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VECTOR FORM OF KUNDU-ECKHAUS EQUATION AND ITS SIMPLEST SOLUTIONS

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Abstract. Nowadays, new vector integrable models of nonlinear optics are actively investigated. This is motivated by a need to transmit more information per unit of time by using polarized waves. In our work we study one of such models and we construct an hierarchy of integrable vector nonlinear differential equations depending on the functional parameter r by using a monodromy matrix. The first equation of this hierarchy for $r = \alpha(\mathbf{p}^t\mathbf{q})$ is a vector analogue of the Kundu-Eckhaus equation. As $\alpha = 0$, the equations of this hierarchy turn into equations of the Manakov system hierarchy. Other values of the functional parameter r correspond to other integrable nonlinear equations. New elliptic solutions to the vector analogue of the Kundu-Eckhaus and Manakov system are presented. We also give an example of a two-gap solution of these equations in the form of a solitary wave. We show that there exist linear transformations of solutions to the vector integrable nonlinear equations into other solutions to the same equations. This statement is true for many vector integrable nonlinear equations. In particular, this is true for multicomponent derivative nonlinear Schrödinger equations and for the Kulish-Sklyanin equation. Therefore, the corresponding Baker-Akhiezer function can be constructed from a spectral curve only up to a linear transformation. In conclusion, we show that the spectral curves of the finite-gap solutions of the Manakov system and the Kundu-Eckhaus vector equation are trigonal curves whose genus is twice the number of phases of the finite-gap solution, that is, in the finite-gap solutions of the Manakov system and the vector analogue of the Kundu-Eckhaus equation, only half of the phases contain the variables t, $z_1, \dots, z_n$. The second half of the phases depends on the parameters of the solutions.

Keywords: Monodromy matrix, spectral curve, derivative nonlinear Schrödinger equation, vector integrable nonlinear equation.

Mathematics Subject Classification: 35Q51, 35Q55

1. Introduction

It is well known that the derived nonlinear Schrodinger equations [1], [2], [3], [4], [5], [6] have numerous applications in various fields of physics and mathematics. In this regard, studies of various types of solutions to these equations are constantly being carried out (see, for example, [7], [8], [9], [10], [11], [12]). At the same time, it should be noted that along with the Kaup-Newell [1] equation

$$ip_z + p_{tt} + i(|p|^2 p)_t = 0, \tag{1.1}$$

Chen-Lee-Liu equation [2]

$$ip_z + p_{tt} + i\,|p|^2 p_t = 0, \tag{1.2}$$

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and Gerdjikov-Ivanov equation [3], [4]

$$ip_z + p_{tt} - ip^2 p_t^* + \frac{1}{2}|p|^4 p = 0, \tag{1.3}$$

there exists the Kundu-Eckhaus equation [13], [14], [15], [16], [17]

$$ip_z + p_{tt} - 2\sigma|p|^2 p + \alpha^2 |p|^4 p + 2i\alpha\partial_t\left(|p|^2\right)p = 0, \quad \sigma = \pm 1. \tag{1.4}$$

Equation (1.4), as well as equations (1.1)–(1.3), contains first derivatives and also has numerous applications.

However, there is a significant difference between equations (1.1)–(1.3) and (1.4). The first three equations are consequences of compatibility conditions of Lax pairs with quadratic in spectral parameter Lax operators. Equation (1.4), in contrast to equations (1.1)–(1.3), is a result of the gauge transformation

$$p = \widehat{p}e^{-i\theta}, \qquad \theta = \alpha\int |p|^2\, dt$$

of a solution $\widehat{p}$ to the nonlinear Schrödinger equation

$$i\widehat{p}_z + \widehat{p}_{tt} - 2\sigma|\widehat{p}|^2\widehat{p} = 0.$$

The ever-growing traffic in networks requires finding ways to increase bandwidth of optical fibers. Therefore, researchers are actively working on vector models of nonlinear optical waves propagation [18], [19], [20]. Many of these models have been known for a long time. One such model is the Manakov system [21], [22], [23], [24], [25], [26]

$$\begin{aligned} \partial_z p_1 &= i\partial_t^2 p_1 - 2i\sigma(|p_1|^2 + |p_2|^2)p_1, \\ \partial_z p_2 &= i\partial_t^2 p_2 - 2i\sigma(|p_1|^2 + |p_2|^2)p_2. \end{aligned} \tag{1.5}$$

We observe that vector nonlinear Schrödinger equations also have derivative forms, one of which is the equations [27], [28], [29], [30], [31]

$$\begin{aligned} i\partial_z p_1 &= -\partial_t^2 p_1 - \frac{2i}{3}\partial_t\left[\left(|p_1|^2 + |p_2|^2\right)p_1\right], \\ i\partial_z p_2 &= -\partial_t^2 p_2 - \frac{2i}{3}\partial_t\left[\left(|p_1|^2 + |p_2|^2\right)p_2\right]. \end{aligned} \tag{1.6}$$

In contrast to the above works, here we investigate a vector analogue of the Kundu-Eckhaus equation. We use a monodromy matrix to derive equations from the vector analogue of the Kundu-Eckhaus equation hierarchy and construct the simplest nontrivial solutions to first equation. We hope that the vector equation we have obtained, as well as the scalar one, will have numerous applications in physics and mathematics.

The work consists of Introduction, four sections, and concluding remarks. In Section 2 we define the Lax operator

$$i\Psi_t + U\Psi = \mathbf{0},$$

which depends on a functional parameter $r \in \mathbb{R}$, and investigate properties of corresponding monodromy matrix. Since the spectral curve equation is a characteristic equation of the monodromy matrix [32], it is not difficult to obtain properties of the spectral curves equations from properties of the monodromy matrix. As in the case of the Manakov system [26], the spectral curves equations are quite cumbersome and we do not provide them in this paper. We only note that, as it was shown in [26], a linear dependence of the functions p_j leads to factorization of the spectral curve equation into separate components. Therefore, from our point of view, solutions with linearly independent p_j are more interesting for studying and using in applications.

In Section 3 we derive stationary equations for multiphase solutions. These equations are analogs of the Novikov equations for the Korteweg-de Vries hierarchy. Also in this section, we define a hierarchy of the second operators of the Lax pair

$$i\Psi_{z_k} + W_k\Psi = \mathbf{0},$$

which depends on the functional parameter r_k: $\partial_t r_k = \partial_{z_k} r$. The Lax pair compatibility conditions give an hierarchy of vector derivative nonlinear Schrödinger equations with an additional functional parameter ϕ: $r = \partial_t\phi$ and $r_k = \partial_{z_k}\phi$. As $r = \alpha(\mathbf{p}^t\mathbf{q})$, these equations are vector analogue of the Kundu-Eckhaus equation and its higher forms. Another choice of the functional parameter leads to other vector nonlinear equations. For $\phi \equiv 0$ these equations turn into equations from the Manakov hierarchy [26]. We note that an existence of a Lax pair makes it possible to use the Darboux transformation to construct new solutions to vector analogue of the Kundu-Eckhaus equation.

In Section 4, we construct one-phase solutions to vector analogue of the Kundu-Eckhaus equation. The first three solutions are expressed in terms of elliptic Jacobi functions, and for $\phi \equiv 0$ they are new elliptic solutions to the Manakov system. Let us recall that elliptic solutions to the Manakov system obtained in [26] were expressed in terms of the Weierstrass functions. Here we construct solutions expressed in terms of the hyperbolic functions. In the end of the section we consider one-phase two-gap solutions. Despite the fact that in the last case the spectral curve has a genus equalling to 2, the corresponding solution is a traveling wave.

In Section 5 we show that there exist linear transformations of solutions to vector integrable nonlinear equations into other solutions to the same equations. Original and transformed solutions are associated with the same spectral curve, but they correspond to different Baker-Akhiezer functions. One of these Baker-Achiezer functions differs from the other by an orthogonal matrix factor. That is, the Baker-Achiezer function for considered vector nonlinear Schrödinger equation is determined up to an orthogonal transformation.

2. Monodromy matrix and its properties

Let first equation of a Lax pair has the form

$$i\Psi_t + U\Psi = \mathbf{0}, \tag{2.1}$$

where

$$U = U_0 + rJ, \qquad U_0 = -\lambda J + Q,$$
$$J = \frac{1}{3}\begin{pmatrix} 2 & \mathbf{0}^t \\ \mathbf{0} & -I \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & \mathbf{p}^t \\ -\mathbf{q} & \mathbf{0} \end{pmatrix},$$

$\mathbf{p}^t = (p_1, p_2)$, $\mathbf{q}^t = (q_1, q_2)$, I is identity matrix, $r \in \mathbb{R}$ is a some function, and λ is a spectral parameter.

Following [32], [26], we assume that there exists a monodromy matrix M such that the matrix function $\widehat{\Psi} = M\Psi$ is also an eigenfunction of Lax operator (2.1). Then the matrix M satisfies the equation

$$iM_t + UM - MU = \mathbf{0}. \tag{2.2}$$

In the case of a finite-gap matrix potential Q, the monodromy matrix M is a polynomial in the spectral parameter λ [32], [26]

$$M = \sum_{j=0}^{n} m_j(t)\lambda^j. \tag{2.3}$$

Substituting (2.3) in (2.2) and simplifying, we get that the matrix M has the following structure

$$M = V_n + \sum_{k=1}^{n-1} c_k V_{n-k} + c_n U_0 + J_n,$$

where $V_1 = \lambda U_0 + V_1^0$,

$$V_{k+1} = \lambda V_k + V_{k+1}^0, \qquad V_k^0 = \begin{pmatrix} -\mathcal{F}_k & \mathbf{H}_k^t \\ \mathbf{G}_k & F_k \end{pmatrix}, \qquad k \geq 1,$$
$$J_n = \begin{pmatrix} -c_{n+1} - c_{n+2} & 0 & 0 \\ 0 & c_{n+1} & c_{n+3} \\ 0 & c_{n+4} & c_{n+2} \end{pmatrix},$$

c_j are some constants, $\mathcal{F}_k = \mathrm{Tr} F_k$.

It follows from equation (2.2) that the entries of the matrices V_k^0 satisfy recurrence relations

$$\begin{aligned} &\mathbf{H}_1 = i\partial_t \mathbf{p} + r\mathbf{p}, \\ &\mathbf{G}_1 = i\partial_t \mathbf{q} - r\mathbf{q}, \\ &F_k = -i\partial_t^{-1}\left(\mathbf{G}_k \mathbf{p}^t + \mathbf{q}\mathbf{H}_k^t\right), \\ &\mathbf{H}_{k+1} = i\partial_t \mathbf{H}_k + r\mathbf{H}_k + \left(F_k^t + \mathcal{F}_k I\right)\mathbf{p}, \\ &\mathbf{G}_{k+1} = -i\partial_t \mathbf{G}_k + r\mathbf{G}_k - \left(F_k + \mathcal{F}_k I\right)\mathbf{q}. \end{aligned} \tag{2.4}$$

In particular, $F_1 = \mathbf{q}\mathbf{p}^t$, $\mathcal{F}_1 = \mathbf{p}^t\mathbf{q} = \mathbf{q}^t\mathbf{p}$,

$$\begin{aligned} \mathbf{H}_2 &= -\partial_t^2 \mathbf{p} + 2ir\partial_t \mathbf{p} + \left(2\mathbf{p}^t\mathbf{q} + r^2 + i\partial_t r\right)\mathbf{p}, \\ \mathbf{G}_2 &= \partial_t^2 \mathbf{q} + 2ir\partial_t \mathbf{q} - \left(2\mathbf{p}^t\mathbf{q} + r^2 - i\partial_t r\right)\mathbf{q}, \\ F_2 &= 2\left(\mathbf{q}\mathbf{p}^t\right) r + i\left(\mathbf{q}\partial_t \mathbf{p}^t - \partial_t \mathbf{q}\mathbf{p}^t\right) = \mathbf{q}\mathbf{H}_1^t - \mathbf{G}_1\mathbf{p}^t, \\ \mathcal{F}_2 &= 2(\mathbf{p}^t\mathbf{q})r + i(\mathbf{q}^t\partial_t\mathbf{p} - \mathbf{p}^t\partial_t\mathbf{q}) = \mathbf{H}_1^t\mathbf{q} - \mathbf{p}^t\mathbf{G}_1, \\ \mathbf{H}_3 &= -i\partial_t^3\mathbf{p} - 3r\partial_t^2\mathbf{p} + 3i\left(\mathbf{p}^t\mathbf{q} + r^2 + i\partial_t r\right)\partial_t\mathbf{p} \\ &\quad + \left(3i\partial_t\mathbf{p}^t\mathbf{q} + r^3 + 6r\mathbf{p}^t\mathbf{q} + 3ir\partial_t r - \partial_t^2 r\right)\mathbf{p}, \\ \mathbf{G}_3 &= -i\partial_t^3\mathbf{q} + 3r\partial_t^2\mathbf{q} + 3i\left(\mathbf{p}^t\mathbf{q} + r^2 - i\partial_t r\right)\partial_t\mathbf{q} \\ &\quad + \left(3i\mathbf{p}^t\partial_t\mathbf{q} - r^3 - 6r\mathbf{p}^t\mathbf{q} + 3ir\partial_r r + \partial_t^2 r\right)\mathbf{q}, \\ F_3 &= 3(\mathbf{q}\mathbf{p}^t)r^2 + 3i(\mathbf{q}\partial_t\mathbf{p}^t - \partial_t\mathbf{q}\mathbf{p}^t)r + \partial_t\mathbf{q}\partial_t\mathbf{p}^t \\ &\quad - \mathbf{q}\partial_t^2\mathbf{p}^t - \partial_t^2\mathbf{q}\mathbf{p}^t + 3(\mathbf{p}^t\mathbf{q})\mathbf{q}\mathbf{p}^t \\ &= \mathbf{q}\mathbf{H}_2^t - \mathbf{G}_2\mathbf{p}^t - \mathbf{G}_1\mathbf{H}_1^t - (\mathbf{p}^t\mathbf{q})\mathbf{q}\mathbf{p}^t, \\ \mathcal{F}_3 &= 3(\mathbf{p}^t\mathbf{q})r^2 + 3i(\mathbf{q}^t\partial_t\mathbf{p} - \mathbf{p}^t\partial_t\mathbf{q})r + 3(\mathbf{p}^t\mathbf{q})^2 \\ &\quad + \partial_t\mathbf{p}^t\partial_t\mathbf{q} - \mathbf{p}^t\partial_t^2\mathbf{q} - \mathbf{q}^t\partial_t^2\mathbf{p} \\ &= \mathbf{q}^t\mathbf{H}_2 - \mathbf{G}_2^t\mathbf{p} - \mathbf{G}_1^t\mathbf{H}_1 - (\mathbf{p}^t\mathbf{q})^2. \end{aligned}$$

For $r \equiv 0$ all above equations turn into corresponding equations for the Manakov system [26].

3. Stationary and evolutionary equations

Equation (2.2) implies the following stationary equations:

$$\begin{aligned} \mathbf{H}_{n+1} + \sum_{k=1}^{n} c_k \mathbf{H}_{n+1-k} + C_n^t \mathbf{p} = \mathbf{0}, \\ \mathbf{G}_{n+1} + \sum_{k=1}^{n} c_k \mathbf{G}_{n+1-k} - C_n \mathbf{q} = \mathbf{0}, \end{aligned} \tag{3.1}$$

where

$$C_n = \begin{pmatrix} 2c_{n+1} + c_{n+2} & c_{n+3} \\ c_{n+4} & c_{n+1} + 2c_{n+2} \end{pmatrix}.$$

All multiphase solutions to evolutionary integrable nonlinear equations are simultaneously solutions to some stationary equations.

In the case of reduction $\mathbf{q} = \sigma \mathbf{p}^*$ ($\sigma = \pm 1$), the identities

$$\mathbf{G}_k^* = -\sigma \mathbf{H}_k, \quad \mathbf{H}_k^* = -\sigma \mathbf{G}_k, \quad F_k^* = F_k^t, \quad \mathcal{F}_k \in \mathbb{R}$$

follow from recurrence relations (2.4). Therefore, the constants c_j in stationary equations (3.1) should satisfy the conditions $c_k \in \mathbb{R}$ ($1 \leqslant k \leqslant n+2$), $c_{n+4} = c_{n+3}^*$.

Let a second operator of a Lax pair have the form

$$i\Psi_{z_k} + W_k \Psi = \mathbf{0} \tag{3.2}$$

where $W_k = V_k + r_k J$. Then the compatibility condition of equations (2.1) and (3.2) as well as equation

$$i\partial_t W_k - i\partial_{z_k} U + U W_k - W_k U = \mathbf{0}$$

yield the evolutionary integrable nonlinear equations

$$i\partial_{z_k} \mathbf{p} = \mathbf{H}_{k+1} - r_k \mathbf{p}, \quad i\partial_{z_k} \mathbf{q} = \mathbf{G}_{k+1} + r_k \mathbf{q} \tag{3.3}$$

and an additional relation

$$\partial_{z_k} r = \partial_t r_k. \tag{3.4}$$

It follows from (3.4) that there exists a function ϕ such that

$$r = \partial_t \phi, \qquad r_k = \partial_{z_k} \phi.$$

The first systems of integrable nonlinear equations from hierarchy (3.3) have the form

$$\begin{aligned} i\partial_{z_1} \mathbf{p} &= -\partial_t^2 \mathbf{p} + 2ir\partial_t \mathbf{p} + \left(2\mathbf{p}^t \mathbf{q} + r^2 + i\partial_t r - r_1\right) \mathbf{p}, \\ i\partial_{z_1} \mathbf{q} &= \partial_t^2 \mathbf{q} + 2ir\partial_t \mathbf{q} - \left(2\mathbf{p}^t \mathbf{q} + r^2 - i\partial_t r - r_1\right) \mathbf{q}, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \partial_{z_2} \mathbf{p} = &- \partial_t^3 \mathbf{p} + 3ir\partial_t^2 \mathbf{p} + 3\left(\mathbf{p}^t \mathbf{q} + r^2 + i\partial_t r\right) \partial_t \mathbf{p} \\ &+ \left(3\partial_t \mathbf{p}^t \mathbf{q} - ir^3 - 6ir\mathbf{p}^t \mathbf{q} + 3r\partial_t r + i\partial_t^2 r + ir_2\right) \mathbf{p}, \\ \partial_{z_2} \mathbf{q} = &-\partial_t^3 \mathbf{q} - 3ir\partial_t^2 \mathbf{q} + 3\left(\mathbf{p}^t \mathbf{q} + r^2 - i\partial_t r\right) \partial_t \mathbf{q} \\ &+ \left(3\mathbf{p}^t \partial_t \mathbf{q} + ir^3 + 6ir\mathbf{p}^t \mathbf{q} + 3r\partial_r r - i\partial_t^2 r - ir_2\right) \mathbf{q}. \end{aligned} \tag{3.6}$$

Equation (3.5) is one of a vector derivative nonlinear Schrödinger equations and (3.6) is a vector modified Korteweg-de Vries equation. Both equations are parametrized by an arbitrary real function ϕ.

By (2.4) and (3.3) we get the identities

$$\partial_{z_k} F_1 = \partial_t F_{k+1} \quad \text{and} \quad \partial_{z_k} \mathcal{F}_1 = \partial_t \mathcal{F}_{k+1}.$$

Therefore, there exist functions Φ and $\widetilde{\phi}$ such that

$$F_1 = \partial_t \Phi, \quad F_{k+1} = \partial_{z_k} \Phi \quad \text{and} \quad \mathcal{F}_1 = \partial_t \widetilde{\phi}, \quad \mathcal{F}_{k+1} = \partial_{z_k} \widetilde{\phi}.$$

Therefore, if we put $\phi = \alpha\widetilde{\phi}$ or

$$r = \alpha\mathcal{F}_1 \quad \text{and} \quad r_k = \alpha\mathcal{F}_{k+1}, \tag{3.7}$$

then equations (3.3), (3.7) determine evolutionary integrable nonlinear equations from the hierarchy of a vector analogue of the Kundu-Eckhaus equation.

In particular, for $k = 1$, or for

$$r = \alpha\mathcal{F}_1 = \alpha(\mathbf{p}^t\mathbf{q}) \quad \text{and} \quad r_1 = \alpha\mathcal{F}_2 = 2\alpha^2(\mathbf{p}^t\mathbf{q})^2 + i\alpha(\mathbf{q}^t\partial_t\mathbf{p} - \mathbf{p}^t\partial_t\mathbf{q}),$$

equation (3.5) becomes

$$\begin{aligned} i\partial_{z_1}\mathbf{p} &= -\partial_t^2\mathbf{p} + 2i\alpha(\mathbf{p}^t\mathbf{q})\partial_t\mathbf{p} + \left(2\mathbf{p}^t\mathbf{q} - \alpha^2(\mathbf{p}^t\mathbf{q})^2 + 2i\alpha\mathbf{p}^t\partial_t\mathbf{q}\right)\mathbf{p}, \\ i\partial_{z_1}\mathbf{q} &= \partial_t^2\mathbf{q} + 2i\alpha(\mathbf{p}^t\mathbf{q})\partial_t\mathbf{q} - \left(2\mathbf{p}^t\mathbf{q} - \alpha^2(\mathbf{p}^t\mathbf{q})^2 - 2i\alpha\partial_t\mathbf{p}^t\mathbf{q}\right)\mathbf{q}. \end{aligned} \tag{3.8}$$

As $\mathbf{q} = S\mathbf{p}^*$, $S = \mathrm{diag}(\sigma_1, \sigma_2)$, $\sigma_j = \pm 1$, equations (3.8) transform into a vector analogue of the Kundu-Eckhaus equation. It is not difficult to see that for $\alpha = 0$ equations (3.8) transform into Manakov system [26].

4. One-phase solutions

4.1. Solutions in elliptic Jacobi functions. For $n = 1$ stationary equations have the form

$$\mathbf{H}_2 + c_1\mathbf{H}_1 + C_1^t\mathbf{p} = \mathbf{0}, \qquad \mathbf{G}_2 + c_1\mathbf{G}_1 - C_1\mathbf{q} = \mathbf{0} \tag{4.1}$$

or (for $c_4 = c_5 = 0$ and $r = \alpha\mathcal{F}_1$, $r_1 = \alpha\mathcal{F}_2$)

$$\begin{aligned} \partial_t^2 p_1 &= i(c_1 + 2\alpha(\mathbf{p}^t\mathbf{q}))\partial_t p_1 + (2c_2 + c_3 + (2 + c_1\alpha)\mathbf{p}^t\mathbf{q} + \alpha^2(\mathbf{p}^t\mathbf{q})^2 + i\alpha\partial_t(\mathbf{p}^t\mathbf{q}))p_1, \\ \partial_t^2 p_2 &= i(c_1 + 2\alpha(\mathbf{p}^t\mathbf{q}))\partial_t p_2 + (c_2 + 2c_3 + (2 + c_1\alpha)\mathbf{p}^t\mathbf{q} + \alpha^2(\mathbf{p}^t\mathbf{q})^2 + i\alpha\partial_t(\mathbf{p}^t\mathbf{q}))p_2, \\ \partial_t^2 q_1 &= -i(c_1 + 2\alpha(\mathbf{p}^t\mathbf{q}))\partial_t q_1 + (2c_2 + c_3 + (2 + c_1\alpha)\mathbf{p}^t\mathbf{q} + \alpha^2(\mathbf{p}^t\mathbf{q})^2 - i\alpha\partial_t(\mathbf{p}^t\mathbf{q}))q_1, \\ \partial_t^2 q_2 &= -i(c_1 + 2\alpha(\mathbf{p}^t\mathbf{q}))\partial_t q_2 + (c_2 + 2c_3 + (2 + c_1\alpha)\mathbf{p}^t\mathbf{q} + \alpha^2(\mathbf{p}^t\mathbf{q})^2 - i\alpha\partial_t(\mathbf{p}^t\mathbf{q}))q_2. \end{aligned} \tag{4.2}$$

Replacing functions p_j and q_j by formulas

$$p_j = \widehat{p}_j e^{i\theta}, \quad q_j = \widehat{q}_j e^{-i\theta}, \quad \partial_t\theta = \frac{1}{2}c_1 + \alpha\mathbf{p}^t\mathbf{q},$$

we obtain the following identities

$$\begin{aligned} \partial_t^2\widehat{p}_1 &= \left(2\widehat{\mathbf{p}}^t\widehat{\mathbf{q}} + 2c_2 + c_3 - \frac{1}{4}c_1^2\right)\widehat{\mathbf{p}}_1, \\ \partial_t^2\widehat{p}_2 &= \left(2\widehat{\mathbf{p}}^t\widehat{\mathbf{q}} + c_2 + 2c_3 - \frac{1}{4}c_1^2\right)\widehat{\mathbf{p}}_2, \\ \partial_t^2\widehat{q}_1 &= \left(2\widehat{\mathbf{p}}^t\widehat{\mathbf{q}} + 2c_2 + c_3 - \frac{1}{4}c_1^2\right)\widehat{\mathbf{q}}_1, \\ \partial_t^2\widehat{q}_2 &= \left(2\widehat{\mathbf{p}}^t\widehat{\mathbf{q}} + c_2 + 2c_3 - \frac{1}{4}c_1^2\right)\widehat{\mathbf{q}}_2. \end{aligned} \tag{4.3}$$

It is easy to see that the functions $\widehat{p}_j$ and $\widehat{q}_j$ are solutions of the same second order linear differential equations. Hence, their products $u_j = \widehat{p}_j\widehat{q}_j$ satisfy the corresponding Appel's equations ([33, Part II, Ch. 14, Ex. 10], [34])

$$\begin{aligned} \partial_t^3 u_1 - (8u_1 + 8u_2 + 8c_2 + 4c_3 - c_1^2)\partial_t u_1 - 4\partial_t(u_1 + u_2)u_1 = 0, \\ \partial_t^3 u_2 - (8u_1 + 8u_2 + 4c_2 + 8c_3 - c_1^2)\partial_t u_2 - 4\partial_t(u_1 + u_2)u_2 = 0. \end{aligned} \tag{4.4}$$

We denote $u_1 + u_2 = u$, $u_1 - u_2 = v$. In these notation, equations (4.4) become

$$\begin{aligned} \partial_t^3 u + (c_1^2 - 6c_2 - 6c_3 - 12u)\partial_t u &= 2(c_2 - c_3)\partial_t v, \\ \partial_t^3 v + (c_1^2 - 6c_2 - 6c_3 - 8u)\partial_t v &= 2(c_2 - c_3 + 2v)\partial_t u. \end{aligned} \tag{4.5}$$

The simplest solutions of equations (4.5) can be obtained as $v = (c_3 - c_2)/2$. In this case, the function u satisfies the equation

$$\partial_t^3 u + (c_1^2 - 6c_2 - 6c_3 - 12u)\partial_t u = 0$$

or

$$\partial_t^2 u + (c_1^2 - 6c_2 - 6c_3)u - 6u^2 = \widehat{c}_1, \tag{4.6}$$

where $\widehat{c}_1$ is an integration constant. Simplifying relation (4.6), we obtain the equation

$$(\partial_t u)^2 = 4u^3 - (c_1^2 - 6c_2 - 6c_3)u^2 + 2\widehat{c}_1 u + \widehat{c}_2, \tag{4.7}$$

where $\widehat{c}_2$ is a second integration constant. It is well known that solutions to equation (4.7) are elliptic functions or their degenerations.

It is easy to verify that one of the non-degenerate solutions to the equation (4.7) has the form

$$u = k^2 \operatorname{sn}^2(t; k) + \frac{1}{12}c_1^2 - \frac{1}{2}(c_2 + c_3) - \frac{1}{3}(1 + k^2), \tag{4.8}$$

where $\operatorname{sn}(t; k)$ is an elliptic Jacobi function [35], [36], which satisies the equation

$$[\operatorname{sn}'(t)]^2 = (1 - \operatorname{sn}^2(t))(1 - k^2 \operatorname{sn}^2(t)).$$

The integration constant for solution (4.8) is equal to

$$\begin{aligned}
\widehat{c}_1 =& \frac{1}{24}c_1^4 - \frac{1}{2}(c_2 + c_3)c_1^2 + \frac{3}{2}(c_2 + c_3)^2 - \frac{2}{3}(1 - k^2 + k^4),\\
\widehat{c}_2 =& -\frac{1}{432}c_1^6 + \frac{1}{24}(c_2 + c_3)c_1^4 - \frac{1}{4}(c_2 + c_3)^2 c_1^2 + \frac{1}{9}(1 - k^2 + k^4)c_1^2\\
&+ \frac{1}{2}(c_2 + c_3)^3 - \frac{2}{3}(c_2 + c_3)(1 - k^2 + k^4) - \frac{4}{27}(2 - 3k^2 - 3k^4 + 2k^6).
\end{aligned}$$

Knowing functions u and v, we obtain functions u_j

$$\begin{aligned}
u_1 &= \frac{1}{2}(u + v) = \frac{k^2}{2}\operatorname{sn}^2(t; k) + \frac{c_1^2}{24} - \frac{c_2}{2} - \frac{1 + k^2}{6},\\
u_2 &= \frac{1}{2}(u - v) = \frac{k^2}{2}\operatorname{sn}^2(t; k) + \frac{c_1^2}{24} - \frac{c_3}{2} - \frac{1 + k^2}{6}.
\end{aligned} \tag{4.9}$$

Thus, functions $\widehat{p}_j$ and $\widehat{q}_j$ are solutions to the equations

$$\begin{aligned}
\partial_t^2 \widehat{p}_1 &= \left(2k^2 \operatorname{sn}^2(t; k) - \frac{2}{3}(1 + k^2) - \frac{1}{12}c_1^2 + c_2\right)\widehat{p}_1,\\
\partial_t^2 \widehat{p}_2 &= \left(2k^2 \operatorname{sn}^2(t; k) - \frac{2}{3}(1 + k^2) - \frac{1}{12}c_1^2 + c_3\right)\widehat{p}_2.
\end{aligned} \tag{4.10}$$

Since functions $\widehat{q}_j$ satisfy the same equations as $\widehat{p}_j$, and the Wronskian of these solutions are constant:

$$W[\widehat{p}_j, \widehat{q}_j] = 2iW_j,$$

functions $\widehat{p}_j$ and $\widehat{q}_j$ are equal to

$$\widehat{p}_j = \sqrt{u_j}\exp\left\{-iW_j \int \frac{dt}{u_j}\right\}, \qquad \widehat{q}_j = \sqrt{u_j}\exp\left\{iW_j \int \frac{dt}{u_j}\right\}, \tag{4.11}$$

where u_j are defined by formulas (4.9). Substituting expressions (4.11) into equation (4.10) and simplifying, we get

$$W_j^2 = \frac{1}{6912}(c_1^2 - 12c_{j+1} - 4 - 4k^2)(c_1^2 - 12c_{j+1} + 8 - 4k^2)(c_1^2 - 12c_{j+1} - 4 + 8k^2). \tag{4.12}$$

It follows from equation (4.12) that there are three cases when $\widehat{p}_j = \widehat{q}_j$ and $\widehat{p}_1 \neq \widehat{p}_2$. If

$$c_2 = \frac{1}{12}(c_1^2 + 8 - 4k^2), \qquad c_3 = \frac{1}{12}(c_1^2 + 8k^2 - 4),$$

then

$$\widehat{p}_1 = \widehat{q}_1 = \frac{i}{\sqrt{2}}\,\mathrm{dn}(t;k), \qquad \widehat{p}_2 = \widehat{q}_2 = \frac{ik}{\sqrt{2}}\,\mathrm{cn}(t;k).$$

In this case, the solution to equations (3.8) has the form

$$p_1 = i\mathfrak{p}_1(t - c_1 z_1)e^{i\theta}, \qquad q_1 = -p_1^*, \qquad p_2 = i\mathfrak{p}_2(t - c_1 z_1)e^{i\theta}, \qquad q_2 = -p_2^*, \tag{4.13}$$

where

$$\mathfrak{p}_1(T) = \frac{1}{\sqrt{2}}\,\mathrm{dn}(T;k), \quad \mathfrak{p}_2(T) = \frac{k}{\sqrt{2}}\,\mathrm{cn}(T;k),$$
$$\theta = \frac{c_1}{2}t + \left(1 - \frac{c_1^2}{4}\right) z_1 + \alpha \int \left(k^2\,\mathrm{sn}^2(t - c_1 z_1;k) - \frac{1+k^2}{2}\right) dt.$$

The magnitudes of solutions (4.13) are shown on Figure 1.

For

$$c_2 = \frac{1}{12}(c_1^2 + 8 - 4k^2), \qquad c_3 = \frac{1}{12}(c_1^2 - 4k^2 - 4),$$

we have

$$\widehat{p}_1 = \widehat{q}_1 = \frac{i}{\sqrt{2}}\,\mathrm{dn}(t;k), \qquad \widehat{p}_2 = \widehat{q}_2 = \frac{k}{\sqrt{2}}\,\mathrm{sn}(t;k).$$

The corresponding solution to equations (3.8) reads as

$$\begin{aligned} p_1 &= i\mathfrak{p}_1(t - c_1 z_1)e^{i\theta}, \qquad & q_1 &= -p_1^*, \\ p_2 &= \mathfrak{p}_2(t - c_1 z_1)e^{i\theta}, \qquad & q_2 &= p_2^*, \end{aligned} \tag{4.14}$$

where

$$\mathfrak{p}_1(T) = \frac{1}{\sqrt{2}}\,\mathrm{dn}(T;k), \quad \mathfrak{p}_2(T) = \frac{k}{\sqrt{2}}\,\mathrm{sn}(T;k),$$
$$\theta = \frac{c_1}{2}t + \left(1 - k^2 - \frac{c_1^2}{4}\right) z_1 + \alpha \int \left(k^2\,\mathrm{sn}^2(t - c_1 z_1;k) - \frac{1}{2}\right) dt.$$

If

$$c_2 = \frac{1}{12}(c_1^2 + 8k^2 - 4), \qquad c_3 = \frac{1}{12}(c_1^2 - 4k^2 - 4),$$

we have

$$\widehat{p}_1 = \widehat{q}_1 = \frac{ik}{\sqrt{2}}\,\mathrm{cn}(t;k), \qquad \widehat{p}_2 = \widehat{q}_2 = \frac{k}{\sqrt{2}}\,\mathrm{sn}(t;k).$$

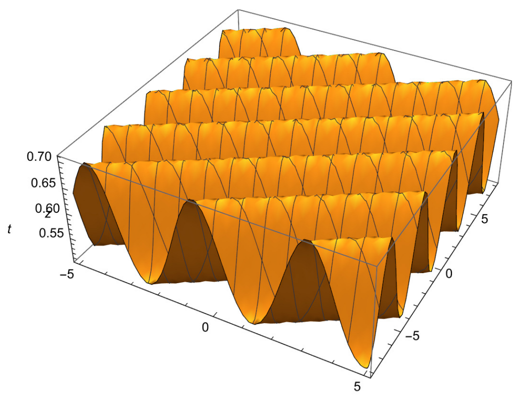
In this case, the solution to equations (3.8) has the form

$$\begin{aligned} p_1 &= i\mathfrak{p}_1(t - c_1 z_1)e^{i\theta}, \qquad & q_1 &= -p_1^*, \\ p_2 &= \mathfrak{p}_2(t - c_1 z_1)e^{i\theta}, \qquad & q_2 &= p_2^*, \end{aligned} \tag{4.15}$$

where

$$\mathfrak{p}_1(T) = \frac{k}{\sqrt{2}}\,\mathrm{cn}(T;k), \quad \mathfrak{p}_2(T) = \frac{k}{\sqrt{2}}\,\mathrm{sn}(T;k),$$
$$\theta = \frac{c_1}{2}t + \left(k^2 - 1 - \frac{c_1^2}{4}\right) z_1 + \alpha \int \left(k^2\,\mathrm{sn}^2(t - c_1 z_1;k) - \frac{k^2}{2}\right) dt.$$

The dependency of solutions (4.13)–(4.15) on the variable z_1 was found from equations (3.8).


The magnitude $\mathfrak{p}_1(t - c_1 z_1)$

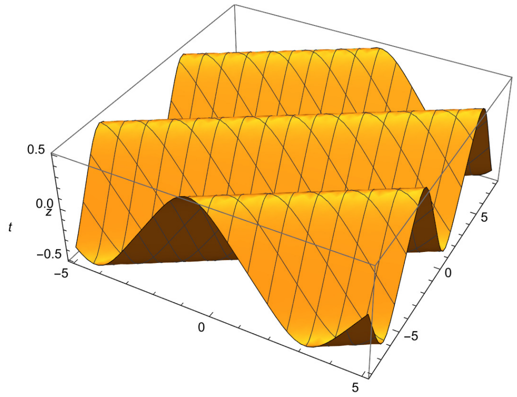

The magnitude $\mathfrak{p}_2(t - c_1 z_1)$

FIGURE 1. The magnitudes of solutions (4.13) for $k = 0.7$, $c_1 = 1$

4.2. Solutions in hyperbolic functions. Equation (4.7) is well-studied. In particular, it has the following solution

$$u = a^2 \tanh^2(at) + \frac{1}{12}\left((c_1^2 - 6c_2 - 6c_3) - 8a^2\right).$$

Integration constants for a given function u are equal to

$$\widehat{c}_1 = \frac{1}{24}\left((c_1^2 - 6c_2 - 6c_3)^2 - 16a^4\right),$$
$$\widehat{c}_2 = \frac{1}{432}\left(4a^2 + (c_1^2 - 6c_2 - 6c_3)\right)^2\left(8a^2 - (c_1^2 - 6c_2 - 6c_3)\right).$$

In this case, functions $\widehat{p}_j$ and $\widehat{q}_j$ satisfy the following equations:

$$\partial_t^2 \widehat{p}_j = \left(2a^2 \tanh^2(at) - \frac{4a^2}{3} - \frac{c_1^2}{12} + c_{j+1}\right)\widehat{p}_j \tag{4.16}$$

and

$$u_j = \widehat{p}_j \widehat{q}_j = \frac{a^2}{2}\tanh^2(at) + \frac{1}{24}\left(c_1^2 - 8a^2 - 12c_{j+1}\right). \tag{4.17}$$

We recall that the functions $\widehat{q}_j$ also satisfy equations (4.16).

Solving equations (4.16) with conditions (4.17), we get

$$\widehat{p}_j = \frac{1}{\sqrt{2}}(k_j + ia\tanh(at))e^{ik_j t}, \qquad \widehat{q}_j = \frac{1}{\sqrt{2}}(k_j - ia\tanh(at))e^{-ik_j t}, \tag{4.18}$$

where

$$k_j^2 = \frac{1}{12}\left(c_1^2 - 8a^2 - 12c_{j+1}\right) \qquad \text{or} \qquad c_{j+1} = \frac{1}{12}\left(c_1^2 - 8a^2 - 12k_j^2\right).$$

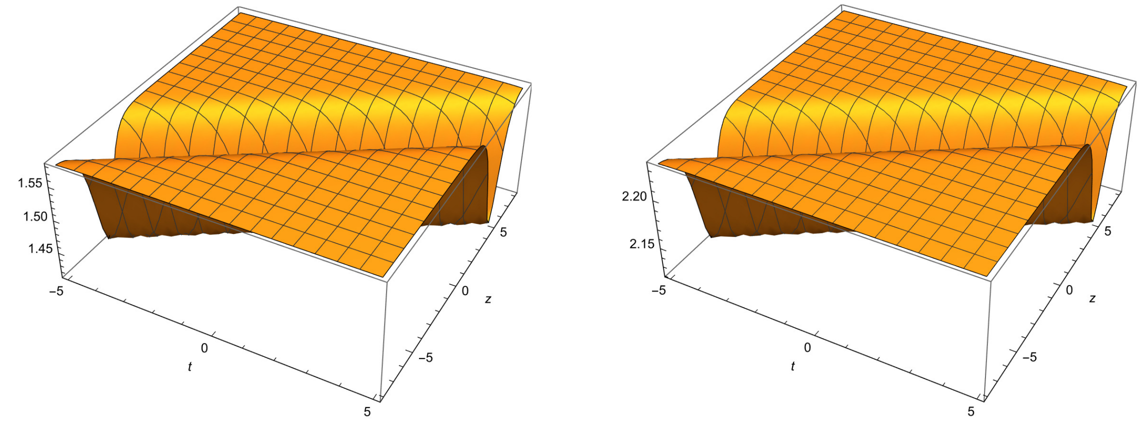
The corresponding solution to equations (3.8) has the form

$$\begin{aligned} p_j &= \frac{1}{\sqrt{2}}\left(k_j + ia\tanh[a(t - c_1 z_1)]\right)e^{i\theta_j}, \qquad q_j = p_j^*, \\ \theta_j &= \left(\frac{c_1}{2} + k_j\right)t + m_j z_1 + \alpha\int\left(a\tanh^2(at - ac_1 z_1) + \frac{k_1^2 + k_2^2}{2a}\right)dt, \end{aligned} \tag{4.19}$$

where

$$m_j = -2a^2 - \frac{1}{4}(c_1 + 2k_j)^2 - k_1^2 - k_2^2 - \alpha(k_1 + k_2)(a^2 + k_1^2 - k_1 k_2 + k_2^2).$$

The magnitudes of solutions (4.19) are shown on Figure 2.



The magnitude $|p_1(t,z_1)|$ The magnitude $|p_2(t,z_1)|$

FIGURE 2. The magnitudes of solutions (4.19) for $a=1$, $k_1=2$, $k_2=3$, $c_1=1$.

4.3. Two-gap one-phase solutions. If $n=1$, $v\neq const$, and $c_3\neq c_2$, then from (4.5) we have

$$v=\frac{1}{2(c_2-c_3)}\left(u_{tt}-6u^2+(c_1^2-6c_2-6c_3)u\right)+C_1, \qquad (4.20)$$

and

$$u_{tttt}+2(c_1^2-6c_2-6c_3-10u)u_{tt}-10\,(u_t)^2+40u^3-12(c_1^2-6c_2-6c_3)u^2 \\ +\left(c_1^4-12c_1^2(c_2+c_3)+8\left(4c_2^2+10c_2c_3+4c_3^3-C_1c_2+C_1c_3\right)\right)u+C_2=0, \qquad (4.21)$$

where C_1 and C_2 are integration constants.

We can rewrite equation (4.21) in the form

$$I_2+2AI_1+(A^2-4B^2+8C_1B)u+C_2=0, \qquad (4.22)$$

where $A=c_1^2-6(c_3+c_2)$, $B=c_3-c_2$,

$$I_2=u_{tttt}-20uu_{tt}-10\,(u_t)^2+40u^3, \quad I_1=u_{tt}-6u^2. \qquad (4.23)$$

It follows from equations (4.22) and (4.23) that the function $2u(t)$ is a two-gap potential of the Schrödinger operator [37, 38]

$$\psi_{tt}-2u\psi=E\psi. \qquad (4.24)$$

It is well known that linear independent solutions to equation (4.24) with two-gap potential $2u(t)$ can be written as

$$\psi_{1,2}=\varepsilon_{1,2}\sqrt{\Psi}\exp\left\{\pm\nu(E)\int\frac{dt}{\Psi}\right\}, \qquad (4.25)$$

where

$$\Psi=E^2+(\gamma_1-u)E+\gamma_2-\gamma_1u-\frac{1}{4}I_1,$$

and

$$\gamma_1=\frac{A}{2}, \qquad \gamma_2=\frac{1}{16}\left(A^2-4B^2+8C_1B\right).$$

Substituting (4.25) in (4.24) and simplifying, we obtain an equation for spectral curve of two-gap potential $u(t)$:

$$\nu^2=E^5+2\gamma_1E^4+(\gamma_1^2+2\gamma_2)E^3+\left(2\gamma_1\gamma_2-\frac{1}{8}C_2\right)E^2+C_3E+C_4, \qquad (4.26)$$

where

$$C_3 = \frac{1}{8}u_t u_{ttt} - \frac{1}{16}u_{tt}^2 + \frac{1}{4}(\gamma_1 - 5u)u_t^2 + \frac{5}{4}u^4 - \gamma_1 u^3 + \gamma_2 u^2 + \frac{1}{8}C_2 u + \gamma_2^2 - \frac{1}{8}\gamma_1 C_2,$$

$$C_4 = \frac{1}{64}u_{ttt}^2 + \frac{1}{8}(\gamma_1 - 3u)u_t u_{ttt} - \frac{1}{8}uu_{tt}^2 + \frac{1}{32}\left(C_2 + 16\gamma_2 u - 24\gamma_1 u^2 + 40u^3 + 2u_t^2\right)u_{tt}$$
$$+ \frac{1}{8}\left(2\gamma_1^2 - 2\gamma_2 - 10\gamma_1 u + 15u^2\right)u_t^2 - 3u^5 + \frac{7}{2}\gamma_1 u^4 - (\gamma_1^2 + 2\gamma_2)u^3$$
$$+ \frac{1}{16}(16\gamma_1\gamma_2 - 3C_2)u^2 + \frac{1}{8}\gamma_1 C_2 u - \frac{1}{8}\gamma_2 C_2.$$

By (4.20) we have

$$u_1 = \frac{1}{2}(u+v) = -\frac{1}{4B}I_1 + \left(\frac{1}{2} - \frac{A}{4B}\right)u + \frac{1}{2}C_1,$$
$$u_2 = \frac{1}{2}(u-v) = \frac{1}{4B}I_1 + \left(\frac{1}{2} + \frac{A}{4B}\right)u - \frac{1}{2}C_1. \tag{4.27}$$

Therefore, functions $\widehat{p}_j$ and $\widehat{q}_j$ are solutions to the equations

$$\partial_t^2\widehat{p}_1 - 2u\widehat{p}_1 = -\frac{1}{4}(A+2B)\widehat{p}_1,$$
$$\partial_t^2\widehat{p}_2 - 2u\widehat{p}_2 = -\frac{1}{4}(A-2B)\widehat{p}_2. \tag{4.28}$$

It follows from (4.24), (4.25), (4.27), and (4.28) that

$$\widehat{p}_1 = \varepsilon_{11}\sqrt{\Psi}\exp\left\{\nu(E)\int\frac{dt}{\Psi}\right\}\Bigg|_{E=E_1}, \qquad \widehat{q}_1 = \varepsilon_{12}\sqrt{\Psi}\exp\left\{-\nu(E)\int\frac{dt}{\Psi}\right\}\Bigg|_{E=E_1},$$
$$\widehat{p}_2 = \varepsilon_{21}\sqrt{\Psi}\exp\left\{\nu(E)\int\frac{dt}{\Psi}\right\}\Bigg|_{E=E_2}, \qquad \widehat{q}_2 = \varepsilon_{22}\sqrt{\Psi}\exp\left\{-\nu(E)\int\frac{dt}{\Psi}\right\}\Bigg|_{E=E_2},$$

where

$$\varepsilon_{11}\varepsilon_{12} = 1/B, \qquad \varepsilon_{21}\varepsilon_{22} = -1/B, \qquad E_1 = -(A+2B)/4, \qquad E_2 = -(A-2B)/4.$$

It is easy to see that the functions $\widehat{p}_j$ and $\widehat{q}_j$ are bounded as E_j satisfies the conditions $\mathrm{Re}(\nu(E_j)) = 0$.

Two-soliton potential of operator (4.24) has the form $(b > a > 0)$

$$u(t) = \frac{(a^2-b^2)(b^2-a^2+a^2\cosh(2bt)+b^2\cosh(2at))}{2(b\cosh(bt)\cosh(at) - a\sinh(bt)\sinh(at))^2}. \tag{4.29}$$

Substituting (4.29) into (4.22), we get

$$A = -2(a^2+b^2), \qquad C_1 = -\frac{(a^2-b^2)^2 - B^2}{2B}, \qquad C_2 = 0.$$

The spectral curve (4.26) of potential (4.29) is determined by the equation

$$\nu^2 = E(E-a^2)^2(E-b^2)^2.$$

Calculating $\nu^2(E_j)$, we have

$$\nu^2(E_1) = \frac{1}{32}(a^2+b^2-B)\left((a^2-b^2)^2 - B^2\right),$$
$$\nu^2(E_2) = \frac{1}{32}(a^2+b^2+B)\left((a^2-b^2)^2 - B^2\right).$$

The conditions $\nu^2(E_j) \leqslant 0$, $j = 1, 2$, imply $B = \pm(b^2 - a^2)$.

If $B=b^2-a^2$, then $E_1=b^2$, $E_2=a^2$,

$$c_2=\frac{1}{12}(c_1^2+8b^2-4a^2),\qquad c_3=\frac{1}{12}(c_1^2+8a^2-4b^2),$$

and

$$\widehat{p}_1=\widehat{q}_1=\frac{ib\sqrt{b^2-a^2}\cosh(at)}{(b\cosh(bt)\cosh(at)-a\sinh(bt)\sinh(at))},\qquad \widehat{q}_1=-\widehat{p}_1^*,$$
$$\widehat{p}_2=\widehat{q}_2=\frac{ia\sqrt{b^2-a^2}\sinh(bt)}{(b\cosh(bt)\cosh(at)-a\sinh(bt)\sinh(at))},\qquad \widehat{q}_2=-\widehat{p}_2^*.$$

The corresponding solution to equations (3.8) has the form

$$\begin{aligned} p_1(t,z_1)&=i\mathfrak{p}_1(t-c_1z_1)e^{i\theta_1(t,z_1)}, && q_1(t,z_1)=-p_1^*(t,z_1),\\ p_2(t,z_1)&=i\mathfrak{p}_2(t-c_1z_1)e^{i\theta_2(t,z_1)}, && q_2(t,z_1)=-p_2^*(t,z_1),\end{aligned} \tag{4.30}$$

where

$$\mathfrak{p}_1(T)=\frac{b\sqrt{b^2-a^2}\cosh(aT+t_a)}{(b\cosh(bT+t_b)\cosh(aT+t_a)-a\sinh(bT+t_b)\sinh(aT+t_a))},$$
$$\mathfrak{p}_2(T)=\frac{a\sqrt{b^2-a^2}\sinh(bT+t_b)}{(b\cosh(bT+t_b)\cosh(aT+t_a)-a\sinh(bT+t_b)\sinh(aT+t_a))},$$

and

$$\theta_1(t,z_1)=\frac{c_1}{2}t+\left(b^2-\frac{1}{4}c_1^2\right)z_1-\alpha\int\left(\mathfrak{p}_1^2(T)+\mathfrak{p}_2^2(T)\right)dt,$$
$$\theta_2(t,z_1)=\frac{c_1}{2}t+\left(a^2-\frac{1}{4}c_1^2\right)z_1-\alpha\int\left(\mathfrak{p}_1^2(T)+\mathfrak{p}_2^2(T)\right)dt.$$

Here (t_a,t_b) is an initial two-dimensional phase. The magnitudes of solutions (4.13) are shown on Figure 3.

5. Orthogonal transformation

Since the matrix J has two equal diagonal entries, an orthogonal transformation of the vectors of solutions to equations (3.3) again gives solutions to these equations. To prove this statement, we consider the equation

$$i\widetilde{\Psi}_t+\widetilde{U}\widetilde{\Psi}=\mathbf{0}, \tag{5.1}$$

where

$$\widetilde{U}=\widetilde{U}_0+rJ,\qquad \widetilde{U}_0=-\lambda J+\widetilde{Q},$$
$$\widetilde{\Psi}=\widetilde{T}\Psi,\qquad \widetilde{T}=\begin{pmatrix}1&\mathbf{0}^t\\ \mathbf{0}&T\end{pmatrix}.$$

It follows from equations (2.1) and (5.1) that $\widetilde{Q}\widetilde{T}=\widetilde{T}Q$. Therefore, the identities

$$\widetilde{\mathbf{q}}=T\mathbf{q},\quad \widetilde{\mathbf{p}}=\left(T^t\right)^{-1}\mathbf{p} \tag{5.2}$$

hold true. Thus, if the matrix T satisfies the condition

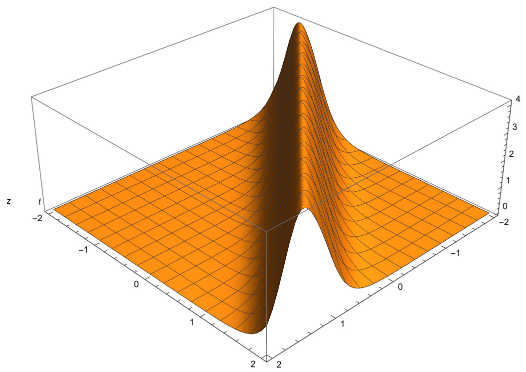
$$TST^\dagger=S, \tag{5.3}$$

then the vectors $\mathbf{p}$ and $\mathbf{q}$ ($\widetilde{\mathbf{p}}$ and $\widetilde{\mathbf{q}}$) are related as

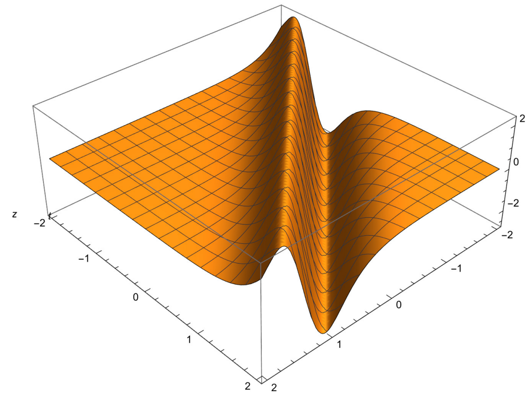
$$\mathbf{q}=S\mathbf{p}^*,\qquad \widetilde{\mathbf{q}}=S\widetilde{\mathbf{p}}^*.$$

The identities

$$\widetilde{\mathbf{G}}_k=T\mathbf{G}_k,\qquad \widetilde{\mathbf{H}}_k=\left(T^t\right)^{-1}\mathbf{H}_k,\qquad \widetilde{F}_k=TF_kT^{-1},\qquad \widetilde{\mathcal{F}}_k=\mathcal{F}_k$$



The magnitude $\mathfrak{p}_1(t - c_1 z_1)$



The magnitude $\mathfrak{p}_2(t - c_1 z_1)$

FIGURE 3. The magnitudes of solutions (4.30) for $a = 3$, $b = 5$, $c_1 = 1$, $t_a = 2$, $t_b = 3$.

follow from recurrence relations (2.4). Therefore, if the vectors $\mathbf{p}$ and $\mathbf{q}$ are solutions to evolutionary equations (3.3), then the vectors $\widetilde{\mathbf{p}}$ and $\widetilde{\mathbf{q}}$ are also solutions to the same equations. Thus, using formula (5.2) with matrix

$$T_1 = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

and solution (4.13), it is possible to construct new elliptic solutions to equation (3.8):

$$\begin{aligned} \widetilde{p}_1 &= \frac{i}{\sqrt{2}} \left(\cos\varphi \, \mathrm{dn}(t - c_1 z_1; k) + k \sin\varphi \, \mathrm{cn}(t - c_1 z_1; k)\right) e^{i\theta}, \quad \widetilde{q}_1 = -\widetilde{p}_1^*, \\ \widetilde{p}_2 &= -\frac{i}{\sqrt{2}} \left(\sin\varphi \, \mathrm{dn}(t - c_1 z_1; k) - k \cos\varphi \, \mathrm{cn}(t - c_1 z_1; k)\right) e^{i\theta}, \quad \widetilde{q}_2 = -\widetilde{p}_2^*. \end{aligned} \tag{5.4}$$

We note that the vectors $\widetilde{p}$ and $\widetilde{q}$ satisfy the stationary equation (3.1) with a non-diagonal matrix

$$\widetilde{C}_n = T C_n T^{-1}.$$

For solution (5.4) the matrix $\widetilde{C}_1$ reads as

$$\widetilde{C}_1 = \frac{1}{2} \begin{pmatrix} 3(c_3 + c_2) - (c_3 - c_2)\cos(2\varphi) & (c_3 - c_2)\sin(2\varphi) \\ (c_3 - c_2)\sin(2\varphi) & 3(c_3 + c_2) + (c_3 - c_2)\cos(2\varphi) \end{pmatrix}.$$

Therefore, the constants $\widetilde{c}_j$ for transformed solutions are equal:

$$\begin{aligned} \widetilde{c}_2 &= \frac{1}{2}(c_3 + c_2) - \frac{1}{2}(c_3 - c_2)\cos(2\varphi), \\ \widetilde{c}_3 &= \frac{1}{2}(c_3 + c_2) + \frac{1}{2}(c_3 - c_2)\cos(2\varphi), \\ \widetilde{c}_4 = \widetilde{c}_5 &= \frac{1}{2}(c_3 - c_2)\sin(2\varphi). \end{aligned}$$

At the same time, since the monodromy matrices of the functions $\widetilde{\Psi}$ and Ψ are similar

$$\widetilde{M} = \widetilde{T} M \widetilde{T}^{-1},$$

an equation of the same spectral curve corresponds to these solutions. Therefore, the Baker-Akhiezer function Ψ can be constructed from a spectral curve only up to a linear transformation $\widetilde{T}$.

6. Discussions and conclusions

In many works devoted to studying finite-gap solutions of the Manakov system (see, for example, [22], [39], [24], [40], [23], [41]), in contrast to our work, the following aspects were not taken into consideration. First, as we showed in [26], if the functions p_j are linearly dependent, then the eigenfunctions of Lax operator (2.1) are determined on two separated spectral curves. Secondly, to the best of the authors' knowledge, other researchers have not previously taken into consideration orthogonal transformations of solutions preserving spectral curves.

And finally, as we have seen in examples discussed in Section 4 , the number of phases of the solution is less than the genus of the corresponding spectral curve. Indeed, it follows from equations (3.1) and (3.3) that the following relations hold:

$$\partial_{z_n}\mathbf{p} = -\sum_{k=1}^{n-1} c_k\partial_{z_{n-k}}\mathbf{p} - c_n\partial_t\mathbf{p} + i\left(r_n + \sum_{k=1}^{n-1} c_k r_{n-k} + c_n r_n\right)\mathbf{p} + iC_n^t\mathbf{p},$$
$$\partial_{z_n}\mathbf{q} = -\sum_{k=1}^{n-1} c_k\partial_{z_{n-k}}\mathbf{q} - c_n\partial_t\mathbf{p} - i\left(r_n + \sum_{k=1}^{n-1} c_k r_{n-k} + c_n r_n\right)\mathbf{q} - iC_n\mathbf{q}.$$

Therefore, the solutions p_j and q_j, up to exponential multipliers, are n-phase functions (functions with n arguments):

$$p_j(t, z_1, \ldots, z_n) = \mathfrak{p}_j(t - c_n z_n, z_1 - c_{n-1}z_n, \ldots, z_{n-1} - c_1 z_n)e^{i\theta_j(t,z_1,\ldots,z_n)},$$
$$q_j(t, z_1, \ldots, z_n) = \mathfrak{q}_j(t - c_n z_n, z_1 - c_{n-1}z_n, \ldots, z_{n-1} - c_1 z_n)e^{-i\theta_j(t,z_1,\ldots,z_n)}.$$

An equation for a spectral curve $\Gamma = \{(\mu, \lambda)\}$ reads as

$$\mathcal{R}(\mu, \lambda) = \det(\mu I - M) = \mu^3 + \mathcal{A}(\lambda)\mu + \mathcal{B}(\lambda) = 0, \tag{6.1}$$

where

$$\mathcal{A}(\lambda) = -\frac{1}{3}\lambda^{2n+2} - \frac{2c_1}{3}\lambda^{2n+1} + \sum_{j=2}^{2n+2} A_j\lambda^{2n+2-j},$$
$$\mathcal{B}(\lambda) = \frac{2}{27}\lambda^{3n+3} + \frac{2c_1}{9}\lambda^{3n+2} + \sum_{j=2}^{3n+3} B_j\lambda^{3n+3-j}.$$

If $n \le 3$, then the discriminant of (6.1), as a polynomial of μ, is

$$\Delta(\lambda) = (c_{n+1} - c_{n+2})^2\lambda^{4n+4} + \sum_{j=1}^{4n+4} D_j\lambda^{4n+4-j}. \tag{6.2}$$

Probably, identity (6.2) is also true for other values of n. It follows from equation (6.2) that when condition $c_{n+1} \neq c_{n+2}$ is fulfilled, the curve Γ has $(4n + 4)$ branching points. Using the Riemann-Hurwitz formula

$$g = \frac{M}{2} - N + 1,$$

where M is a number of branching points, N is a number of sheets of a covering, we get that the genus of the spectral curve Γ is equal

$$g = \frac{4n+4}{2} - 3 + 1 = 2n.$$

Thus, to construct finite-gap solutions to the Manakov system or to the vector Kundu-Eckhaus equation, it is necessary to use trigonal curves, the genus of which is twice the number of phases of solutions. That is, in the finite-gap solutions of the Manakov system and the vector Kundu-Eckhaus equation, only half of the phases involves the variables t, $z_1, \ldots, z_n$. The second half of the phases depends on the parameters of the solutions.

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