m-2}}{2^{2m+1}(m-1)!\, m!} \\ & - \frac{1}{2\pi} (y-x, \nu(y)) \sum_{m=1}^{n} \frac{(-1)^m k^{2m} |x-y|^{2m-2}}{2^{2m} (m!)^2}. \end{aligned}$$

By straightforward calculations we find:

$$\begin{aligned} \frac{\partial V_n(x,y)}{\partial \nu(x)} = & \frac{1}{2\pi} \frac{(y-x,\nu(x))(y-x,\nu(y))}{|x-y|^2} \sum_{m=1}^{n} \frac{(-1)^m k^{2m} |x-y|^{2m-2}}{2^{2m-1}(m-1)!\, m!} \\ & - \left(\frac{i}{4} - \frac{C}{2\pi} - \frac{1}{2\pi} \ln \frac{k\ |x-y|}{2} \right) (\nu(y), \nu(x)) \sum_{m=1}^{n} \frac{(-1)^m k^{2m} |x-y|^{2m-2}}{2^{2m-1}(m-1)!\, m!} \\ & + \left(\frac{i}{4} - \frac{C}{2\pi} - \frac{1}{2\pi} \ln \frac{k\ |x-y|}{2} \right) (y-x, \nu(y)) \\ & \cdot (x-y, \nu(x)) \sum_{m=2}^{n} \frac{(-1)^m k^{2m} |x-y|^{2m-4}}{2^{2m-2}(m-2)!\, m!} \\ & + (\nu(y), \nu(x)) \sum_{m=1}^{n} \left(\sum_{l=1}^{m} \frac{1}{l} \right) \frac{(-1)^{m+1} k^{2m} |x-y|^{2m-2}}{2^{2m+1}(m-1)!\, m!} \\ & - (x-y, \nu(x))\ (y-x, \nu(y)) \sum_{m=2}^{n} \left(\sum_{l=1}^{m} \frac{1}{l} \right) \frac{(-1)^{m+1} k^{2m} |x-y|^{2m-4}}{2^{2m}(m-2)!\, m!} \\ & + \frac{1}{2\pi} (\nu(y), \nu(x))\ (y-x, \nu(y)) \sum_{m=1}^{n} \frac{(-1)^m k^{2m} |x-y|^{2m-2}}{2^{2m} (m!)^2} \\ & - \frac{1}{2\pi} (x-y, \nu(x))\ (y-x, \nu(y)) \sum_{m=2}^{n} \frac{(-1)^m (m-1) k^{2m} |x-y|^{2m-4}}{2^{2m-1} (m!)^2}. \end{aligned}$$

The following theorem holds true.

Theorem 3.1. *Let $L \subset \mathbb{R}^2$ be a simple closed Lyapunov curve with the exponent $0 < \alpha \leqslant 1$, $\rho(x)$ be a continuously differentiable function on L and*

$$\int_0^{\operatorname{diam} L} \frac{\omega(\operatorname{grad} \rho,\, t)}{t} dt < +\infty.$$

Then the expression

$$(T_n\rho)(x(\tau_l)) = \frac{2\ (b-a)}{n} \sum_{\substack{j=1 \\ j \neq l}}^{n} \frac{\partial V_n(x(\tau_l), x(\tau_j))}{\partial \nu(x(\tau_l))} \sqrt{(x_1'(\tau_j))^2 + (x_2'(\tau_j))^2}\, \rho(x(\tau_j))$$

$$- \frac{2\,(b-a)}{\pi\, n} \sum_{\substack{j=1\\ j\neq l}}^{n} \frac{\left(x\left(\tau_l\right)-x\left(\tau_j\right),\, \nu\left(x\left(\tau_j\right)\right)\right)\,\left(x\left(\tau_l\right)-x\left(\tau_j\right), \nu\left(x\left(\tau_l\right)\right)\right)}{\left|x\left(\tau_l\right)-x\left(\tau_j\right)\right|^4}$$

$$\cdot \sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2}\left(\rho\left(x\left(\tau_j\right)\right)-\rho\left(x\left(\tau_l\right)\right)\right)$$

$$+ \frac{b-a}{\pi\, n} \sum_{j\in Q_l} \frac{\left(\nu\left(x\left(\tau_j\right)\right), \nu\left(x\left(\tau_l\right)\right)\right)}{\left|x\left(\tau_j\right)-x\left(\tau_l\right)\right|^2} \sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2}\left(\rho\left(x\left(\tau_j\right)\right)-\rho\left(x\left(\tau_l\right)\right)\right)$$

at the nodes $x\left(\tau_l\right)$, $l=\overline{1,\, n}$, *is a quadrature formula for* $\left(T\rho\right)(x)$ *and the following estimates hold:*

$$\max_{l=\overline{1,\, n}} \left|\left(T\rho\right)\left(x\left(\tau_l\right)\right)-\left(T_n\rho\right)\left(x\left(\tau_l\right)\right)\right| \leqslant M\left[\,\|\rho\|_\infty\, n^{-\alpha}+\|\mathrm{grad}\ \rho\|_\infty\, n^{-\frac{\alpha}{1+\alpha}} + \int\limits_0^{n^{-\frac{1}{1+\alpha}}} \frac{\omega(\mathrm{grad}\ \rho,\, t)}{t}\, dt\right]$$

if $0<\alpha<1$ *and*

$$\max_{l=\overline{1,\, n}} \left|\left(T\rho\right)\left(x\left(\tau_l\right)\right)-\left(T_n\rho\right)\left(x\left(\tau_l\right)\right)\right| \leqslant M\left[\frac{\|\rho\|_\infty \ln n}{n}+\frac{\|\mathrm{grad}\ \rho\|_\infty}{\sqrt{n}} + \int\limits_0^{1/\sqrt{n}} \frac{\omega(\mathrm{grad}\ \rho,\, t)}{t}\, dt\right]$$

if $\alpha=1$.

Proof. It was proved in work [16] that if the function $\rho(x)$ is continuously differentiable on L and

$$\int\limits_0^{\mathrm{diam}\, L} \frac{\omega(\mathrm{grad}\,\rho,\, t)}{t} dt < +\infty,$$

then the expression

$$\left(\frac{\partial W_0}{\partial \nu}\right)^n \left(x\left(\tau_l\right)\right) = -\frac{b-a}{\pi\, n} \sum_{\substack{j=1\\ j\neq l}}^{n} \frac{\left(x\left(\tau_l\right)-x\left(\tau_j\right),\, \nu\left(x\left(\tau_j\right)\right)\right)\,\left(x\left(\tau_l\right)-x\left(\tau_j\right),\, \nu\left(x\left(\tau_l\right)\right)\right)}{\left|x\left(\tau_l\right)-x\left(\tau_j\right)\right|^4}$$

$$\cdot \sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2}\ \left(\rho\left(x\left(\tau_j\right)\right)-\rho\left(x\left(\tau_l\right)\right)\right)$$

$$+\frac{b-a}{2\pi\, n} \sum_{j\in Q_l} \frac{\left(\nu\left(x\left(\tau_j\right)\right),\, \nu\left(x\left(\tau_l\right)\right)\right)}{\left|x\left(\tau_j\right)-x\left(\tau_l\right)\right|^2}$$

$$\cdot \sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2}\ \left(\rho\left(x\left(\tau_j\right)\right)-\rho\left(x\left(\tau_l\right)\right)\right)$$

at the nodes $x\left(\tau_l\right)$, $l=\overline{1,\, n}$, is a quadrature formula for the integral $\frac{\partial W_0(x)}{\partial \nu(x)}$ and the following estimates hold:

$$\max_{p=\overline{1,\, n}} \left|\frac{\partial W_0\left(x\left(\tau_l\right)\right)}{\partial \nu\left(x\left(\tau_l\right)\right)}-\left(\frac{\partial W_0}{\partial \nu}\right)^n\left(x\left(\tau_l\right)\right)\right| \leqslant M\left[\,\|\rho\|_\infty\, n^{-\alpha}+\|\mathrm{grad}\ \rho\|_\infty\, n^{-\frac{\alpha}{1+\alpha}}\right.$$

$$+\int\limits_0^{n^{-\frac{1}{1+\alpha}}} \frac{\omega(\text{grad }\rho,\,t)}{t}\,dt\Bigg]$$

if $0<\alpha<1$ and

$$\max_{p=\overline{1,\,n}}\left|\frac{\partial W_0\left(x\left(\tau_l\right)\right)}{\partial\nu\left(x\left(\tau_l\right)\right)}-\left(\frac{\partial W_0}{\partial\nu}\right)^n\left(x\left(\tau_l\right)\right)\right|\leqslant M\left[\frac{\|\rho\|_\infty\ln n}{n}+\frac{\|\text{grad }\rho\|_\infty}{\sqrt{n}}\right.$$
$$\left.+\int\limits_0^{1/\sqrt{n}}\frac{\omega(\text{grad }\rho,\,t)}{t}\,dt\right]$$

if $\alpha=1$.

Now let us show that the expression

$$\left(\frac{\partial W_1}{\partial\nu}\right)^n\left(x\left(\tau_p\right)\right)=\frac{b-a}{n}\sum_{\substack{j=1\\ j\neq p}}^n\frac{\partial V_n\left(x\left(\tau_p\right),x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}\sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2}\,\rho\left(x\left(\tau_j\right)\right)$$

at the nodes $x\left(\tau_p\right)$, $p=\overline{1,\,n}$, is a quadrature formula for the integral $\frac{\partial W_1(x)}{\partial\nu(x)}$. It is easy to see that

$$\frac{\partial W_1\left(x\left(\tau_p\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}-\left(\frac{\partial W_1}{\partial\nu}\right)^n\left(x\left(\tau_p\right)\right)=\int\limits_{L_p}\frac{\partial V\left(x\left(\tau_p\right),\,y\right)}{\partial\nu\left(x\right)}\,\rho\left(y\right)dL_y$$
$$+\sum_{\substack{j=1\\ j\neq p}}^n\int\limits_{L_j}\left(\frac{\partial V\left(x\left(\tau_p\right),\,y\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}-\frac{\partial V_n\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}\right)\rho\left(y\right)dL_y$$
$$+\sum_{\substack{j=1\\ j\neq p}}^n\int\limits_{L_j}\frac{\partial V_n\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}\,\left(\rho\left(y\right)-\rho\left(x\left(\tau_j\right)\right)\right)dL_y$$
$$+\sum_{\substack{j=1\\ j\neq p}}^n\int\limits_{t_{j-1}}^{t_j}\frac{\partial V_n\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}$$
$$\cdot\left(\sqrt{\left(x_1'\left(t\right)\right)^2+\left(x_2'\left(t\right)\right)^2}-\sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2}\right)\rho\left(x\left(\tau_j\right)\right)dt.$$

We denote the terms in this identity by $\delta_1^n\left(x\left(\tau_p\right)\right)$, $\delta_2^n\left(x\left(\tau_p\right)\right)$, $\delta_3^n\left(x\left(\tau_p\right)\right)$ and $\delta_4^n\left(x\left(\tau_p\right)\right)$, respectively.

Taking into consideration (2.6) and the formula for calculating a curvilinear integral, we obtain:

$$\left|\delta_1^n\left(x\left(\tau_p\right)\right)\right|\leqslant M\ \|\rho\|_\infty\int\limits_0^{R(n)}\left|\ln\tau\right|d\tau\leqslant M\ \|\rho\|_\infty\,R\left(n\right)\ \left|\ln R\left(n\right)\right|.$$

Let $y\in L_j$ and $j\neq p$. In view of Lemma 3.1 and inequality (2.3) it is obvious that

$$\left|\left|x\left(\tau_p\right)-y\right|^q-\left|x\left(\tau_p\right)-x\left(\tau_j\right)\right|^q\right|\leqslant MqR\left(n\right)\left(\text{diam}\,L\right)^{q-1},$$
$$\left|\left(\nu\left(y\right),\nu\left(x\left(\tau_p\right)\right)\right)-\left(\nu\left(x\left(\tau_j\right)\right),\nu\left(x\left(\tau_p\right)\right)\right)\right|\leqslant M(R(n))^\alpha,$$
$$\left|\left(x\left(\tau_p\right)-y,\nu\left(y\right)\right)-\left(x\left(\tau_p\right)-x\left(\tau_j\right),\nu\left(y\right)\right)\right|=\left|\left(x\left(\tau_j\right)-y,\nu\left(y\right)\right)\right|\leqslant M(R(n))^{1+\alpha},$$
$$\left|\left(x\left(\tau_p\right)-y,\nu\left(x\left(\tau_p\right)\right)\right)-\left(x\left(\tau_p\right)-x\left(\tau_j\right),\nu\left(x\left(\tau_p\right)\right)\right)\right|=\left|\left(x\left(\tau_j\right)-y,\nu\left(x\left(\tau_p\right)\right)\right)\right|$$

$$\leqslant \left|\left(x\left(\tau_j\right)-y, \nu\left(x\left(\tau_j\right)\right)\right)\right|+\left|\left(x\left(\tau_j\right)-y, \nu\left(x\left(\tau_p\right)\right)-\nu\left(x\left(\tau_j\right)\right)\right)\right| \leqslant M\left|y-x\left(\tau_p\right)\right|^\alpha R(n)$$

and

$$\begin{aligned}
\left|\ln \left(k\left|x\left(\tau_p\right)-y\right|\right)-\ln \left(k\left|x\left(\tau_p\right)-x\left(\tau_j\right)\right|\right)\right| &= \left|\ln \frac{\left|x\left(\tau_p\right)-x\left(\tau_j\right)\right|}{\left|x\left(\tau_p\right)-y\right|}\right| \\
&= \left|\ln \left(1+\frac{\left|x\left(\tau_p\right)-x\left(\tau_j\right)\right|-\left|x\left(\tau_p\right)-y\right|}{\left|x\left(\tau_p\right)-y\right|}\right)\right| \\
&\leqslant \left|\ln \left(1+\frac{\left|x\left(\tau_j\right)-y\right|}{\left|x\left(\tau_p\right)-y\right|}\right)\right| \leqslant M \frac{R(n)}{\left|x\left(\tau_p\right)-y\right|},
\end{aligned}$$

where $q \in \mathbb{N}$. Then it follows from inequalities (2.4) and (2.5) that

$$\left|\frac{\partial V\left(x\left(\tau_p\right), y\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}-\frac{\partial V\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}\right| \leqslant M\left((R(n))^\alpha\left|\ln \left|x\left(\tau_p\right)-y\right|\right|+\frac{R(n)}{\left|x\left(\tau_p\right)-y\right|}\right).$$

Also taking into consideration the inequality

$$\left|\frac{\partial V\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}-\frac{\partial V_n\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}\right| \leqslant M \frac{\left|\ln \left|x\left(\tau_p\right)-y\right|\right|}{n!}, \tag{3.1}$$

we obtain:

$$\begin{aligned}
\left|\frac{\partial V\left(x\left(\tau_p\right), y\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}-\frac{\partial V_n\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}\right| \leqslant M\Bigg(& (R(n))^\alpha\left|\ln \left|x\left(\tau_p\right)-y\right|\right|+\frac{R(n)}{\left|x\left(\tau_p\right)-y\right|} \\
& +\frac{\left|\ln \left|x\left(\tau_p\right)-y\right|\right|}{n!}\Bigg).
\end{aligned}$$

Finally we obtain that if $0<\alpha<1$, then

$$\begin{aligned}
\left|\delta_2^n\left(x\left(\tau_p\right)\right)\right| &\leqslant M\|\rho\|_{\infty}\left((R(n))^\alpha \int_{r(n)}^{\operatorname{diam} L}|\ln \tau| d \tau+R(n) \int_{r(n)}^{\operatorname{diam} L} \frac{d \tau}{\tau}+\frac{1}{n!} \int_{r(n)}^{\operatorname{diam} L}|\ln \tau| d \tau\right) \\
&\leqslant M\|\rho\|_{\infty}\left((R(n))^\alpha+\frac{1}{n!}\right),
\end{aligned}$$

and if $\alpha=1$, then

$$\left|\delta_2^n\left(x\left(\tau_p\right)\right)\right| \leqslant M\|\rho\|_{\infty}\left(R(n)|\ln R(n)|+\frac{1}{n!}\right).$$

Let $y \in L_j$ and $j \neq p$. Since in view of Lemma 3.1 and inequalities (2.6) and (3.1) we obviously have

$$\begin{aligned}
\left|\frac{\partial V_n\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}\right| &\leqslant \left|\frac{\partial V\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}\right| \\
&\quad+\left|\frac{\partial V\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}-\frac{\partial V_n\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}\right| \\
&\leqslant M\left(\left|\ln \left|x\left(\tau_p\right)-x\left(\tau_j\right)\right|\right|+\frac{1}{\left|x\left(\tau_p\right)-y\right|^{1-\alpha} n!}\right), \quad \forall n \in \mathrm{N},
\end{aligned} \tag{3.2}$$

then

$$\left|\delta_3^n\left(x\left(\tau_p\right)\right)\right| \leqslant 2 \omega(\rho, R(n)) \sum_{\substack{j=1 \\ j \neq p}}^n \int_{L_j}\left|\frac{\partial V_n\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}\right| d L_y$$

$$\leqslant 2\omega\left(\rho, R\left(n\right)\right) \int\limits_L \left| \frac{\partial V_n\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)} \right| dL_y \leqslant M\omega\left(\rho, R\left(n\right)\right).$$

Moreover, taking into consideration Lemma 3.1 and inequality (3.2) and

$$\left| \sqrt{\left(x_1'\left(t\right)\right)^2 + \left(x_2'\left(t\right)\right)^2} - \sqrt{\left(x_1'\left(\tau_j\right)\right)^2 + \left(x_2'\left(\tau_j\right)\right)^2} \right| \leqslant M\left(R\left(n\right)\right)^\alpha, \quad \forall t \in \left[t_{j-1}, t_j\right],$$

we obtain:

$$\left|\delta_4^n\left(x\left(\tau_p\right)\right)\right| \leqslant M\left\|\rho\right\|_\infty \left(R\left(n\right)\right)^\alpha \sum_{\substack{j=1\\ j\neq p}}^n \int\limits_{t_{j-1}}^{t_j} \left| \frac{\partial V_n\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)} \right| dt$$

$$\leqslant M\left\|\rho\right\|_\infty \left(R\left(n\right)\right)^\alpha \sum_{\substack{j=1\\ j\neq p}}^n \int\limits_{L_j} \left| \frac{\partial V_n\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)} \right| dL_y$$

$$\leqslant M\left\|\rho\right\|_\infty \left(R\left(n\right)\right)^\alpha \int\limits_L \left| \frac{\partial V_n\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)} \right| dL_y \leqslant M\left\|\rho\right\|_\infty \left(R\left(n\right)\right)^\alpha.$$

As a result, summing up the obtained estimates for the expressions $\delta_1^n\left(x\left(\tau_p\right)\right)$, $\delta_2^n\left(x\left(\tau_p\right)\right)$, $\delta_3^n\left(x\left(\tau_p\right)\right)$ and $\delta_4^n\left(x\left(\tau_p\right)\right)$ and taking into consideration relation $R\left(n\right) \sim \frac{1}{n}$, we see that if $0 < \alpha < 1$, then

$$\max_{p=\overline{1,n}} \left| \frac{\partial W_1\left(x\left(\tau_p\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)} - \left(\frac{\partial W_1}{\partial \nu}\right)^n \left(x\left(\tau_p\right)\right) \right| \leqslant M\left(\omega\left(\rho, 1/n\right) + \left\|\rho\right\|_\infty \frac{1}{n^\alpha}\right),$$

while if $\alpha = 1$, then

$$\max_{p=\overline{1,n}} \left| \frac{\partial W_1\left(x\left(\tau_p\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)} - \left(\frac{\partial W_1}{\partial \nu}\right)^n \left(x\left(\tau_p\right)\right) \right| \leqslant M\left(\omega\left(\rho, 1/n\right) + \left\|\rho\right\|_\infty \frac{\ln n}{n}\right).$$

Finally, summing up the constructed quadrature formulas for the integrals $\frac{\partial W_0(x)}{\partial \nu(x)}$ and $\frac{\partial W_1(x)}{\partial \nu(x)}$ in the nodes $x\left(\tau_l\right)$, $l = \overline{1, n}$, we complete the proof. $\square$

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