ISSN 2304-0122 Ufa Mathematical Journal. Vol. 15. No 2 (2023). P. 42-54.

doi:10.13108/2023-15-2-42

ASYMPTOTICS FOR SOLUTIONS OF PROBLEM ON OPTIMALLY DISTRIBUTED CONTROL IN CONVEX DOMAIN WITH SMALL PARAMETER AT ONE OF HIGHER DERIVATIVES

A.R. DANILIN

Abstract. We consider a problem on optimally distributed control in a planar strictly convex domain with a smooth boundary and a small parameter at one of the higher derivatives in the elliptic operator. On the boundary of the domain the homogeneous Dirichlet condition is imposed, while the control is additively involved in an inhomogeneity. As a set of admissible controls we use a unit ball in the corresponding space of square integrable functions. The solutions of the studied boundary value problem are treated in the generalized sense as elements of some Hilbert space. As the optimality criterion, we employ the sum of squared norm of the deviation of a state from a prescribed one and the squared norm of the control with some coefficient. Such structure of the optimality criterion allows, if this is needed, to strengthen the role of the first or the second term in this criterion. In the first case it is more important to achieve a prescribed state, while in this second case it is more important to minimize the resource expenses. We study in details the asymptotics of the problem generated by the differential operator with a small coefficient at one of the higher derivatives, to which a zero order differential operator is added.

Keywords: small parameter, optimal control, boundary value problems for systems of partial differential equations, asymptotic expansions.

Mathematical Subject Classification: 35C20, 35B25, 76M45, 93C70.

1. Introduction

The paper is devoted to studying the asymptotics of solution to a problem on an optimally distributed control [1] in a planar strictly convex domain with a smooth boundary and a small parameter at one of the higher derivative in an elliptic operator. Such operators are typical for the steady processes of heat conduction and diffusion in layered media, when the propagation of heat (diffusion) has significantly different coefficients in perpendicular directions (in a layer and when moving to a new layer) [2, Ch. III, Sect. 1, Item 3].

Asymptotics of the solutions of the Dirichlet problem for such elliptic equations in such domains was studied in [3], [4]. An asymptotics of the distributed control for an operator with a small coefficient at the Laplace operator in an essentially different domain was considered in [8], [9], while the case of a similar domain was treated in [10]. We also note that the study of the problems on optimal control described by partial differential equations is permanently relevant, see, for instance, [5]–[7] and the references therein.

A.R. Danilin, Asymptotics for solutions to problem on optimally distributed control in convex domain with small parameter at one of higher derivatives.

Submitted July 20, 2022.

2. General formulation of problem and optimality conditions

Let $\Omega\subset\mathbb{R}^2$ be a bounded strictly convex domain with a smooth boundary $\Gamma:=\Gamma$ (Ω is a manifold of the class C^∞ with an edge).

We consider the following problem on a distributed control [1, Ch. 2, Sect. 2, Eqs. (2.8)–(2.9)]

$$\mathcal{L}_\varepsilon z_\varepsilon := -\varepsilon^2\frac{\partial^2 z_\varepsilon}{\partial x^2}-\frac{\partial^2 z_\varepsilon}{\partial y^2}+a(x,y)z_\varepsilon = f(x,y)-u_\varepsilon(x,y),\qquad (x,y)\in\Omega,\quad z_\varepsilon\in H_0^1(\Omega), \quad (2.1)$$

$$J(u):=\|z_\varepsilon-z_d\|^2+\beta^{-1}\|u\|\longrightarrow\inf,\quad u\in\mathcal{U}, \quad (2.2)$$

$$\mathcal{U}=\mathcal{U}(1),\quad \text{where}\quad \mathcal{U}(r):=\{u\in L_2(\Omega):\|u\|\leqslant r\}. \quad (2.3)$$

Here $\varepsilon>0$, $H_0^1(\Omega)$ is the Sobolev space of differentiable functions with zero trace on the boundary $\partial\Omega$ (see, for instance, [11]), $\|\cdot\|$ is the norm in the space $L_2(\Omega)$,

$$f,z_d,a\in C^\infty(\overline{\Omega_\delta}),\quad a(x,y)\geqslant\alpha^2>0\quad\text{as}\quad (x,y)\in\overline{\Omega_\delta}, \quad (2.4)$$

where $\delta>0$ and Ω_δ is a δ-neighbourhood of the domain Ω.

The scalar product in $L_2(\Omega)$ is denoted by $(\cdot,\cdot)$. A solution of equation (2.1) is treated in the weak sense, that is, for each $v\in H_0^1(\Omega)$ the identities hold:

$$\varepsilon^2\left(\frac{\partial z}{\partial x},\frac{\partial v}{\partial x}\right)+\left(\frac{\partial z}{\partial y},\frac{\partial v}{\partial y}\right)+(a(x,y)z,v)=(f+u,v).$$

By (2.4) for all small $\varepsilon>0$ the relation is true:

$$\begin{aligned}(\mathcal{L}_\varepsilon v,v)=&\varepsilon^2\left\|\frac{\partial v}{\partial x}\right\|^2+\left\|\frac{\partial v}{\partial y}\right\|^2+(a(x,y)v,v)\\ \geqslant&\varepsilon^2\left\|\frac{\partial v}{\partial x}\right\|^2+\left\|\frac{\partial v}{\partial y}\right\|^2+\alpha^2\|v\|^2\geqslant\varepsilon^2\|v\|^2_{H_0^1(\Omega)}.\end{aligned} \quad (2.5)$$

In this case the unique optimal control $u_\varepsilon(\cdot)$ and a corresponding $z_\varepsilon(\cdot)$ in problem (2.1)–(2.3) are characterized as follows: there exist $p_\varepsilon\in H_0^1(\Omega)$ such that [1, Ch. 2, Sect. 2, Eqs. (2.10)]

$$\begin{cases}\mathcal{L}_\varepsilon z_\varepsilon=f(x,y)+u_\varepsilon, & \mathcal{L}_\varepsilon p_\varepsilon-z_\varepsilon=-z_d(x,y), \quad (x,y)\in\Omega, \quad z_\varepsilon,\ p_\varepsilon\in H_0^1(\Omega),\\ z_\varepsilon=0,\quad p_\varepsilon=0, & (x,y)\in\Gamma,\end{cases}$$

$$\forall\widetilde{v}\in\mathcal{U}\qquad \big(p+\beta^{-1}u_{opt},(\widetilde{v}-u_{opt})\big)\geqslant 0. \quad (2.6)$$

As it was shown in Lemma 1 in [12], in this case condition (2.6) is equivalent to the following one:

$$\big(u_\varepsilon=-\lambda_\varepsilon p_\varepsilon\big)\wedge\big(\lambda_\varepsilon\in(0;\beta]\big)\wedge\Big(\lambda_\varepsilon\|p_\varepsilon\|\leqslant 1\Big)\wedge\Big((\beta-\lambda_\varepsilon)\cdot\big(1-\lambda_\varepsilon\|p_\varepsilon\|\big)=0\Big). \quad (2.7)$$

Thus, the original problem is reduced to the system of equations

$$\begin{cases}\mathcal{L}_\varepsilon z_\varepsilon+\lambda_\varepsilon p_\varepsilon=f(x,y), & (x,y)\in\Omega,\\ \mathcal{L}_\varepsilon p_\varepsilon-z_\varepsilon=-z_d(x,y), & z_\varepsilon,\ p_\varepsilon\in H_0^1(\Omega),\\ z_\varepsilon=0,\qquad p_\varepsilon=0, & (x,y)\in\Gamma,\end{cases} \quad (2.8)$$

depending on a scalar parameter λ_ε with additional condition (2.7).

The aim of the work is to study the behavior of z_ε, p_ε and λ_ε as $\varepsilon\to 0$ and to find complete asymptotic expansions for these quantities as $\varepsilon\to 0$.

In what follows we often denote by the letter K, sometimes with subscripts, various positive constants which depend only on the domain Ω and the function $a(x,y)$.

3. Apriori estimates

The first part of inequality (2.5) implies the following lemma.

Lemma 3.1. *Let the function $a(\cdot)$ satisfies the conditions in (2.4). If $f \in L_2(\Omega)$ and $\mathcal{L}_\varepsilon z_\varepsilon = f$, then the inequality*

$$\{\alpha^2\|z_\varepsilon\|, \alpha\varepsilon\|z'_x\|, \alpha\|z'_y\|\} \leqslant \|f\| \tag{3.1}$$

holds true, where $z'_x := \frac{\partial z}{\partial x}$ and $z'_y := \frac{\partial z}{\partial y}$.

Corollary 3.1. *Let conditions (2.4) hold. If λ_ε, z_ε and p_ε satisfy (2.8), (2.7), then there exists a constant $\lambda_* > 0$ such that for all small $\varepsilon > 0$ the inequalities $\lambda_\varepsilon \geqslant \lambda_*$ and*

$$\|z_\varepsilon\| \leqslant \alpha^{-2}(\|f\| + 1), \qquad \|p_\varepsilon\| \leqslant \alpha^{-2}\|z_d\| + \alpha^{-4}(\|f\| + 1)$$

hold true.

Proof. Two latter inequalities are implied by (3.1), while the first inequality follows from the boundedness of $\|p_\varepsilon\|$ and (2.7). □

Together with (2.7) we consider a system

$$\begin{cases} \mathcal{L}_\varepsilon z_{\varepsilon,\lambda} + \lambda p_{\varepsilon,\lambda} = f_{\varepsilon,1}(x,y), & \mathcal{L}_\varepsilon p_{\varepsilon,\lambda} - z_{\varepsilon,\lambda} = f_{\varepsilon,2}(x,y), \qquad (x,y) \in \Omega, \\ z = 0, & p = 0, \qquad (x,y) \in \Gamma. \end{cases} \tag{3.2}$$

Theorem 3.1. *Problem (3.2) is uniquely solvable for all $f_{\varepsilon,i} \in L_2(\Omega)$ and $\varepsilon > 0$, $i = 1, 2$, and its solution satisfies z, $p \in H^2(\Omega)$. If $f_{\varepsilon,i} \in C^\infty(\overline{\Omega})$, then $z_{\varepsilon,\lambda}$, $p_{\varepsilon,\lambda} \in C^\infty(\Omega)$.*

Proof. This theorem can be proved similarly to Theorem 1 from [13]. □

As $f_{\varepsilon,1} = f$ and $f_{\varepsilon,2} = -z_d$, we denote the solution of system (3.2) by $z_{\varepsilon,\lambda,d}$, $p_{\varepsilon,\lambda,d}$.

Remark 3.1. *We note that if $\beta\|p_{\varepsilon,\beta,d}\| \leqslant 1$, then $z_{\varepsilon,\beta,d} = z_\varepsilon$ and $p_{\varepsilon,\beta,d} = p_\varepsilon$, while restrictions for the control are not essential.*

Let $z_{\varepsilon,\lambda}, p_{\varepsilon,\lambda} \in H_0^1(\Omega)$ be solutions of system (3.2), then for all $\varepsilon > 0$ the identity $(\mathcal{L}_\varepsilon z_{\varepsilon,\lambda}, p_{\varepsilon,\lambda}) = (z_{\varepsilon,\lambda}, \mathcal{L}_\varepsilon p_{\varepsilon,\lambda})$ holds. Then by (3.2) we obtain

$$\|z_{\varepsilon,\lambda}\|^2 + \lambda\|p_{\varepsilon,\lambda}\|^2 = (f_{\varepsilon,1}, p_{\varepsilon,\lambda}) - (f_{\varepsilon,2}, z_{\varepsilon,\lambda}). \tag{3.3}$$

By (3.3), the norms $\|z_{\varepsilon,\lambda}\|$ and $\|p_{\varepsilon,\lambda}\|$ satisfy a quadratic inequality

$$\|z_{\varepsilon,\lambda}\|^2 + \lambda\|p_{\varepsilon,\lambda}\|^2 \leqslant \|f_{\varepsilon,1}\| \cdot \|p_{\varepsilon,\lambda}\| + \|f_{\varepsilon,2}\| \cdot \|z_{\varepsilon,\lambda}\|,$$

which yields

$$\|z_{\varepsilon,\lambda}\| \leqslant \|f_{\varepsilon,2}\| + \frac{\|f_{\varepsilon,1}\|}{2\sqrt{\lambda}}, \qquad \|p_{\varepsilon,\lambda}\| \leqslant \frac{\|f_{\varepsilon,1}\|}{\lambda} + \frac{\|f_{\varepsilon,2}\|}{2\sqrt{\lambda}}, \tag{3.4}$$

By (3.4) and (2.5) we get apriori estimates for the derivatives

$$\varepsilon^2\left\|\frac{\partial}{\partial x}z_{\varepsilon,\lambda}\right\|^2 + \left\|\frac{\partial}{\partial y}z_{\varepsilon,\lambda}\right\|^2 \leqslant \|z_{\varepsilon,\lambda}\|(\|f_{\varepsilon,1}\| + \lambda\|p_{\varepsilon,\lambda}\|),$$

$$\varepsilon^2\left\|\frac{\partial}{\partial x}p_{\varepsilon,\lambda}\right\|^2 + \left\|\frac{\partial}{\partial y}p_{\varepsilon,\lambda}\right\|^2 \leqslant \|p_{\varepsilon,\lambda}\|(\|f_{\varepsilon,2}\| + \|z_{\varepsilon,\lambda}\|).$$

Thus, if $\lambda \in [\lambda_*, \lambda^*] \subset (0, +\infty)$, there exists $K > 0$ such that for each small $\varepsilon > 0$ the inequalities

$$\begin{aligned} \varepsilon\left\|\frac{\partial}{\partial x}z_{\varepsilon,\lambda}\right\| + \left\|\frac{\partial}{\partial y}z_{\varepsilon,\lambda}\right\| + \|z_{\varepsilon,\lambda}\| \leqslant K(\|f_{\varepsilon,1}\| + \|f_{\varepsilon,2}\|), \\ \varepsilon\left\|\frac{\partial}{\partial x}p_{\varepsilon,\lambda}\right\| + \left\|\frac{\partial}{\partial y}p_{\varepsilon,\lambda}\right\| + \|p_{\varepsilon,\lambda}\| \leqslant K(\|f_{\varepsilon,1}\| + \|f_{\varepsilon,2}\|). \end{aligned} \tag{3.5}$$

hold.

4. Limiting relations

We consider a “limiting” for (3.2) problem

$$\begin{cases} \mathcal{L}_0 z_{0,\lambda} + \lambda p_{0,\lambda} = f_{0,1}(x,y), & \\ \mathcal{L}_0 p_{0,\lambda} - z_{0,\lambda} = f_{0,2}(x,y), & (x,y) \in \Omega, \\ z = 0, \quad p = 0, & (x,y) \in \Gamma, \quad \lambda \geqslant 0, \end{cases} \tag{4.1}$$

where the operator $\mathcal{L}_0$ is obtained from $\mathcal{L}_\varepsilon$ once we formally let $\varepsilon = 0$:

$$\mathcal{L}_0 v := -\frac{\partial^2 v}{\partial y^2} + a(x,y)v, \qquad v \in H_0^1(\Omega).$$

Since the domain Ω *is strictly convex*, there exist points $M_i = (x_i, y_i) \in \Gamma$, $i = 1, 2$, at which the equation of the tangentials to Γ reads as $x = x_i$, respectively. The points M_i partition the boundary Γ into two parts Γ_j, a lower one ($j = 1$) and an upper one ($j = 2$). Both these parts are the graphs of the functions $\varphi_j(x)$, $x \in [x_1; x_2]$. At the same time,

$$\varphi_j(x) \in C([x_1; x_2]) \cap C^\infty(x_1; x_2), \qquad \varphi_j(x_i) = y_i, \qquad \varphi_j'(x_i - (-1)^i 0) = \infty. \tag{4.2}$$

In the vicinity of the points M_i there exists one more parametrization of the boundary Γ, namely, $x = \psi_i(y)$, respectively. We observe that ψ_1 is convex ($\psi_1'' \geqslant 0$), while ψ_2 is concave ($\psi_1'' \leqslant 0$) and $\psi_i'(y_i) = 0$.

In what follows we assume that

$$x_1 = y_1 = 0, \qquad \psi_1''(y_1) > 0, \qquad \psi_2''(y_2) < 0. \tag{4.3}$$

Theorem 4.1. *Let conditions (2.4), (4.2) hold and $f_{0,1}, f_{0,1} \in C^\infty(\overline{\Omega_\delta})$. Then problem (4.1) is uniquely solvable and its solutions are infinitely differentiable in $\overline{\Omega} \setminus \{M_1, M_2\}$. At the same time, for each segment $[\lambda_*, \lambda^*] \subset (0, +\infty)$ there exists $K > 0$ such that for all $(x,y) \in \Omega$ and all continuous in Ω functions $f_{0,1}$, $f_{0,2}$ the apriori inequality*

$$\begin{aligned} |z_{0,\lambda}(x,y)| + \left|\frac{\partial}{\partial y} z_{0,\lambda}(x,y)\right| + |p_{0,\lambda}(x,y)| + \left|\frac{\partial}{\partial y} p_{0,\lambda}(x,y)\right| \\ \leqslant K \int\limits_{\varphi_1(x)}^{\varphi_2(x)} (|f_{0,1}(x,y)| + |f_{0,2}(x,y)|)\, dy \end{aligned} \tag{4.4}$$

holds true.

Proof. If $\lambda = 0$, then by the first equation in system (4.1) we obtain

$$(f_{0,1}, z_{0,0}) = (\mathcal{L}_0 z_{0,0}, z_{0,0}) = \left\|\frac{\partial}{\partial y} z_{0,0}\right\|^2 + (a(x,y) z_{0,0}, z_{0,})) \geqslant \alpha^2 \|z_{0,\lambda}\|^2$$

and this yields

$$\alpha^2 \|z_{0,0}\| \leqslant \|f_{0,1}\|, \qquad \alpha^2 \|p_{0,0}\| \leqslant \|f_{0,2}\| + \alpha^{-2} \|f_{0,1}\|. \tag{4.5}$$

Similarly to (3.3) we show that each solution $z_{0,\lambda}$, $p_{0,\lambda}$ of system (4.1) satisfies the identity

$$\|z_{0,\lambda}\|^2 + \lambda \|p_{0,\lambda}\|^2 = (f_{0,1}, p_{0,\lambda}) - (f_{0,2}, z_{0,\lambda}),$$

and hence, for $\lambda \in [\lambda_*, \lambda^*] \subset (0, +\infty)$ there exists $K > 0$ such that

$$\|z_{0,\lambda}\| + \|p_{0,\lambda}\| \leqslant K(\|f_{0,1}\| + \|f_{0,2}\|). \tag{4.6}$$

The equations in (4.1) is a system of ordinary differential equations in y smoothly depending on the parameter x. Since this system is Fredholm, we consider its solution as $f_1 = 0$ and $f_2 = 0$.

By (4.6) under these conditions we obtain that $z_{0,\lambda} = 0$ and $p_{0,\lambda} = 0$. Thus, problem (4.1) is solvable for each $x \in (x_1; x_2)$, while by the theorem on solutions depending on a parameter we obtain the smoothness of $z_{0,\lambda}$ and $p_{0,\lambda}$ in Ω.

Applying Theorem 3.1 from [14, Ch. XII, Sect. 3] to system (4.1), in view of the continuity in Ω_δ of the fundamental matrix of the linear system associated with system (4.1) we obtain apriori estimates (4.4). The proof is complete. $\square$

As $f_{0,1} = f$ and $f_{0,2} = -z_d$, we denote the solution of system (4.1) by $z_{0,\lambda,d}$, $p_{0,\lambda,d}$.

Theorem 4.2. *Let conditions (2.4) and (4.2) be satisfied. Then*

$$\|z_{\varepsilon,\lambda,d} - z_{0,\lambda,d}\| \to 0, \qquad \|p_{\varepsilon,\lambda,d} - p_{0,\lambda,d}\| \to 0, \qquad \text{as} \quad \varepsilon \to 0.$$

Proof. We consider the functions $\widetilde{z}_r := \chi_r(x) z_{0,\lambda,d}$ and $\widetilde{p}_r := \chi_r(x) p_{0,\lambda,d}$, where r is an auxiliary small positive parameter, while $\chi_r(x)$ is a cut-off function, that is,

$$\chi_r(\cdot) \in C^\infty(\mathbb{R}), \qquad |\chi_r(x)| \leqslant 1 \quad \text{for all} \quad x \in \mathbb{R},$$

$$\chi_r(x) = \begin{cases} 0, & x \in (-\infty; r/2) \cup (x_2 - r/2; +\infty), \\ 1, & x \in [r; x_2 - r]. \end{cases}$$

The functions $\widetilde{z}_r$, $\widetilde{p}_r$ are infinitely differentiable on $\overline{\Omega}$, vanish on Γ and satisfy the identities

$$\mathcal{L}_0 \widetilde{z}_r + \lambda \widetilde{p}_r = \chi_r(x) f \text{and} \quad \mathcal{L}_0 \widetilde{p}_r - \widetilde{z}_r = -\chi_r(x) z_d.$$

This is why

$$\mathcal{L}_\varepsilon \widetilde{z}_r + \lambda \widetilde{p}_r = \chi_r(x) f + \varepsilon^2 \frac{\partial^2}{\partial x^2} \widetilde{z}_r, \qquad \mathcal{L}_\varepsilon \widetilde{p}_r - \widetilde{z}_r = -\chi_r(x) z_d + \varepsilon^2 \frac{\partial^2}{\partial x^2} \widetilde{p}_r.$$

Since $\widetilde{z}_r$, $\widetilde{p}_r \in C^\infty(\overline{\Omega})$, there exists $K_r > 0$ such that

$$\left\| \frac{\partial^2}{\partial x^2} \widetilde{z}_r \right\| \leqslant K_r, \qquad \left\| \frac{\partial^2}{\partial x^2} \widetilde{p}_r \right\| \leqslant K_r.$$

We denote $\widetilde{z}_{\varepsilon,r} := z_{\varepsilon,\lambda,d} - \widetilde{z}_r$, $\widetilde{p}_{\varepsilon,r} := p_{\varepsilon,\lambda,d} - \widetilde{p}_r$. Then

$$\mathcal{L}_\varepsilon \widetilde{z}_{\varepsilon,r} + \lambda \widetilde{p}_{\varepsilon,r} = (1 - \chi_r(x)) f + \varepsilon^2 \frac{\partial^2}{\partial x^2} \widetilde{z}_r, \quad \mathcal{L}_\varepsilon \widetilde{p}_{\varepsilon,r} - \widetilde{z}_{\varepsilon,r} = -(1 - \chi_r(x)) z_d + \varepsilon^2 \frac{\partial^2}{\partial x^2} \widetilde{p}_r.$$

By (3.5) the inequalities hold

$$\begin{aligned} \|\widetilde{z}_{\varepsilon,r}\| &\leqslant K \|1 - \chi_r\| (\|f\| + \|z_d\|) + 2\varepsilon^2 K_r, \\ \|\widetilde{p}_{\varepsilon,r}\| &\leqslant K \|1 - \chi_r\| (\|f\| + \|z_d\|) + 2\varepsilon^2 K_r. \end{aligned} \tag{4.7}$$

Since

$$z_{\varepsilon,\lambda,d} - z_{0,\lambda,d} = \widetilde{z}_{\varepsilon,r} + (1 - \chi_r(x)) z_{0,\lambda,d}, \qquad p_{\varepsilon,\lambda,d} - p_{0,\lambda,d} = \widetilde{p}_{\varepsilon,r} + (1 - \chi_r(x)) p_{0,\lambda,d},$$

then by (4.7)

$$\begin{aligned} 0 &\leqslant \liminf_{\varepsilon \to +0} \|z_{\varepsilon,\lambda,d} - z_{0,\lambda,d}\| \leqslant \limsup_{\varepsilon \to +0} \|z_{\varepsilon,\lambda,d} - z_{0,\lambda,d}\| \\ &\leqslant K \|1 - \chi_r\| (\|f\| + \|z_d\|) + \|1 - \chi_r\| \cdot \|z_{0,\lambda,d}\|, \\ 0 &\leqslant \liminf_{\varepsilon \to +0} \|p_{\varepsilon,\lambda,d} - p_{0,\lambda,d}\| \leqslant \limsup_{\varepsilon \to +0} \|p_{\varepsilon,\lambda,d} - p_{0,\lambda,d}\| \\ &\leqslant K \|1 - \chi_r\| (\|f\| + \|z_d\|) + \|1 - \chi_r\| \cdot \|p_{0,\lambda,d}\|. \end{aligned}$$

But $\|1 - \chi_r\| \longrightarrow 0$ as $r \to +0$ and this is why

$$\lim_{r \to +0} \Big(K \|1 - \chi_r\| (\|f\| + \|z_d\|) + \|1 - \chi_r\| \cdot \|p_{0,\lambda,d}\| \Big) = 0,$$

and hence, $\|z_{\varepsilon,\lambda,d} - z_{0,\lambda,d}\| \to 0$ and $\|p_{\varepsilon,\lambda,d} - p_{0,\lambda,d}\| \to 0$ as $\varepsilon \to 0$. The proof is complete. □

Corollary 4.1. *Let conditions (2.4) and (4.2) be satisfied. Then*

1. *If*

$$\beta\|p_{0,\beta,d}\| < 1, \tag{4.8}$$

then $\lambda_\varepsilon = \beta$ for all small $\varepsilon > 0$, that is, the restrictions for the control in problem (2.1)–(2.3) are not essential and $\|z_\varepsilon - z_{0,\beta,d}\| \to 0$, $\|p_\varepsilon - p_{0,\beta,d}\| \to 0$ as $\varepsilon \to 0$.

2. *If $\beta\|p_{0,\beta,d}\| > 1$, then for all small $\varepsilon > 0$ the restrictions for the control in problem (2.1)–(2.3) are essential and $\lambda_\varepsilon\|p_\varepsilon\| = 1$ for all such ε.*

Proof. If $\beta\|p_{0,\beta,d}\| < 1$, then for all small $\varepsilon > 0$ the inequality $\beta\|p_{\varepsilon,\beta,d}\| < 1$ holds and by Remark 3.1 the identities $z_{\varepsilon,\beta,d} = z_\varepsilon$ and $p_{\varepsilon,\beta,d} = p_\varepsilon$ are valid. The proof is complete. □

Lemma 4.1. *Let*

$$\mathcal{L}_0 z_d \neq f \quad \text{and} \quad \beta\|p_{0,\beta,d}\| > 1. \tag{4.9}$$

Then there exists a unique $\lambda_0 \in (0,\beta)$ such that

$$\lambda_0\|p_{0,\lambda_0,d}\| = 1. \tag{4.10}$$

Proof. In (4.1) we pass from the functions $z_{0,\lambda,d}$ and $p_{0,\lambda,d}$ to the functions $Z_\lambda := z_{0,\lambda,d}$ and $P_\lambda := \lambda p_{0,\lambda,d}$. Then $\{Z_\lambda, P_\lambda\}$ satisfies the system

$$\begin{cases} \mathcal{L}_0 Z_\lambda + P_\lambda = f(x,y), \qquad \mathcal{L}_0 P_\lambda - \lambda Z_\lambda = -\lambda z_d(x,y), \\ Z_\lambda|_\Gamma = 0 = P_\lambda|_\Gamma. \end{cases} \tag{4.11}$$

We consider a function $\mathcal{F}(\lambda) := \|P_\lambda\|^2$. Then $\mathcal{F}'(\lambda) = 2(P_\lambda, \frac{\partial}{\partial\lambda}P_\lambda)$.

Let $\widetilde{Z}_\lambda := \frac{\partial}{\partial\lambda}Z_\lambda$, and $\widetilde{P}_\lambda := \frac{\partial}{\partial\lambda}P_\lambda$. Then by the theorem on the differentiability in the parameter of the solution to a system of ordinary differential equations, the pair $\{\widetilde{Z}_\lambda, \widetilde{P}_\lambda\}$ is a solution of the system

$$\begin{cases} \mathcal{L}_0 \widetilde{Z}_\lambda + \widetilde{P}_\lambda = 0, \qquad \mathcal{L}_0 \widetilde{P}_\lambda - \lambda \widetilde{Z}_\lambda = \lambda(\widetilde{Z}_\lambda - z_d(x,y)), \\ \widetilde{Z}_\lambda|_\Gamma = 0 = \widetilde{P}_\lambda|_\Gamma. \end{cases} \tag{4.12}$$

Let us show that $\widetilde{P}_\lambda \neq 0$ for all $\lambda > 0$. Otherwise by (4.12) we obtain that $\widetilde{Z}_\lambda = 0$ and $z_\lambda := \widetilde{Z}_\lambda = z_d$. But in this case it follows from (4.1) that $\mathcal{L}_0 p_{0,\lambda,d} = 0$ and hence, $p_{0,\lambda,d} = 0$ and $\mathcal{L}_0 z_{0,\lambda,d} = \mathcal{L}_0 z_d = f$, which contradicts condition (4.9).

By (4.11) and (4.12) we have:

$$\begin{aligned}(P_\lambda, \widetilde{P}_\lambda) &= -(\mathcal{L}_0\widetilde{Z}_\lambda, P_\lambda) = -(\widetilde{Z}_\lambda, \mathcal{L}_0^* P_\lambda) = -(\widetilde{Z}_\lambda, \lambda(z_\lambda - z_d(x,y)) \\ &= -(\widetilde{Z}_\lambda, \mathcal{L}_0^*\widetilde{P}_\lambda - \lambda\widetilde{Z}_\lambda) = \lambda\|\widetilde{Z}_\lambda\|^2 + \|\widetilde{P}_\lambda\|^2 > 0.\end{aligned}$$

This is why $\mathcal{F}'(\lambda) > 0$ and the function $\mathcal{F}(\lambda)$ strictly increases, is continuous on $[0,\beta]$ and $\mathcal{F}(0) = 0$, while $\mathcal{F}(\beta) > 1$. This is why there exists a unique $\lambda_0 \in (0,\beta)$ such that $1 = \mathcal{F}(\lambda_0) = \lambda_0\|p_{0,\lambda_0,d}\|$. The proof is complete. □

Theorem 4.3. *Let conditions (4.9) be satisfied. Then*

$$\lambda_\varepsilon \longrightarrow \lambda_0, \quad \|z_\varepsilon - z_{0,\lambda_0,d}\| \to 0, \qquad \|p_\varepsilon - p_{0,\lambda_0,d}\| \to 0 \qquad \text{as} \quad \varepsilon \to 0,$$

where λ_0 satisfies (4.10).

Proof. First of all by the assumptions of the theorem for all sufficiently small $\varepsilon > 0$ the identity holds:

$$\lambda_\varepsilon\|p_\varepsilon\| = 1. \tag{4.13}$$

Let $\lambda_0 \in [0,\beta]$ be a partial limit of $\{\lambda_\varepsilon\}$, that is, $\lambda_{\varepsilon_k} \longrightarrow \lambda_0$ for some $\{\lambda_{\varepsilon_k}\}$ such that $\varepsilon_k \longrightarrow +0$. In the rest of the proof, to avoid bulky writing, we shall omit the subscript k of ε_k.

We consider the functions $Z_\varepsilon := z_\varepsilon - z_{\varepsilon,\lambda_0,d}$ and $P_\varepsilon := p_\varepsilon - p_{\varepsilon,\lambda_0,d}$. Since

$$\mathcal{L}_\varepsilon Z_\varepsilon + \lambda_0 P_\varepsilon = (\lambda_0 - \lambda_\varepsilon) p_\varepsilon, \qquad \mathcal{L}_\varepsilon P_\varepsilon - Z_\varepsilon = 0,$$

then by (3.5)

$$\|Z_\varepsilon\| \leqslant K|\lambda_0 - \lambda_\varepsilon| \cdot \|p_\varepsilon\|, \qquad \|P_\varepsilon\| \leqslant K|\lambda_0 - \lambda_\varepsilon| \cdot \|p_\varepsilon\|,$$

and by using Corollary 3.1 we see that $\|Z_\varepsilon\| \longrightarrow 0$ and $\|P_\varepsilon\| \longrightarrow 0$. Applying Theorem 4.2, we obtain:

$$\|z_\varepsilon - z_{0,\lambda_0,d}\| \to 0 \qquad \text{and} \qquad \|p_\varepsilon - p_{0,\lambda_0,d}\| \to 0.$$

Passing then to the limit in (4.13), we find that λ_0 solves equation (4.10). By the uniqueness of solution to equation (4.10) this shows that λ_0 is the unique partial limit of λ_ε and therefore, $\lambda_\varepsilon \longrightarrow \lambda_0$. The proof is complete. □

5. Approximation theorems

In order to justify asymptotic expansions for solutions of problem (2.8), (2.7), we need theorems on estimates for the deviation of the exact solution $\{z_\varepsilon, p_\varepsilon, \lambda_\varepsilon\}$ of this problem from that of the approximating problem

$$\begin{cases} \mathcal{L}_\varepsilon \widetilde{z}_{\varepsilon,\gamma} + \widetilde{\lambda}_{\varepsilon,\gamma} \widetilde{p}_{\varepsilon,\gamma} = f(x,y) + f_{1,\gamma}, & (x,y) \in \Omega, \\ \mathcal{L}_\varepsilon \widetilde{p}_{\varepsilon,\gamma} - \widetilde{z}_{\varepsilon,\gamma} = f_{2,\gamma} - z_d(x,y), & \widetilde{z}_{\varepsilon,\gamma},\, \widetilde{p}_{\varepsilon,\gamma} \in H_0^1(\Omega), \\ \widetilde{z}_{\varepsilon,\gamma} = 0, \qquad \widetilde{p}_{\varepsilon,\gamma} = 0, & (x,y) \in \Gamma, \end{cases} \tag{5.1}$$

in the case when as $\varepsilon \to 0$

$$f_{j,\gamma} \in C^\infty(\overline{\Omega}), \qquad \|f_{j,\gamma}\| = O\big(\varepsilon^\gamma\big), \qquad j = 1,2, \tag{5.2}$$

and we also need an approximation of condition (2.7).

If for all sufficiently small $\varepsilon > 0$ the restrictions on the control in the original problem are not essential, then $\widetilde{\lambda}_{\varepsilon,\gamma} = \beta$. Otherwise an additional approximation condition reads as

$$\widetilde{\lambda}_{\varepsilon,\gamma} \|\widetilde{p}_{\varepsilon,\gamma}\| = 1 + O\big(\varepsilon^\gamma\big). \tag{5.3}$$

If $\widetilde{\lambda}_{\varepsilon,\gamma} = \beta$, then (3.5) provide needed estimates for the approximation errors.

Theorem 5.1. *Let $\widetilde{\lambda}_{\varepsilon,\gamma} = \beta$. Then*

$$\varepsilon \left\| \frac{\partial}{\partial x}(z_\varepsilon - \widetilde{z}_{\varepsilon,\gamma}) \right\| + \left\| \frac{\partial}{\partial y}(z_\varepsilon - \widetilde{z}_{\varepsilon,\gamma}) \right\| + \|(z_\varepsilon - \widetilde{z}_{\varepsilon,\gamma})\| = O(\varepsilon^\gamma),$$

$$\varepsilon \left\| \frac{\partial}{\partial x}(p_\varepsilon - \widetilde{p}_{\varepsilon,\gamma}) \right\| + \left\| \frac{\partial}{\partial y}(p_\varepsilon - \widetilde{p}_{\varepsilon,\gamma}) \right\| + \|(p_\varepsilon - \widetilde{p}_{\varepsilon,\gamma})\| = O(\varepsilon^\gamma)$$

as $\varepsilon \to 0$, where $\widetilde{z}_{\varepsilon,\gamma}$ and $\widetilde{p}_{\varepsilon,\gamma}$ is a solution of problem (5.1), (5.2).

In the case when the restrictions for the control are not essential, that is, approximation condition (5.3) holds, to prove an approximation theorem we need an auxiliary statement on the dependence of the optimal $u_{\varepsilon,r}$ in problem (2.1)–(2.2) on r under the condition $\mathcal{U} = \mathcal{U}_r$ and $\|u_{\varepsilon,r}\| = r$.

Theorem 5.2. *Let conditions (2.4) be satisfied, and $u_{\varepsilon,r}$ be a solution of problem (2.1), (2.2) with $\mathcal{U} = \mathcal{U}_r$ and $\|u_{\varepsilon,r}\| = r$ for all $r \in [r_*; r^*]$. Then*

$$\forall\, r, r' \in [r_*; r^*], \quad \forall\, \varepsilon \in (0; \varepsilon_0], \quad \|u_r - u_{r'}\| \leqslant K|r - r'| \tag{5.4}$$

for some $K > 0$ and $\varepsilon_0 > 0$.

Proof. Let $z_{\varepsilon,0}$ be the solution of problem (2.1) with $u = 0$, and an operator $A_\varepsilon : L_2(\Omega) \to L_2(\Omega)$ maps the function u_ε into the solution of problem (2.1) with $f = 0$. Then $z_\varepsilon = z_{\varepsilon,0} + A_\varepsilon u_\varepsilon$ and the quality functional becomes

$$J(u_\varepsilon) = \|A_\varepsilon u_\varepsilon + v_0\|^2 + \beta^{-1}\|u_\varepsilon\|^2,$$

where $v_0 := z_{\varepsilon,0} - z_d$. By Theorem 3 in [15]

$$\|u_r - u_{r'}\| \leqslant K_1 \cdot |r - r'| \cdot \|A_\varepsilon\|^2 \cdot \big(\|A_\varepsilon\| + \|v_0\|\big)^4.$$

By estimates (3.1) and the definition of $\|A_\varepsilon\|$ we get $\|A\| \leqslant K_2$. At the same time $\|v_0\| \leqslant K_3$, where K_2 and K_3 are constants determined by (3.1). Thus, $\|u_r - u_{r'}\| \leqslant K|r - r'|$ for all sufficiently small $\varepsilon > 0$. The proof is complete. □

Theorem 5.3. *Let identity (4.13) hold for all sufficiently small $\varepsilon > 0$. Then*

$$\varepsilon\left\|\frac{\partial}{\partial x}(z_\varepsilon - \widetilde{z}_{\varepsilon,\gamma})\right\| + \left\|\frac{\partial}{\partial y}(z_\varepsilon - \widetilde{z}_{\varepsilon,\gamma})\right\| + \|(z_\varepsilon - \widetilde{z}_{\varepsilon,\gamma})\| = O(\varepsilon^\gamma),$$

$$\varepsilon\left\|\frac{\partial}{\partial x}(p_\varepsilon - \widetilde{p}_{\varepsilon,\gamma})\right\| + \left\|\frac{\partial}{\partial y}(p_\varepsilon - \widetilde{p}_{\varepsilon,\gamma})\right\| + \|(p_\varepsilon - \widetilde{p}_{\varepsilon,\gamma})\| = O(\varepsilon^\gamma),$$

$$|\lambda_\varepsilon - \widetilde{\lambda}_{\varepsilon,\gamma}| = O(\varepsilon^\gamma),$$

as $\varepsilon \to 0$, where $\widetilde{z}_{\varepsilon,\gamma}$, $\widetilde{p}_{\varepsilon,\gamma}$ and $\widetilde{\lambda}_{\varepsilon,\gamma}$ are the solution of problem (5.1)–(5.3).

Proof. This theorem can be proved similarly to Theorem 4 in [15] taking into consideration (3.1) and (5.4). □

The approximation theorems show that the construction of an asymptotic expansion for the solution of problem (2.8), (2.7) is reduced to constructing its formal asymptotic solution [16, Ch. I, Sect. 1].

6. Outer asymptotic expansion

In contrast to [10], since for $\varepsilon = 0$ system (2.8) remains a system of second order ordinary differential equations smoothly depending on the parameter x, by means of the outer expansion we succeed to satisfy the boundary conditions without an exponentially decaying boundary layer.

We seek outer expansions for z_ε and p_ε and an expansion for λ_ε as

$$z_{out} := \sum_{k=0}^{+\infty} \varepsilon^{2k} z_k(x,y), \quad p_{out} := \sum_{k=0}^{+\infty} \varepsilon^{2k} p_k(x,y), \quad \lambda := \sum_{k=0}^{+\infty} \varepsilon^{2k}\lambda_k. \tag{6.1}$$

We substitute these series into system (2.8) and equate the terms of like smallness order. As a result, for determining the functions z_k, p_k and constants λ_k, we obtain the equations

$$\begin{cases} \mathcal{L}_0 z_0 + \lambda_0 p_0 = f(x,y), \mathcal{L}_0 p_0 - z_0 = -z_d(x,y), \\ \mathcal{L}_0 z_k + \lambda_0 p_k + \lambda_k p_0 = F_{1,k}, \mathcal{L}_0 p_k - z_k = F_{2,k}, \quad k \geqslant 1, \\ z_k|_\Gamma = 0 = p_k|_\Gamma, k \geqslant 0, \end{cases} \tag{6.2}$$

where an operator $\mathcal{L}_0$ is obtained from $\mathcal{L}_\varepsilon$ once we formally let $\varepsilon = 0$:

$$\mathcal{L}_0 := -\frac{\partial^2}{\partial y^2} + a(x,y), \quad F_{1,k} = \frac{\partial^2 z_{k-1}}{\partial x^2} - \sum_{l=1}^{k-1}\lambda_l p_{k-l}, \quad F_{2,k} = \frac{\partial^2 p_{k-1}}{\partial x^2}. \tag{6.3}$$

By Theorem 4.1, system (6.2), (6.3) possesses an unique solution for a given set $\{\lambda_k\}$.

Thus, the outer expansion has been constructed for a given set $\{\lambda_k\}$. By its construction, this expansion is a formal asymptotic solution for problem (2.8) in subdomains of the domain Ω, in which series (6.1) keep their asymptotic property.

We are going to show that these series are valid in the entire domain Ω. In order to do this, we consider the asymptotics of the functions z_k and p_k as $(x,y)\to M_i$, $i=1,2$. The arguing for all such neighbourhoods are similar and this is why we consider only the neighbourhood of the point $M_1=(0,0)$.

Let

$$\psi_1(y)\overset{as}{=}c^{-2}y^2\left(1+\sum_{k=1}^{+\infty}c_ky^k\right),\quad y\to 0.$$

Then the functions φ_j determining Γ_j have the following asymptotic expansions as $x\to+0$:

$$\varphi_1(x)\overset{as}{=}-cx^{1/2}+\sum_{s=2}^{+\infty}c_s(-x^{1/2})^s,\quad \varphi_2(x)\overset{as}{=}cx^{1/2}+\sum_{s=2}^{+\infty}c_s(-x^{1/2})^s. \tag{6.4}$$

By $\sigma(x)$, sometimes with subscripts, we shall denote smooth on $(0;x_1)$ functions having a power asymptotic expansion as $x\to+0$, which can be differentiated infinitely many times. By $\sigma(x,y)$, sometimes with subscripts, we shall denote smooth in Ω functions having a power asymptotic expansion as $\Omega\ni(x,y)\to(0,0)$, which can be differentiated infinitely many times.

We shall group the terms in the asymptotic representation of the functions $\sigma(x,y)$ into homogeneous of $(2,1)$-degree polynomials, where $\deg_{(2,1)}(x^sy^r):=2s+r$. Homogeneous of $(2,1)$-degree n polynomials are denoted by P_n. Such polynomial reads as

$$P_n(x,y)=\sum_{s:\,n\geqslant 2s\geqslant 0}\gamma_sx^sy^{n-2s}.$$

Thus, as $\Omega\ni(x,y)\to(0,0)$,

$$\sigma(x,y)\overset{as}{=}\sum_{n=0}^{+\infty}P_n(x,y).$$

As for usual homogeneous polynomials of degree n,

$$P_n(x,y)P_m(x,y)=P_{n+m}(x,y).$$

We note that $P_n(x,y)=x^{n/2}Q(y/\sqrt{x})$, where $Q(\eta)$ is some polynomial of η.

If $\sigma(x,y)\overset{as}{=}\sum\limits_{n=k}^{+\infty}P_n(x,y)$ as $\Omega\ni(x,y)\to(0,0)$, to stress this fact, we shall employ the notation $\sigma(x,y;k)$.

By (6.4), the functions φ_1 and φ_2 can be represented as

$$\varphi_1(x)=-x^{1/2}\sigma_1(x)+x\sigma_2(x),\qquad \varphi_2(x)=x^{1/2}\sigma_1(x)+x\sigma_2(x),\qquad \sigma_1(0)=c. \tag{6.5}$$

It follows from (6.5) that the following asymptotic estimates hold:

$$\begin{gathered}\left|\frac{\varphi_i(x)}{\sqrt{x}}\right|=c+O(x)\quad\text{as}\quad x\to+0,\quad i=1,2,\\ \sigma(x,y;n)=O\Big(x^{n/2}\Big)\quad\text{as}\quad \Omega\ni(x,y)\to(0;0)\quad\text{uniformly in } y.\end{gathered} \tag{6.6}$$

Lemma 6.1. *If $w_k(x,y)$ is a solution of the problem*

$$\mathcal{L}_{0,0}w_k:=\frac{\partial^2w_k}{\partial y^2}=y^k,\qquad (x,y)\in\Omega,\quad w_k\Big|_\Gamma=0,$$

then

$$w_k(x,y)=\gamma_ky^{k+2}+yx^{[(k+1)/2]}\sigma_3(x,y)+x^{[(k+2)/2]}\sigma_4(x,y). \tag{6.7}$$

Here $[b]$ is the integer part of a number b and $\gamma_k = 1/((k+1)(k+2))$.

Proof. By an explicit formula, the solution of the considered equation $w_k(x,y)$ is of the form

$$\frac{w_k(x,y)}{\gamma_k} = y^{k+2} - y\frac{\varphi_2(x)^{k+2} - \varphi_1(x)^{k+2}}{\varphi_2(x) - \varphi_1(x)} + \frac{\varphi_1(x)\varphi_2(x)\big(\varphi_2(x)^{k+1} - \varphi_1(x)^{k+1}\big)}{\varphi_2(x) - \varphi_1(x)}.$$

Since by (6.5)

$$\varphi_2(x) - \varphi_1(x) = 2\sqrt{x}\sigma_1(x), \quad \varphi_2(x)\varphi_1(x) = x^2\sigma_2(x)^2 - x\sigma_1(x)^2,$$

and for a natural number m

$$\varphi_2(x)^m - \varphi_1(x)^m = 2\sqrt{x}\sum_{s=0}^{2s\leqslant m} C_m^{2s+1} x^{m-s-1}\sigma_1(x)^{2s+1}\sigma_2(x)^{m-2s-1},$$

then by standard procedures with power asymptotic series we obtain formula (6.7). □

Corollary 6.1. *If $w_k(x,y)$ is the solution of the problem*

$$\mathcal{L}_{0,0}w_k = P_k(x,y), \quad (x,y)\in\Omega, \quad w_k\big|_\Gamma = 0,$$

then

$$w_k(x,y) = \sigma(x,y;k+1).$$

Proof. By the linear property of the considered problem, it is sufficient to confirm this fact for a monomial, that is, for $P_k(x,y) = x^s y^{k-2s}$. In this case by formula (6.7) we obtain

$$w_k(x,y) = \gamma_k x^s y^{k+2} + yx^{[(k+1+2s)/2]}\sigma_3(x,y) + x^{[(k+2+2s)/2]}\sigma_4(x,y).$$

Considering even and odd k, we see that the relation

$$w_k(x,y) = \sigma(x,y;k+1)$$

is true. The proof is complete. □

Lemma 6.2. *If $v(x,y)$, $w(x,y)$ is the solution of problem (4.1) with $f_{0,i}(x,y) = \sigma_i(x,y;k)$, $i=1,2$, then*

$$v(x,y) = \sigma_1(x,y;k+1), \qquad w(x,y) = \sigma_2(x,y;k+1).$$

Proof. We first of all observe that $a(x,y) = \sigma(x,y;0)$. Let

$$f_{0,i}(x,y) \overset{as}{=} \sum_{n=k}^{+\infty} P_{n,i,1}(x,y), \quad l=1,2,$$

where $P_{n,i,1}(x,y)$ are homogeneous of $(2,1)$-degree n polynomials, while $v_1(x,y)$ and $w_1(x,y)$ is the solution of the problem

$$\mathcal{L}_{0,0}v_1 = P_{k,1,1}(x,y), \qquad \mathcal{L}_{0,0}w_1 = P_{k,2,1}(x,y), \qquad (x,y)\in\Omega, \qquad v_1\big|_\Gamma = 0 = w_1\big|_\Gamma.$$

Then $v_1 = \sigma_{v,1}(x,y;k+1)$, $w_1 = \sigma_{w,1}(x,y;k+1)$ by Corollary 6.1, and the functions $V_1 := v - v_1$ and $W_1 := w - w_1$ satisfy the system

$$\mathcal{L}_0 V_1 + \lambda W_1 = f_{0,1} - v_1 - a(x,y)v_1 - \lambda w_1 = \sigma_{1,1}(x,y;k+1) \overset{as}{=} \sum_{n=k+1}^{+\infty} P_{n,1,2}(x,y),$$

$$\mathcal{L}_0 W_1 - V_1 = f_{0,2} - w_1 - a(x,y)w_1 + v_1 = \sigma_{2,1}(x,y;k+1) \overset{as}{=} \sum_{n=k+1}^{+\infty} P_{n,2,2}(x,y),$$

$$V_1\Big|_\Gamma = 0 = W_1\Big|_\Gamma.$$

Now we consider the solution $v_2(x,y)$ and $w_2(x,y)$ of the problem

$$\mathcal{L}_{0,0}v_2 = P_{k+1,1,2}(x,y) \qquad \mathcal{L}_{0,0}w_2 = P_{k+1,2,2}(x,y), \qquad (x,y)\in\Omega, \qquad v_1\big|_{\Gamma} = 0 = w_1\big|_{\Gamma}.$$

Then the functions $V_1 := v - v_1 - v_2$ and $W_1 := w - w_1 - w_2$ solve the system

$$\mathcal{L}_0 V_2 + \lambda W_2 = \sigma_{1,2}(x,y;k+2), \qquad \mathcal{L}_0 W_1 - V_1 = \sigma_{2,2}(x,y;k+1), \qquad V_1\big|_{\Gamma} = 0 = W_1\big|_{\Gamma}.$$

Continuing this process, we obtain asymptotic series $V := \sum\limits_{n=1}^{+\infty} v_n$ and $W := \sum\limits_{n=1}^{+\infty} w_n$, which are a formal asymptotic solution of problem (4.1). By estimates (4.4) and (6.6) the constructed series are asymptotic expansions for the solutions of problem (4.1). The proof is complete. □

Theorem 6.1. *Let conditions (2.4) and (4.3) hold. Then for each set* $\{\lambda_k\}_{k=0}^{\infty} \subset [\lambda_*;\lambda^*] \subset (0;+\infty)$ *the solution of system (6.2), (6.3) read as* $z_k = \sigma_{z,k}(x,y)$, $p_k = \sigma_{p,k}(x,y)$.

Proof. By Lemma 6.2 we have $z_0 = \sigma_{z,0}(x,y)$ and $p_0 = \sigma_{p,0}(x,y)$. Since by the definition a function of for $\sigma(x,y)$ $(\partial/\partial x)\sigma(x,y)$ is again a function of form $\sigma(x,y)$, then

$$\mathcal{L}_0 z_1 = \frac{\partial^2}{\partial x^2} z_0 - \lambda_1 p_0 = \sigma_{z,0,1}(x,y), \qquad \mathcal{L}_0 p_1 = \frac{\partial^2}{\partial x^2} p_0 = \sigma_{p,0,1}(x,y),$$

and hence, $z_1 = \sigma_{z,1}(x,y)$ and $p_1 = \sigma_{p,1}(x,y)$. Then we continue arguing by induction and this completes the proof. □

Thus, the outer expansion is valid everywhere in Ω and by Corollary 3.1 this is an asymptotic expansion of the solution of problem (2.8) for each fixed λ_ε with an asymptotic expansion of form $\sum\limits_{k=0}^{+\infty} \varepsilon^{2k}\lambda_k$.

We note that for a single equation with $\psi_1(y) = y^2$ a similar result was obtained in [3, Thm. 2]. At the same time, if $\psi_1(y) = y^4$, then the functions in the outer expansion have increasing singularities as $\Omega \ni (x,y) \to (0;0)$, see [3, Sect. 1.2].

7. Complete asymptotics for solution

1. Let condition (4.8) hold. In this case the identity $\lambda_\varepsilon = \beta$ is true and hence, $\lambda_0 = \beta$, $\lambda_k = 0$ as $k > 0$. This is why by Theorem 5.1 we arrive at the following theorem.

Theorem 7.1. *Let conditions (2.4), (4.3) and (4.8) hold. Then outer expansion (6.1) with* $\lambda_0 = \beta$, $\lambda_k = 0$ *for* $k > 0$ *is an asymptotic expansion for the solution of problem (2.8) with* $\lambda_\varepsilon = \beta$ *as* $\varepsilon \to +0$.

2. Let condition (4.9) hold. In this case the conditions determining $\{\lambda_k\}$ are generated by the asymptotic identity $\Lambda^2\|p_{out}\|^2 \overset{as}{=} 1$ as $\varepsilon \to +0$.

Equating the coefficients at the like powers of ε, we obtain the identities

$$\lambda_0\|p_0\| = 1, \qquad 2\lambda_0\lambda_k\|p_0\|^2 + 2\lambda_0^2(p_0,p_k) = h_k, \qquad k\in\mathbb{N}, \tag{7.1}$$

where h_k is completely determined by previous z_l, p_l and λ_l, $0 \leqslant l < k$. Thus, as λ_0 we choose a unique solution of the equation $\lambda\|p_{0,\lambda,d}\| = 1$, which is ensured by Lemma 4.1.

As in [10], for $k > 0$ it is convenient to represent z_k and p_k as $z_k = \widetilde{z}_k + \lambda_k\widetilde{z}$, $p_k = \widetilde{p}_k + \lambda_k\widetilde{p}$, where $\widetilde{z}_k$, $\widetilde{p}_k$ is the solution of the problem

$$\begin{cases} \mathcal{L}_0\widetilde{z}_k + \lambda_0\widetilde{p}_k = F_{1,k}, \qquad \mathcal{L}_0\widetilde{p}_k - \widetilde{z}_k = F_{2,k}, & k \geqslant 1,\\ \widetilde{z}_k|_\Gamma = 0 = \widetilde{p}_k|_\Gamma, & k \geqslant 0, \end{cases} \tag{7.2}$$

while $\widetilde{z}$, $\widetilde{p}$ is the solution of the problem

$$\begin{cases} \mathcal{L}_0\widetilde{z} + \lambda_0\widetilde{p} + p_0 = 0, \qquad \mathcal{L}_0\widetilde{p} - \widetilde{z} = 0, \\ \widetilde{z}|_\Gamma = 0 = \widetilde{p}|_\Gamma. \end{cases} \tag{7.3}$$

We observe that the functions $\widetilde{z}_k$, $\widetilde{p}_k$ are completely determined by previous z_l, p_l and λ_l, $0 \leqslant l < k$. Under such representation, equations (7.1) for determining λ_k become

$$\lambda_0\|p_0\| = 1, \qquad 2\lambda_k\Big(\lambda_0\|p_0\|^2 + \lambda_0^2(p_0,\widetilde{p})\Big) = \widetilde{h}_k, \qquad k \in \mathbb{N}. \tag{7.4}$$

Lemma 7.1. *If condition (4.9) hold, then*

$$\lambda_0\|p_0\|^2 + \lambda_0^2(p_0,\widetilde{p}) > 0.$$

Proof. We note that

$$\widetilde{z} = \frac{\partial}{\partial\lambda} z_{0,\lambda,d}\Big|_{\lambda=\lambda_0}, \qquad \widetilde{p} = \frac{\partial}{\partial\lambda}_{0,\lambda,d}\Big|_{\lambda=\lambda_0}.$$

This is why by Lemma 4.1

$$0 < \frac{\partial}{\partial\lambda}\Big(\lambda^2\|p_{0,\lambda,d}\|^2\Big)\Big|_{\lambda=\lambda_0} = 2\lambda_0\|p_0\|^2 + 2\lambda_0^2(p_0,\widetilde{p}).$$

The proof is complete. □

Thus, problems (6.2)–(6.3), (7.2)–(7.4) are uniquely solvable and outer expansion (6.1) is a formal asymptotic solution of problem (2.8), (2.7). Thus, in this case we also have a final theorem.

Theorem 7.2. *Let conditions (2.4), (4.3) and (4.9 hold. Then outer expansion (6.1) with the coefficients determined by problems (6.2)–(6.3), (7.2)–(7.4) is an asymptotic expansion for solution of problem (2.8) as $\varepsilon \to +0$ with an additional condition $\lambda_\varepsilon\|p_\varepsilon\| = 1$.*

Conclusion

We note that under condition (4.3) considered problem (2.1)–(2.3) turns out to be regular and the asymptotic expansion for its solution coincides with the outer expansion. However, as $\psi(y) = y^4$, the outer expansion is no longer valid everywhere in Ω and for constructing the asymptotic expansion for the solution of the considered problem in this case one has to employ the method of matching asymptotic expansions [16].

BIBLIOGRAPHY

1. J.L. Lions. *Optimal control of systems governed by partial differential equations.* Springer-Verlag, Berlin (1971).
2. A.N. Tikhonov, A.A. Samarskii. *Equations of mathematical physics.* Nauka, Moscow (1977). [Pergamon Press, Oxford (1963).]
3. E.F. Lelikova. *On the asymptotics of a solution of a second order elliptic equation with small parameter at a higher derivative* // Trudy IMM UrO RAN. **9**:1, 107–120 (2003). [Proc. Steklov Inst. Math. **2003**, Suppl. 1. S129–S143 (2003).]
4. A.M. Il'in, E.F. Lelikova. *On asymptotic approximations of solutions of an equation with a small parameter* // Alg. An. **22**:6, 109–126 (2010). [St. Petersburg Math. J. **22**:6, 927–939 (2011).]
5. C. Eduardo. *A review on sparse solutions in optimal control of partial differential equations* // SeMA J. **74**:3, 319–344 (2017).
6. H. Lou, J. Yong. *Second-order necessary conditions for optimal control of semilinear elliptic equations with leading term containing controls* // Math. Control Relat. Fields. **8**:1, 57–88 (2018).

7. M. Betz Livia. *Second-order sufficient optimality conditions for optimal control of nonsmooth, semilinear parabolic equations* // SIAM J. Control Optim. **57**:6, 4033–4062 (2019).
8. A.R. Danilin. *Approximation of a singularly perturbed elliptic problem of optimal control* // Matem. Sb. **191**:10, 3–12 (2000). [Sb. Math. **191**:10, 1421–1431 (2000).]
9. A.R. Danilin. *Asymptotic behaviour of solutions of a singular elliptic system in a rectangle* // Matem. Sb. **194**:1, 31–60 (2003). [Sb. Math. **194**:1, 31–61 (2003).]
10. A.R. Danilin. *Asymptotics of the solution of a singular optimal distributed control problem with essential constraints in a convex domain* // Diff. Uravn. **56**:2, 256–268 (2020). [Diff. Equat. **56**:2, 251–263 (2020).]
11. J.L. Lions, E. Magenes. *Non-homogeneous boundary value problems and applications. V. I.* Springer-Verlag, Berlin (1972).
12. A.R. Danilin, A.P. Zorin. *Asymptotics of a solution to an optimal boundary control problem* // Trudy IMM UrO RAN. **15**:4, 95–107 (2009). [Proc. Steklov Inst. Math. **269**, Suppl. 1, S81–S94 (2010).]
13. A.R. Danilin. *Optimal boundary control in a small concave domain* // Ufimskij Matem. Zhurn. **4**:2, 87–100 (2012). (in Russian).
14. P. Hartman. *Ordinary differential equations*. John Wiley & Sons, New York (1964).
15. A.R. Danilin. *Solution asymptotics in a problem of optimal boundary control of a flow through a part of the boundary* // Trudy IMM UrO RAN. **20**:4, 116–127 (2014). [Proc. Steklov Inst. Math. **292**. Suppl. 1, 55–66 (2016).]
16. A.M. Il'in. *Matching of asymptotic expansions of solutions of boundary value problems.* Nauka, Moscow (1989). [Amer. Math. Soc. Providence, RI (1992).]

Alexey Rufimovich Danilin,
N.N. Krasovskii Institute of Mathematics and Mechanics,
Ural Branch of RAS,
Kovalevskaya str. 16,
620108, Ekaterinburg, Russia
E-mail: dar@imm.uran.ru