ISSN 2304-0122 Ufa Mathematical Journal. Vol. 15. No 2 (2023). P. 74-84.

doi:10.13108/2023-15-2-74

SHARP JACKSON–STECHKIN TYPE INEQUALITIES IN HARDY SPACE H_2 AND WIDTHS OF FUNCTIONAL CLASSES

M.Sh. SHABOZOV, Z.Sh. MALAKBOZOV

Abstract. In this work we obtain sharp Jackson–Stechkin type inequalities relating the best joint polynomial approximation of functions analytic in the unit disk and a special generalization of the continuity modulus, which is defined by means of the Steklov function.

While solving a series of problems in the theory on approximation of periodic functions by trigonometric polynomials in the space L_2, a modification of the classical definition of the continuity modulus of mth order generated by the Steklov function was employed by S.B. Vakarchuk, M.Sh. Shabozov and A.A. Shabozova . Here the proposed construction is employed for defining a modification of the continuity modulus of mth order for functions analytic in the unit disk generated by the Steklov function in the Hardy space H_2.

By using this smoothness characteristic we solve a problem on finding a sharp constant in the Jackson–Stechkin type inequalities for joint approximation of the functions and their intermediate derivatives.

For the classes of function, averaged with a weight, the generalized continuity moduli of which are bounded by a given majorant, we find exact values of various n-widths. We also solve the problem on finding sharp upper bounds for best joint approximations of the mentioned classes of functions in the Hardy space H_2.

Keywords: Jackson–Stechkin type inequalities, continuity modulus, Steklov function, n-width, Hardy space.

Mathematics Subject Classification: 30E05, 30E10, 42A10.

1. Introduction

Extremal problems on the best polynomial approximation of functions analytic in the circle were studied in many papers, see, for example, [1]–[18] and the references therein. Among these problems, one of the most important is on finding exact constants in Jackson–Stechkin type inequalities in various normed spaces. We recall that by inequalities of the Jackson–Stechkin type in the considered normed space we mean ones, in which the approximation of a function by a finite-dimensional subspace is estimated in terms of some characteristic of the smoothness of the function or of its given derivative.

Recently, in solving a series of problems in the approximation theory, as a characteristic of the smoothness of a function, various modifications of the classical definition of the modulus of continuity are used. For instance, in the case of approximation of 2π-periodic functions, instead of classical shift operator $T_h f(x) = f(x+h)$, in [19], [20] the Steklov function $S_h(f)$ was used. This article continues these studied and provides a generalization and development of ideas presented in works [19]–[21].

M.Sh. Shabozov, Z.Sh. Malakbozov, Sharp Jackson–Stechkin type inequalities in Hardy space H_2 and widths of functional classes.

Submitted May 4, 2022.

Let $\mathbb{N}$, $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$, $\mathbb{C}$ be respectively the set of natural, non-negative integer and complex numbers, $U := \{z \in \mathbb{C} : |z| < 1\}$ be an open unit circle in $\mathbb{C}$ and $A(U)$ be the set of functions analytic in U.

An analytic in the unit disk $U := \{z \in \mathbb{C} : |z| < 1\}$ function

$$f(z) = \sum_{k=0}^{\infty} c_k(f) z^k, \quad z = \rho e^{it}, \quad 0 \leqslant \rho < 1, \tag{1.1}$$

is said to belong to the Hardy space H_2 [17] if

$$\|f\|_2 := \|f\|_{H_2} = \lim_{\rho\to 1-0} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{it})|^2 dt\right)^{1/2} < \infty. \tag{1.2}$$

It is well-known, see, for instance, [17], that the integral in (1.2) does not increases as ρ increases and almost everywhere on the circumference $|z| = 1$ there exist angular values $f(e^{it}) := F(t)$. At the same time, $F \in L_2 := L_2[0, 2\pi]$ and

$$\|f\|_2 := \|F\|_{L_2} = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 dt\right)^{1/2}. \tag{1.3}$$

We define the derivative of a function $f \in A(U)$ of rth order as usually:

$$f^{(r)}(z) := \frac{d^{(r)}f}{dz^r} = \sum_{k=r}^{\infty} k(k-1)\cdots(k-r+1)c_k(f)z^{k-r}, \quad r \in \mathbb{N}, \tag{1.4}$$

while the angular value of the derivative is denoted by $f^{(r)}(e^{it})$. For the sake of brevity we introduce the notation

$$\alpha_{n,m} := n(n-1)\cdots(n-m+1) = \frac{n!}{(n-m)!}, \quad n, m \in \mathbb{N}, \quad n \geqslant m.$$

Here we let $\alpha_{n,0} \equiv 1$, $\alpha_{n,1} = n$, $n \in \mathbb{N}$. Now we shortly write identity (1.4) as

$$f^{(r)}(z) = \sum_{k=r}^{\infty} \alpha_{k,r}\, c_k(f) z^{k-r}. \tag{1.5}$$

Hereinafter, by the symbol $H_2^{(r)} (r \in \mathbb{Z}_+,\ H_2^{(0)} = H_2)$ we denote the set of functions $f \in A(U)$ belonging to the Hardy space H_2, the derivative of which of rth order $f^{(r)}(z)$ also belongs to H_2, that is,

$$H_2^{(r)} := \Big\{f \in H_2 : \|f^{(r)}\|_2 < \infty\Big\}.$$

Let $\mathcal{P}_{n-1}$ be the subspace of complex algebraic polynomials of degree at most $n-1$. Since for $f \in H_2^{(r)}$, all its derivatives $f^{(s)}$, $s = 1, 2, \ldots, r-1$, also belong to the space H_2, see [18], then it is of a natural interest to find exact values of joint approximations for the functions f and their derivatives $f^{(s)}$, $s \geqslant 2$, $s = \overline{1, r-1}$,

$$E_{n-s-1}(f^{(s)})_2 := \inf\Big\{\|f^{(s)} - p_{n-1}^{(s)}\|_2 : p_{n-1} \in \mathcal{P}_{n-1}\Big\}$$

on some subset $\mathfrak{M}^{(r)} \subseteq H_2^{(r)}$ or on the class $H_2^{(r)}$. Thus, we need to find an exact value of the quantity

$$\mathcal{E}_{n-s-1}^{(s)}(\mathfrak{M})_2 := \sup\Big\{E_{n-s-1}(f^{(s)})_2 : f \in \mathfrak{M}\Big\}. \tag{1.6}$$

Since in the present work we use only the norms in the spaces H_2 and L_2, in view of relation (1.3) hereinafter we omit the subscripts of the norms $\|\cdot\|_2$ and $\|\cdot\|_{L_2}$. In the same way we do for the quantities defined by means of these norms, for instance, instead of $E_{n-s-1}\big(f^{(s)}\big)_2$, $\mathcal{E}^{(s)}_{n-s-1}(\mathfrak{M})_2$ we write $E_{n-s-1}\big(f^{(s)}\big)$, $\mathcal{E}^{(s)}_{n-s-1}(\mathfrak{M})$.

2. Auxiliary statements

In what follows we shall make use the following known statements.

Lemma 2.1 ([21]). *Let $f \in H_2^{(r)}$, $r,n \in \mathbb{N}$, $n > r$. Then for each $s \in \mathbb{Z}_+$, $0 \leqslant s \leqslant r$ the inequality holds:*

$$E_{n-s-1}\big(f^{(s)}\big) = \left(\sum_{k=n}^{\infty} \alpha_{k,s}^2 |c_k(f)|^2\right)^{1/2}. \tag{2.1}$$

Lemma 2.2 ([21]). *For an arbitrary function $f \in H_2^{(r)}$, $r \in \mathbb{N}$, for all $n \in \mathbb{N}$, $s \in \mathbb{Z}_+$, obeying the condition $n > r \geqslant s$ the inequality holds:*

$$E_{n-s-1}\big(f^{(s)}\big) \leqslant \frac{\alpha_{n,s}}{\alpha_{n,r}} E_{n-r-1}\big(f^{(r)}\big). \tag{2.2}$$

There exists a function $g \in H_2^{(r)}$, for which inequality (2.2) becomes the identity.

Let

$$S_h f(e^{ix}) = \frac{1}{2h}\int_{x-h}^{x+h} f(e^{it})dt, \quad h > 0 \tag{2.3}$$

be the Steklov function of the boundary value $f(\rho e^{it})$ of the function $f \in H_2$. We let $S_{h,k}(f) := S_h\left(S_{h,k-1}(f)\right)$, where $k \in \mathbb{N}$ and $S_{h,0}(f) \equiv f$, $\mathbb{E}$ is the identity mapping in the space H_2. Following [20], we denote the first and higher order differences by the relations

$$\begin{aligned}
\widetilde{\Delta}_h^1 f(e^{ix}) =& S_h f(e^{ix}) - f(e^{ix}) = (S_h - \mathbb{E}) f(e^{ix}),\\
\widetilde{\Delta}_h^m f(e^{ix}) =& \widetilde{\Delta}_h^1\big(\widetilde{\Delta}_h^{m-1} f(e^{ix})\big) = \big(S_h - \mathbb{E}\big)^m f(e^{ix}) = \sum_{k=0}^{m} (-1)^{m-k}\binom{m}{k} S_{h,k}(f(e^{ix})),
\end{aligned}$$

where $m = 2, 3, \ldots$. Using the introduced notations, we consider a smoothness characteristics of a function $f \in H_2$:

$$\widetilde{\omega}_m(f,t) := \widetilde{\omega}_m(f,t)_2 = \sup\left\{\big\|\widetilde{\Delta}_h^m f(e^{i(\cdot)})\big\| : 0 < h \leqslant t\right\}, \tag{2.4}$$

which we call a generalized modulus of continuity of mth order. Hereinafter we let

$$\operatorname{sinc} t := \begin{cases} \dfrac{\sin t}{t} & \text{as} \quad t \neq 0,\\ 1 & \text{as} \quad t = 0. \end{cases}$$

Since in view of identities (2.3) and (1.1)

$$\begin{aligned}
\widetilde{\Delta}_h^1(f, e^{ix}) &= \frac{1}{2h}\int_0^h \left\{f(e^{i(x+t)}) + f(e^{i(x-t)}) - 2f(e^{ix})\right\} dt\\
&= \sum_{k=1}^{\infty} c_k(f) e^{ikx} \cdot \frac{1}{2h}\int_0^h \left\{e^{ikt} + e^{-ikt} - 2\right\} dt\\
&= \sum_{k=1}^{\infty} c_k(f) e^{ikx} \cdot \frac{1}{h}\int_0^h (\cos kt - 1) dt = -\sum_{k=1}^{\infty} c_k(f) e^{ikx}(1 - \operatorname{sinc} kh)
\end{aligned}$$

and by induction for each $m \in \mathbb{N}$, $m \geqslant 2$, we have

$$\widetilde{\Delta}_h^m(f, e^{ix}) = (-1)^m \sum_{k=1}^{\infty} c_k(f) e^{ikx} (1 - \operatorname{sinc} kh)^m, \tag{2.5}$$

by applying the Parseval identity to (2.5) we obtain

$$\|\widetilde{\Delta}_h^m(f)\| = \sum_{k=1}^{\infty} |c_k(f)|^2 (1 - \operatorname{sinc} kh)^{2m}.$$

This allows us to write an explicit form for quantity (2.4)

$$\widetilde{\omega}_m(f, t) = \sup \left\{ \left(\sum_{k=1}^{\infty} |c_k(f)|^2 (1 - \operatorname{sinc} kh)^{2m} \right)^{1/2} \; : \; 0 < h \leqslant t \right\}. \tag{2.6}$$

It follows from identities (1.5) and (1.1) that the coefficients $c_k(f^{(r)})$ in the Maclaurin series of the derivative $f^{(r)}$ and the coefficients $c_k(f)$ in the Maclaurin series of the function f are related by the identity

$$c_k(f^{(r)}) := \alpha_{k,r}\, c_k(f). \tag{2.7}$$

Taking into consideration (2.7) and (2.6), for an arbitrary function $f \in H_2^{(r)}$ we have:

$$\widetilde{\omega}_m(f^{(r)}, t) = \sup \left\{ \left(\sum_{k=r}^{\infty} \alpha_{k,r}^2 |c_k(f)|^2 (1 - \operatorname{sinc}(k-r)h)^{2m} \right)^{1/2} \; : \; 0 < h \leqslant t \right\}. \tag{2.8}$$

Lemma 2.3. *Let $m, n \in \mathbb{N}$, $r, s \in \mathbb{Z}_+$, $n > r \geqslant s$. Then for each $t \in (0, \frac{3\pi}{4(n-r)}]$ the inequality*

$$\widetilde{\omega}_m(f^{(r)}, t) \geqslant \frac{\alpha_{n,r}}{\alpha_{n,s}} (1 - \operatorname{sinc}(n-r)t)^m E_{n-s-1}(f^{(s)}) \tag{2.9}$$

holds true. This inequality is sharp in the sense that there exists a function $f_0 \in H_2^{(r)}$, for which this inequality becomes the identity.

Proof. Using the fact that for $0 < nh \leqslant \frac{3\pi}{4}$ [22]

$$\max\{\operatorname{sinc} x : 0 < |t| \leqslant n\tau\} = \operatorname{sinc} n\tau,$$
$$\min\{(1 - \operatorname{sinc} u)^m : u \geqslant nt\} = (1 - \max_{u \geqslant nt} \operatorname{sinc} u)^m = (1 - \operatorname{sinc} n\tau)^m,$$

by (2.8) for an arbitrary function $f \in H_2^{(r)}$ we obtain

$$\begin{aligned}
\widetilde{\omega}_m^2(f^{(r)}, t) &\geqslant \sum_{k=n}^{\infty} \alpha_{k,r}^2 |c_k(f)|^2 (1 - \operatorname{sinc}(k-r)h)^{2m} \\
&\geqslant (1 - \operatorname{sinc}(n-r)t)^{2m} \cdot \sum_{k=n}^{\infty} \alpha_{k,r}^2 |c_k(f)|^2 \\
&= (1 - \operatorname{sinc}(n-r)t)^{2m} \cdot \sum_{k=n}^{\infty} \left(\frac{\alpha_{k,r}}{\alpha_{k,s}} \right)^2 \alpha_{k,s}^2 |c_k(f)|^2 \\
&\geqslant (1 - \operatorname{sinc}(n-r)t)^{2m} \cdot \min_{k \geqslant n} \left(\frac{\alpha_{k,r}}{\alpha_{k,s}} \right)^2 \cdot \sum_{k=n}^{\infty} \alpha_{k,s}^2 |c_k(f)|^2 \\
&= (1 - \operatorname{sinc}(n-r)t)^{2m} \cdot \min_{k \geqslant n} \left(\frac{\alpha_{k,r}}{\alpha_{k,s}} \right)^2 \cdot E_{n-s-1}^2(f^{(s)}).
\end{aligned} \tag{2.10}$$

It was proved in [22] that for $k \geqslant n > r \geqslant s$

$$\min_{k\geqslant n} \frac{\alpha_{k,r}}{\alpha_{k,s}} = \frac{\alpha_{n,r}}{\alpha_{n,s}}, \tag{2.11}$$

and this is why, taking into consideration (2.11), by (2.10) we obtain (2.9). For a function $f_0(z) = z^n \in H_2^{(r)}$, which due to identities (2.1) and (2.8) satisfies

$$E_{n-s-1}(f_0^{(s)}) = \alpha_{n,s}, \qquad \widetilde{\omega}_m(f_0^{(r)}, t) = \alpha_{n,r}(1 - \operatorname{sinc}(n-r)t)^m, \tag{2.12}$$

by (2.12) we obtain:

$$\begin{aligned}\widetilde{\omega}_m(f_0^{(r)}, t) = \alpha_{n,r}(1 - \operatorname{sinc}(n-r)t)^m &= \frac{\alpha_{n,r}}{\alpha_{n,s}}(1 - \operatorname{sinc}(n-r)t)^m \alpha_{n,s} \\ &= \frac{\alpha_{n,r}}{\alpha_{n,s}}(1 - \operatorname{sinc}(n-r)t)^m E_{n-s-1}(f_0^{(s)}),\end{aligned}$$

which implies the statement of the lemma. The proof is complete. □

In what follows by a weight function on the segment $[0, h]$ we mean a non-negative summable function q, which is non-equivalent to the zero on the same segment. The following theorem holds true.

Theorem 2.1. *Let $m, n \in \mathbb{N}$, $r, s \in \mathbb{Z}_+$, $n > r \geqslant s$, $0 < p \leqslant \infty$, $0 < h \leqslant \frac{3\pi}{4(n-r)}$ and q be a weight function on the segment $[0, h]$. Then the identity holds:*

$$\sup_{f\in H_2^{(r)}} \frac{(\alpha_{n,r}/\alpha_{n,s}) \cdot E_{n-s-1}(f^{(s)})}{\left\{\int_0^h \widetilde{\omega}_m^p(f^{(r)}, t) q(t) dt\right\}^{1/p}} = \left\{\int_0^h (1 - \operatorname{sinc}(n-r)t)^{mp} q(t) dt\right\}^{-1/p}. \tag{2.13}$$

Proof. We take the pth $(0 < p \leqslant \infty)$ power of the both sides of inequality (2.9), multiply by the weight function q and integrate from 0 to h, where $0 < h \leqslant \frac{3\pi}{4(n-r)}$. Taking then the root of the power $1/p$, from the obtained identity we pass to the inequality

$$\left(\int_0^h \widetilde{\omega}_m^p(f^{(r)}, t) q(t) dt\right)^{1/p} \geqslant (\alpha_{n,r}/\alpha_{n,s}) \cdot E_{n-s-1}(f^{(s)}) \left(\int_0^h (1 - \operatorname{sinc}(n-r)t)^{mp} q(t) dt\right)^{1/p}.$$

The obtained inequality holds true for an arbitrary function $f \in H_2^{(r)}$ and this is why it implies an upper bound for the quantity in the left hand side of identity (2.13):

$$\sup_{f\in H_2^{(r)}} \frac{(\alpha_{n,r}/\alpha_{n,s}) \cdot E_{n-s-1}(f^{(s)})}{\left(\int_0^h \widetilde{\omega}_m^p(f^{(r)}, t) q(t) dt\right)^{1/p}} \leqslant \left(\int_0^h (1 - \operatorname{sinc}(n-r)t)^{mp} q(t) dt\right)^{-1/p}. \tag{2.14}$$

In order to obtain a similar lower bound for the mentioned quantity, we consider a function $f_0(z) = z^n \in H_2^{(r)}$, which was introduced in the proof of Lemma 2.3 and for which identities (2.12) hold. Using identities (2.12), we write the lower bound

$$\begin{aligned}\sup_{f\in H_2^{(r)}} \frac{(\alpha_{n,r}/\alpha_{n,s}) \cdot E_{n-s-1}(f^{(s)})}{\left(\int_0^h \widetilde{\omega}_m^p(f^{(r)}, t) q(t) dt\right)^{1/p}} &\geqslant \frac{(\alpha_{n,r}/\alpha_{n,s}) \cdot E_{n-s-1}(f_0^{(s)})}{\left(\int_0^h \widetilde{\omega}_m^p(f_0^{(r)}, t) q(t) dt\right)^{1/p}} \\ &= \left(\int_0^h (1 - \operatorname{sinc}(n-r)t)^{mp} q(t) dt\right)^{-1/p}.\end{aligned} \tag{2.15}$$

We obtain needed identity (2.13) by comparing upper bound (2.14) with lower bound (2.15) and this completes the proof of the theorem. □

Theorem 2.1 implies a series of corollaries.

Corollary 2.1. *Suppose that, under the assumptions of Theorem 2.1, $m, n \in \mathbb{N}$, $r, s \in \mathbb{Z}_+$, $n > r \geqslant s$, $p = 1/m$, $0 < h \leqslant \frac{3\pi}{4(n-r)}$, $q(t) \equiv 1$. Then*

$$\sup_{f \in H_2^{(r)}} \frac{(\alpha_{n,r}/\alpha_{n,s}) \cdot E_{n-s-1}(f^{(s)})}{\left(\int_0^h \widetilde{\omega}_m^{1/m}(f^{(r)}, t)dt\right)^m} = \left\{\frac{n-r}{(n-r)h - Si(n-r)h}\right\}^m,$$

where $Si(t) := \int_0^t \operatorname{sinc} u\,du$ is the integral sine.

If, under the same assumptions, $q(t) = t$, by (2.15) we then have

$$\sup_{f \in H_2^{(r)}} \frac{(\alpha_{n,r}/\alpha_{n,s}) \cdot E_{n-s-1}(f^{(s)})}{\left(\int_0^h t\,\widetilde{\omega}_m^{1/m}(f^{(r)}, t)dt\right)^m} = \frac{(n-r)^{2m}}{2^m}\left\{\left[\frac{(n-r)h}{2}\right]^2 - \sin^2\left[\frac{(n-r)h}{2}\right]\right\}^{-m}. \tag{2.16}$$

In particular, it follows from (2.16) with $h = \frac{\pi}{2(n-r)}$ that

$$\sup_{f \in H_2^{(r)}} \frac{(\alpha_{n,r}/\alpha_{n,s}) \cdot E_{n-s-1}(f^{(s)})}{\left((n-r)^2 \int_0^{\pi/2(n-r)} t\,\widetilde{\omega}_m^{1/m}(f^{(r)}, t)dt\right)^m} = \left(\frac{4}{\pi^2 - 8}\right)^m.$$

3. Widths of some classes of functions

In order to formulate further results, we first introduce needed notation and definitions. Let $\mathcal{B}$ be a unit ball in the space H_2; $\mathcal{M}$ be a convex centrally symmetric subset in H_2; $\Lambda_n \subset H_2$ be an n-dimensional subspace; $\Lambda^n \subset H_2$ be a subspace of codimension n; $\mathcal{L} : H_2 \to \Lambda_n$ be a continuous linear operator mapping the elements of the space H_2 into Λ_n; $\mathcal{L}^\perp : H_2 \to \Lambda_n$ be a continuous operator of linear projecting of H_2 onto the subspace Λ_n. The quantities

$$\begin{aligned}
b_n(\mathcal{M}, H_2) &:= \sup\{\sup\{\varepsilon > 0;\ \varepsilon\mathcal{B} \cap \Lambda_{n+1} \subset \mathcal{M}\} : \Lambda_{n+1} \subset H_2\},\\
d_n(\mathcal{M}, H_2) &:= \inf\{\sup\{\inf\{\|f - g\|_2 : g \in \Lambda_n\} : f \in \mathcal{M}\} : \Lambda_n \subset H_2\},\\
\delta_n(\mathcal{M}, H_2) &:= \inf\{\sup\{\inf\{\|f - \mathcal{L}(f)\|_2 : f \in \mathcal{M}\} : \mathcal{L}H_2 \subset \Lambda_n\} : \Lambda_n \subset H_2\},\\
d^n(\mathcal{M}, H_2) &:= \inf\{\sup\{\|f\|_2 : f \in \mathcal{M} \cap \Lambda_n\} : \Lambda_n \subset H_2\},\\
\Pi_n(\mathcal{M}, H_2) &:= \inf\big\{\inf\big\{\sup\big\{\|f - \mathcal{L}^\perp(f)\|_2 : f \in \mathcal{M}\big\} : \mathcal{L}^\perp H_2 \subset \Lambda_n\big\} : \Lambda_n \subset H_2\big\}
\end{aligned}$$

are respectively called *Bernstein, Kolmogorov, linear, Gelfand, projection n-width.* In the Hilbert space H_2 these quantities are related as follows, see [23], [24]:

$$b_n(\mathcal{M}, H_2) \leqslant d^n(\mathcal{M}, H_2) \leqslant d_n(\mathcal{M}, H_2) = \delta_n(\mathcal{M}, H_2) = \Pi_n(\mathcal{M}, H_2). \tag{3.1}$$

Using smoothness characteristics (2.4), we define the following classes of functions. Let $\Phi(t)$, $t \in \mathbb{R}_+$, be a continuous non-decreasing function such that $\Phi(0) = 0$. By the symbol $W_p^{(r)}(\omega_m, \Phi)$, $0 < p \leqslant \infty$, $r \in \mathbb{Z}_+$, we denote the class of functions $f \in H_2^{(r)}$, which for each $t \in \mathbb{R}_+$ obeys the inequality

$$\left(\frac{1}{t}\int_0^t \widetilde{\omega}_m^p(f^{(r)}, \tau)d\tau\right)^{1/p} \leqslant \Phi(t).$$

By t_* we denote the point, at which the function $\operatorname{sinc} t$ attains its minimal value on $\mathbb{R}_+$. This point t_*, $4.49 < t_* < 4.51$, is the smallest positive root of the equation $t = \tan t$. Following [19], we introduce the notation

$$(1-\operatorname{sinc} t)_* := \begin{cases} 1-\operatorname{sinc} t & \text{as} \quad 0 \leqslant t \leqslant t_*, \\ 1-\operatorname{sinc} t_* & \text{as} \quad t_* \leqslant t < \infty. \end{cases}$$

We also let

$$E_{n-1}(\mathfrak{M}) := \sup\Big\{E_{n-1}(f) : f \in \mathfrak{M}\Big\},$$

where $\mathfrak{M}$ is some class of functions in H_2.

Theorem 3.1. *Let $m, n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n > r$, $0 < p \leqslant \infty$ and the function Φ for all values $t \in \mathbb{R}_+$ obeys the restriction*

$$\left(\frac{\Phi(t)}{\Phi(\pi/(n-r))}\right)^p \geqslant \frac{\pi}{2(n-r)t} \frac{\displaystyle\int_0^{(n-r)t} (1-\operatorname{sinc}\tau)_*^{mp}\, d\tau}{\displaystyle\int_0^{\pi/2} (1-\operatorname{sinc}\tau)^{mp}\, d\tau}. \tag{3.2}$$

Then the identities hold

$$\begin{aligned} \lambda_n(W_p^{(r)}(\omega_m, \Phi); H_2) &= E_{n-1}(W_p^{(r)}(\omega_m, \Phi)) \\ &= \left(\frac{2}{\pi}\int_0^{\pi/2} (1-\operatorname{sinc}\, t)^{mp} dt\right)^{-1/p} \cdot \frac{1}{\alpha_{n,r}} \cdot \Phi\Big(\frac{\pi}{2(n-r)}\Big), \end{aligned} \tag{3.3}$$

where $\lambda_n(\cdot)$ is an arbitrary of the aforementioned n-widths. The set of majorants Φ obeying condition (3.2) *is non-empty.*

Proof. Using relation (2.13), in which we let $s = 0$, $q(t) \equiv 1$, $h = \frac{\pi}{2(n-r)}$, for an arbitrary function $f \in H_2^{(r)}$ we write an upper bound for the quantity $E_{n-1}(f)$:

$$\begin{aligned} E_{n-1}(f)_2 &\leqslant \frac{1}{\alpha_{n,r}} \left(\int\limits_0^{\pi/2(n-r)} \Big(1-\operatorname{sinc}(n-r)t\Big)^{mp} dt\right)^{-1/p} \left(\int\limits_0^{\pi/2(n-r)} \widetilde{\omega}_m^p(f^{(r)}, t)\, dt\right)^{1/p} \\ &= \frac{1}{\alpha_{n,r}} \left(\frac{2}{\pi}\int\limits_0^{\pi/2} (1-\operatorname{sinc} \mathrm{t})^{mp}\, dt\right)^{-1/p} \left(\frac{2(n-r)}{\pi}\int\limits_0^{\pi/2(n-r)} \widetilde{\omega}_m^p(f^{(r)}, t)\, dt\right)^{1/p}. \end{aligned} \tag{3.4}$$

Taking into consideration the definition of the class $W_p^{(r)}(\widetilde{\omega}_m, \Phi)$, on the base of relation (2.16) between the n-widths and inequality (3.4), we find:

$$\begin{aligned} \lambda_n(W_p^{(r)}(\widetilde{\omega}_m, \Phi), H_2) &\leqslant E_{n-1}(W_p^{(r)}(\widetilde{\omega}_m, \Phi)) \\ &\leqslant \frac{1}{\alpha_{n,r}} \left\{\frac{2}{\pi}\int_0^{\pi/2} (1-\operatorname{sinc} t)^{mp} dt\right\}^{-1/p} \cdot \Phi\Big(\frac{\pi}{2(n-r)}\Big). \end{aligned} \tag{3.5}$$

To obtain lower bounded for the aforementioned n-widths, it is sufficient to estimate from below the Bernstein n-width of the conisdered class. In order to do this, we introduce a ball

$$\mathcal{B}_{n+1} := \left\{p_n \in \mathcal{P}_n : \|p_n\| \leqslant \frac{1}{\alpha_{n,r}} \left(\frac{2}{\pi}\int\limits_0^{\pi/2} (1-\operatorname{sinc} t)^{mp} dt\right)^{-1/p} \Phi\Big(\frac{\pi}{2(n-r)}\Big)\right\}.$$

By formula (2.8) for an arbitrary function $f \in H_2^{(r)}$ we have

$$\|\widetilde{\Delta}_h^m(p_n^{(r)},\cdot)\|^2 = \sum_{k=r}^{n} \alpha_{k,r}^2 |c_k(p_n)|^2 (1-\operatorname{sinc}(k-r)h)^{2m}$$
$$\leqslant \alpha_{n,r}^2 \sum_{k=r}^{n} (1-\operatorname{sinc}(k-r)h)^{2m} |c_k(p_n)|^2 \leqslant \alpha_{n,r}^2 (1-\operatorname{sinc}(n-r)h)_*^{2m} \cdot \|p_n\|^2.$$

This yields

$$\widetilde{\omega}_m^p(p_n^{(r)},\tau) \leqslant (\alpha_{n,r})^p (1-\operatorname{sinc}(n-r)\tau)_*^{mp} \|p_n\|^p. \tag{3.6}$$

Using inequality (3.6) and restrictions (3.2), for an arbitrary polynomial $p_n \subset \mathcal{B}_{n+1}$ we write

$$\frac{1}{t}\int_0^t \widetilde{\omega}_m^p(p_n^{(r)},\tau)\,d\tau \leqslant (\alpha_{n,r})^p \|p_n\|^p \frac{1}{t}\int_0^t (1-\operatorname{sinc}(n-r)\tau)_*^{mp}\,d\tau$$
$$\leqslant \left(\frac{2}{\pi}\int_0^{\pi/2} (1-\operatorname{sinc} t)^{mp} dt\right)^{-1} \frac{1}{(n-r)t}\int_0^{(n-r)t} (1-\operatorname{sinc}\tau)_*^{mp}\,d\tau \cdot \Phi^p\Big(\frac{\pi}{2(n-r)}\Big)$$
$$\leqslant \Phi^p(t).$$

Therefore, $\mathcal{B}_{n+1} \subset W_p^{(r)}(\widetilde{\omega}_m,\Phi)$, and using relations (3.1) and the definition of the Bernstein n-width, we obtain

$$\begin{aligned}\lambda_n(W_p^{(r)}(\widetilde{\omega}_m,\Phi);H_2) &\geqslant b_n(W_p^{(r)}(\widetilde{\omega}_m,\Phi),H_2) \geqslant b_n(\mathcal{B}_{n+1},H_2)\\ &\geqslant \frac{1}{\alpha_{n,r}}\left(\frac{2}{\pi}\int_0^{\pi/2}(1-\operatorname{sinc} t)^{mp}dt\right)^{-1/p} \Phi\Big(\frac{\pi}{2(n-r)}\Big).\end{aligned} \tag{3.7}$$

Comparing inequalities (3.5) and (3.7), we obtain identity (3.3). The proof is complete. □

It was shown in [19], that the set of majorants obeying restriction (3.2) is non-empty and for instance, this restriction is satisfied by the majorant $\Phi_*(t) := t^{m\alpha/2}$, where

$$\alpha := \frac{(\pi-2)^2}{2\pi\int_0^{\pi/2}(1-\operatorname{sinc}\tau)^2 d\tau} - 1.$$

4. Solution to extremal problem (1.6) for class $W_p^{(r)}(\omega_m,\Phi)$

There is a certain interest is in studying the behavior of the quantities $E_{n-1}(f^{(s)})$, $s = 0,1,\ldots,r$, on the class of functions $W_p^{(r)}(\widetilde{\omega}_m,\Phi)$, $m\in\mathbb{N}$, $r\in\mathbb{Z}_+$, $0<p\leqslant\infty$. In other words, one needs to find an exact value of quantity (1.6) as $\mathfrak{M}^{(r)} = W_p^{(r)}(\omega_m,\Phi)$.

Theorem 4.1. *Let $m,n\in\mathbb{N}$, $r,s\in\mathbb{Z}_+$, $n>r\geqslant s$. If for each $t\in(0,2\pi]$ the majorant Φ obeys restriction (3.2), then for each $s=0,1,2\ldots,r$ the identity holds:*

$$\mathcal{E}_{n-s-1}^{(s)}\big(W_p^{(r)}(\omega_m,\Phi)\big) = \frac{\alpha_{n,s}}{\alpha_{n,r}}\left\{\frac{2}{\pi}\int_0^{\pi/2}(1-\operatorname{sinc}\ t)^{mp}dt\right\}^{-1/p}\Phi\Big(\frac{\pi}{2(n-r)}\Big). \tag{4.1}$$

Proof. By inequality (2.15) with $q(t) \equiv 1$ and $h = \pi/2(n-r)$ for an arbitrary function $f \in H_2^{(r)}$ we obtain

$$E_{n-s-1}(f^{(s)}) \leqslant \frac{\alpha_{n,s}}{\alpha_{n,r}} \left(\int\limits_0^{\pi/2(n-r)} (1 - \operatorname{sinc}(n-r)t)^{mp} dt \right)^{-1/p} \left(\int\limits_0^{\pi/2(n-r)} \widetilde{\omega}_m^p (f^{(r)}, t)\, dt \right)^{1/p}$$

$$= \frac{\alpha_{n,s}}{\alpha_{n,r}} \left(\frac{2}{\pi} \int\limits_0^{\pi/2} (1 - \operatorname{sinc} t)^{mp} dt \right)^{-1/p} \left(\frac{2(n-r)}{\pi} \int\limits_0^{\pi/2(n-r)} \widetilde{\omega}_m^p (f^{(r)}, t)\, dt \right)^{1/p}.$$

Taking into consideration the definition of the class $W_p^{(r)}(\omega_m, \Phi)$, we hence have

$$\mathcal{E}_{n-s-1}^{(s)}\big(W_p^{(r)}(\omega_m, \Phi)\big) \leqslant \frac{\alpha_{n,s}}{\alpha_{n,r}} \left(\frac{2}{\pi} \int\limits_0^{\pi/2} (1 - \operatorname{sinc} t)^{mp}\, dt \right)^{-1/p} \Phi\Big(\frac{\pi}{2(n-r)}\Big). \tag{4.2}$$

It has been established in the proof of Theorem 3.1 that the set of algebraic complex-valued polynomials $p_n \in \mathcal{P}_n$ obeying the condition

$$\|p_n\| \leqslant \frac{1}{\alpha_{n,r}} \left(\frac{2}{\pi} \int_0^{\pi/2} (1 - \operatorname{sinc} t)^{mp} dt \right)^{-1/p} \Phi\Big(\frac{\pi}{2(n-r)}\Big),$$

belongs to the class $W_p^{(r)}(\omega_m, \Phi)$. We consider the function

$$g(z) = \frac{1}{\alpha_{n,r}} \left(\frac{2}{\pi} \int_0^{\pi/2} (1 - \operatorname{sinc} t)^{mp} dt \right)^{-1/p} \Phi\Big(\frac{\pi}{2(n-r)}\Big) z^n.$$

For each $s = 0, 1, \ldots, r$ this function satisfies

$$\begin{aligned} g^{(s)}(z) &= \frac{\alpha_{n,s}}{\alpha_{n,r}} \left(\frac{2}{\pi} \int_0^{\pi/2} (1 - \operatorname{sinc} t)^{mp} dt \right)^{-1/p} \Phi\Big(\frac{\pi}{2(n-r)}\Big) z^{n-s}, \\ E_{n-s-1}(g^{(s)}) &= \frac{\alpha_{n,s}}{\alpha_{n,r}} \left(\frac{2}{\pi} \int_0^{\pi/2} (1 - \operatorname{sinc} t)^{mp} dt \right)^{-1/p} \Phi\Big(\frac{\pi}{2(n-r)}\Big) \end{aligned} \tag{4.3}$$

and since

$$\|g\| = \frac{1}{\alpha_{n,r}} \left(\frac{2}{\pi} \int_0^{\pi/2} (1 - \operatorname{sinc} t)^{mp} dt \right)^{-1/p} \Phi\Big(\frac{\pi}{2(n-r)}\Big),$$

then $g \in W_p^{(r)}(\omega_m, \Phi)$, and this is why by (4.3) we have

$$\begin{aligned} \mathcal{E}_{n-s-1}^{(s)}\big(W_p^{(r)}(\omega_m, \Phi)\big) &\geqslant E_{n-s-1}(g^{(s)}) \\ &= \frac{\alpha_{n,s}}{\alpha_{n,r}} \cdot \left(\frac{2}{\pi} \int_0^{\pi/2} (1 - \operatorname{sinc} t)^{mp} dt \right)^{-1/p} \Phi\Big(\frac{\pi}{2(n-r)}\Big). \end{aligned} \tag{4.4}$$

Comparing relations (4.2) and (4.4), we arrive at required identities (4.1). The proof is complete. □

BIBLIOGRAPHY

1. K.I. Babenko. *Best approximations to a class of analytic functions* // Izv. AN SSSR. Ser. Matem. **22**:5, 631–640 (1958). (in Russian).
2. L.V. Taikov. *On the best linear methods of approximating functions of classes* $\mathcal{B}^r$ *and* $\mathcal{H}^r$ // Uspekhi Matem. Nauk. **18**:4, 183–189 (1963). (in Russian).
3. L.V. Taikov. *Best approximation in the mean of certain classes of analytic functions* // Matem. Zamet. **1**:2, 155–162 (1967). [Math. Notes. **1**:2, 104–109 (1967).]
4. L.V. Taikov. *Diameters of certain classes of analytic functions* // Matem. Zamet. **22**:2, 285–295 (1977). [Math. Notes. **22**:2, 650–656 (1977).]
5. M.Z. Dveirin, I.V. Chebanenko. *On polynomial approximation of analytic functions in Banach spaces* // in "Theory of mappings and approximation of functions", Coll. Sci. Works. Inst. Appl. Math. Mekh. AN USSR, 62–73 (1983). (in Russian).
6. N.Aynulloev, L.V. Taikov. *Best approximation in the sense of Kolmogorov of classes of functions analytic in the unit disc* // Matem. Zamet. **40**:3, 341–351 (1986). [Math. Notes. **40**:3, 699–705 (1986).]
7. L.V. Taikov. *Some sharp inequalities in the theory of the approximation of functions* // Anal. Math. **2**:1, 77–85 (1976). (in Russian).
8. S.B. Vakarchuk. *Diameters of certain classes of functions analytic in the unit disc I, II* // Ukrain. Matem. Zhurn. **42**:7, 873–881 (1990); **42**:8, 1019–1026 (1990). [Ukrain. Math. J. **42**:7, 769–778 (1990); **42**:7, 907–914 (1990).]
9. S.B. Vakarchuk. *Best linear methods of approximation and widths of classes of analytic functions in a disk* // Matem. Zamet. **57**:1, 30–39 (1995). [Math. Notes. **57**:1, 21–27 (1995).]
10. S.B. Vakarchuk. *On the best linear approximation methods and the widths of certain classes of analytic functions* // Matem. Zamet. **65**:2, 186–193 (1999). [Math. Notes. **65**:2, 153–158 (1990).]
11. S.B. Vakarchuk. *Exact values of widths of classes of analytic functions on the disk and best linear approximation methods* // Matem. Zamet. **72**:5, 665–669 (2002). [Math. Notes. **72**:5, 615–619 (2002).]
12. S.B. Vakarchuk, V.I. Zabutnaya. *Best linear approximation methods for functions of Taikov classes in the Hardy spaces* $H_{q,\rho}$, $q \geqslant 1$, $0 < \rho \leqslant 1$ // Matem. Zamet. **85**:3, 323–329 (2009). [Math. Notes. **85**:3, 322–327 (2009).]
13. M.Sh. Shabozov, O.Sh. Shabozov. *Widths of some classes of analytic functions in the Hardy space* H_2 // Matem. Zamet. **68**:5, 796–800 (2000). [Math. Notes. **68**:5, 675–679 (2000).]
14. M.Sh. Shabozov, G.A. Yusupov. *Best approximation and width of some classes of analytic functions* // Dokl. RAN. **382**:6, 747–749 (2002). [Dokl. Math. **65**:1, 111–113 (2002).]
15. M.Sh. Shabozov, Kh.Kh. Pirov. *Exact constants in Jackson-type inequalities for approximation of analytic functions from* H_p, $1 \leqslant p \leqslant 2$ // Dokl. RAN. **394**:4, 19–24 (2004). [Dokl. Math. **69**:1, 76–78 (2004).]
16. M.Sh. Shabozov, M.R. Langarshoev. *Best linear approximation methods for some classes of analytic functions on the unit disk* // Sibir. Matem. Zhurn. **60**:6, 1414–1423 (2019). [Sib. Math. J. **60**:6, 1101-1108 (2019).]
17. I.I. Privalov. *Boundary properties of analytic functions.* GITTL, Moscow (1950). (in Russian).
18. S.B. Vakarchuk, M.B. Vakarchuk. *Inequalities of Kolmogorov type for analytic functions of one and two complex variables and their applications to approximation theory* // Ukrain. Matem. Zhurn. **63**:12, 1579–1601 (2011). [Ukr. Math. J. **63**:12, 1795–1819 (2012).]
19. S.B. Vakarchuk. *Generalized smoothness characteristics in Jackson-type inequalities and widths of classes of functions in* L_2 // Matem. Zamet. **98**:4, 511–529 (2015). [Math. Notes. **98**:4, 572-588 (2015).]
20. M.Sh. Shabozov, A.A. Shabozova. *Sharp inequalities of Jackson-Stechkin type for periodic functions in* L_2 *differentiable in the Weyl sense* // Trudy Inst. Matem. Mekh. URO RAN. **25**:4, 255–264 (2019). (in Russian).

21. M.Sh. Shabozov, G.A. Yusupov, J.J. Zargarov. *On the best simultaneous polynomial approximation of functions and their derivatives in Hardy spaces* // Trudy Inst. Matem. Mekh. URO RAN. **27**:4, 239–254 (2021). (in Russian).
22. L.V. Taikov. *Inequalities containing best approximations and the modulus of continuity of functions in L_2* // Matem. Zamet. **20**:3, 433–438 (1976). [Math. Notes. **20**:3, 797–800 (1976).]
23. V.M. Tikhomirov. *Some issues of theory of approximation.* Moscow State Univ. Publ., Moscow (1976). (in Russian).
24. A. Pinkus. *n-Widths in approximation theory.* Springer-Verlag, Berlin (1985).

Mirgand Shabozovich Shabozov,
Tajik National University,
Rudaki av., 17,
734025, Dushanbe, Tajikistan
E-mail: shabozov@mail.ru

Zokir Sherozovich Malakbozov,
Tajik National University,
Rudaki av., 17,
734025, Dushanbe, Tajikistan
E-mail: malakbozov1994@mail.ru