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ELLIPTIC DIFFERENTIAL-DIFFERENCE PROBLEMS IN HALF-SPACES: CASE OF SUMMABLE FUNCTIONS

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Abstract. We study the Dirichlet problem in the half-space for elliptic equations involving, apart of differential operators, the shift operators acting in tangential (spatial-like) variables, that is, in independent variables varying in entire real line. The boundary function in the problem is supposed to be summable, which in the classical case corresponds to the situation, in which only solutions with finite energy are possible.

We consider two principally different cases: the case, in which the studied equation involves superpositions of differential operators and the shift operators and the case, when it involves their sums, that is, it is an equation with nonlocal potentials.

For both types of problems we construct an integral representation of the solution to this problem in the sense of generalized functions and we prove that its infinitely smoothness in an open half-space (i.e., outside the boundary hyperplane) and tends uniformly to zero together with all its derivatives as a time-like variable tends to infinity; this time-like variable is a single independent variable varying on the positive half-axis. The rate of this decay is power-law; the degree is equal to the sum of the dimension of the space-like independent variable and the order of the derivative of the solution.

The most general current results are presented: shifts of independent variables are allowed in arbitrary (tangential) directions, and if there are several shifts, no conditions of commensurability are imposed on their values.

Thus, just as in the classical case, problems with summable boundary functions fundamentally differ from the previously studied problems with essentially bounded boundary functions: the latter, as previously established, admit solutions having no limit when a time-like variable tends to infinity, and the presence or absence of such a limit is determined by the Repnikov-Eidelman stabilization condition.

Keywords: elliptic differential-difference equations, problems in half-space, summable boundary functions.

Mathematics Subject Classification: 35R10, 35J25

1. Introduction

Traditionally, boundary value problems *in the half-space* are treated as natural for *non-stationary* equations: the only independent variable varying over the half-line is naturally treated as a time, while all other independent variables are considered as spatial. The data imposed on the boundary of the domain, that is, on the hyperplane orthogonal to the mentioned half-line, are treated respectively as initial data. However, for elliptic equations, well-posed problems are well-known in the half-space. Moreover, in many cases the *spatial* variable chosen in such way (that is, the equation remains the same and only its isotropy in the domain is

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violated) gain *time-like* properties, in particular, the resolving operator possesses a semi-group property and there is a stabilization of solution as the aforementioned time-like tends to infinity.

For a classical case of *differential* elliptic equations this phenomenon is known for at least six decades, see, for instance, [1], [2]. Rather recently it was found to be a feature of *differential-difference* equations, see [3], that is, for the case, when on an unknown function, apart of the differential operators, the shift operators act as well.

An interest to such, and wider, to *functional-difference* partial differential equations, appear nowadays in entire world starting from pioneering work [4]. This is due to numerous applications not covered by classical models of mathematical physics as well as due to pure theoretical reasons: a *nonlocal* nature of such equations produces principally new phenomena not arising in the classical case of differential equations. Various methods of studying, which proved to be effective in the theory of differential equations, turn out to be non-applicable; for instance, this concerns all methods based on the maximum principle, since in contrast to differential equations, the studied equation relates the values of the sought function at *different points*. Hence, one needs to develop qualitatively new methods, see [5]–[8] and the references therein.

Both in the differential and differential-difference case, the problems in the half-space, the problems in the half-space, both of elliptic and parabolic types, divide into two classes: problems with *bounded* boundary functions and problems *summable* boundary functions. This natural difference in the formulation of problem is principal: it generates solutions with principally different sets of properties. In particular, constant solutions are possible only for the problem of the first of these classes. On the other hand, only solutions with finite energy are possible for problems of second class. In other words, the necessary and sufficient Repnikov-Eidelman stabilization condition, according to which the solution can have a limit, which is, generally speaking, non-zero, and it also can have no such solution, holds only for the problems in the first class. The solutions of the problems in the second class always have a zero limit, and the study is mostly concentrated on the decay rate of the solution.

At present, the study of problems of the second class is a little behind. The aim of the present work is to systematically expose the results obtained by present days for the mentioned problems, that is, for problems with summable boundary data. The paper is organized as follows. The problems for equations containing the *sums* of differential operators and shift operators and for the equations containing *superpositions* are studied separately. Such approach is motivated by a nonlocal nature of the mentioned operators established in the previous studies of differential-difference equations of all types. In each of these two sections we first solve in details a model problem and then we expose the results for its maximal at present generalization.

2. Equations with superpositions of operators

In the half-space $\{(x,y)|\, x\in\mathbb{R}^n,\ y>0\}$ we consider the Dirichlet problem for the following model equation

$$u_{x_1x_1}(x,y)+au_{x_1x_1}(x_1+h,x_2,\dots,x_n,y)+\sum_{j=2}^{n}u_{x_jx_j}(x,y)+u_{yy}(x,y)=0, \tag{2.1}$$

where $|a|<1$ and h is an arbitrary real parameter.

Under the mentioned restriction, the following functions are well-defined in $\mathbb{R}^n$:

$$\varphi(\xi):=\varphi(\xi_1,\dots,\xi_n):=\sqrt{|\xi|^4+2a|\xi|^2\xi_1^2\cos h\xi_1+a^2\xi_1^4}\,, \tag{2.2}$$

$$G_1(\xi):=\sqrt{\frac{\varphi(\xi)+|\xi|^2+a\xi_1^2\cos h\xi_1}{2}}\,, \tag{2.3}$$

$$G_2(\xi):=\sqrt{\frac{\varphi(\xi)-|\xi|^2-a\xi_1^2\cos h\xi_1}{2}}\,. \tag{2.4}$$

The following statement holds true.

Lemma 2.1. *If $|a| < 1$, then the function*

$$\mathcal{E}(x, y) := \int\limits_{\mathbb{R}^n} e^{-yG_1(\xi)} \cos\left[x \cdot \xi - yG_2(\xi)\right] d\xi \tag{2.5}$$

is well-defined in the half-space $\mathbb{R}^n \times (0, +\infty)$ and in the classical sense it solves equation (2.1).

Proof. In order to prove the first statement of the lemma, the radical expression in (2.3) is estimated from below by $(1 - |a|)|\xi|^2$, while to prove its second statement functions (2.3) and (2.4) are represented as $\rho(\xi) \cos\theta(\xi)$ and $\rho(\xi) \sin\theta(\xi)$, respectively, where

$$\rho(\xi) = \left[\left(|\xi|^2 + a\xi_1^2 \cos h\xi_1\right)^2 + a^2\xi_1^4 \sin^2 h\xi_1\right]^{\frac{1}{4}}, \quad \theta(\xi) = \frac{1}{2} \arctan \frac{a\xi_1^2 \sin h\xi_1}{|\xi|^2 + a\xi_1^2 \cos h\xi_1}. \tag{2.6}$$

Taking into consideration the values of arc tangent, we conclude that $|\theta(\xi)| \leqslant \dfrac{\pi}{4}$ and hence, $\cos\theta(\xi) > 0$, and $\cos 2\theta(\xi) \geqslant 0$, this implies that

$$\cos\theta(\xi) = \sqrt{\frac{1 + \cos\left(\arctan \frac{a\xi_1^2 \sin h\xi_1}{|\xi|^2 + a\xi_1^2 \cos h\xi_1}\right)}{2}}.$$

Then, applying the formula, $\arctan x = \arccos \dfrac{1}{\sqrt{1 + x^2}}$, we obtain that

$$\cos\theta(\xi) = \frac{1}{\sqrt{2}} \left[1 + \frac{1}{\sqrt{1 + \frac{a^2\xi_1^4 \sin^2 h\xi_1}{\left(|\xi|^2 + a\xi_1^2 \cos h\xi_1\right)^2}}}\right]^{\frac{1}{2}},$$

and $\sin\theta(\xi)$ can be calculated similarly.

Now we can substitute function (2.5) into equation (2.1):

$$\begin{aligned}
\mathcal{E}_{x_j x_j}(x, y) &= -\int\limits_{\mathbb{R}^n} \xi_j^2 e^{-yG_1(\xi)} \cos\left[x \cdot \xi - yG_2(\xi)\right] d\xi, \ j = \overline{1, n}, \\
\mathcal{E}_{yy}(x, y) &= \int\limits_{\mathbb{R}^n} \left[G_1^2(\xi) - G_2^2(\xi)\right] e^{-yG_1(\xi)} \cos\left[x \cdot \xi - yG_2(\xi)\right] d\xi \\
&\quad - 2\int\limits_{\mathbb{R}^n} G_1(\xi) G_2(\xi) e^{-yG_1(\xi)} \sin\left[x \cdot \xi - yG_2(\xi)\right] d\xi.
\end{aligned} \tag{2.7}$$

In view of the non-negativity of $\cos 2\theta(\xi)$, by the identity

$$2G_1(\xi)G_2(\xi) = \rho^2(\xi) \sin 2\theta(\xi)$$

we get that

$$2G_1(\xi)G_2(\xi) = \frac{\rho^2(\xi) \tan 2\theta(\xi)}{\sqrt{1 + \tan^2 2\theta(\xi)}} = \frac{a\rho^2(\xi)\xi_1^2 \sin h\xi_1}{\sqrt{\left(|\xi|^2 + a\xi_1^2 \cos h\xi_1\right)^2 + \xi_1^4 \left(a \sin h\xi_1\right)^2}} = a\xi_1^2 \sin h\xi_1,$$

and

$$G_1^2(\xi) - G_2^2(\xi) = \rho^2(\xi)\cos 2\theta(\xi) = \rho^2(\xi)\sqrt{1 - \frac{\tan^2 2\theta(\xi)}{1+\tan^2 2\theta(\xi)}}$$
$$= \rho^2(\xi)\frac{1}{\sqrt{1+\tan^2 2\theta(\xi)}} = |\xi|^2 + a\xi_1^2 \cos h\xi_1.$$

This implies that

$$\sum_{j=1}^{n} \mathcal{E}_{x_j x_j}(x,y) + \mathcal{E}_{yy}(x,y) + a\mathcal{E}_{x_1x_1}(x_1+h_k, x_2, \dots, x_n, y)$$
$$= \int\limits_{\mathbb{R}^n} \left[-\sum_{j=1}^{n} \xi_j^2 + |\xi|^2 + a\xi_1^2 \cos h\xi_1 \right] e^{-yG_1(\xi)} \cos\left[x\cdot\xi - yG_2(\xi)\right] d\xi$$
$$- a\int\limits_{\mathbb{R}^n} \xi_1^2 \sin h\xi_1 e^{-yG_1(\xi)} \sin\left[x\cdot\xi - yG_2(\xi)\right] d\xi - a\int\limits_{\mathbb{R}^n} \xi_1^2 e^{-yG_1(\xi)} \cos\left[x\cdot\xi - yG_2(\xi) + h\xi_1\right] d\xi$$
$$= a\int\limits_{\mathbb{R}^n} \xi_1^2 \cos h\xi_1 e^{-yG_1(\xi)} \cos\left[x\cdot\xi - yG_2(\xi)\right] d\xi - a\int\limits_{\mathbb{R}^n} \xi_1^2 \sin h\xi_1 e^{-yG_1(\xi)} \sin\left[x\cdot\xi - yG_2(\xi)\right] d\xi$$
$$- a\int\limits_{\mathbb{R}^n} \xi_1^2 \cos h\xi_1 e^{-yG_1(\xi)} \cos\left[x\cdot\xi - yG_2(\xi)\right] d\xi + a\int\limits_{\mathbb{R}^n} \xi_1^2 \sin h\xi_1 e^{-yG_1(\xi)} \sin\left[x\cdot\xi - yG_2(\xi)\right] d\xi = 0.$$

The proof is complete. □

Now let $u_0 \in L_1(\mathbb{R}^n)$. Then the following statement holds.

Theorem 2.1. *If $u_0 \in L_1(\mathbb{R}^n)$ and $|a| < 1$, then the function*

$$u(x,y) = \int\limits_{\mathbb{R}^n} \mathcal{E}(x-\xi, y) u_0(\xi) d\xi \tag{2.8}$$

is infinitely differentiable in $\mathbb{R}^n \times (0,+\infty)$ and satisfies equation (2.1) in this half-space.

Proof. We take into consideration Lemma 2.1 and we majorate the function $\mathcal{E}(x,y)$ and its derivatives of arbitrary order:

$$\int\limits_{\mathbb{R}^n} |\xi|^m e^{-y|\xi|\sqrt{1-|a|}} d\xi = \frac{1}{(1-|a|)^{\frac{m+n}{2}} y^{m+n}} \int\limits_{\mathbb{R}^n} |\eta|^m e^{-|\eta|} d\eta = \frac{const}{y^{m+n}} \int\limits_0^\infty \rho^{m+n-1} e^{-\rho} d\rho = \frac{const}{y^{m+n}}.$$

The proof is complete. □

The found majorant gives also an asymptotic estimate for solution and all its derivatives.

Theorem 2.2. *If $u_0 \in L_1(\mathbb{R}^n)$, then solution (2.8) and each its derivative are infinitely differentiable in $\mathbb{R}^n \times (0,+\infty)$ and each of these functions tend to zero as $y \to +\infty$ uniformly in $x \in \mathbb{R}^n$.*

A maximal generalization of equation (2.1) achieved by present days, to the best of author's knowledge, is as follows, see [9]:

$$\sum_{j=1}^{n} u_{x_j x_j}(x,y) + u_{yy}(x,y) + \sum_{j=1}^{n} a_j u_{x_j x_j}(x+h_j, y) = 0, \tag{2.9}$$

where $h_j := (h_{j1}, \dots, h_{jn})$, $j = \overline{1,n}$, are arbitrary vectors in $\mathbb{R}^n$. In this case, we impose the following condition on the coefficients of the equation

$$a_0 := \max_{j=\overline{1,n}} |a_j| < 1, \tag{2.10}$$

while the functions G_1 and G_2 are defined as

$$G_1(\xi) = \rho(\xi)\cos\theta(\xi), \qquad G_2(\xi) = \rho(\xi)\sin\theta(\xi), \tag{2.11}$$

where

$$\begin{aligned} \rho(\xi) &= \left[\left(|\xi|^2 + \sum_{j=1}^{n} a_j\xi_j^2 \cos h_j\cdot\xi\right)^2 + \left(\sum_{j=1}^{n} a_j\xi_j^2 \sin h_j\cdot\xi\right)^2\right]^{\frac{1}{4}}, \\ \theta(\xi) &= \frac{1}{2}\arctan\frac{\sum\limits_{j=1}^{n} a_j\xi_j^2\sin h_j\cdot\xi}{|\xi|^2 + \sum\limits_{j=1}^{n} a_j\xi_j^2\cos h_j\cdot\xi}. \end{aligned} \tag{2.12}$$

In this case the statements of Theorems 2.1-2.2 hold also for equation (2.9).

3. Equations with sums of operators

In the half-plane $\left\{(x,y)\middle| x\in\mathbb{R}^n, y>0\right\}$ we consider a model equation

$$u_{x_1x_1}(x,y) - au(x_1+h, x_2, \dots, x_n, y) + \sum_{j=2}^{n} u_{x_jx_j}(x,y) + u_{yy}(x,y) = 0 \tag{3.1}$$

under the condition that

$$0 < a \leqslant \frac{2}{h^2}, \tag{3.2}$$

where h is an arbitrary real parameter.

In this case we introduce

$$\rho(\xi) = \left(|\xi|^4 + 2a|\xi|^2\cos h\xi_1 + a^2\right)^{\frac{1}{4}}, \qquad \theta(\xi) = \frac{1}{2}\arctan\frac{a\sin h\xi_1}{|\xi|^2 + a\cos h\xi_1}, \tag{3.3}$$

while the functions $G_{\left\{\frac{1}{2}\right\}}(\xi)$ and, respectively, $\mathcal{E}(x,y)$ are still defined by formulas (2.11) and (2.5), respectively.

In order to estimate the function $G_1(\xi)$ from below, we take into consideration that the values of the arc tangent are located in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Hence, $|\theta(\xi)| < \frac{\pi}{4}$, that is, $\cos\theta(\xi) > 0$ and $\cos 2\theta(\xi) > 0$. Therefore, $\cos\theta(\xi)$ can be represented in the form $\sqrt{\frac{1+\cos 2\theta(\xi)}{2}}$, while $\cos 2\theta(\xi)$ is represented as $\frac{1}{\sqrt{1+\tan^2 2\theta(\xi)}}$. Since

$$2\theta(\xi) = \arctan\frac{a\sin h\xi_1}{|\xi|^2 + a\cos h\xi_1},$$

the identity

$$\tan 2\theta(\xi) = \frac{a\sin h\xi_1}{|\xi|^2 + a\cos h\xi_1} \tag{3.4}$$

holds true. The positivity of the latter denominator is ensured by condition (3.2). Indeed, it is bounded from below by a function of one variable $f(\xi_1) := \xi_1^2 + a\cos h\xi_1$. Its derivative $f'(\xi_1)$ is equal to $2\xi_1 - ah\sin h\xi_1 = \xi_1\left(2 - ah^2\frac{\sin h\xi_1}{h\xi_1}\right)$, and hence, it is non-negative on

$[0,+\infty)$. Therefore, $f(\xi_1)$ is a non-decreasing on the positive semi-axis function. This is why it is bounded from below by the quantity $f(0)=a>0$ on this semi-axis. Finally, since $f(\xi_1)$ is an even function, the latter estimate remains true on the entire axis.

By this positivity we conclude that

$$\begin{aligned}\cos 2\theta(\xi) &= \left(1+\frac{a^2\sin^2 h\xi_1}{[|\xi|^2+a\cos h\xi_1]^2}\right)^{-\frac{1}{2}} = \sqrt{\frac{[|\xi|^2+a\cos h\xi_1]^2}{[|\xi|^2+a\cos h\xi_1]^2+a^2\sin^2 h\xi_1}} \\ &= \frac{|\xi|^2+a\cos h\xi_1}{\sqrt{|\xi|^4+2a|\xi|^2\cos h\xi_1+a^2}} = \frac{|\xi|^2+a\cos h\xi_1}{\rho^2(\xi)}.\end{aligned} \tag{3.5}$$

Then $\cos\theta(\xi)=\frac{1}{\sqrt{2}}\left[1+\frac{|\xi|^2+a\cos h\xi_1}{\rho^2(\xi)}\right]^{\frac{1}{2}}$ and, therefore,

$$G_1(\xi)=\rho(\xi)\frac{1}{\sqrt{2}}\left[1+\frac{|\xi|^2+a\cos h\xi_1}{\rho^2(\xi)}\right]^{\frac{1}{2}} = \sqrt{\frac{\rho^2(\xi)+|\xi|^2+a\cos h\xi_1}{2}}. \tag{3.6}$$

Since

$$\rho^4(\xi)=|\xi|^4+2a|\xi|^2\cos h\xi_1+a^2 \geqslant |\xi|^4-2a|\xi|^2+a^2=(|\xi|^2-a)^2,$$

the inequality $\rho^2(\xi)\geqslant|\xi|^2-a$ holds under the condition that $|\xi|\geqslant\sqrt{a}$. Thus, as $|\xi|\geqslant\sqrt{a}$, the radical expression in (3.6) is bounded from below by the function $\frac{|\xi|^2-a+|\xi|^2-a}{2}=|\xi|^2-a$.

Using the found estimate for the radical expression in (3.6), we conclude that for each positive y the absolute value of the integrand in (2.5) is majorated by an integrable function $e^{-y\sqrt{|\xi|^2-a}}$ in the exterior of the ball of radius $\sqrt{a}$ centered at the origin; inside this ball it is majorized by the unity. Thus, the function $\mathcal{E}(x,y)$ is well-defined in $\mathbb{R}^n\times(0,+\infty)$. Differentiating formally the function $\mathcal{E}(x,y)$ arbitrary many times in each variable under the integral, we obtain just additional integrand factors of at most polynomial growth in ξ, which have no singularities. The absolute convergence of the obtained integrals is justified in the same way as in the case of the function $\mathcal{E}(x,y)$, just the majorants are to be replaced by $a^{\frac{m}{2}}$ and $|\xi|^{\frac{m}{2}}e^{-y\sqrt{|\xi|^2-a}}$, respectively, here m is the order of the derivative. Therefore, the aforementioned formal differentiation under the integral is possible and all derivatives of the function $\mathcal{E}(x,y)$ are also well-defined in the half-space $\mathbb{R}^n\times(0,+\infty)$.

In view of identities (3.4)-(3.5), we obtain the relations

$$G_1^2(\xi)-G_2^2(\xi)=\rho^2(\xi)\left[\cos^2\theta(\xi)-\sin^2\theta(\xi)\right]=\rho^2(\xi)\cos 2\theta(\xi)=|\xi|^2+a\cos h\xi_1$$

and

$$2G_1(\xi)G_2(\xi)=2\rho^2(\xi)\cos^2\theta(\xi)\sin^2\theta(\xi)=\rho^2(\xi)\sin 2\theta(\xi)\rho^2(\xi)\tan 2\theta(\xi)\cos 2\theta(\xi)=a\sin h\xi_1.$$

Applying now identities (2.7), we find the Laplacian of function (2.5):

$$\sum_{j=1}^{n}\mathcal{E}_{x_jx_j}(x,y)+\mathcal{E}_{yy}(x,y)=\int\limits_{\mathbb{R}^n}\left(-\sum_{j=1}^{n}\xi_j^2+|\xi|^2+a\cos h\xi_1\right)e^{-yG_1(\xi)}\cos\left[x\cdot\xi-yG_2(\xi)\right]d\xi$$
$$-a\int\limits_{\mathbb{R}^n}\sin h\xi_1 e^{-yG_1(\xi)}\sin\left[x\cdot\xi-yG_2(\xi)\right]d\xi$$
$$=a\int\limits_{\mathbb{R}^n}e^{-yG_1(\xi)}\Big(\cos\left[x\cdot\xi-yG_2(\xi)\right]\cos h\xi_1-\sin\left[x\cdot\xi-yG_2(\xi)\right]\sin h\xi_1\Big)d\xi$$
$$=a\int\limits_{\mathbb{R}^n}e^{-yG_1(\xi)}\cos\left[(x\cdot\xi+h\xi_1)-yG_2(\xi)\right]d\xi$$
$$=a\int\limits_{\mathbb{R}^n}e^{-yG_1(\xi)}\cos\left[(x_1+h,x_2,\dots,x_n)\cdot\xi-yG_2(\xi)\right]d\xi=a\mathcal{E}(x_1+h,x',y),$$

where $x'=(x_2,\dots,x_n)\in\mathbb{R}^{n-1}$, and hence, under condition (3.2) the function (2.5) satisfies equation (3.1).

Now we employ the above found majorants of the integrand in (2.5) and of its derivatives in order to estimate function (2.5) and its derivatives:

$$\left|\int\limits_{\mathbb{R}^n}|\xi|^m e^{-yG_1(\xi)}\cos\left[x\cdot\xi-yG_2(\xi)\right]\,d\xi\right|=\left|\int\limits_{B(2\sqrt{a})}|\xi|^m e^{-yG_1(\xi)}d\xi+\int\limits_{\mathbb{R}^n\setminus B(2\sqrt{a})}|\xi|^m e^{-yG_1(\xi)}\,d\xi\right|$$
$$\leqslant\left|\int\limits_{B(2\sqrt{a})}|\xi|^m\,d\xi\right|+\left|\int\limits_{\mathbb{R}^n\setminus B(2\sqrt{a})}|\xi|^m e^{-y\sqrt{|\xi|^2-a}}\,d\xi\right|$$
$$=:C(a)+C(a,y),$$

where $B(r)$ is a ball of radius r centered at the origin. Thus, under condition (3.2), the statements of Theorems (2.1)–(2.2), in which $\rho(\xi)$ and $\theta(\xi)$ are given by formulas (3.3) hold also for equation (3.1).

To the best of the author's knowledge, a maximal generalization of equation (3.1) achieved by present days is as follows [10]:

$$\sum_{j=1}^{n}u_{x_jx_j}(x,y)-au(x+h,y)+u_{yy}(x,y)=0,\tag{3.7}$$

where h is an arbitrary vector in $\mathbb{R}^n$ obeying the inequality

$$0<a|h|^2\leqslant\frac{\pi^2}{4}.\tag{3.8}$$

For such equation the functions G_1 and G_2 and, respectively, the kernel $\mathcal{E}$ are still defined by relations (2.11), while the functions $\rho(\xi)$ and $\theta(\xi)$ are introduced as

$$\rho(\xi)=\left(|\xi|^4+2a|\xi|^2\cos h\cdot\xi+a^2\right)^{\frac{1}{4}},\qquad\theta(\xi)=\frac{1}{2}\arctan\frac{a\sin h\cdot\xi}{|\xi|^2+a\cos h\cdot\xi}.\tag{3.9}$$

If condition (3.8) is satisfied, then the statements of Theorems 2.1–2.2 are also true for equation (3.7).

4. Construction of Poisson kernel

We are going to show how to find the Poisson kernel $\mathcal{E}$ such that the constructed in this work solutions are convolutions with this kernel. As an example, we shall do this for model equation (2.1).

Together with this equation we consider the boundary condition

$$u\Big|_{y=0} = u_0(x), \qquad x \in \mathbb{R}^n. \tag{4.1}$$

We formally apply the Fourier transform in the n-dimensional variable x to problem (2.1), (4.1) and we obtain the following initial problem for an *ordinary* differential equation:

$$\frac{d^2\widehat{u}}{dy^2} = \left(|\xi|^2 + a\xi_1^2 e^{-ih\xi_1}\right)\widehat{u}, \qquad y \in (0, +\infty), \tag{4.2}$$

$$\widehat{u}(0;\xi) = \widehat{u}_0(\xi). \tag{4.3}$$

This is not a Cauchy problem: the equation is of second order but there is just a single initial condition.

The characteristic equation of the obtained linear second order ordinary differential equation depending on an n-dimensional parameter ξ possesses two roots:

$$\pm\sqrt{|\xi|^2 + a\xi_1^2 e^{-ih\xi_1}} = \pm\sqrt{|\xi|^2 + a\xi_1^2 \cos h\xi_1 - ai\xi_1^2 \sin h\xi_1} = \pm\rho(\cos\theta + i\sin\theta),$$

where $\rho(\xi)$ and $\theta(\xi)$ are determined by relations (2.6).

We solve problem (4.2)-(4.3) choosing appropriately a "free" arbitrary constant, which arises since the number of initial condition is less than the order of the equation. Then we formally apply the inverse Fourier transform to the obtained solution and this gives a convolution of the boundary function with the function

$$e^{-y\rho(\xi)\cos\theta(\xi)}\cos[x\cdot\xi - y\rho(\xi)\sin\theta(\xi)],$$

that is, exactly, with the integrand in integral (2.5).

5. Fulfillment of boundary condition

In order to show that function (2.8) has the boundary value $u_0(x)$ on the hyperplane $\{y = 0\}$ in the sense of generalized function, we employ the same scheme as in [11, Rem. 2]. Namely, the boundary condition is treated in the Gelfand-Shilov sense, see [12, Sect. 10], a solution is sought in the class of generalized functions of an n-dimensional variable x depending on the real parameter y, twice differentiable in this parameter on the positive semi-axis and continuous in it at the origin, see, for instance, [13, Sect. 9, Item 5]. Thus, outside the boundary hyperplane the constructed solution is smooth (classical) and at the same time boundary condition (4.1) is treated as a limiting relation $u(\cdot, y) \to u_0$ in the topology of generalized functions of the variable x as a real parameter y tends to zero from the right.

Thus, the following statements hold.

Theorem 5.1. *If* $u_0 \in L_1(\mathbb{R}^n)$ *and Condition* (2.10) *is satisfied, then function* (2.8)*, where the functions* $G_1(\xi)$ *and* $G_2(\xi)$ *are defined by identities* (2.11)*, while the functions* $\rho(\xi)$ *and* $\theta(\xi)$ *are defined by identities* (2.12)*, satisfies problem* (2.9), (4.1) *in the sense of generalized functions in the Gelfand-Shilov sense, is infinitely differentiable in the half-space* $\mathbb{R}^n \times (0, +\infty)$ *and together with each its derivative it tends to zero as* $y \to +\infty$ *uniformly in* $x \in \mathbb{R}^n$.

Theorem 5.2. *If* $u_0 \in L_1(\mathbb{R}^n)$ *and Condition* (3.8)*, is satisfied, then function* (2.8)*, where the functions* $G_1(\xi)$ *and* $G_2(\xi)$ *are defined by identities* (2.11)*, while the functions* $\rho(\xi)$ *and* $\theta(\xi)$ *are defined by identities* (3.9)*, satisfies problem* (3.7),(4.1) *in the sense of generalized functions in the Gelfand-Shilov sense, is infinitely differentiable in the half-space* $\mathbb{R}^n \times (0, +\infty)$ *and together with each its derivative it tends to zero as* $y \to +\infty$ *uniformly in* $x \in \mathbb{R}^n$.

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