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AVERAGING OF RANDOM AFFINE TRANSFORMATIONS OF FUNCTIONS DOMAIN

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Abstract. We study the averaging of Feynman-Chernoff iterations of random operator-valued strongly continuous functions, the values of which are bounded linear operators on separable Hilbert space. In this work we consider averaging for a certain family of such random operator-valued functions. Linear operators, being the values of the considered functions, act in the Hilbert space of square integrable functions on a finite-dimensional Euclidean space and they are defined by random affine transformations of the functions domain. At the same time, the compositions of independent identically distributed random affine transformations are a non-commutative analogue of random walk.

For an operator-valued function being an averaging of Feynman-Chernoff iterations, we prove an upper bound for its norm and we also establish that the closure of the derivative of this operator-valued function at zero is a generator a strongly continuous semigroup. In the work we obtain sufficient conditions for the convergence of the mathematical expectation of the sequence of Feynman-Chernoff iterations to the semigroup resolving the Cauchy problem for the corresponding Fokker-Planck equation.

Keywords: Feynman-Chernoff iterations, Chernoff theorem, operator-valued random process, Fokker-Planck equation.

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1. Introduction

The theory of statistical properties of products of independent random matrices and compositions of independent random transformations was intensively developed in the second half of the 20th century; its main foundations can be found, for example, in works [1]–[5].

In this paper we study averagings of Feynman-Chernoff iterations for a certain class of random operator-valued processes with values in the algebra of bounded linear operators on a separable Hilbert space. Linear operators, which are the values of the considered random processes, act in the Hilbert space of square-integrable functions on a finite-dimensional Euclidean space and are defined by random affine transformations of their variable. In this case, the compositions of independent identically distributed random affine transformations are a non-commutative analogue of random walks.

Mathematical models with compositions of random operator-valued functions arise in problems of classical and quantum mechanics for systems in random nonstationary fields [6]–[13]. The averaged dynamics of such systems is of both theoretical and practical interest from the point of view of analyzing the mean values of observables. In particular, it is important to understand to what extent the averaging of solutions of some evolutionary equation with non-stationary parameters is related to the solution of the equation averaged over these parameters. The use of the averaging procedure for this purpose by constructing Chernoff equivalent semigroups is a very efficient method, which was developed in [14]–[16].

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In this paper, for a certain class of operator-valued functions generated by transformations of their domains, we obtain sufficient conditions for the convergence of the mathematical expectation of the sequence of Feynman-Chernoff iterations of random affine transformations of the function variable to a semigroup resolving the Cauchy problem for the corresponding Fokker-Planck equation. In comparison with the results of work [17], we consider a wider class of random transformations and we omit the condition of the independence of the random linear part of the affine transformation and the shift transformation.

The structure of this work is as follows. The introduction is followed by a second section containing the necessary preliminary information including Chernoff theorem and the employed definitions of a random operator and the expectation of a random operator. In the third section we define a considered class of random operator-valued functions and we prove auxiliary lemmas. In the fourth section we formulate and prove Theorem 4.1, which is the main result of this paper, on the convergence of a sequence of Feynman-Chernoff iterations averagings to the corresponding Chernoff averaging semigroup.

2. Preliminaries

Let X be a Banach space, $B(X)$ be the space of linear bounded operators in X. We also introduce the notation $\mathbb{R}_+ = [0, +\infty)$.

An operator-valued function $F(t) : \mathbb{R}_+ \to B(X)$ is called strongly continuous if for each $u_0 \in X$ and each $t_0 \geqslant 0$ the identity

$$\lim_{t\to t_0} \|F(t)u_0 - F(t_0)u_0\|_X = 0 \tag{2.1}$$

holds.

We introduce a notation $C_s(\mathbb{R}_+, B(X))$ for a topological vector space of strongly continuous operator-valued functions $U(t) : [0, +\infty) \to B(X)$.

The topology τ_s in $C_s(\mathbb{R}_+, B(X))$ is generated by the family of semi-norms

$$\Phi_{T,v}(U) = \sup_{t\in[0,T]} \|U(t)v\|_X, \quad \forall T > 0, \quad \forall v \in X. \tag{2.2}$$

We observe that if $U, \{U_n\}_{n=0}^{\infty} \in C_s(\mathbb{R}_+, B(X))$, then

$$U_n \xrightarrow{\tau_s} U \quad \Leftrightarrow \quad \lim_{n\to\infty} \sup_{t\in[0,T]} \|U_n(t)v - U(t)v\|_X = 0, \quad \forall T > 0, \quad \forall v \in X. \tag{2.3}$$

An operator-valued function $U(t) : \mathbb{R}_+ \to B(X)$ is called a semigroup if $U(0) = I$ (the identity operator) and $U(t_1 + t_2) = U(t_1) \circ U(t_2)$, $\forall t_1, t_2 \in \mathbb{R}_+$. A semigroup is called strongly continuous or a C_0-semigroup if for each $u_0 \in X$ the identity holds

$$\lim_{t\to 0} \|U(t)u_0 - u_0\|_X = 0. \tag{2.4}$$

The proof the main result of this work employ essentially Chernoff theorem [18]. To formulate this theorem, we introduce the notion of the Chernoff equivalence.

Definition 2.1. *We say that a strongly continuous operator-valued function $F(t) : \mathbb{R}_+ \to B(X)$ is Chernoff equivalent to a strongly continuous semigroup $U(t) : \mathbb{R}_+ \to B(X)$ if $F^n\left(\frac{t}{n}\right) \xrightarrow{\tau_s} U(t)$.*

In terms of the above notations, the Chernoff theorem is formulated as follows.

Theorem (Chernoff, 1968). *Let an operator-valued function $F(t) \in C_s(\mathbb{R}_+, B(X))$ satisfies the conditions*

1. *$F(0)$ is the identity operator,*
2. *$\|F(t)\|_{B(\mathcal{H})} \leqslant e^{\alpha t}$, $t \geqslant 0$ for some $\alpha > 0$,*

3. the operator F_0' is closable and its closure is generator of strongly continuous semigroup $U(t)$.

Then the function $F(t)$ is Chernoff equivalent to the semigroup $U(t)$.

Let $\mathcal{H}$ be a separable Hilbert space with a scalar product $(\cdot\,,\,\cdot)$. We are going to introduce the notion of a random operator in $\mathcal{H}$ and its mathematical expectation. Let a triple $(\Omega, \mathcal{A}, \mu)$ be a probability space.

Definition 2.2. *A mapping $\hat{A} : \Omega \to B(\mathcal{H})$ is called a random operator in $\mathcal{H}$ if the functions $(\hat{A}u, v)$ are $(\Omega, \mathcal{A})$-measurable, that is, they are random variables, for all $u, v \in \mathcal{H}$.*

Definition 2.3. *A mathematical expectation (or averaging) of a random operator $\hat{A}$ is an operator $\mathbb{E}\hat{A} \in B(\mathcal{H})$ such that*

$$(\mathbb{E}\hat{A}u, v) = \mathbb{E}(\hat{A}u, v), \quad \forall u, v \in \mathcal{H}. \tag{2.5}$$

Sufficient conditions ensuring the existence of the averaging for a random operator can be found, for instance, in work [19].

3. Random affine transformations of functions domain

We consider a random operator-valued function $F(t) : \mathbb{R}_+ \to \mathrm{Aff}(\mathbb{R}^n)$ with values in the group of affine transformations of a finite-dimensional Euclidean space:

$$F(t)\vec{x} = e^{A\sqrt{t}+Bt+R(t^{\frac{3}{2}})}\vec{x} + \vec{h}\sqrt{t} + \vec{g}t + \vec{r}(t^{\frac{3}{2}}), \qquad t \in \mathbb{R}_+, \qquad \vec{x} \in \mathbb{R}^n, \tag{3.1}$$

where for $i, j \in 1, \ldots, n$ the entries $\{A^i{}_j\}$, $\{B^i{}_j\}$, $\{h^i\}$, $\{g^i\}$ are real random variables, while $\{R^i{}_j(s)\}$ and $\{r^i(s)\}$ are random continuously differentiable functions vanishing at the point $s = 0$. All random variables are supposed to be jointly distributed on the probability space $(\Omega, \mathcal{A}, \mu)$.

For an arbitrary fixed $\omega \in \Omega$ the following representation is true for the function $F(t)$.

Lemma 3.1. *In some neighbourhood of the zero a function $F(t)$ of form (3.1) can be represented as a composition $F_2(t) \circ F_1(t)$, where*

$$\begin{aligned} F_1(t)\vec{x} &= e^{Bt+R_0(t^{\frac{3}{2}})}\vec{x} + \vec{g}t + \vec{r}_0(t^{\frac{3}{2}}), \\ F_2(t)\vec{x} &= e^{A\sqrt{t}}\vec{x} + \vec{h}\sqrt{t}, \end{aligned} \tag{3.2}$$

where $\{(R_0)^i{}_j(s)\}$ and $\{(r_0)^i(s)\}$ are some continuously differentiable functions vanishing at the point $s = 0$.

Proof. By the Baker-Campbell-Hausdorff formula, see, for instance, [20], for sufficiently small t the identity holds

$$e^{-A\sqrt{t}}e^{A\sqrt{t}+Bt+R(t^{\frac{3}{2}})} = e^{Bt+R_0(t^{\frac{3}{2}})}, \tag{3.3}$$

and $\{(R_0)^i{}_j(s)\}$ are also continuously differentiable and vanish at the point $s = 0$. Then as $\vec{r}_0(t^{\frac{3}{2}}) = \vec{r}(t^{\frac{3}{2}}) - e^{A\sqrt{t}}\vec{g}t$ we obtain

$$F(t)\vec{x} = e^{A\sqrt{t}}\left(e^{+Bt+R(t^{\frac{3}{2}})}\vec{x} + \vec{g}t\right) + \vec{h}\sqrt{t} + \vec{r}(t^{\frac{3}{2}}) - e^{A\sqrt{t}}\vec{g}t = F_2(t) \circ F_1(t)\vec{x}. \tag{3.4}$$

The proof is complete. □

A random operator-valued function $F(t)$ (3.1) generates a random operator-valued function $\hat{U}_F(t)$, $t \geqslant 0$, with values in the space of linear bounded operators acting in $\mathcal{H} = L_2(\mathbb{R}^n)$ such that for each fixed $\omega \in \Omega$ the identity holds:

$$\hat{U}_F(t, \omega)u(x) = u(F(t, \omega)x), \quad \forall u \in \mathcal{H} \tag{3.5}$$

According to Theorem 1 from work [17] under the condition

$$\int\limits_{\Omega} \left|\det\left(e^{A\sqrt{t}+Bt+R(t)}\right)\right|^{\frac{1}{2}} d\mu(\omega) < +\infty \tag{3.6}$$

there exists a mathematical expectation

$$\mathbb{E}\hat{U}_F(t)u = \int\limits_{\Omega} u \circ F(t) d\mu(\omega) \in \mathcal{H}, \qquad \forall u \in \mathcal{H}. \tag{3.7}$$

Let the following condition be satisfied:

A1. For each $t \in [0, T]$ the operators A, B and $R'(t)$ take values in the ball of a radius $\rho_0 < +\infty$ in the space $B(\mathbb{R}^n)$ with probability 1;
A2. The distribution of a random vector $(\{A^i{}_j\})$ is discrete and symmetric;
A3. $\mathbb{E}h^i = 0$;
A4. An operator A is diagonalizable with the probability 1;
A5. $\operatorname{tr} A = 0$ with the probability 1;
A6. The covariation matrix of a random vector $(\{A^i{}_j\}, \{B^i{}_j\}, \{h^i\}, \{g^i\})$ is positive definite.

In Condition A6 we mean the following. We consider the entries of the matrices A, B and the components of vectors $\vec{h}$, $\vec{g}$ as a single random vector of dimension $2n^2 + 2n$. The covariation matrix of such vector is positive semi-definite by its definition. In Condition A6 we impose the condition that is strictly positive definite. This yields that the second moments $(\{A^i{}_j\}, \{B^i{}_j\}, \{h^i\}, \{g^i\})$ are strictly positive and their pairwise correlations are strictly less than 1 otherwise the covariation matrix would be degenerate.

Lemma 3.2. *Suppose that we are given a random operator-valued function F of form (3.1), for which Conditions A1-A2 hold. Then for some positive α the estimate holds:*

$$\|\mathbb{E}\hat{U}_F(t)\|_{B(\mathcal{H})} \leqslant e^{\alpha t}. \tag{3.8}$$

Proof. We have:

$$\begin{aligned}
\|\mathbb{E}\hat{U}_F(t)\|^2_{B(\mathcal{H})} &= \sup_{\|u\|_{\mathcal{H}}=1} \int\limits_{\mathbb{R}^n} \left| \int\limits_{\Omega} u(F(t)x) d\mu(\omega) \right|^2 dx \\
&\overset{①}{\leqslant} \sup_{\|u\|_{\mathcal{H}}=1} \int\limits_{\mathbb{R}^n} \int\limits_{\Omega} |u(F(t)x)|^2 d\mu(\omega) dx \overset{②}{=} \sup_{\|u\|_{\mathcal{H}}=1} \int\limits_{\Omega} \int\limits_{\mathbb{R}^n} |u(F(t)x)|^2 dx d\mu(\omega) \\
&\overset{③}{=} \sup_{\|u\|_{\mathcal{H}}=1} \int\limits_{\Omega} \int\limits_{\mathbb{R}^n} |u(x)|^2 \left|\det(F^{-1}(t))\right| dx d\mu(\omega) = \left|\mathbb{E}\left(\det F^{-1}(t)\right)\right| \\
&= \mathbb{E}e^{-\operatorname{tr}(A\sqrt{t}+Bt+R(t))}.
\end{aligned} \tag{3.9}$$

In the above calculations in the corresponding passages we have employed

① Cauchy-Schwarz inequality,
② Fubini theorem,
③ Theorem on the change of the variable in the Lebesgue integral.

By the Taylor formula we obtain:

$$\|\mathbb{E}\hat{U}_F(t)\|_{B(\mathcal{H})} \leqslant \left(\mathbb{E}e^{-\operatorname{tr}(A\sqrt{t}+Bt+R(t))}\right)^{\frac{1}{2}} = \left(\mathbb{E}\left(1 - A^i{}_i\sqrt{t} - B^i{}_i t - o(t)\right)\right)^{\frac{1}{2}}, \tag{3.10}$$

and $\mathbb{E}A^i{}_i = 0$ since the random variables $A^i{}_i$ are bounded (A1) and symmetrically distributed (A2), while the mathematical expectation of the remainder is $o(t)$ by Condition A1. Then for some $\alpha > 0$ and all $t \in [0, T]$ we have:

$$\|\mathbb{E}\hat{U}_F(t)\|_{B(L_2(\mathbb{R}))} \leqslant \left(1 + t\left(-\mathbb{E}B^i{}_i + o(t)\right)\right)^{\frac{1}{2}} \leqslant 1 + \alpha t \leqslant e^{\alpha t}. \tag{3.11}$$

The proof is complete. □

Lemma 3.3. *Suppose that we are given a random operator-valued function F of form (3.1), for which Conditions A1-A3 hold. Then for each $u \in C_0^\infty(\mathbb{R}^n)$*

$$\begin{aligned}(\mathbb{E}\hat{U}_F)'_0 u =& \mathbb{E}(B^i{}_j x^j + \frac{1}{2}A^i{}_k A^k{}_j x^j + g^i)\partial_i u \\ &+ \frac{1}{2}\mathbb{E}\left(A^i{}_k A^j{}_l x^k x^l + 2A^i{}_k h^j x^k + h^i h^j\right)\partial_i \partial_j u.\end{aligned} \tag{3.12}$$

Proof. We are going to find the value $(\mathbb{E}\hat{U}_F)'_0 u = \lim_{t\to 0}\left(\frac{\mathbb{E}\hat{U}_F(t)u - u}{t}\right)$ for $u \in C_0^\infty(\mathbb{R}^n)$. According to the Taylor formula we have:

$$\begin{aligned}&\mathbb{E}\hat{U}_F(t)u(x) = \mathbb{E}u(F(t)x) \\ &= \mathbb{E}\left(u(x) + \partial_i u(x)\left(F(t)x - x\right)^i + \frac{1}{2}\partial_i\partial_j u(x)\left(F(t)x - x\right)^i\left(F(t)x - x\right)^j + r(t)\right) \\ &= u(x) + \partial_i u(x)\mathbb{E}\left(F(t)x - x\right)^i + \frac{1}{2}\partial_i\partial_j u(x)\mathbb{E}((F(t)x - x)^i\left(F(t)x - x\right)^j) + \mathbb{E}(r(t)),\end{aligned} \tag{3.13}$$

where

$$r(t) = \partial_i\partial_j\partial_k u(\zeta)(F(t)x - x)^i(F(t)x - x)^j(F(t)x - x)^k, \tag{3.14}$$

and ζ is located between 0 and $(F(t)x - x)$ and depends of ω.

In its turn for $\mathbb{E}(F(t)x - x)$ by the Taylor formula we obtain:

$$\begin{aligned}\mathbb{E}(F(t)x - x)^i =& \mathbb{E}\left((A^i{}_j\sqrt{t} + B^i{}_j t) + \frac{1}{2}(A^i{}_k\sqrt{t} + B^i{}_k t)(A^k{}_j\sqrt{t} + B^k{}_j t) + o(t)\right)x^j \\ &+ \mathbb{E}h^i\sqrt{t} + \mathbb{E}g^i t = \mathbb{E}\left(B^i{}_j x^j + \frac{1}{2}A^i{}_k A^k{}_j x^j + g^i\right)t + o(t).\end{aligned} \tag{3.15}$$

Then similarly for the monomials of degree 2 and 3 we have:

$$\begin{aligned}&\mathbb{E}((F(t)x - x)^i\left(F(t)x - x\right)^j) \\ &\quad = \mathbb{E}(((A^i{}_j\sqrt{t} + B^i{}_j t) + \frac{1}{2}(A^i{}_k\sqrt{t} + B^i{}_k t)(A^k{}_j\sqrt{t} + B^k{}_j t) + o(t))x^j + h^i\sqrt{t} + g^i t)^2 \\ &\quad = \mathbb{E}\left(A^i{}_k A^j{}_l x^k x^l + 2A^i{}_k h^j x^k + h^i h^j\right)t + o(t),\end{aligned} \tag{3.16}$$

$$\mathbb{E}((F(t)x - x)^i(F(t)x - x)^j(F(t)x - x)^k) = o(t). \tag{3.17}$$

In formulas (3.15)–(3.17) we have taken into consideration Conditions A1-A3 in the formulation of the theorem and at the same time all remainders of order $o(t)$ are defined and uniformly Lipschitz in ω on the segment $[0, (F(t)x - x)]$. Expansions (3.14)–(3.17) then imply the statement of the lemma. The proof is complete. □

By Lemma 3.1 in some neighbourhood of the zero the function $F(t)$ can be represented as a composition $F_2(t) \circ F_1(t)$ of form (3.2). Random functions $F_1(t)$ and $F_2(t)$ also generate operator-valued functions $\mathbb{E}\hat{U}_{F_1}(t)$ and $\mathbb{E}\hat{U}_{F_2}(t)$. At the same time, the derivatives $(\mathbb{E}\hat{U}_{F_1})'_0$ and $(\mathbb{E}\hat{U}_{F_2})'_0$ are operators, the domains of which are subspaces, on which respectively $\mathbb{E}\hat{U}_{F_1}(t)$ and $\mathbb{E}\hat{U}_{F_2}(t)$ are differentiable at zero.

We define the operators $\hat{H}_j$, $j=1,2$, as follows. For $u\in C_0^\infty$

$$\hat{H}_1 u=(\mathbb{E}\hat{U}_{F_1})'_0 u=\mathbb{E}\left({B^i}_j x^j+g^i\right)\partial_i u, \tag{3.18}$$

$$\hat{H}_2 u=(\mathbb{E}\hat{U}_{F_2})'_0 u=\frac{1}{2}\mathbb{E}({A^i}_k x^k+h^i)\partial_i({A^j}_l x^l+h^j)\partial_j u. \tag{3.19}$$

At the same time, the operator $(\mathbb{E}\hat{U}_{F_2})'_0$ defined on the space C_0^∞ defines a closable non-positive quadratic form κ_2. As the domain of the operator $\hat{H}_2$ we take that of the Friedrichs extension of the operator $(\mathbb{E}\hat{U}_{F_2})'_0 :\ C_0^\infty\to\mathcal{H}$, that is, of the operator associated with the closure of the quadratic form κ_2.

By Conditions A1-A3, operator (3.18) is defined on $D(\hat{H}_2)$ since there exist constants c_1, $c_2>0$ such that

$$(\hat{H}_1 u,\hat{H}_1 u)\leqslant c_1\|u\|^2_{\mathcal{H}}+c_2|(u,\hat{H}_2 u)| \qquad \forall u\in D(\hat{H}_2). \tag{3.20}$$

This is why we let $D(\hat{H}_1)=D(\hat{H}_2)$.

We also observe that by Condition A4, the function $F_2(t)$ can be represented as a composition of a random orthogonal transformation S_1, a random sefl-adjoint transformation S_2 and a shift S_3 by a random vector $\vec{h}\sqrt{t}$. Thus,

$$\hat{U}_{F_2}(t)u=\hat{U}_{S_3}(\sqrt{t})\circ\hat{U}_{S_2}(\sqrt{t})\circ\hat{U}_{S_1}(\sqrt{t})u, \quad u\in\mathcal{H}, \quad t\geqslant 0. \tag{3.21}$$

At the same time, the operator S_3 does not commute with S_1 and S_2 and in order to obtain F_2 as a result of composition, the operator S_3 should act last.

It was established in [17] that for each $i=1,2,3$ and each $\omega\in\Omega$ one-parametric families of the operators $\hat{U}_{S_i(\omega)}(t)$, $t\geqslant 0$, form a strongly continuous unitary group of the operators in the space $\mathcal{H}$, which under Condition A5 possesses an anti-Hermitian generator $\hat{L}_{S_i(\omega)}$. Then by Theorem 1 in work [21] for each $i=1,2,3$ and each $u\in D((\hat{L}_i(\omega))^2)$ the identities hold:

$$\hat{U}_{S_i(\omega)}(t)u=u+t\hat{L}_{S_i(\omega)}u+\frac{t^2}{2}\hat{L}^2_{S_i(\omega)}u+\hat{R}_{S_i}(t,\omega)u, \quad \omega\in\Omega, \quad t\geqslant 0, \tag{3.22}$$

and for each $\omega\in\Omega$

$$\lim_{t\to 0}\frac{1}{t^2}\|\hat{R}_{S_i}(t,\omega)u\|_{\mathcal{H}}=0. \tag{3.23}$$

This is why by Conditions A1-A5 we obtain:

$$\mathbb{E}\hat{U}_{S_i(\omega)}(t)u=u+t^2\hat{H}_{S_i}u+\hat{R}_i(t)u, \qquad t\geqslant 0, \qquad \lim_{t\to 0}\frac{1}{t^2}\|\hat{R}_i(t)u\|_H=0. \tag{3.24}$$

Here $\hat{H}_{S_i}u=\frac{1}{2}\mathbb{E}\hat{L}^2_{S_i}u$ for all $u\in C_0^\infty$.

The operators $\hat{H}_{S_j} :\ C_0^\infty\to\mathcal{H}$, $j=1,2,3$, are densely defined and are non-positive. As it was shown in [21], [22], the Friedrichs extensions of the operators $\hat{H}_{S_j} :\ C_0^\infty\to\mathcal{H}$, $j=1,2,3$, are generators $\hat{H}_{S_j}$ of strongly continuous contracting semigroups $e^{t\hat{H}_{S_j}}$ in the space $\mathcal{H}$.

In terms of the above notations the following statement holds.

Lemma 3.4. *Suppose that we are given a random operator-valued function F of form (3.1), which obeys Conditions A1-A6. Then the operator $\hat{H}_2$ defined as the Friedrichs extension of the operator $(\mathbb{E}\hat{U}_{F_2}(t))'_0 :\ \bigcap_{i=1}^3 D(\hat{H}_{S_i})\to\mathcal{H}$ has the domain $D(\hat{H}_2)=\bigcap_{i=1}^3 D(\hat{H}_{S_i})$ and is non-positive and self-adjoint.*

Proof. By (3.22), on the domain $(\mathbb{E}\hat{U}_{F_2}(t))'_0$ the identity

$$(\mathbb{E}\hat{U}_{F_2}(t))'_0 u=\mathbb{E}\left(\frac{1}{2}\hat{L}^2_{S_1}+\frac{1}{2}\hat{L}^2_{S_2}+\frac{1}{2}\hat{L}^2_{S_3}+\hat{L}_{S_2}\circ\hat{L}_{S_1}+\hat{L}_{S_3}\circ\hat{L}_{S_2}+\hat{L}_{S_3}\circ\hat{L}_{S_1}\right)u \tag{3.25}$$

is valid.

The anti-Hermitian property of the operators L_{S_i} implies that $(\mathbb{E}\hat{U}_{F_2}(t))'_0$ is a non-positive operator

$$\begin{aligned}(u,(\mathbb{E}\hat{U}_{F_2}(t))'_0u) =& \frac{1}{2}\mathbb{E}((u,\hat{L}^2_{S_1}u)+(u,\hat{L}^2_{S_2}u)+(u,\hat{L}^2_{S_2}u)\\ &+2(u,\hat{L}_{S_2}\circ\hat{L}_{S_1}u)+2(u,\hat{L}_{S_3}\circ\hat{L}_{S_3}u)+2(u,\hat{L}_{S_3}\circ\hat{L}_{S_1}u))\\ =&-\mathbb{E}\|(\hat{L}_{S_1}+\hat{L}_{S_2}+\hat{L}_{S_3})u\|^2_{\mathcal{H}}\leqslant 0.\end{aligned} \tag{3.26}$$

Similar to arguing in work [17] we consider a quadratic form

$$\beta(u,u)=-(u,\hat{H}_2|_{C_0^\infty}v)=\int_{\mathbb{R}^n}\mathbb{E}\left(A^i{}_kA^j{}_lx^kx^l+2A^i{}_kh^jx^k+h^ih^j\right)\partial_iu\partial_ju\,dx, \tag{3.27}$$

which obeys the following representation

$$\begin{aligned}\beta(u,u)&=\int_{\mathbb{R}^n}\mathbb{E}|A^i{}_kx^k\partial_iu+h^i\partial_iu|^2dx=\int_{\mathbb{R}^n}\mathbb{E}|(\nabla u,A\vec{x}+\vec{h})|^2\,dx\\ &=\int_{\mathbb{R}^n}|\nabla u|^2\mathbb{E}(\vec{e},A\vec{x}+\vec{h})^2dx=\int_{\mathbb{R}^n}|\nabla u|^2\operatorname{Var}(|\vec{x}|(\vec{e},A\vec{e'})+(\vec{e},\vec{h}))\,dx\\ &=\int_{\mathbb{R}^n}|\nabla u|^2(|\vec{x}|^2\operatorname{Var}(\vec{e},A\vec{e'})+2\rho|\vec{x}|\sqrt{\operatorname{Var}(\vec{e},A\vec{e'})\operatorname{Var}(\vec{e},\vec{h})}+\operatorname{Var}(\vec{e},\vec{h}))\,dx,\end{aligned} \tag{3.28}$$

where $\vec{e}=\frac{\nabla u}{|\nabla u|}$, $\vec{e'}=\frac{\vec{x}}{|\vec{x}|}$, ρ is the linear correlation coefficient of the random variables $(\vec{e},A\vec{e'})$ and $(\vec{e},\vec{h})$. We note that if the random variables $(\vec{e},A\vec{e'})$ and $(\vec{e},\vec{h})$, which are linear combinations of the random variables $\{A^i{}_j\}$ and $\{h^i\}$, respectively, had been linearly independent, then the covariation matrix of a random vectors $(\{A^i{}_j\},\{h^k\})$ would have been degenerate, which contradict Condition A6. This implies that $\exists\gamma>0:|\rho|<1-\gamma$.

Then by (3.28) we obtain the inequalities

$$-\left(u,C_1(\hat{H}_{S_1}+\hat{H}_{S_2}+\hat{H}_{S_3})u\right)\leqslant\beta(u,u)\leqslant-\left(u,C_2(\hat{H}_{S_1}+\hat{H}_{S_2}+\hat{H}_{S_3})u\right), \tag{3.29}$$

for some $C_1,C_2>0$. This implies that the domain of the closure $\beta(u,u)$ coincides with the domain of the closure of the form $-(u,(\hat{H}_{S_1}+\hat{H}_{S_2}+\hat{H}_{S_3})u)$ as well as $\bigcap_{i=1}^3D(\hat{H}_{S_i})\subset D(\hat{H}_2)$.

On $\bigcap_{i=1}^3D(\hat{H}_{S_i})$ the quadratic form of the operator $-\hat{H}_2$ majorizes those of the operators $-\hat{H}_{S_i}$. Hence, the domain of the closure of the quadratic form of the operator $-\hat{H}_2|_{\bigcap_{i=1}^3D(\hat{H}_{S_i})}$ is contained in the domain of the closures of the quadratic forms of the operators $-\hat{H}_{S_i}|_{\bigcap_{i=1}^3D(\hat{H}_{S_i})}$. We recall that for each $i=1,2,3$ the operator $-\hat{H}_{S_i}$ is the Friedrichs extension of the operator $-\hat{H}_{S_i}|_{C_0^\infty}$ [21]. Therefore, the domain of the Friedrichs extension of the operator $\hat{H}_2|_{\bigcap_{i=1}^3D(\hat{H}_{S_i})}$ is contained in each of the domains $D(\hat{H}_{S_i})$ and hence,

$$D(\hat{H}_2)\subset\bigcap_{i=1}^3D(\hat{H}_{S_i}).$$

Therefore, the operator $\hat{H}_2$ defined as the Friedrichs extension of the operator $(\mathbb{E}\hat{U}_{F_2}u(t))'|_{t=0}:\bigcap_{i=1}^3D(\hat{H}_{S_i})\to\mathcal{H}$ has the domain $D(\hat{H}_2)=\bigcap_{i=1}^3D(\hat{H}_{S_i})$ and is self-adjoint. The proof is complete. □

Lemma 3.5. *Suppose that we are given a random operator-valued function F of form (3.1), which obeys Conditions A1-A6. Then for each $u\in D(\hat{H}_2)$ there exists a derivative*

$$(\mathbb{E}\hat{U}_F)'_0u=\hat{H}_2u+\hat{H}_1u.$$

Proof. For each $u \in D(\hat{H}_2)$ we have

$$\begin{aligned}(\mathbb{E}\hat{U}_F)'_0 u =&(\mathbb{E}(\hat{U}_{F_2} \circ \hat{U}_{F_1}))'_0 u = (\mathbb{E}(\hat{U}_{F_2} \circ \hat{U}_{F_1} - \hat{U}_{F_1} + \hat{U}_{F_1}))'_0 u \\ =& \left(\mathbb{E}((\hat{U}_{F_2} - \hat{I}) \circ \hat{U}_{F_1})\right)'_0 u + \left(\mathbb{E}\hat{U}_{F_1}\right)'_0 u = \left(\mathbb{E}\left(\hat{H}_2 + \hat{R}(t)\right)\right)'_0 u + \hat{H}_1 u \\ =&\hat{H}_2 u + \hat{H}_1 u,\end{aligned} \tag{3.30}$$

where $\lim_{t\to 0} \frac{1}{t}\|\hat{R}(t)u\|_H = 0$. The proof is complete. □

Lemma 3.6. *Suppose that we are given a random operator-valued function F of form (3.1), which obeys Conditions A1-A6. Then the closure of the operator $(\mathbb{E}\hat{U}_F)'_0$ is a generator of a strongly continuous semigroup in the space $\mathcal{H}$.*

Proof. In the above introduced notation the closure of the operator $(\mathbb{E}\hat{U}_F)'_0$ can be represented as $\hat{H}_1 + \hat{H}_2 = \hat{H}$.

Completing the square in formula (3.28), we obtain an estimate for $\beta(u,u)$:

$$\beta(u,u) \geqslant \int_{\mathbb{R}^n} |\nabla u|^2 (1-\rho^2) \operatorname{Var}(\vec{e}, \vec{h}) dx \geqslant \alpha \| |\nabla u| \|^2, \tag{3.31}$$

where α is some positive constant independent of u.

Since the sesquilinear form β is positive, then the operator $\hat{H}_2$ is a generator of a contracting semigroup in $\mathcal{H}$.

The quadratic form of the operator $\hat{I} + (-\hat{H}_2)$ majorizes that of the operator $\hat{H}_1$ since the following chains of inequalities holds

$$|(\hat{H}_1 v, v)| \leqslant \left\|\mathbb{E}\left(B^i{}_j\right)\right\|_{B(\mathbb{R}^n)} \| |\vec{x}| |\nabla v| \|_{L_2(\mathbb{R})} \|v\|_{L_2(\mathbb{R})} \leqslant \frac{1}{\epsilon} C((v,v) - \epsilon(\hat{H}_2 v, v)) \tag{3.32}$$

for an arbitrary $\epsilon \in (0,1]$. Since $\hat{H} = \hat{H}_1 + \hat{H}_2$ and the quadratic form of the operator $\hat{I} - \hat{H}_2$ majorizes that of the operator $\hat{H}_1$ by inequality (3.32), then we can apply the theorem on the perturbation of the generator of the semigroup, see, for instance, [23]. Therefore, the operator $\hat{H}$ is a generator of a strongly continuous semigroup in the space $\mathcal{H}$. □

4. Feynman-Chernoff iterations

For a sequence $\{F_k(t)\}$, $k \in \mathbb{N}$, of independent identically distributed random functions of form (3.1) and an arbitrary non-negative t we define the sequences of Feynman-Chernoff iterations:

$$F_n\left(\frac{t}{n}\right) \circ \ldots \circ F_1\left(\frac{t}{n}\right), \quad n \in \mathbb{N}. \tag{4.1}$$

Definition 4.1. *A Chernoff averaging semigroup for a random operator-valued function $\hat{U}_F$ generated by a function F of form (3.1) is a semigroup $\hat{W}_F$, the generator of which is the closure of the operator $(\mathbb{E}\hat{U}_F)'_0$.*

The semigroup $\hat{W}_F$ is generated by the solutions of the Cauchy problem for the Fokker-Planck equation

$$\frac{\partial u}{\partial t} = \hat{H}u, \qquad u|_{t=0} = u_0 \tag{4.2}$$

for an arbitrary $u_0 \in \mathcal{H}$ and the operator $\hat{H}$ on the domain $(\mathbb{E}\hat{U}_F)'_0$ is given by the differential expression

$$\hat{H} = \mathbb{E}(B^i{}_j x^j + \frac{1}{2} A^i{}_k A^k{}_j x^j + g^i)\partial_i + \frac{1}{2}\mathbb{E}\left(A^i{}_k A^j{}_l x^k x^l + 2A^i{}_k h^j x^k + h^i h^j\right)\partial_i \partial_j. \tag{4.3}$$

Theorem 4.1. *Suppose that we are given a random operator-valued function F of form (3.1), which obeys Conditions A1-A6. Then the sequence of the averagings of the Feynman-Chernoff iterations converges in the topology of $C_s(\mathbb{R}_+, B(\mathcal{H}))$ to a corresponding Chernoff averaging semigroup:*

$$\mathbb{E}\hat{U}_{F_n(\frac{t}{n})\circ\ldots\circ F_1(\frac{t}{n})} \xrightarrow{\tau_s} \hat{W}_F(t). \tag{4.4}$$

Proof. According to Lemmas 2 and 3 from work [17] and by the independence of the terms in the sequence $\{F_k(t)\}$ for all $t > 0$, $n \in \mathbb{N}$ the identity holds:

$$\mathbb{E}\hat{U}_{F_n(\frac{t}{n})\circ\ldots\circ\mathbf{F}_1(\frac{t}{n})} = \mathbb{E}\hat{U}_{F_n(\frac{t}{n})} \circ \ldots \circ \mathbb{E}\hat{U}_{F_1(\frac{t}{n})}. \tag{4.5}$$

A strong continuity of the function $\mathbb{E}\hat{U}_F(t)$ follows from the fact that it is the integral with respect to the probability measure of the function, the values of which strongly continuous operator functions.

The rest of the proof is in fact checking Conditions 1-3 of the Chernoff theorem for the function $\mathbb{E}\hat{U}_F(t)$:

1. $\mathbb{E}\hat{U}_F(0)$ is the identity operator by its construction.
2. By Lemma 3.2, $\|\mathbb{E}\hat{U}_F(t)\|_{B(\mathcal{H})} \leqslant e^{\alpha t}$ for some positive α.
3. By Lemma 3.5, the closure of the operator $(\mathbb{E}\hat{U}_F)'_0$ is the generator of a strongly continuous semigroup $\hat{W}_F$ in the space $\mathcal{H}$.

The proof is complete. □

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