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ON GELFAND-SHILOV SPACES

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Abstract. In this work we follow the scheme of constructing of Gelfand-Shilov spaces S_α and S^β by means of some family of separately radial weight functions in $\mathbb{R}^n$ and define two spaces of rapidly decreasing infinitely differentiable functions in $\mathbb{R}^n$. One of them, namely, the space $\mathcal{S}_{\mathcal{M}}$ is an inductive limit of countable-normed spaces

$$\mathcal{S}_{\mathcal{M}_\nu} = \left\{ f \in C^\infty(\mathbb{R}^n) : \|f\|_{m,\nu} = \sup_{\substack{x\in\mathbb{R}^n, \beta\in\mathbb{Z}_+^n, \\ \alpha\in\mathbb{Z}_+^n : |\alpha|\leq m}} \frac{|x^\beta (D^\alpha f)(x)|}{\mathcal{M}_\nu(\beta)} < \infty, \, m \in \mathbb{Z}_+ \right\}.$$

Similarly, starting with the normed spaces

$$\mathcal{S}_m^{\mathcal{M}_\nu} = \left\{ f \in C^\infty(\mathbb{R}^n) : \rho_{m,\nu}(f) = \sup_{x\in\mathbb{R}^n, \alpha\in\mathbb{Z}_+^n} \frac{(1+\|x\|)^m |(D^\alpha f)(x)|}{\mathcal{M}_\nu(\alpha)} < \infty \right\}$$

we introduce the space $\mathcal{S}^{\mathcal{M}}$. We show that under certain natural conditions on weight functions the Fourier transform establishes an isomorphism between spaces $\mathcal{S}_{\mathcal{M}}$ and $\mathcal{S}^{\mathcal{M}}$.

Keywords: Gelfand-Shilov spaces, Fourier transform, convex functions.

Mathematics Subject Classification: 46F05, 46A13, 42B10

Introduction

In the mid-1950s, families of S-type spaces of infinitely differentiable functions in $\mathbb{R}^n$ were introduced, which, along with the Schwarz space, became one of the central objects of the theory of generalized functions and the theory of partial differential equations and they had significant applications in the theory of pseudodifferential operators and time-frequency analysis. Their study was initiated in works by G.E. Shilova [1], I.M. Gelfand and G.E. Shilov [2]-[4]. They characterized S-type spaces in terms of the Fourier transform of functions and then they applied the resulting description to study the uniqueness of the Cauchy problem for partial differential equations and their systems.

An essential development the theory of spaces of S-type was done in works by M.A. Soloviev in studying problems of nonlocal field theory. In particular, he obtained [5, Sec. 4] a description of the image of the space $S_b(\mathbb{R}^n)$ under the Fourier transform; this space consists of the functions $f \in C^\infty(\mathbb{R}^n)$ obeying the inequalities

$$|x^\beta (D^\alpha f)(x)| \leq C_\alpha \mu^{|\beta|} b_{|\beta|}, \quad x \in \mathbb{R}^n, \quad \beta \in \mathbb{Z}_+^n,$$

for some $C > 0$ and $\mu > 0$ depending on f and $\alpha \in \mathbb{Z}_+^n$, where, as usually, for the multi-index $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n$, $|\beta| = \beta_1 + \dots + \beta_n$, under the condition that a monotonically non-decreasing sequence $(b_k)_{k=0}^\infty$ of numbers $b_k > 0$ satisfy the condition: there exist numbers $B > 0$

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and $h > 0$ such that

$$b_{k+1} \le Bh^k b_k, \qquad k = 0, 1, \dots$$

One of the aims of the present paper is to generalize this result to a wider class of Gelfand-Shilov spaces of such type.

1. Spaces $\mathcal{S}_{\mathcal{M}}$ and $\mathcal{S}^{\mathcal{M}}$. Main results.

Let $\mathcal{M} = \{\mathcal{M}_\nu\}_{\nu=1}^\infty$ be an arbitrary family of functions $\mathcal{M}_\nu : \mathbb{Z}_+^n \to \mathbb{R}$ such that for each $\nu \in \mathbb{N}$:

i_1) there exist numbers $a_1 = a_1(\nu) > 0$, $a_2 = a_2(\nu) > 0$ such that

$$\mathcal{M}_\nu(\alpha) \ge a_1 a_2^{|\alpha|}, \qquad \alpha \in \mathbb{Z}_+^n;$$

i_2) $\lim\limits_{|\alpha|\to+\infty} \dfrac{\mathcal{M}_{\nu+1}(\alpha)}{\mathcal{M}_\nu(\alpha)} = +\infty.$

We define the space $S_{\mathcal{M}}$ following the scheme of constructing the Gelfand-Shilov space S_α [3, Ch. 4]. For each $\nu \in \mathbb{N}$ and $m \in \mathbb{Z}_+$ let

$$\mathcal{S}_{m,\mathcal{M}_\nu} = \left\{ f \in C^m(\mathbb{R}^n) : \|f\|_{m,\nu} = \sup_{\substack{x\in\mathbb{R}^n, \beta\in\mathbb{Z}_+^n, \\ \alpha\in\mathbb{Z}_+^n : |\alpha|\le m}} \frac{|x^\beta (D^\alpha f)(x)|}{\mathcal{M}_\nu(\beta)} < \infty \right\}.$$

We also denote $\mathcal{S}_{\mathcal{M}_\nu} := \bigcap\limits_{m=0}^\infty \mathcal{S}_{m,\mathcal{M}_\nu}$. The class $\mathcal{S}_{\mathcal{M}_\nu}$ is non-empty: it contains compactly supported functions with the support in $[-a_2, a_2]^n$. We equip $\mathcal{S}_{\mathcal{M}_\nu}$ with the topology defined by the family of norms $\|\cdot\|_{m,\nu}$ ($m \in \mathbb{Z}_+$). By Condition i_2), the space $\mathcal{S}_{\mathcal{M}_\nu}$ is continuously embedded into $\mathcal{S}_{\mathcal{M}_{\nu+1}}$ for each $\nu \in \mathbb{N}$. We let $\mathcal{S}_{\mathcal{M}} := \bigcup\limits_{\nu=1}^\infty \mathcal{S}_{\mathcal{M}_\nu}$. Being equipped with usual summation and multiplication by complex numbers, $\mathcal{S}_{\mathcal{M}}$ is a linear space. We equip $\mathcal{S}_{\mathcal{M}}$ with the topology of inner inductive limit [6] of the spaces $\mathcal{S}_{\mathcal{M}_\nu}$.

Let us define the space $\mathcal{S}^{\mathcal{M}}$. By $\nu \in \mathbb{N}$, $m \in \mathbb{Z}_+$, we introduce the space

$$\mathcal{S}_m^{\mathcal{M}_\nu} = \left\{ f \in C^\infty(\mathbb{R}^n) : \rho_{m,\nu}(f) = \sup_{x\in\mathbb{R}^n, \alpha\in\mathbb{Z}_+^n} \frac{(1+\|x\|)^m |(D^\alpha f)(x)|}{\mathcal{M}_\nu(\alpha)} < \infty \right\}.$$

An equivalent topology in $\mathcal{S}_m^{\mathcal{M}_\nu}$ can be introduced by means of the norms

$$q_{m,\nu}(f) = \sup_{\substack{x\in\mathbb{R}^n, \alpha\in\mathbb{Z}_+^n, \\ \beta\in\mathbb{Z}_+^n : |\beta|\le m}} \frac{|x^\beta (D^\alpha f)(x)|}{\mathcal{M}_\nu(\alpha)}.$$

It is obvious that the normed space $\mathcal{S}_{m+1}^{\mathcal{M}_\nu}$ is continuously embedded into $\mathcal{S}_m^{\mathcal{M}_\nu}$. Let $\mathcal{S}^{\mathcal{M}_\nu} := \bigcap\limits_{m=0}^\infty \mathcal{S}_m^{\mathcal{M}_\nu}$. We equip the space $\mathcal{S}^{\mathcal{M}_\nu}$ by the topology defined by the family of the norms $\rho_{m,\nu}$ ($m \in \mathbb{Z}_+$). In view of Condition i_2), the space $\mathcal{S}^{\mathcal{M}_\nu}$ is continuously embedded into $\mathcal{S}^{\mathcal{M}_{\nu+1}}$. We let $\mathcal{S}^{\mathcal{M}} := \bigcup\limits_{\nu=1}^\infty \mathcal{S}^{\mathcal{M}_\nu}$. In $\mathcal{S}^{\mathcal{M}}$ we introduce a topology of inner inductive limit of the spaces $\mathcal{S}^{\mathcal{M}_\nu}$. The space $\mathcal{S}^{\mathcal{M}}$ is constructed by analogy with the Gelfand-Shilov space S^β [3, Ch. 4].

We shall employ the following definition of the Fourier transform $\hat{f}$ of a function $f \in S(\mathbb{R}^n)$:

$$\hat{f}(x) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(\xi) e^{i\langle x,\xi\rangle}\, d\xi, \quad x \in \mathbb{R}^n.$$

The following theorem holds true.

Theorem 1.1. *Let a family $\mathcal{M}$ be such that for each $\nu \in \mathbb{N}$:*
i_3) *there exists a number $d_\nu > 0$ such that for all $\alpha \in \mathbb{Z}_+^n$, $\beta \in \mathbb{Z}_+^n \cap [0,1]^n$*

$$\mathcal{M}_\nu(\alpha + \beta) \leq d_\nu \mathcal{M}_{\nu+1}(\alpha);$$

i_4) *for each $m \in \mathbb{N}$ there exists a number $d_{\nu,m} > 0$ such that*

$$\mathcal{M}_{\nu+1}(\alpha) \geq d_{\nu,m} \mathcal{M}_\nu(\alpha) \prod_{k=1}^{n} (1 + \alpha_k)^m, \qquad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n.$$

Then the mapping $\mathcal{F} : f \in S_{\mathcal{M}} \to \hat{f}$ is an isomorphism between the spaces $\mathcal{S}_{\mathcal{M}}$ and $\mathcal{S}^{\mathcal{M}}$.

Corollary 1.1. *Under the assumptions of Theorem 1.1 the Fourier transform is an isomorphism between the spaces $\mathcal{S}^{\mathcal{M}}$ and $\mathcal{S}_{\mathcal{M}}$.*

Remark 1.1. *If $(b_k)_{k=0}^\infty$ is a monotonically non-decreasing sequence of numbers $b_k > 0$ such that for some $B > 0$ and $h > 1$, $b_{k+1} \leq Bh^k b_k$ for all $k \in \mathbb{Z}_+$, then the family $\{h^{\nu n |\alpha|} b_{|\alpha|}\}_{\nu=1}^\infty$ satisfies Conditions $i_1) - -i_4)$. In this case the space $\mathcal{S}_{\mathcal{M}}$ coincides with the space $S_b(\mathbb{R}^n)$.*

Remark 1.2. *If a monotonically non-decreasing sequence $(b_k)_{k=0}^\infty$ of numbers $b_k > 0$ is such that $\lim\limits_{k\to\infty} \left(\frac{b_{k+1}}{b_k}\right)^{\frac{1}{k}} = 1$, then the family $\{(\sigma - 2^{-\nu})^{|\alpha|} b_{|\alpha|}\}_{\nu=1}^\infty$ with $\sigma > 0$ satisfies Conditions $i_1)-$ $-i_4)$.*

Let $\mathcal{H}$ be an arbitrary family of non-negative functions h_ν in $\mathbb{R}^n$ such that for each $\nu \in \mathbb{N}$:
H_1) $h_\nu(x) = h_\nu(|x_1|, \ldots, |x_n|)$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$;
H_2) there exist numbers $Q_1 = Q_1(\nu) > 0$, $Q_2 = Q_2(\nu) > 0$ such that

$$h_\nu(x) \leq \sum_{1 \leq j \leq n : x_j \neq 0} x_j \ln \frac{x_j}{Q_1} + Q_2, \qquad x = (x_1, \ldots, x_n) \in [0, \infty)^n;$$

H_3) $\lim\limits_{x\to\infty} (h_\nu(x) - h_{\nu+1}(x)) = +\infty$.

We observe that the functions $\mathcal{M}_\nu(\alpha) = \alpha! e^{-h_\nu(\alpha)}$, $\alpha \in \mathbb{Z}_+^n$, where $h_\nu \in \mathcal{H}$, satisfy Conditions i_1) and i_2) imposed for the functions of family $\mathcal{M}$. Thus, if $\mathcal{M} = \{\alpha! e^{-h_\nu(\alpha)}\}_{\nu \in \mathbb{N}}$, then the space $\mathcal{S}_{\mathcal{M}}$ consists of the functions $f \in C^\infty(\mathbb{R}^n)$, for which for some $\nu \in \mathbb{N}$ and for each $\alpha \in \mathbb{Z}_+^n$ there exists a number $K_\alpha > 0$ such that

$$|x^\beta (D^\alpha f)(x)| \leq K_\alpha \beta! e^{-h_\nu(\beta)}, \qquad x \in \mathbb{R}^n, \qquad \beta \in \mathbb{Z}_+^n,$$

while the space $\mathcal{S}^{\mathcal{M}}$ consists of the functions $f \in C^\infty(\mathbb{R}^n)$, for which for some $\nu \in \mathbb{N}$ and for each $\beta \in \mathbb{Z}_+^n$ there exists a number $L_\beta > 0$ such that

$$|x^\beta (D^\alpha f)(x)| \leq L_\beta \alpha! e^{-h_\nu(\alpha)}, \qquad x \in \mathbb{R}^n,$$

for all $\alpha \in \mathbb{Z}_+^n$.

In order to select a particular case of the family $\mathcal{M}$, we shall denote the space $\mathcal{S}_{\mathcal{M}}$ by $\mathbb{S}_{\mathcal{H}}$, the space $\mathcal{S}^{\mathcal{M}_\nu}$ is denoted by $\mathbb{S}(h_\nu)$, and the space $\mathcal{S}^{\mathcal{M}}$ is denoted by $\mathbb{S}^{\mathcal{H}}$.

Then Theorem 1.1 implies one more corollary.

Corollary 1.2. *Let the family $\mathcal{M}$ consist of the functions $\mathcal{M}_\nu(\alpha) = \alpha! e^{-h_\nu(\alpha)}$, $\alpha \in \mathbb{Z}_+^n$, where the functions $h_\nu \in \mathcal{H}$ satisfy additional conditions:*
H_4) *for each $\nu \in \mathbb{N}$ there exists a number $\tau_\nu > 0$ such that*

$$h_\nu(x + y) - h_{\nu+1}(x) \geq \sum_{k=1}^{n} \ln(1 + x_k) - \tau_\nu$$

for all $x = (x_1, \ldots, x_n) \in [0, \infty)^n$, $y \in [0,1]^n$;

H_5) for all $\nu, m \in \mathbb{N}$ there exists a number $\tau_{\nu,m} > 0$ such that

$$h_\nu(x) - h_{\nu+1}(x) \geq m \sum_{k=1}^{n} \ln(1 + x_k) - \tau_{\nu,m}$$

for all $x = (x_1, \ldots, x_n) \in [0, \infty)^n$, $y \in [0, 1]^n$.

Then the mapping $\mathcal{F} : f \in \mathbb{S}_{\mathcal{H}} \to \hat{f}$ is an isomorphism between the spaces $\mathbb{S}_{\mathcal{H}}$ and $\mathbb{S}^{\mathcal{H}}$.

Indeed, Condition H_4) ensures the validity of Condition i_3), while Condition H_5) guarantees Condition i_4).

Corollary 1.3. *Let the family $\mathcal{M}$ consist of the functions $\mathcal{M}_\nu(\alpha) = \alpha! e^{-h_\nu(\alpha)}$, $\alpha \in \mathbb{Z}_+^n$, where non-decreasing in each variable on $[0, \infty)^n$ functions $h_\nu \in \mathcal{H}$ satisfy Condition H_5).*

Then the mapping $\mathcal{F} : f \in \mathbb{S}_{\mathcal{H}} \to \hat{f}$ is an isomorphism between the spaces $\mathbb{S}_{\mathcal{H}}$ and $\mathbb{S}^{\mathcal{H}}$.

It is interesting to consider the case, when all functions in the family $\mathcal{H}$ obey the condition

H_6) $\lim\limits_{x\to\infty} \dfrac{h_\nu(x)}{\|x\|} = +\infty$ ($\|x\|$ is the Euclidean norm of $x \in \mathbb{R}^n$).

The matter is that in this case, whatever a function f is $\mathbb{S}^{\mathcal{H}}$, for each $\varepsilon > 0$ there exists a number $c_\varepsilon(f) > 0$ such that

$$|(D^\alpha f)(x)| \leq c_\varepsilon(f) \varepsilon^{|\alpha|} \alpha!, \qquad x \in \mathbb{R}^n, \qquad \alpha \in \mathbb{Z}_+^n,$$

and therefore, f admits a unique continuation to an entire function in $\mathbb{C}^n$. We denote this continuation by F_f, while by $\mathcal{A}$ we denote the mapping $f \in \mathbb{S}^{\mathcal{H}} \to F_f$. In a natural way there arises a problem on describing the image $\mathbb{S}^{\mathcal{H}}$ under the mapping $\mathcal{A}$. Its solution is obtained under additional conditions for $\mathcal{H}$, see Theorem 1.2. Let us introduce notation and definitions involved in the formulation and proof of Theorem 1.2. For an arbitrary function $g : \mathbb{R}^n \to (-\infty, +\infty)$ such that $\lim\limits_{x\to\infty} \dfrac{g(x)}{\|x\|} = +\infty$ by g^* and $\tilde{g}$ we denote functions defined on $\mathbb{R}^n$ by the rule:

$$g^*(x) = \sup_{\alpha \in \mathbb{Z}^n} (\langle \alpha, x\rangle - g(\alpha)), \quad x \in \mathbb{R}^n,$$
$$\tilde{g}(x) = \sup_{y \in \mathbb{R}^n} (\langle x, y\rangle - g(y)), \qquad x \in \mathbb{R}^n.$$

The function $\tilde{g}$ is called the Young-Fenchel transform of the function g [7]. Now for each $\nu \in \mathbb{N}$ we define a function φ_ν in $\mathbb{R}^n$ letting

$$\varphi_\nu(x) = h_\nu^*(\ln^+ |x_1|, \ldots, \ln^+ |x_n|), \qquad x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

where $\ln^+ t = 0$ as $t \in [0, 1)$ and $\ln^+ t = \ln t$ as $t \in [1, \infty)$. Since a convex in $\mathbb{R}^n$ function h_ν^* take finite value, then it is continuous in $\mathbb{R}^n$ [8, Sec. 11]. Thus, the function φ_ν is continuous in $\mathbb{R}^n$. It is obvious that its restriction on $[0, \infty)^n$ does not decrease in each variable. In view of Condition H_2) for some $Q_3 = Q_3(\nu) > 0$ the inequality

$$\varphi_\nu(x) \geq \frac{Q_1}{e} \sum_{k=1}^{n} |x_k| - Q_3, \qquad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

holds true. Owing to Conditions H_6) and H_3),

$$\lim_{x\to\infty} (h_{\nu+1}^*(x) - h_\nu^*(x)) = +\infty.$$

Therefore,

$$\lim_{x\to\infty} (\varphi_{\nu+1}(x) - \varphi_\nu(x)) = +\infty. \tag{1.1}$$

Then, for arbitrary $\nu \in \mathbb{N}$ and $m \in \mathbb{Z}_+$ we introduce the space

$$\mathcal{P}_m(\varphi_\nu) = \left\{ f \in H(\mathbb{C}^n) : p_{\nu,m}(f) = \sup_{z \in \mathbb{C}^n} \frac{|f(z)|(1+\|z\|)^m}{e^{\varphi_\nu(Im\, z)}} < \infty \right\}.$$

It is obvious that the space $\mathcal{P}_{m+1}(\varphi_\nu)$ is continuously embedded into $\mathcal{P}_m(\varphi_\nu)$. Let $\mathcal{P}(\varphi_\nu)$ be the intersection of the spaces $\mathcal{P}_m(\varphi_\nu)$. We equip $\mathcal{P}(\varphi_\nu)$ with a topology of projective limit of the spaces $\mathcal{P}_m(\varphi_\nu)$. By (1.1) the space $\mathcal{P}(\varphi_\nu)$ is continuously embedded into $\mathcal{P}(\varphi_{\nu+1})$. We denote the family $\{\varphi_\nu\}_{\nu=1}^\infty$ by Φ. Let $\mathcal{P}(\Phi) := \bigcup\limits_{\nu=1}^{\infty} \mathcal{P}(\varphi_\nu)$. We equip $\mathcal{P}(\Phi)$ by the topology of interior inductive limit of the spaces $\mathcal{P}(\varphi_\nu)$. In Section 4 we prove the following theorem.

Theorem 1.2. *Let the functions of the family $\mathcal{H}$ be convex and apart of Condition H_6) satisfy the conditions:*

H_7) for each $a > 0$ there exists a number $l_{\nu,a} > 0$ such that

$$h_{\nu+1}(x+y) \le h_\nu(x) + l_{\nu,a}, \qquad x \in [0,\infty)^n, \qquad y \in [0,a]^n;$$

H_8) for each $\nu \in \mathbb{N}$ there exists a number $s \in \mathbb{N}$ such that

$$\sum_{|\alpha| \ge 0} e^{h_{\nu+s}(\alpha) - h_\nu(\alpha)} < \infty.$$

Then the mapping $\mathcal{A}$ is an isomorphism between the spaces $\mathbb{S}^{\mathcal{H}}$ and $\mathcal{P}(\Phi)$.

By these two theorems the following statement holds true.

Theorem 1.3. *Let the functions of the family $\mathcal{H}$ be convex and satisfy Conditions $H_5)-H_7$). Then the mapping $\mathcal{AF}$ is an isomorphism between the spaces $\mathbb{S}_{\mathcal{H}}$ and $\mathcal{P}(\Phi)$.*

2. Auxiliary result

In the proof of Theorem 1.2 we shall need a corollary from the following statement.

Proposition 2.1. *Let the functions of the family $\mathcal{H}$ satisfy Conditions H_6) and H_7), while $m \in \mathbb{N}$ is arbitrary and $\tilde{m} = (m, \dots, m) \in \mathbb{N}^n$. Then for each $\nu \in \mathbb{N}$*

$$h^*_{\nu+1}(x) \ge h^*_\nu(x) + \langle x, \tilde{m}\rangle - l_{\nu,m}, \quad x \in \mathbb{R}^n_+,$$

where $l_{\nu,m}$ is the same as in Condition H_7).

Proof. Let $m \in \mathbb{N}$ and $x \in \mathbb{R}^n_+$. Then

$$\begin{aligned} h^*_{\nu+1}(x) &= \sup_{\alpha \in \mathbb{Z}^n} (\langle x, \alpha\rangle - h_{\nu+1}(\alpha)) = \sup_{\alpha \in \mathbb{Z}^n_+} (\langle x, \alpha\rangle - h_{\nu+1}(\alpha)) \\ &\ge \sup_{\alpha \ge \tilde{m}} (\langle x, \alpha\rangle - h_{\nu+1}(\alpha)) = \sup_{\alpha \in \mathbb{Z}^n_+} (\langle x, \alpha + \tilde{m}\rangle - h_{\nu+1}(\alpha + \tilde{m})). \end{aligned}$$

Employing Condition H_7) on $\mathcal{H}$, we then have

$$\begin{aligned} h^*_{\nu+1}(x) &\ge \langle x, \tilde{m}\rangle + \sup_{\alpha \in \mathbb{Z}^n_+} (\langle x, \alpha\rangle - h_\nu(\alpha)) - l_{\nu,m} \\ &= \langle x, \tilde{m}\rangle + \sup_{\alpha \in \mathbb{Z}^n} (\langle x, \alpha\rangle - h_\nu(\alpha)) - l_{\nu,m} = h^*_\nu(x) + \langle x, \tilde{m}\rangle - l_{\nu,m}. \end{aligned}$$

The proof is complete. □

Under the assumptions of Proposition 2.1 the following corollary holds.

Corollary 2.1. *For all $\nu, m \in \mathbb{N}$*

$$\varphi_\nu(x) + m \ln(1 + \|x\|) \le \varphi_{\nu+1}(x) + b_{\nu,m} > 0, \quad x \in \mathbb{R}^n,$$

where $b_{\nu,m} = l_{\nu,m} + 2mn \ln 2$.

3. Proof of Theorem 1.1

Let us show first that the mapping $\mathcal{F}$ acts from $\mathcal{S}_{\mathcal{M}}$ into $\mathcal{S}^{\mathcal{M}}$. Let $g \in \mathcal{S}_{\mathcal{M}}$. Then $g \in \mathcal{S}_{\mathcal{M}_\nu}$ for some $\nu \in \mathbb{N}$. This is why whatever $m \in \mathbb{Z}_+$ is, for all $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \le m$, $\mu \in \mathbb{Z}_+^n$, $x \in \mathbb{R}^n$ the inequality holds:

$$|x^\mu (D^\gamma g)(x)| \le \|g\|_{m,\nu} \mathcal{M}_\nu(\mu). \tag{3.1}$$

Let us show that $\hat{g} \in \mathcal{S}^{\mathcal{M}}$. Let $\xi \in \mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n$ be arbitrary. We denote $\kappa_s := \min(\alpha_s, \beta_s)$ for $s = 1, \dots, n$ and $\kappa := (\kappa_1, \dots, \kappa_n)$. Since

$$(i\xi)^\beta (D^\alpha \hat{g})(\xi) = \frac{(-1)^{|\beta|}}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}_+^n : j \le \kappa} C_\beta^j (D^{\beta - j} g)(x) (D^j (ix)^\alpha) e^{i\langle x, \xi\rangle}\, dx,$$

then

$$|\xi^\beta (D^\alpha \hat{g})(\xi)| \le \frac{1}{(\sqrt{2\pi})^n} \sum_{j \in \mathbb{Z}_+^n : j \le \kappa} C_\beta^j \int\limits_{\mathbb{R}^n} |(D^{\beta - j} g)(x)| |D^j (x^\alpha)|\, dx. \tag{3.2}$$

According to [5], if $u \in S(\mathbb{R}^n)$, then for all $\mu, j \in \mathbb{Z}_+^n$ the inequality holds:

$$\int\limits_{\mathbb{R}^n} |D^j (x^\mu)| |u(x)|\, dx \le \sqrt{2} \int\limits_{\mathbb{R}^n} |x^\mu| |(D^j u)(x)|\, dx. \tag{3.3}$$

Employing this inequality, by (3.2) we obtain:

$$|\xi^\beta (D^\alpha \hat{g})(\xi)| \le \frac{\sqrt{2}}{(\sqrt{2\pi})^n} \sum_{j \in \mathbb{Z}_+^n : j \le \kappa} C_\beta^j \int\limits_{\mathbb{R}^n} |x^\alpha (D^\beta g)(x)|\, dx. \tag{3.4}$$

We continue estimate 3.4) following [5]. Namely,

1) we represent $\int\limits_{\mathbb{R}^n} |x^\alpha (D^\beta g)(x)|\, dx$ as a sum of 2^n integrals over non-intersecting sets $\mathbb{R}^n$ described by n inequalities of form $|x_k| \le 1$ or $|x_k| > 1$;

2) in the integrals over the sets, in description of which the inequality $|x_k| > 1$ is involved, we multiply and divide the integrand by x_k^2.

Then by (3.4), employing inequality (3.1), we obtain that

$$|\xi^\beta (D^\alpha \hat{g})(\xi)| \le \frac{(\sqrt{2})^{3n+1}}{(\sqrt{\pi})^n} 2^{|\beta|} \|g\|_{|\beta|,\nu} \sup_{\substack{\omega = (\omega_1, \dots, \omega_n) \in \mathbb{Z}_+^n : \\ \omega_j \le 2, j = 1, \dots, n}} \mathcal{M}_\nu(\alpha + \omega).$$

Then, owing to Condition i_3) on $\mathcal{M}$, we have

$$|\xi^\beta (D^\alpha \hat{g})(\xi)| \le C_1 \|g\|_{|\beta|,\nu} 2^{|\beta|} \mathcal{M}_{\nu+2}(\alpha),$$

where $C_1 = \frac{(\sqrt{2})^{3n+1}}{(\sqrt{\pi})^n} d_\nu d_{\nu+1}$. Then for each $k \in \mathbb{Z}_+$ we can find a constant $C_2 > 0$ such that

$$(1 + \|\xi\|)^k |(D^\alpha \hat{g})(\xi)| \le C_2 \|g\|_{k,\nu} \mathcal{M}_{\nu+2}(\alpha), \qquad \alpha \in \mathbb{Z}_+^n. \tag{3.5}$$

Therefore, $\hat{g} \in \mathcal{S}^{\mathcal{M}_{\nu+2}}$. Thus, $\hat{g} \in \mathcal{S}^{\mathcal{M}}$. By inequality (3.5),

$$\rho_{k,\nu+2}(\hat{g}) \le C_2 \|g\|_{k,\nu}, \quad g \in \mathcal{S}_{\mathcal{M}_\nu}, \quad k \in \mathbb{Z}_+.$$

This implies that the mapping $\mathcal{F}$ acts continuously from $S_{\mathcal{M}}$ into $\mathcal{S}^{\mathcal{M}}$.

It is obvious that the linear mapping $\mathcal{F}$ acts injectively from $\mathcal{S}_{\mathcal{M}}$ into $\mathcal{S}^{\mathcal{M}}$.

We are going to show that $\mathcal{F}$ is a mapping onto. Let $F \in \mathcal{S}^{\mathcal{M}}$. Then $F \in \mathcal{S}^{\mathcal{M}_\nu}$ for some $\nu \in \mathbb{N}$. This is why whatever $m \in \mathbb{Z}_+$, for all $\gamma \in \mathbb{Z}_+^n$, $x \in \mathbb{R}^n$,

$$(1 + \|x\|)^m |(D^\gamma F)(x)| \le \rho_{m,\nu}(F) \mathcal{M}_\nu(\gamma). \tag{3.6}$$

We let $f(x) := \hat{F}(-x)$, $x \in \mathbb{R}^n$. Then for each $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n$, $\xi \in \mathbb{R}^n$,

$$(i\xi)^\beta (D^\alpha f)(\xi) = \frac{(-1)^{|\alpha|}}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} D^\beta (F(x)(ix)^\alpha) e^{-i\langle x,\xi\rangle}\, dx.$$

That is,

$$(i\xi)^\beta (D^\alpha f)(\xi) = \frac{(-1)^{|\alpha|}}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}_+^n : j\le\kappa} C_\beta^j (D^{\beta-j}F)(x)(D^j(ix)^\alpha) e^{-i\langle x,\xi\rangle}\, dx,$$

where $\kappa := (\kappa_1, \dots, \kappa_n)$, $\kappa_s := \min(\alpha_s, \beta_s)$ for $s = 1, \dots, n$. This implies

$$|\xi^\beta (D^\alpha f)(\xi)| \le \frac{1}{(\sqrt{2\pi})^n} \sum_{j\in\mathbb{Z}_+^n : j\le\kappa} C_\beta^j \int_{\mathbb{R}^n} |(D^{\beta-j}F)(x)||D^j(x^\alpha)|\, dx.$$

Using inequality (3.3), we get:

$$|\xi^\beta (D^\alpha f)(\xi)| \le \frac{\sqrt{2}}{(\sqrt{2\pi})^n} \sum_{j\in\mathbb{Z}_+^n : j\le\kappa} C_\beta^j \int_{\mathbb{R}^n} |(D^{\beta}F)(x)||x^\alpha|\, dx.$$

This yields:

$$|\xi^\beta (D^\alpha f)(\xi)| \le \frac{\sqrt{2}}{(\sqrt{2\pi})^n} \sum_{j\in\mathbb{Z}_+^n : j\le\kappa} C_\beta^j \int_{\mathbb{R}^n} |(D^{\beta}F)(x)|(1+\|x\|)^{|\alpha|}\, dx.$$

Let $m \in \mathbb{Z}_+$ be arbitrary. Then, for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le m$,

$$|\xi^\beta (D^\alpha f)(\xi)| \le \frac{\sqrt{2}}{(\sqrt{2\pi})^n} \sum_{j\in\mathbb{Z}_+^n : j\le\kappa} C_\beta^j \int_{\mathbb{R}^n} |(D^{\beta}F)(x)|(1+\|x\|)^{m+2n} \frac{dx}{\prod\limits_{k=1}^n (1+x_k^2)}.$$

Employing estimate (3.6), for each $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le m$ we have

$$\begin{aligned}|\xi^\beta (D^\alpha f)(\xi)| &\le \sqrt{2}\left(\sqrt{\frac{\pi}{2}}\right)^n \rho_{m+2n,\nu}(F)\mathcal{M}_\nu(\beta) \sum_{j\in\mathbb{Z}_+^n : j\le\kappa} C_\beta^j \\ &\le \sqrt{2}\left(\sqrt{\frac{\pi}{2}}\right)^n \rho_{m+2n,\nu}(F)(m+1)^n(1+\beta_1)^m \cdots (1+\beta_n)^m \mathcal{M}_\nu(\beta).\end{aligned}$$

Finally, employing Condition i_4) on $\mathcal{M}$, we obtain that for some $C_3 = C_3(\nu, m) > 0$ for $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le m$ and all $\beta \in \mathbb{Z}_+^n$

$$|\xi^\beta (D^\alpha f)(\xi)| \le C_3 \rho_{m+2n,\nu}(F)\mathcal{M}_{\nu+1}(\beta), \quad \xi \in \mathbb{R}^n. \tag{3.7}$$

Therefore, $f \in \mathcal{S}_{\mathcal{M}_{\nu+1}}$. Hence, $f \in \mathcal{S}_{\mathcal{M}}$. It is clear that $\hat{f} = F$. Thus, the mapping $\mathcal{F}$ acts from $\mathcal{S}_{\mathcal{M}}$ onto $\mathcal{S}^{\mathcal{M}}$. Estimate (3.7) means that

$$\|f\|_{m,\nu+1} \le C_3 \rho_{m+2n,\nu}(F), \quad F \in \mathcal{S}^{\mathcal{M}_\nu}.$$

It implies that the inverse mapping $\mathcal{F}^{-1}$ is continuous.

The proven facts imply that the mapping $\mathcal{F}$ is an isomorphism between $\mathcal{S}_{\mathcal{M}}$ and $\mathcal{S}^{\mathcal{M}}$.

4. Proof of Theorem 1.2

Let $f \in \mathbb{S}^{\mathcal{H}}$. We are going to prove $F_f u \in \mathcal{P}(\Phi)$. Let $m \in \mathbb{Z}_+$ be arbitrary. Employing the expansion $F_f(z)$ ($z = x + iy$, $x, y \in \mathbb{R}^n$) into the Taylor series about the point x and the fact that $f \in \mathbb{S}(h_\nu)$ for some $\nu \in \mathbb{N}$, we have:

$$(1+\|z\|)^m |F_f(z)| \le \rho_{m,\nu}(f)(1+\|y\|)^m \sum_{|\alpha|\ge 0} e^{-h_\nu(\alpha)} \prod_{j=1}^n (|y_j|^+)^{\alpha_j}$$
$$\le B_{\nu,s}\rho_{m,\nu}(f)(1+\|y\|)^m e^{\sup\limits_{t=(t_1,\dots,t_n)\in\mathbb{R}^n}(t_1 \ln^+ |y_1| + \dots + t_n \ln^+ |y_n| - h_{\nu+s}(t))},$$

where $B_{\nu,s} := \sum\limits_{|\alpha|\ge 0} e^{h_{\nu+s}(\alpha)-h_\nu(\alpha)}$, s is from Condition H_8). Therefore,

$$(1+\|z\|)^m |F_f(z)| \le B_{\nu,s}\rho_{m,\nu}(f)(1+\|y\|)^m e^{\varphi_{\nu+s}(Im\, z)}, \quad z \in \mathbb{C}^n.$$

By this estimate, employing Corollary 1.1, we obtain that, for some $K_{\nu,m} > 0$,

$$(1+\|z\|)^m |F_f(z)| \le K_{\nu,m}\rho_{m,\nu}(f) e^{\varphi_{\nu+s+1}(Im\, z)}, \qquad z \in \mathbb{C}^n.$$

That is,

$$p_{\nu+s+1,m}(F_f) \le K_{\nu,m}\rho_{m,\nu}(f), \quad f \in \mathbb{S}(h_\nu).$$

In view of the arbitrariness of $m \in \mathbb{Z}_+$, $F_f \in \mathcal{P}(\varphi_{\nu+s+1})$. Thus, $F_f \in \mathcal{P}(\Phi)$. Moreover, the latter inequality means that the linear mapping $\mathcal{A}$ is continuous.

It is obvious that $\mathcal{A}$ is a one-to-one correspondence from $\mathbb{S}^{\mathcal{H}}$ into $\mathcal{P}(\Phi)$.

The mapping $\mathcal{A}$ is surjective. Indeed, let $F \in \mathcal{P}(\Phi)$. Then $F \in \mathcal{P}(\varphi_\nu)$ for some $\nu \in \mathbb{N}$. Let $m \in \mathbb{Z}_+$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$. Employing the Cauchy integral formula and the non-decreasing of φ_ν in each variable $[0,\infty)^n$, we obtain, proceeding as in the proof of Theorem 1 in [9], that for each $R \in (0,\infty)^n$ and each $x \in \mathbb{R}^n$

$$(1+\|x\|)^m |(D^\alpha F)(x)| \le \frac{\alpha! p_{\nu,m}(F)(1+\|R\|)^m e^{\varphi_\nu(R)}}{R^\alpha}.$$

Then, emplyoing Corollary 1.1, we have:

$$(1+\|x\|)^m |(D^\alpha F)(x)| \le e^{b_{\nu,m}} \alpha! p_{\nu,m}(F) \frac{e^{\varphi_{\nu+1}(R)}}{R^\alpha}.$$

For the brevity we let $\varphi_{\nu+1}[e](r) := \varphi_{\nu+1}(e^{r_1}, \dots, e^{r_n})$, $r = (r_1, \dots, r_n) \in \mathbb{R}^n$. Then

$$(1+\|x\|)^m |(D^\alpha F)(x)| \le e^{b_{\nu,m}} \alpha! p_{\nu,m}(F) \inf_{R\in(0,\infty)^n} \frac{e^{\varphi_{\nu+1}(R)}}{R^\alpha}$$
$$= \frac{e^{b_{\nu,m}} \alpha! p_{\nu,m}(F)}{\exp(\sup\limits_{r\in\mathbb{R}^n}(\langle\alpha, r\rangle - \varphi_{\nu+1}[e](r)))} \le \frac{e^{b_{\nu,m}} \alpha! p_{\nu,m}(F)}{\exp(\sup\limits_{r\in\mathbb{R}^n_+}(\langle\alpha, r\rangle - \varphi_{\nu+1}[e](r)))}$$
$$= \frac{e^{b_{\nu,m}} \alpha! p_{\nu,m}(F)}{\exp(\sup\limits_{r=(r_1,\dots,r_n)\in\mathbb{R}^n_+}(\langle\alpha, r\rangle - h^*_{\nu+1}(\ln^+ e^{r_1}, \dots, \ln^+ e^{r_n})))}$$
$$= \frac{e^{b_{\nu,m}} \alpha! p_{\nu,m}(F)}{\exp(\sup\limits_{r\in\mathbb{R}^n_+}(\langle\alpha, r\rangle - h^*_{\nu+1}(r)))} = \frac{e^{b_{\nu,m}} \alpha! p_{\nu,m}(F)}{\exp(\sup\limits_{r\in\mathbb{R}^n}(\langle\alpha, r\rangle - h^*_{\nu+1}(r)))}$$
$$= e^{b_{\nu,m}} \alpha! p_{\nu,m}(F) \exp(-\widetilde{h^*_{\nu+1}}(\alpha))) = e^{b_{\nu,m}} \alpha! p_{\nu,m}(F) \exp(-h_{\nu+1}(\alpha)).$$

In the end of this estimate we have employed the fact that due to the convexity of the function $h_{\nu+1}$ we have $\widetilde{h^*_{\nu+1}}(\alpha) = h_{\nu+1}(\alpha)$ for each $\alpha \in \mathbb{Z}^n$ according to Proposition 1 in [10]. By the obtained estimate it follows that

$$\rho_{m,\nu+1}(F_{|\mathbb{R}^n}) \leq e^{b_{\nu,m}} p_{\nu,m}(F), \qquad F \in \mathcal{P}(\varphi_\nu). \tag{4.1}$$

Therefore, $F_{|\mathbb{R}^n} \in \mathbb{S}(h_{\nu+1})$. Thus, $F_{|\mathbb{R}^n} \in \mathbb{S}^{\mathcal{H}}$. It is obvious that $\mathcal{A}(F_{|\mathbb{R}^n}) = F$ and inequality (4.1) ensures the continuity of the mapping $\mathcal{A}^{-1}$. Thus, the mapping $\mathcal{A}$ is an isomorphism between $\mathbb{S}^{\mathcal{H}}$ and $\mathcal{P}(\varPhi)$.

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