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ANALYSIS OF A THERMO-ELASTO-VISCOPLASTIC CONTACT PROBLEM WITH WEAR AND DAMAGE

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Abstract. This paper presents a quasistatic problem of a thermo-elaso-visco-plastic body in frictional contact with a moving foundation. The contact is modelled with the normal compliance condition and the associated law of dry friction. The model takes into account wear of the contact surface of the body caused by the friction and which is described by the Archard law. The mechanical damage of the material, caused by excessive stress or strain, is described by the damage function, the evolution of which is determined by a parabolic inclusion. We list the assumptions on the data and derive a variational formulation of the mechanical problem. Existence and uniqueness of the weak solution for the problem is proved using the theory of evolutionary variational inequalities, parabolic variational inequalities, first order evolution equation and Banach fixed point.

Keywords: Thermo-elasto-viscoplastic material, damage, wear, frictional contact, existence and uniqueness, fixed point arguments, weak solution.

Mathematics Subject Classification: 74H20, 74H25, 74M15, 74F05, 74R20, 49J40.

1. Introduction

Thermo-mechanical contact problems are of essential importance in analysis and design of structural elements interacting through localized or distributed contact forces and undergoing monotonic or oscillatory relative slip or sliding motion. The elevated temperature regimes are then generated either by environmental conditions or by frictional dissipation at the contact interface. The frictional sliding motion usually generates progressive wear coupled with such effects as localized plastic deformation, damage growth, oxidation and phase transformation, wear debris, etc. in the contact layer. The wear is defined as the material loss or change in surface texture occurring when two surfaces of mechanical components contact each other. As the contact process evolves, the contacting surfaces evolve too, via their wear. There exists a large engineering and mathematical literature devoted to this topic. The wear phenomenon subjects of numerous experimental and theoretical studies. We mention here the references [1], [3], [17], [18], [26], [27], [28], [30], [33], [42]. Numerical methods for wear problems with application to implanted knee joints were developed in [27]. An original analytical approach to wear was performed in [11]. General models for frictional contact with wear can be found in [33, 42] as well as in survey [41]. The mathematical analysis of various models of frictional contact with wear, including existence and uniqueness results of the weak solution, was carried out in [20], [21], [23], [25], [29]. Thermo-mechanic contact problems with the evolution of the temperature parameter with wear were treated in [5, 12, 13, 36, 37, 38, 39, 40]. The damage subject is extremely important in design engineering, since it directly affects the useful life of

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the designed structure or component. There exists a very large engineering literature on it. Models taking into account the influence of the internal damage of the material on the contact process were investigated mathematically. General novel models for damage were derived in [8], [10] from the virtual power principle. Mathematical analysis of one-dimensional damage models can be found in [7], [9]. The three-dimensional case has been investigated in [4], [17]. The damage function ζ is restricted to have values between zero and one. As $\zeta = 1$, there is no damage in the material, for $\zeta = 0$ the material is completely damaged, while in the case $0 < \zeta < 1$ there is a partial damage and the system has a reduced load carrying capacity. Quasistatic contact problems with damage were investigated in [7, 14, 16, 24].

The paper is organized as follows. In Section 2 we introduce the notation and mention some preliminary material. In Section 3 we describe the frictional contact problem, state the assumptions on the data and derive its variational formulation. Finally, in Section 4 we establish the existence of a weak solution to the model.

2. Notation and preliminaries

As it has been already mentioned in the previous section, we start by introducing the notation together with some preliminary results. For further details we refer to [6], [16], [22]. Throughout the paper $d \in \{1, 2, 3\}$ and $\mathbb{S}^d$ represents the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The zero element of the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$ will be denoted by $\mathbf{0}$. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i, \qquad \|\mathbf{v}\| = (\mathbf{v} \cdot \mathbf{v})^{1/2} \qquad \forall \mathbf{u} = (u_i), \quad \mathbf{v} = (v_i) \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \tau_{ij}, \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2} \qquad \forall \boldsymbol{\sigma} = (\sigma_{ij}), \quad \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d,$$

where the subscripts i, j range between 1 and d and, unless stated otherwise, the summation convention over repeated indices is adopted.

The symbol Ω denotes a bounded domain of $\mathbb{R}^d$ with a Lipschitz continuous boundary Γ and Γ_1, Γ_2, Γ_3 represents a partition of Γ into three measurable parts such that $\operatorname{meas}(\Gamma_1) > 0$. We use $\mathbf{x} = (x_i)$ for a generic point in $\Omega \cup \Gamma$. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable $\mathbf{x} \in \Omega \cup \Gamma$, for instance, $f_{,i} = \partial f / \partial x_i$.

We use standard notation for Sobolev and Lebesgue spaces associated to Ω and Γ. In particular, we use the spaces $L^2(\Omega)^d$, $L^2(\Gamma_2)^d$, $L^2(\Gamma_3)^d$ and $H^1(\Omega)^d$, endowed with their canonical inner products and associated norms. Moreover, we recall that for an element $\mathbf{v} \in H^1(\Omega)^d$ we sometimes write $\mathbf{v}$ for the trace $\gamma \mathbf{v} \in L^2(\Gamma)^d$ of $\mathbf{v}$ to Γ. In addition, we consider the following spaces:

$$H = L^2(\Omega)^d = \{\mathbf{u} = (u_i) : u_i \in L^2(\Omega)\}, \qquad Q = \{\boldsymbol{\sigma} = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$
$$H_1 = \{\mathbf{u} = (u_i) : \boldsymbol{\varepsilon}(\mathbf{u}) \in Q\}, \qquad Q_1 = \{\boldsymbol{\sigma} \in Q / \operatorname{Div} \boldsymbol{\sigma} \in H\},$$

where $\boldsymbol{\varepsilon} : H_1 \to Q$ is the deformation operator defined by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \qquad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad 1 \leqslant i, j \leqslant d, \qquad \forall \mathbf{u} \in H_1. \tag{2.1}$$

By $\operatorname{Div} : Q_1 \to H$ we denote the divergence operator given by the formulae

$$\operatorname{Div} \boldsymbol{\sigma} = (\operatorname{Div} \boldsymbol{\sigma})_i = (\sigma_{ij,j}), \qquad 1 \leqslant i, j \leqslant d, \qquad \forall \boldsymbol{\sigma} \in Q_1. \tag{2.2}$$

The spaces H, Q, H_1 and Q_1 are real Hilbert spaces endowed with the canonical inner products

$$(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega u_i v_i dx \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in H,$$

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_Q=\int_\Omega \sigma_{ij}\tau_{ij}dx \qquad \forall \boldsymbol{\sigma},\boldsymbol{\tau}\in Q,$$
$$(\boldsymbol{u},\boldsymbol{v})_{H_1}=(\boldsymbol{u},\boldsymbol{v})_H+(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \qquad \forall \boldsymbol{u},\boldsymbol{v}\in H_1,$$
$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{Q_1}=(\boldsymbol{\sigma},\boldsymbol{\tau})_Q+(\mathrm{Div}\,\boldsymbol{\sigma},\mathrm{Div}\,\boldsymbol{\tau})_H \qquad \forall \boldsymbol{\sigma},\boldsymbol{\tau}\in Q_1.$$

The associated norms on these spaces are denoted by $\|\cdot\|_H$, $\|\cdot\|_Q$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{Q_1}$, respectively. Let V the closed subspace of $H^1(\Omega)^d$ defined by

$$V=\{\mathbf{v}\in H^1(\Omega)^d:\mathbf{v}=\mathbf{0} \quad \text{on } \Gamma_1\},$$

endowed with the inner product

$$(\mathbf{u},\mathbf{v})_V=(\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v}))_Q \qquad \forall \mathbf{u},\mathbf{v}\in V, \tag{2.3}$$

and let $\|\cdot\|_V$ be the associated norm:

$$\|\mathbf{v}\|_V=\|\boldsymbol{\varepsilon}(\mathbf{v})\|_Q \qquad \forall \mathbf{u}\in V. \tag{2.4}$$

Since $\mathrm{meas}(\Gamma_1)>0$, Korn's inequality holds and there exists a constant $c_k>0$, that depends only on Ω and Γ_1, such that

$$\|\boldsymbol{\varepsilon}(\mathbf{v})\|_Q\geqslant c_k\|\mathbf{v}\|_{H^1(\Omega)^d} \qquad \forall \mathbf{v}\in V.$$

A proof of Korn's inequality may be found in ([19],p.79). It follows that $\|\cdot\|_{H^1(\Omega)^d}$ and $\|\cdot\|_V$ are equivalent norms on V and therefore $(V,\|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace Theorem, there exists a constant $c_0>0$ which depends on Ω, Γ_1 and Γ_3 such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d}\leqslant c_0\|\mathbf{v}\|_V \qquad \forall \mathbf{v}\in V. \tag{2.5}$$

Next, we introduce the closed subspace E of $H^1(\Omega)$ defined by

$$E=\{\omega\in H^1(\Omega):\,\omega=0 \text{ on } \Gamma_1\cup\Gamma_2\},$$

endowed with the canonical inner product of $H^1(\Omega)$. Moreover, by the Sobolev trace Theorem, there exists a constant $c_1>0$ which depends on Ω, Γ_1, Γ_2 and Γ_3 such that

$$\|\omega\|_{L^2(\Gamma_3)}\leqslant c_1\|\omega\|_E \qquad \forall \omega\in E. \tag{2.6}$$

The following Friedrichs-Poincaré inequality holds on E:

$$\|\nabla\omega\|_H\geqslant c_F\|\omega\|_E \qquad \forall \omega\in E. \tag{2.7}$$

The space $L^2(\Omega)$ is identified with its dual and with a subspace of the dual E' of E, that is, $E\subset L^2(\Omega)\subset E'$, and we say that the inclusions above define a Gelfand triple. The notation $\langle\cdot,\cdot,\rangle_{E',E}$ represents the duality pairing between E' and E.

For any element $\mathbf{v}\in V$ we denote by v_ν and $\mathbf{v}_\tau$ its normal and tangential components on the boundary Γ given by

$$v_\nu=\mathbf{v}\cdot\boldsymbol{\nu}, \qquad \mathbf{v}_\tau=\mathbf{v}-v_\nu\boldsymbol{\nu}.$$

In a similar manner, we recall that the normal and tangential components of the stress field $\boldsymbol{\sigma}$ on the boundary Γ are defined by

$$\sigma_\nu=(\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}, \qquad \boldsymbol{\sigma}_\tau=\boldsymbol{\sigma}\boldsymbol{\nu}-\sigma_\nu\boldsymbol{\nu}.$$

Finally, it is well known that if $\boldsymbol{\sigma}$ is a regular function, then the following Green formula holds:

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\mathbf{v})\,dx+\int_\Omega \mathrm{Div}\,\boldsymbol{\sigma}\cdot\mathbf{v}\,dx=\int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\mathbf{v}\,da \qquad \forall \mathbf{v}\in V. \tag{2.8}$$

Let $T > 0$. For every real Banach space X we use the classical notation for the spaces $C(0,T;X)$ and $C^1(0,T;X)$ for the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norm

$$\|\mathbf{f}\|_{C(0,T;X)} = \max_{t\in[0,T]} \|\mathbf{f}(t)\|_X, \qquad \|\mathbf{f}\|_{C^1(0,T;X)} = \max_{t\in[0,T]} (\|\mathbf{f}(t)\|_X + \|\dot{\mathbf{f}}(t)\|_X),$$

respectively. Moreover, we use the dot above to indicate the derivative with respect to the time variable.

Finally, for $k \in \mathbb{N}$ and $p \in [1,\infty]$, we use the standard notation for the Lebesgue spaces and for the Sobolev spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, $1 \leqslant p \leqslant \infty$, $1 \leqslant k$. Moreover, if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(\,\cdot\,,\,\cdot\,)_{X_1\times X_2}$.

3. Mechanical and variational formulations

Let $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ be a domain occupied by a thermo-viscoelastic-viscoplastic body with a Lipschitz boundary Γ which is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, the measure of Γ_1 is strictly positive and, in addition, Γ_3 is a plane. By $[0,T]$, $T > 0$, we denote the time interval of an interest. We admit an external heat source q_{th} applied in $\Omega \times (0,T)$. The body is clamped on $\Gamma_1 \times (0,T)$, so the displacement field vanishes there. A surface traction of density $\mathbf{f}_2$ act on $\Gamma_2 \times (0,T)$, and a body force of density $\mathbf{f}_0$ is applied in $\Omega \times (0,T)$. The body is in frictional contact on Γ_3 with a moving obstacle, the so-called foundation. In this paper we assume that the material behavior follows a thermo-viscoelastic-viscoplastic constitutive law with damage given by

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)) - \mathcal{M}(\theta(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \zeta(s))ds, \tag{3.1}$$

$$\dot{\theta}(t) - \operatorname{div}(\mathcal{K}\nabla\theta(t)) = -\mathcal{M}\cdot\nabla\dot{\mathbf{u}}(t) + q_{th}(t), \tag{3.2}$$

where, hereinafter $\boldsymbol{\sigma}$ denotes the stress tensor, $\mathbf{u}$ represents the displacement field, $\dot{\mathbf{u}}$ is the velocity, $\boldsymbol{\varepsilon}(\mathbf{u})$ is the linearized strain tensor, θ is the temperature field, $\mathcal{M}$ represents the thermal expansion tensor, $\mathcal{A}$ and $\mathcal{B}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, and $\mathcal{G}$ is a nonlinear constitutive function which describes the viscoplastic behavior of the material, where ζ is an internal variable describing the damage of the material caused by viscoplastic deformations.

The evolution of the temperature field θ is governed by a heat equation obtained from the conservation of energy and defined by (3.2), where $\mathcal{K} = (k_{ij})$ represents the thermal conductivity tensor, $\operatorname{div}(\mathcal{K}\nabla\theta) = (k_{ij}\theta_{,i})$ and q_{th} represents the density of volume heat sources.

The inclusion used for the evolution of the damage field is

$$\dot{\zeta} - \Delta\zeta + \partial\varphi_K(\zeta) \ni \psi(\boldsymbol{\sigma}, \varepsilon(\mathbf{u}), \theta, \zeta),$$

where K denotes the set of admissible damage functions defined as

$$K = \{\vartheta \in H^1(\Omega) : 0 \leqslant \vartheta \leqslant 1 \text{ a.e. in } \Omega\},$$

$\partial\varphi_K$ denotes the subdifferential of the indicator function φ_K and ψ is a given constitutive function which describes the sources of the damage in the system.

We now briefly describe the boundary conditions on the contact surface Γ_3, based on the model derived in [34], [35]. We assume that a wear of the contact surface occurs as a result of the relative slip between the surface and the foundation and the frictional forces modify the asperities on the surface by removing small amounts of the material and possibly rearranging the remaining undulations. We introduce a wear function $w : \Gamma_3 \times [0,T] \to \mathbb{R}$ which will measure the wear of the contact surface Γ_3. The wear is identified as the normal depth of the material that is lost. We use the modified version of Archard law:

$$\dot{w}(t) = -k_w\|\mathbf{v}^*\|\sigma_\nu(t). \tag{3.3}$$

where $k_w > 0$ is a wear coefficient and $\mathbf{v}^*$ is the tangential velocity of the foundation. For the sake of simplicity we assume in the rest of the section that the motion of the foundation is uniform, that is, $\mathbf{v}^*$ does not vary in time. The soft material is thermo-viscoelastic-viscoplastic and could wear. Therefore, we assume that $\sigma_\nu(t)$ satisfies a normal compliance contact condition with wear, that is

$$-\sigma_\nu(t) = p_\nu(u_\nu(t) - w(t)) \qquad \text{on } \Gamma_3, \tag{3.4}$$

where p_ν represents the normal compliance function, this condition shows that at each moment t, the reaction of the soft layer depends on the current value of the penetration represented by $u_\nu(t) - w(t)$. Indeed, we assume that a wear process of the foundation takes place and the debris are immediately removed from the system. Thus, the penetration becomes $u_\nu(t) - w(t)$ instead of $u_\nu(t)$ as in the case without wear.

The corresponding generalization of Coulomb law of dry friction may be stated as

$$\|\boldsymbol{\sigma}_\tau(t)\| \leqslant p_\tau(u_\nu(t) - w(t)) \qquad \text{on} \quad \Gamma_3, \tag{3.5}$$

$$\|\boldsymbol{\sigma}_\tau(t)\| < p_\tau(u_\nu(t) - w(t)) \quad \Rightarrow \quad \dot{\mathbf{u}}_\tau(t) = \mathbf{v}^*, \tag{3.6}$$

$$\|\boldsymbol{\sigma}_\tau(t)\| = p_\tau(u_\nu(t) - w(t)) \quad \Rightarrow \quad \dot{\mathbf{u}}_\tau(t) = \mathbf{v}^* - \alpha\boldsymbol{\sigma}_\tau(t), \quad \alpha \geqslant 0. \tag{3.7}$$

Here p_τ represents the tangential compliance function (called the friction bound) and $\dot{\mathbf{u}}_\tau$ is the tangential velocity of the body. Equations (3.6) and (3.7) can be interpreted physically in the following way. Condition (3.6) characterizes the behavior of the body in the so-called stick zone. It implies that as the tangential stress is insufficient, the boundary sticks to the foundation and moves at the same velocity as the foundation. Condition (3.7) describes the so-called slip zone, that is, as the tangential stress reaches its greatest value, the boundary does not move in tandem with the foundation. The scalar $\alpha \geqslant 0$ is a multiplier indicating the relative direction of the slip between the body and the foundation. Finally, the associated temperature boundary condition on Γ_3 reads as

$$-k_{ij}\frac{\partial\theta}{\partial x_i}\nu_j = k_e(\theta - \theta_F) - k_\tau(\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|) \qquad \text{on} \quad \Gamma_3 \times (0,T), \tag{3.8}$$

where θ_F is the temperature of the foundation, k_e is the heat exchange coefficient between the body and the obstacle, and $k_\tau : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$ is a given tangential function.

To simplify the notation, we do not indicate explicitly the dependence of various functions on the variables $x \in \Omega \cup \Gamma$ and $t \in [0,T]$. Then, the classical formulation of the thermo-mechanical problem is as follows.

Problem $\mathcal{P}$. *Find a displacement field $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$, a temperature $\theta : \Omega \times [0,T] \to \mathbb{R}$, a damage $\zeta : \Omega \times [0,T] \to \mathbb{R}$ and a wear function $w : \Gamma_3 \times [0,T] \to \mathbb{R}$ such that*

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}) - \mathcal{M}\theta + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \zeta(s))ds \qquad \text{in } \Omega \times (0,T), \tag{3.9}$$

$$\dot{\theta} - \operatorname{div}(\mathcal{K}\nabla\theta) = -\mathcal{M} \cdot \nabla\dot{\mathbf{u}} + q_{th} \qquad \text{in} \quad \Omega \times (0,T), \tag{3.10}$$

$$\dot{\zeta} - \Delta\zeta + \partial\varphi_K(\zeta) \ni \psi(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}), \theta, \zeta) \qquad \text{in} \quad \Omega \times (0,T), \tag{3.11}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0} \qquad \text{in } \Omega \times (0,T), \tag{3.12}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma_1 \times (0,T), \tag{3.13}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \mathbf{f}_2 \qquad \text{on } \Gamma_2 \times (0,T), \tag{3.14}$$

$$\theta = 0 \qquad \text{on } \Gamma_1 \cup \Gamma_2 \times (0,T), \tag{3.15}$$

$$\begin{cases} -\sigma_\nu = p_\nu(u_\nu - w), & \|\boldsymbol{\sigma}_\tau\| \leqslant p_\tau(u_\nu - w), \\ \|\boldsymbol{\sigma}_\tau\| < p_\tau(u_\nu - w) & \Rightarrow \quad \dot{\mathbf{u}}_\tau = \mathbf{v}^*, \\ \|\boldsymbol{\sigma}_\tau\| = p_\tau(u_\nu - w) & \Rightarrow \quad \dot{\mathbf{u}}_\tau = \mathbf{v}^* - \alpha\boldsymbol{\sigma}_\tau \qquad \alpha > 0, \end{cases} \quad \text{on} \quad \Gamma_3 \times (0,T), \tag{3.16}$$

$$-k_{ij}\frac{\partial \theta}{\partial x_i}\nu_j = k_e(\theta - \theta_F) - k_\tau(\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|) \qquad \text{on} \quad \Gamma_3 \times (0,T), \tag{3.17}$$

$$\dot{w} = k_w\|\mathbf{v}^*\|p_\nu(u_\nu - w) \quad \text{on} \quad \Gamma_3 \times (0,T), \tag{3.18}$$

$$\frac{\partial \zeta}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \times (0,T), \tag{3.19}$$

$$w(0) = 0 \quad \text{on} \quad \Gamma_3, \tag{3.20}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \theta(0) = \theta_0, \qquad \zeta(0) = \zeta_0 \qquad \text{in} \quad \Omega. \tag{3.21}$$

Here equations (3.9) and (3.10) represent the thermo-viscoelastic-viscoplastic constitutive law with damage introduced in the first section, the evolution of the damage is governed by the inclusion of parabolic type given by relation (3.11). Equation (3.12) represents the equilibrium equations for the stress. Equalities (3.13) and (3.14) are the displacement-traction boundary conditions, respectively. Condition (3.16) describes the frictional contact with normal compliance and wear described above on the potential contact surface Γ_3. Equation (3.18) is an ordinary differential equation which describes the evolution of the wear function. Relation (3.19) describes a homogeneous Neumann boundary condition, where $\dfrac{\partial \zeta}{\partial \nu}$ is the normal derivative of ζ. In equation (3.20), the identity $w(0) = 0$ means that at the initial moment the body is not subject to any prior wear. Next, the functions u_0, θ_0 and ζ_0 in (3.21) are the initial data. In the study of the mechanical problem $\mathcal{P}$, we consider the following assumptions.

The viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies the following conditions:

$$\begin{cases} \text{(a) There exists } L_{\mathcal{A}} > 0 \text{ such that } \|\mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_2)\| \leqslant L_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\ \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{a.e. } \mathbf{x} \in \Omega; \\ \text{(b) There exists } m_{\mathcal{A}} > 0 \text{ such that} (\mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geqslant m_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\ \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega; \\ \text{(c) The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega \text{ for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d; \\ \text{(d) The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x},\mathbf{0}) \text{ belongs to } Q. \end{cases} \tag{3.22}$$

The elasticity operator $\mathcal{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies the following conditions:

$$\begin{cases} \text{(a) There exists a constant } L_{\mathcal{B}} > 0 \text{ such that} \\ \|\mathcal{B}(\mathbf{x},\boldsymbol{\varepsilon}_1) - \mathcal{B}(\mathbf{x},\boldsymbol{\varepsilon}_2)\| \leqslant L_{\mathcal{B}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{a.e.} \quad \mathbf{x} \in \Omega; \\ \text{(b) The mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x},\boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega \text{ for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d; \\ \text{(c) The mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x},\mathbf{0}) \text{ belongs to } Q. \end{cases} \tag{3.23}$$

The viscoplastic operator $\mathcal{G}:\Omega\times\mathbb{S}^d\times\mathbb{S}^d\times\mathbb{R}\to\mathbb{S}^d$ satisfies the following conditions:

$$\begin{cases} \text{(a) There exists a constant } L_{\mathcal{G}}>0 \text{ such that} \\ \|\mathcal{G}(\mathbf{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1,\zeta_1)-\mathcal{G}(\mathbf{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2,\zeta_2)\|\leqslant L_{\mathcal{G}}(\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\|+\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|+|\zeta_1-\zeta_2|) \\ \text{for all } \boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \zeta_1,\zeta_2\in\mathbb{R},\ \text{a.e. } \mathbf{x}\in\Omega. \\ \text{(b) The mapping } \mathbf{x}\mapsto\mathcal{G}(\mathbf{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon},\zeta) \text{ is Lebesgue measurable on } \Omega, \\ \ \text{for all } \boldsymbol{\sigma},\boldsymbol{\varepsilon}\in\mathbb{S}^d \text{ and } \zeta\in\mathbb{R}. \\ \text{(c) The mapping } \mathbf{x}\mapsto\mathcal{G}(\mathbf{x},\mathbf{0},\mathbf{0},0) \text{ belongs to } Q. \end{cases} \tag{3.24}$$

The damage source function $\psi:\Omega\times\mathbb{S}^d\times\mathbb{S}^d\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies the following conditions:

$$\begin{cases} \text{(a) There exists a constant } L_\psi>0 \text{ such that} \\ |\psi(\mathbf{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1,\theta_1,\zeta_1)-\psi(\mathbf{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2,\theta_2,\zeta_2)| \\ \ \leqslant L_\psi(\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\|+\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|+|\theta_1-\theta_2|+|\zeta_1-\zeta_2|) \\ \text{for all } \boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\theta_1,\theta_2,\zeta_1,\zeta_2\in\mathbb{R},\ \text{a.e.}\quad \mathbf{x}\in\Omega; \\ \text{(b) The mapping } \mathbf{x}\mapsto\psi(\mathbf{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon},\theta,\zeta) \text{ is Lebesgue measurable on } \Omega, \\ \ \text{for all } \boldsymbol{\sigma},\boldsymbol{\varepsilon}\in\mathbb{S}^d \text{ and } \theta,\zeta\in\mathbb{R}; \\ \text{(c) The mapping } \mathbf{x}\mapsto\psi(\mathbf{x},\mathbf{0},\mathbf{0},0,0) \text{ belongs to } L^2(\Omega). \end{cases} \tag{3.25}$$

The normal compliance function $p_\nu:\Gamma_3\times\mathbb{R}\to\mathbb{R}_+$ satisfies the following conditions:

$$\begin{cases} \text{(a) There exists } L_\nu>0 \text{ such that } |p_\nu(\mathbf{x},u_1)-p_\nu(\mathbf{x},u_2)|\leqslant L_\nu\,|u_1-u_2| \\ \text{for all } u_1,\ u_2\in\mathbb{R},\ \text{a.e. } \mathbf{x}\in\Gamma_3; \\ \text{(b) The mapping } \mathbf{x}\mapsto p_\nu(\mathbf{x},u) \text{ is measurable on } \Gamma_3 \text{ for all } u\in\mathbb{R}; \\ \text{(c) The mapping } \mathbf{x}\mapsto p_\nu(\mathbf{x},0) \text{ blongs to } L^2(\Gamma_3). \end{cases} \tag{3.26}$$

The tangential compliance function $p_\tau:\Gamma_3\times\mathbb{R}\to\mathbb{R}_+$ satisfies the following conditions:

$$\begin{cases} \text{(a) There exists } L_\tau>0 \text{ such that } |p_\tau(\mathbf{x},u_1)-p_\tau(\mathbf{x},u_2)|\leqslant L_\tau\,|u_1-u_2| \\ \text{for all } u_1,\ u_2\in\mathbb{R},\ \text{a.e.}\quad \mathbf{x}\in\Gamma_3; \\ \text{(b) The mapping } \mathbf{x}\mapsto p_\tau(\mathbf{x},u) \text{ is measurable on } \Gamma_3 \text{ for all } u\in\mathbb{R}; \\ \text{(c) The mapping } \mathbf{x}\mapsto p_\tau(\mathbf{x},0) \text{ blongs to } L^2(\Gamma_3). \end{cases} \tag{3.27}$$

The tangential function $k_\tau:\Gamma_3\times\mathbb{R}_+\to\mathbb{R}_+$ satisfies the following conditions:

$$\begin{cases} \text{(a) There exists } M_\tau>0 \text{ such that } |k_\tau(\mathbf{x},r_1)-k_\tau(\mathbf{x},r_2)|\leqslant M_\tau\,|r_1-r_2| \\ \text{for all } r_1,\ r_2\in\mathbb{R}_+,\ \text{a.e. } \mathbf{x}\in\Gamma_3; \\ \text{(b) The mapping } \mathbf{x}\mapsto k_\tau(\mathbf{x},0)\in L^2(\Gamma_3) \text{ is measurable on } \Gamma_3,\ \forall\ r\in\mathbb{R}_+. \end{cases} \tag{3.28}$$

A particular example of a tangential function k_τ is given by

$$k_\tau(\mathbf{x},r)=\rho(\mathbf{x})r,\qquad \mathbf{x}\in\Gamma_3,$$

where $\rho\in L^\infty(\Gamma_3;\mathbb{R}_+)$ represents some rate coefficient for the gradient of the temperature.

The thermal expansion tensor $\mathcal{M}=(m_{ij}):\Omega\times\mathbb{R}^d\to\mathbb{R}^d$, the heat sources density q_{th} and the thermal conductivity tensor $\mathcal{K}:\Omega\times\mathbb{R}^d\to\mathbb{R}^d$ satisfy the following conditions:

$$\mathcal{M}=(m_{ij}),\ m_{ij}=m_{ji}\in L^\infty(\Omega),\qquad q_{th}\in C([0,T];L^2(\Omega)), \tag{3.29}$$

$$\mathcal{K}=(k_{ij})=(k_{ji})\in L^\infty(\Omega), \tag{3.30}$$

$$\exists c_k>0,\ \text{such that } k_{ij}\xi_i\xi_j\geqslant c_k\xi_i\xi_j\qquad \forall\xi\in\mathbb{R}^d. \tag{3.31}$$

The densities of forces are assumed to have the following regularity:

$$\mathbf{f}_0 \in C([0,T]; L^2(\Omega)^d), \qquad \mathbf{f}_2 \in C([0,T]; L^2(\Gamma_2)^d). \tag{3.32}$$

We assume that the initial conditions satisfies the following condition:

$$\mathbf{u}_0 \in V, \qquad \theta_0 \in E, \quad \zeta_0 \in K, \qquad \theta_F \in L^2(0,T; L^2(\Gamma_3)), \qquad k_e \in L^\infty(\Omega; \mathbb{R}_+) \tag{3.33}$$

Finally, we define mappings

$$a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}, \qquad P : V \times V \times L^2(\Gamma_3) \to \mathbb{R},$$
$$\mathbf{f} : V \times V \to \mathbb{R}, \qquad S : [0,T] \to \mathbb{R},$$

and the functions $\mathcal{Z} : E \to E'$ and $\mathcal{R} : V \to E'$, respectively, by

$$a(\zeta, \xi) = \int_\Omega \nabla\zeta \cdot \nabla\xi \, dx, \tag{3.34}$$

$$P(w, \mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} p_\nu(u_\nu - w) v_\nu da + \int_{\Gamma_3} p_\tau(u_\nu - w)\|\mathbf{v}_\tau - \mathbf{v}^*\| da, \tag{3.35}$$

$$\mathbf{f}(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{f}_0.\mathbf{v} dx + \int_{\Gamma_2} \mathbf{f}_2 \cdot \mathbf{v} \, da, \tag{3.36}$$

$$\langle S(t), \mu\rangle_{E'\times E} = \int_\Omega q_{th}(t)\mu dx + \int_{\Gamma_3} k_e \theta_F(t)\mu \, dx, \tag{3.37}$$

$$\langle \mathcal{Z}\tau, \mu\rangle_{E'\times E} = \int_\Omega k_{ij} \frac{\partial\tau}{\partial x_j}\frac{\partial\mu}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau\mu \, da, \tag{3.38}$$

$$\langle \mathcal{R}\mathbf{v}, \mu\rangle_{E'\times E} = -\int_\Omega (\mathcal{M}.\nabla\mathbf{v})\mu dx + \int_{\Gamma_3} k_\tau(\|\dot{\mathbf{v}}_\tau - \mathbf{v}^*\|)\mu \, da. \tag{3.39}$$

We end this section with the remark that Problem $\mathcal{P}$ represents the classical formulation of the frictional problem. In general, this problem has no classical solution, which has all the necessary classical derivatives. For this reason, as usual in the analysis of frictional contact problems, there is a need to associate to Problem $\mathcal{P}$ a new problem, a so-called variational formulation. Using standard arguments we obtain the variational formulation of the thermo-mechanical problem (3.9)-(3.21).

Problem $\mathcal{P}^V$. *Find a displacement field $\mathbf{u} : [0,T] \to V$, a stress field $\boldsymbol{\sigma} : [0,T] \to Q_1$, a temperature field $\theta : [0,T] \to E$, a damage field $\zeta : [0,T] \to H^1(\Omega)$ and a wear function $w : [0,T] \to L^2(\Gamma_3)$ such that for all $t \in [0,T]$*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)) - \mathcal{M}\theta(t) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \zeta(s))ds, \tag{3.40}$$

$$\begin{aligned}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + P(w(t), \mathbf{u}(t), \mathbf{v}) - P(w(t), \mathbf{u}(t), \dot{\mathbf{u}}(t)) \\ \geqslant (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \qquad \forall \mathbf{v} \in V,\end{aligned} \tag{3.41}$$

$$\begin{aligned}(\dot{\zeta}(t), \vartheta - \zeta(t))_{L^2(\Omega)} + a(\zeta(t), \vartheta - \zeta(t)) \\ \geqslant (\psi(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)), \theta(t), \zeta(t)), \vartheta - \zeta(t))_{L^2(\Omega)}, \quad \zeta(t) \in K, \ \forall\vartheta \in K,\end{aligned} \tag{3.42}$$

$$\dot{\theta}(t) + \mathcal{Z}\theta(t) = \mathcal{R}\dot{u}(t) + S(t) \quad \text{in} \quad E', \tag{3.43}$$

$$\dot{w}(t) = k_w \|\mathbf{v}^*(t)\| p_\nu(u_\nu(t) - w(t)), \tag{3.44}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \theta(0) = \theta_0, \quad \zeta(0) = \zeta_0, \quad w(0) = w_0. \tag{3.45}$$

We note that variational Problem $\mathcal{P}^V$ is formulated in terms of the displacement field, stress field, temperature field, damage field and wear function. The functions $\mathbf{u}$, $\boldsymbol{\sigma}$, θ, ζ and w satisfying (3.40)-(3.45) are called weak solution to contact problem $\mathcal{P}$. The existence of the unique solution to Problem $\mathcal{P}^V$ is stated and proved in the next section.

4. An existence and uniqueness result

Theorem 4.1. *Let assumptions* (3.22)-(3.33) *hold. Then there exists a unique solution* $(\mathbf{u}, \boldsymbol{\sigma}, \theta, \zeta, w)$ *to problem* $\mathcal{P}^V$. *Moreover, this solution possesses the following properties:*

$$\mathbf{u} \in C^1(0, T; V), \tag{4.1}$$

$$\boldsymbol{\sigma} \in C([0, T]; Q_1), \tag{4.2}$$

$$\theta \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; E) \cap W^{1,2}(0, T; E'), \tag{4.3}$$

$$\zeta \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \tag{4.4}$$

$$w \in C^1([0, T]; L^2(\Gamma_3)). \tag{4.5}$$

The proof of theorem 4.1 consists in several steps, which we shall provide in what follows.

Throughout this section the assumptions of Theorem 4.1 are supposed to hold true. By C we denote generic positive constants which may depend on Ω, Γ_1, Γ_2, Γ_3, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{M}$, p_ν, p_τ and T but are independent of t and the rest of input data, and whose value may change from place to place.

First, let $(\lambda, \boldsymbol{\eta}, \boldsymbol{\xi}) \in C([0, T]; L^2(\Gamma_3) \times Q \times Q)$ be given and consider the following variational problem.

Problem $\mathcal{P}^V_{\lambda\eta\xi}$: *Find a displacement field* $\mathbf{u}_{\lambda\eta\xi} : [0, T] \to V$ *and a stress field* $\boldsymbol{\sigma}_{\lambda\eta\xi} : [0, T] \to Q_1$ *such that for all* $t \in [0, T]$

$$\boldsymbol{\sigma}_{\lambda\eta\xi}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_{\lambda\eta\xi}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}_{\lambda\eta\xi}(t)) + \boldsymbol{\eta}(t) + \boldsymbol{\xi}(t), \tag{4.6}$$

$$\begin{aligned} &(\boldsymbol{\sigma}_{\lambda\eta\xi}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_{\lambda\eta\xi}(t)))_Q + P(\lambda(t), \mathbf{u}_{w\eta\xi}(t), \mathbf{v}) \\ &\qquad - P(\lambda(t), \mathbf{u}_{\lambda\eta\xi}(t), \dot{\mathbf{u}}_{\lambda\eta\xi}(t)) \geqslant (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}_{\lambda\eta\xi}(t))_V \qquad \forall \mathbf{v} \in V, \end{aligned} \tag{4.7}$$

$$\mathbf{u}_{\lambda\eta\xi}(0) = \mathbf{u}_0. \tag{4.8}$$

For Problem $\mathcal{P}^V_{\lambda\eta\xi}$ we have the following result.

Lemma 4.1. *There exists a unique solution* $(\mathbf{u}_{\lambda\eta\xi}, \boldsymbol{\sigma}_{\lambda\eta\xi})$ *to Problem* $\mathcal{P}^V_{\lambda\eta\xi}$ *and it has its regularity expressed in* (4.1)–(4.2).

Proof. We use the Riesz representation theorem to define the operators $A : V \to V$, $B : V \to V$ and the function $\mathbf{F} : [0, T] \to V$ by

$$(A\mathbf{u}, \mathbf{v})_V = (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q, \tag{4.9}$$

$$(B\mathbf{u}, \mathbf{v})_V = (\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q, \tag{4.10}$$

$$(\mathbf{F}(t), \mathbf{v})_V = (\mathbf{f}(t), \mathbf{v})_V - (\boldsymbol{\eta}(t) + \boldsymbol{\xi}(t), \boldsymbol{\varepsilon}(\mathbf{v}))_Q, \tag{4.11}$$

for all $\mathbf{u}, \mathbf{v} \in V$ and $t \in [0, T]$. Assumptions (3.22)(b), (3.22)(c) and (3.22) imply that the operator A is a strongly monotone and Lipschitz continuous operator on V and B is a Lipschitz continuous operator on V. For $\mathbf{u} \in V$ and $\lambda \in L^2(\Gamma_3)$, the functional $P(\lambda, \mathbf{u}, \cdot)$ is convex and lower semicontinuous on V. We use (2.5), (3.26) and (3.27) to find

$$P(\lambda, \mathbf{u}_1, \mathbf{v}_2) - P(\lambda, \mathbf{u}_1, \mathbf{v}_1) + P(\lambda, \mathbf{u}_2, \mathbf{v}_1) - P(\lambda, \mathbf{u}_2, \mathbf{v}_2) \leqslant c_0^2(L_\nu + L_\tau)\|\mathbf{u}_1 - \mathbf{u}_2\|_V \|\mathbf{v}_1 - \mathbf{v}_2\|_V, \tag{4.12}$$

for all $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in V$. Moreover, using (3.32) it is easy to see that the function $\mathbf{f}$ defined by (3.36) satisfies $\mathbf{f} \in C([0,T];V)$ and, keeping in mind that $(\boldsymbol{\eta}, \boldsymbol{\xi}) \in C([0,T];Q \times Q)$, we deduce from (4.11) that $\mathbf{F} \in C([0,T];V)$. It follows from a class of abstract evolutionary variational inequalities (see for example [15]) that there exists a unique function $\boldsymbol{u}_{\lambda\boldsymbol{\eta}\boldsymbol{\xi}} \in C([0,T];V)$.

We use relation (4.6), assumptions (3.22), (3.23) and the regularity of the functions $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ with the properties of the deformation tensor $\boldsymbol{\varepsilon}$ to obtain that $\boldsymbol{\sigma}_{\lambda\boldsymbol{\eta}\boldsymbol{\xi}} \in C([0,T];Q)$.

We choose a function $\mathbf{v} = \mathbf{v}_{\lambda\boldsymbol{\eta}\boldsymbol{\xi}}(t) \pm z$ in (4.7), where $z \in \mathcal{D}(\Omega)^d$ is arbitrary and we use definitions (3.35) and (3.36) to obtain

$$\operatorname{Div} \boldsymbol{\sigma}_{\lambda\boldsymbol{\eta}\boldsymbol{\xi}}(t) + \boldsymbol{f}_0(t) = \mathbf{0}, \tag{4.13}$$

for all $t \in [0,T]$. The regularity of the functions $\boldsymbol{f}_0$, the relation (4.13) and since $\boldsymbol{\sigma}_{\lambda\boldsymbol{\eta}\boldsymbol{\xi}} \in C([0,T];Q)$ show that $\boldsymbol{\sigma}_{\lambda\boldsymbol{\eta}\boldsymbol{\xi}} \in C([0,T];Q_1)$. The proof is complete. □

In the second step, let $\mu \in C([0,T];L^2(\Omega))$ be given and consider the following variational problem for the damage field.

Problem $\mathcal{P}_\mu^V$. *Find a damage field $\zeta_\mu : [0,T] \to H^1(\Omega)$ such that*

$$\begin{aligned}&(\dot{\zeta}_\mu(t), \vartheta - \zeta_\mu(t))_{L^2(\Omega)} + a(\zeta_\mu(t), \vartheta - \zeta_\mu(t)) \\ &\qquad \geqslant (\mu(t), \vartheta - \zeta_\mu(t))_{L^2(\Omega)} \qquad \zeta(t) \in K, \quad \forall \vartheta \in K, \quad \text{a.e.} \quad t \in [0,T],\end{aligned} \tag{4.14}$$

$$\zeta_\mu(0) = \zeta_0. \tag{4.15}$$

Lemma 4.2. *Problem $\mathcal{P}_\mu^V$ has a unique solution ζ_μ which satisfies the regularity expressed in* (4.4). *Moreover, if ζ_i is the solution of Problem $\mathcal{P}_{\mu_i}^V$ corresponding to $\mu_i \in C([0,T];L^2(\Omega))$, $i = 1,2$, then there exists $C > 0$ such that*

$$\|\zeta_1(t) - \zeta_2(t)\|^2_{L^2(\Omega)} \leqslant C \int_0^t \|\mu_1(s) - \mu_2(s)\|^2_{L^2(\Omega)} ds \quad \forall t \in [0,T]. \tag{4.16}$$

Proof. To solve $\mathcal{P}_\mu^V$, we use a classical existence and uniqueness result on parabolic variational inequalities (see, for instance, [2]). By (4.14) we get:

$$(\dot{\zeta}_1 - \dot{\zeta}_2, \zeta_1 - \zeta_2)_{L^2(\Omega)} + a(\zeta_1 - \zeta_2, \zeta_1 - \zeta_2) \leqslant (\mu_1 - \mu_2, \mu_1 - \mu_2)_{L^2(\Omega)} \qquad \forall t \in [0,T].$$

Integrating the above inequality with respect to time and using the initial conditions $\zeta_1(0) = \zeta_2(0) = \zeta_0$ and the inequality $a(\zeta_1 - \zeta_2, \zeta_1 - \zeta_2) \geqslant 0$, we find

$$\frac{1}{2}\|\zeta_1(t) - \zeta_2(t)\|^2_{L^2(\Omega)} \leqslant \int_0^t (\mu_1(s) - \mu_2(s), \zeta_1(s) - \zeta_2(s))_{L^2(\Omega)}\, ds \qquad \forall t \in [0,T],$$

which implies that

$$\|\zeta_1(t) - \zeta_2(t)\|^2_{L^2(\Omega)} \leqslant \int_0^t |\mu_1(s) - \mu_2(s)|^2_{L^2(\Omega)} ds + \int_0^t \|\zeta_1(s) - \zeta_2(s)\|^2_{L^2(\Omega)}\, ds.$$

This inequality combined with Gronwall inequality leads to (4.16) which completes the proof. □

At the third step, we define the operator $\mathcal{L} : C([0,T];Q) \to C([0,T];Q)$ by

$$\mathcal{L}\boldsymbol{G} = \mathcal{G}(\boldsymbol{\sigma}_{\lambda\boldsymbol{\eta}\boldsymbol{\xi}}, \boldsymbol{\varepsilon}(\mathbf{u}_{\lambda\boldsymbol{\eta}\boldsymbol{\xi}}), \zeta_\mu) \quad \text{where} \quad \boldsymbol{\xi}(t) = \int_0^t \boldsymbol{G}(s)\, ds \quad \forall \boldsymbol{G} \in C([0,T];Q), \tag{4.17}$$

and $\boldsymbol{\sigma}_{\lambda\eta\xi}$ is the stress field

$$\boldsymbol{\sigma}_{\lambda\eta\xi} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_{\lambda\eta\xi}) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}_{\lambda\eta\xi}) + \boldsymbol{\eta} + \boldsymbol{\xi}. \tag{4.18}$$

We observe that since $(\boldsymbol{\eta}, \boldsymbol{\xi}) \in C([0,T]; Q \times Q)$, it is straightforward to see that $\boldsymbol{\sigma}_{\lambda\eta\xi} \in C([0,T]; Q)$. We hence obtain the following result.

Lemma 4.3. *The operator $\mathcal{L}$ has a unique fixed point $\boldsymbol{G}^* \in C([0,T]; Q)$.*

Proof. The continuity of $\mathcal{L}\boldsymbol{G}$ is a straightforward implication of the continuity of $\boldsymbol{\sigma}_{\lambda\eta\xi}$, $\mathbf{u}_{\lambda\eta\xi}$ and ζ_μ and (3.24) . Moreover, let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in C([0,T]; Q)$ and $\boldsymbol{G}_1, \boldsymbol{G}_2 \in C([0,T]; Q)$ be their corresponding integrals in time and let $\mu_1, \mu_2 \in C([0,T]; L^2(\Omega))$. For the sake of simplicity, we use the notation $\mathbf{u}_{\lambda\eta\xi_i} = \mathbf{u}_i$, $\dot{\mathbf{u}}_{\lambda\eta\xi_i} = \dot{\mathbf{u}}_i$, $\boldsymbol{\sigma}_{\lambda\eta\xi_i} = \boldsymbol{\sigma}_i$ and $\zeta_{\mu_i} = \zeta_i$ for $i = 1, 2$. Given $t \in [0,T]$ by (3.24) we find that

$$\|\mathcal{L}\boldsymbol{G}_1(t) - \mathcal{L}\boldsymbol{G}_2(t)\|_Q \leqslant C(\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_Q + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V + \|\zeta_1(t) - \zeta_2(t)\|_{L^2(\Omega)}). \tag{4.19}$$

It follows from (4.17) and (4.18) that

$$\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_Q^2 \leqslant C\bigg(\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_V^2 + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + \int_0^t |\boldsymbol{G}_1(s) - \boldsymbol{G}_2(s)|_Q^2\, ds\bigg). \tag{4.20}$$

Moreover by (4.6) and (4.7) we obtain

$$\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1(t)) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2(t)), \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1(t)) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2)(t))_Q \\
&\quad + (\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}_1(t)) - \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}_2(t)), \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1(t)) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2(t)))_Q \\
&\leqslant P(\lambda, \mathbf{u}_1(t), \dot{\mathbf{u}}_2(t)) - P(\lambda, \mathbf{u}_1(t), \dot{\mathbf{u}}_1(t)) + P(\lambda, \mathbf{u}_2(t), \dot{\mathbf{u}}_1(t)) \\
&\quad - P(\lambda, \mathbf{u}_2(t), \dot{\mathbf{u}}_2(t)) - (\boldsymbol{\xi}_1(t) - \boldsymbol{\xi}_2(t), (\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1(t)) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2(t)))_Q.
\end{aligned} \tag{4.21}$$

Also it follows from (3.22)(b), (3.23)(a) and (4.12) that

$$\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_V \leqslant C(\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V + \|\boldsymbol{\xi}_1(t) - \boldsymbol{\xi}_2(t)\|_Q). \tag{4.22}$$

Using this result, the inequality $2ab \leqslant a^2 + b^2$ and (4.17), we obtain

$$\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_V^2 \leqslant C(\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + \int_0^t \|\boldsymbol{G}_1(s) - \boldsymbol{G}_2(s)\|_Q^2 ds). \tag{4.23}$$

Since $\mathbf{u}_1(0) = \mathbf{u}_2(0) = \mathbf{u}_0$, we have

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 \leqslant C \int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 ds. \tag{4.24}$$

It follows from (4.23), (4.24) and (4.16) that

$$\begin{aligned}
\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_V^2 + \|\zeta_1(t) - \zeta_2(t)\|_{L^2(\Omega)}^2 \leqslant C\bigg(&\int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2\, ds \\
&+ \int_0^t \|\mu_1(s) - \mu_2(s)\|_{L^2(\Omega)}^2\, ds \\
&+ \int_0^t \|\boldsymbol{G}_1(s) - \boldsymbol{G}_2(s)\|_Q^2\, ds\bigg).
\end{aligned} \tag{4.25}$$

By using the Gronwall Lemma, we find that

$$\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_V^2 + \|\zeta_1(t) - \zeta_2(t)\|_{L^2(\Omega)}^2 \leqslant C\int_0^t \|\boldsymbol{G}_1(s) - \boldsymbol{G}_2(s)\|_Q^2\,ds. \tag{4.26}$$

This inequality combined with (4.20), (4.24), (4.26), (4.19) lead us to the following estimate:

$$\|\mathcal{L}\boldsymbol{G}_1(t) - \mathcal{L}\boldsymbol{G}_2(t)\|_Q^2 \leqslant C\int_0^t \|\boldsymbol{G}_1(s) - \boldsymbol{G}_2(s)\|_Q^2\,ds. \tag{4.27}$$

Reiterating inequality (4.27) n times, we get:

$$\|\mathcal{L}^n\boldsymbol{G}_1 - \mathcal{L}^n\boldsymbol{G}_2\|_{C([0,T];Q)}^2 \leqslant \frac{C^nT^n}{n!}\|\boldsymbol{G}_1 - \boldsymbol{G}_2\|_{C([0,T];Q)}^2. \tag{4.28}$$

Therefore, for n large enough, $\mathcal{L}^n$ is a contractive operator on the Banach $C([0,T];Q)$ and we conclude that there exists a unique $\boldsymbol{G}^* \in C([0,T];Q)$ such that $\mathcal{L}\boldsymbol{G}^* = \boldsymbol{G}^*$. The proof is complete. □

At the forth step, let $(\lambda, \boldsymbol{\eta}) \in C([0,T]; L^2(\Gamma_3) \times Q)$ be given and we consider the following auxiliary problem.

Problem $\mathcal{P}_{\lambda\boldsymbol{\eta}}^V$. *Find a displacement field $\mathbf{u}_{\lambda\boldsymbol{\eta}} : [0,T] \to V$, a stress field $\boldsymbol{\sigma}_{\lambda\boldsymbol{\eta}} : [0,T] \to Q_1$ and a temperature field $\theta_{\lambda\boldsymbol{\eta}} : [0,T] \to E$ such that*

$$\boldsymbol{\sigma}_{\lambda\boldsymbol{\eta}}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_{\lambda\boldsymbol{\eta}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}_{\lambda\boldsymbol{\eta}}(t)) + \boldsymbol{\eta}(t) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}_{\lambda\boldsymbol{\eta}}(s), \boldsymbol{\varepsilon}(\mathbf{u}_{\lambda\boldsymbol{\eta}}(s)), \zeta_\mu(s))\,ds, \tag{4.29}$$

$$\begin{aligned}(\boldsymbol{\sigma}_{\lambda\boldsymbol{\eta}}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_{\lambda\boldsymbol{\eta}}(t)))_Q &+ P(\lambda(t), \mathbf{u}_{\lambda\boldsymbol{\eta}}(t), \mathbf{v}) \\ &- P(\lambda(t), \mathbf{u}_{\lambda\boldsymbol{\eta}}(t), \dot{\mathbf{u}}_{\lambda\boldsymbol{\eta}}(t)) \geqslant (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}_{\lambda\boldsymbol{\eta}}(t))_V \quad \forall \mathbf{v} \in V,\end{aligned} \tag{4.30}$$

$$\dot{\theta}_{\lambda\boldsymbol{\eta}}(t) + \mathcal{Z}\theta_{\lambda\boldsymbol{\eta}}(t) = \mathcal{R}\dot{\boldsymbol{u}}_{\lambda\boldsymbol{\eta}}(t) + S(t) \quad \text{in} \quad E', \tag{4.31}$$

$$\mathbf{u}_{\lambda\boldsymbol{\eta}}(0) = \mathbf{u}_0, \qquad \theta_{\lambda\boldsymbol{\eta}}(0) = \theta_0. \tag{4.32}$$

Lemma 4.4. *There exists a unique solution $\{\mathbf{u}_{\lambda\boldsymbol{\eta}}, \boldsymbol{\sigma}_{\lambda\boldsymbol{\eta}}, \theta_{\lambda\boldsymbol{\eta}}\}$ to the auxiliary problem $\mathcal{P}_{\lambda\boldsymbol{\eta}}^V$ obeying regularity* (4.1)–(4.3). *Moreover, if $\{\mathbf{u}_i, \boldsymbol{\sigma}_i, \theta_i\}$ and ζ_i represents the solutions of Problems $\mathcal{P}_{\lambda\boldsymbol{\eta}_i}^V$ and $\mathcal{P}_{\zeta_i}^V$, respectively, for $(\boldsymbol{\eta}_i, \mu_i) \in C([0,T]; Q \times L^2(\Omega))$, $i = 1, 2$, then there exists $C > 0$ such that*

$$\begin{aligned}\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_Q^2 \leqslant & C\Big(\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_V^2 + \int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2\,ds \\ & + \|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_Q^2 + \int_0^t \|\zeta_1(s) - \zeta_2(s)\|_{L^2(\Omega)}^2\,ds\Big),\end{aligned} \tag{4.33}$$

$$\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_V^2 \leqslant C\Big(\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_Q^2 + \int_0^t \|\zeta_1(s) - \zeta_2(s)\|_{L^2(\Omega)}^2\,ds\Big), \tag{4.34}$$

$$\|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 \leqslant C\int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2\,ds. \tag{4.35}$$

Proof. Let $\boldsymbol{G}^* \in C([0,T];Q)$ be the fixed point of the operator $\mathcal{L}$ defined by (4.17) and $\mathbf{u}_{\lambda\boldsymbol{\eta}} = \mathbf{u}_{\lambda\boldsymbol{\eta}\boldsymbol{\xi}^*}$, $\boldsymbol{\sigma}_{\lambda\boldsymbol{\eta}} = \boldsymbol{\sigma}_{\lambda\boldsymbol{\eta}\boldsymbol{\xi}^*}$ be the solution to problem $\mathcal{P}^V_{\lambda\boldsymbol{\eta}\boldsymbol{\xi}}$ obtained in Lemma 4.1 for $\boldsymbol{\xi} = \boldsymbol{\xi}^* = \int_0^t \boldsymbol{G}^*(s)ds$. Equation $\boldsymbol{\mathcal{L}}\boldsymbol{G}^* = \boldsymbol{G}^*$ combined with (4.17) shows that $\mathbf{u}_{\lambda\boldsymbol{\eta}}, \boldsymbol{\sigma}_{\lambda\boldsymbol{\eta}}$ satisfies (4.29) and (4.30). Then conditions (4.32), regularities (4.1) and (4.2) follow from Lemma 4.1. Using now the displacement field $\mathbf{u}_{\lambda\boldsymbol{\eta}}$ obtained in lemma 4.4, we obtain (4.31) and we know that the inclusion mapping of $(E, \|\cdot\|_E)$ into $(L^2(\Omega), \|\cdot\|_{L^2(\Omega)})$ is continuous and dense, we can write the Gelfand evolution triple

$$E \subset L^2(\Omega) \subset E'.$$

The operator $\mathcal{Z}$ is linear and coercive. Using Friedrichs-Poincaré inequality, we have

$$\langle \mathcal{Z}\tau, \tau\rangle_{E\times E'} \geqslant C\|\tau\|^2_E. \tag{4.36}$$

By (3.30) and (3.38) for all $\tau, \omega \in E$, we have

$$\langle \mathcal{Z}\tau, \omega\rangle_{E'\times E} \leqslant \sum_{i,j=1}^{d} \|k_{i,j}\|_{L^\infty(\Omega)}\|\tau_{,i}\|_{L^2(\Omega)}\|\omega_{,i}\|_{L^2(\Omega)} + k_e\|\tau\|_{L^2(\Gamma_3)}\|\mu\|_{L^2(\Gamma_3)}.$$

Using (2.6), we find

$$\langle \mathcal{Z}\tau, \omega\rangle_{E'\times E} \leqslant C\|\tau\|_E\|\omega\|_E. \tag{4.37}$$

On the other hand, from the definitions of $\mathcal{R}$, S and the regularity of $\dot{u}_{\lambda\boldsymbol{\eta}}$ we deduce that

$$\varphi_{\lambda\boldsymbol{\eta}} = \mathcal{R}\dot{u}_{\lambda\boldsymbol{\eta}} + S \in L^2(0,T;E'). \tag{4.38}$$

Since $\theta_0 \in E$, from inequalities (4.36), (4.37) and regularity (4.38), it follows that the operator $\mathcal{Z}$ is hemicontinuous and monotone. Then by using classical arguments of functional analysis concerning parabolic equations (see, for instance, [31]), we prove easily the existence and uniqueness of $\theta_{\lambda\boldsymbol{\eta}}$ satisfying

$$\begin{cases} \theta_{\lambda\boldsymbol{\eta}} \in C([0,T];L^2(\Omega)) \cap L^2(0,T;E) \cup W^{1,2}(0,T;E'), \\ \dot{\theta}_{\lambda\boldsymbol{\eta}}(t) + \mathcal{Z}\theta_{\lambda\boldsymbol{\eta}}(t) = \varphi_{\lambda\boldsymbol{\eta}}(t) \quad \text{in} \quad E', \\ \theta_{\lambda\boldsymbol{\eta}}(0) = \theta_0. \end{cases} \tag{4.39}$$

Consider now $(\boldsymbol{\eta}_1, \mu_1), (\boldsymbol{\eta}_2, \mu_2) \in C([0,T]; Q \times L^2(\Omega))$ and for $i = 1, 2$, we denote

$$\mathbf{u}_{\lambda\boldsymbol{\eta_i}} = \mathbf{u}_i, \quad \boldsymbol{\sigma}_{\lambda\boldsymbol{\eta_i}} = \boldsymbol{\sigma_i}, \quad \theta_i = \theta_{w\boldsymbol{\eta_i}}, \quad \zeta_{\mu_i} = \zeta_i.$$

Using (4.29) and (3.24), we have

$$\begin{aligned} \|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|^2_Q \leqslant C\bigg(& \|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|^2_V + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|^2_V \\ & + \|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|^2_Q + \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|^2_Q\,ds \\ & + \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|^2_V ds + \int_0^t \|\zeta_1(s) - \zeta_2(s)\|^2_{L^2(\Omega)}\,ds\bigg). \end{aligned} \tag{4.40}$$

Using a Gronwall argument and (4.24) in the above inequality, we deduce (4.33).

We use (4.29), (4.30), (4.33) and arguments similar to those used in the proof of (4.23) to obtain

$$\|\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2\|^2_V \leqslant C\bigg(\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|^2_V + \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|^2_V\,ds$$

$$+\int_0^t \|\dot{\mathbf{u}}_1(s)-\dot{\mathbf{u}}_2(s)\|_V^2 ds+\int_0^t \|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_Q^2 ds$$
$$+\int_0^t \|\zeta_1(s)-\zeta_2(s)\|_{L^2(\Omega)}^2 ds\Big).$$

This inequality combined with (4.24), (4.16) and Gronwall inequality leads us to (4.34). Now for $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\in C([0,T];V)$ we have for $t\in[0,T]$:

$$\begin{aligned}\langle\dot\theta_1(t)-\dot\theta_2(t),\theta_1(t)-\theta_2(t)\rangle_{E'\times E}+\langle\mathcal{Z}\theta_1(t)-\mathcal{Z}\theta_2(t),\theta_1(t)-\theta_2(t)\rangle_{E'\times E}\\ =\langle\mathcal{R}\dot u_1(t)-\mathcal{R}\dot u_2(t),\theta_1(t)-\theta_2(t)\rangle_{E'\times E}.\end{aligned}\tag{4.41}$$

Then by integrating over $(0,t)$, we get (4.35) by using (3.39), (3.29), (3.30) and (3.31). The proof is complete. □

Finally, as a consequence of these results and by using the properties of the operator $\mathcal{M}$, and the function ψ, for $t\in[0,T]$ we consider the element

$$\Pi_\lambda(\boldsymbol{\eta},\mu)(t)=(\Pi_\lambda^0(\boldsymbol{\eta})(t),\Pi_\lambda^1(\boldsymbol{\eta},\mu)(t))\in Q\times L^2(\Omega),\tag{4.42}$$

defined by the identities

$$\Pi_\lambda^0(\boldsymbol{\eta})(t)=-\mathcal{M}(\theta_{\lambda\boldsymbol{\eta}}(t)),\tag{4.43}$$
$$\Pi_\lambda^1(\boldsymbol{\eta},\mu)(t)=\psi(\boldsymbol{\sigma}_{\lambda\boldsymbol{\eta}}(t),\mathbf{u}_{\lambda\boldsymbol{\eta}}(t),\theta_{\lambda,\boldsymbol{\eta}}(t),\zeta_\mu(t)).\tag{4.44}$$

We have the following result.

Lemma 4.5. *For $(\boldsymbol{\eta},\mu)\in C([0,T];Q\times L^2(\Omega))$ the function $\Pi_\lambda(\boldsymbol{\eta},\mu):[0,T]\to Q\times L^2(\Omega)$ is continuous and there is a unique element $(\boldsymbol{\eta}^*,\mu^*)\in C([0,T];Q\times L^2(\Omega))$ such that $\Pi_\lambda(\boldsymbol{\eta}^*,\mu^*)=(\boldsymbol{\eta}^*,\mu^*)$.*

Proof. Let $(\boldsymbol{\eta},\mu)\in C([0,T];Q\times L^2(\Omega))$ and $t_1,t_2\in[0,T]$. Using (4.43) and (3.29), we have

$$\begin{aligned}\|\Pi_\lambda^0(\boldsymbol{\eta})(t_1)-\Pi_\lambda^0(\boldsymbol{\eta})(t_2)\|_Q&\leqslant\|\mathcal{M}(\theta_{\lambda\boldsymbol{\eta}}(t_1))-\mathcal{M}(\theta_{\lambda\boldsymbol{\eta}}(t_2))\|_Q\\&\leqslant L_{\mathcal{M}}\|\theta_{\lambda\boldsymbol{\eta}}(t_1)-\theta_{\lambda\boldsymbol{\eta}}(t_2)\|_{L^2(\Omega)},\end{aligned}\tag{4.45}$$

where $L_{\mathcal{M}}=\sup\limits_{i,j}\|m_{i,j}\|_{L^2(\Omega)}$. Then due to the regularity of $\theta_{\lambda\boldsymbol{\eta}}$ stated in Lemma 4.3 we deduce from (4.45) that $\Pi_\lambda^0(\boldsymbol{\eta})\in C([0,T];Q)$. By similar arguments, from (4.44), (2.4) and (3.25) it follows that

$$\begin{aligned}\|\Pi_\lambda^1(\boldsymbol{\eta},\mu)(t_1)-\Pi_\lambda^1(\boldsymbol{\eta},\mu)(t_2)\|_{L^2(\Omega)}\leqslant&C\big(\|\mathbf{u}_{\lambda\boldsymbol{\eta}}(t_1)-\mathbf{u}_{\lambda\boldsymbol{\eta}}(t_2)\|_V\\&+\|\theta_{\lambda\boldsymbol{\eta}}(t_1)-\theta_{\lambda\boldsymbol{\eta}}(t_2)\|_V+\|\zeta_\mu(t_1)-\zeta_\mu(t_2)|_{L^2(\Omega)}\big)\end{aligned}\tag{4.46}$$

Therefore, $\Pi_\lambda^1(\boldsymbol{\eta},\mu)\in C([0,T];L^2(\Omega))$ and $\Pi_\lambda(\boldsymbol{\eta},\mu)\in C([0,T];Q\times L^2(\Omega))$.

Let now $(\boldsymbol{\eta}_1,\mu_1),(\boldsymbol{\eta}_2,\mu_2)\in C([0,T];Q\times L^2(\Omega))$. We use the notation

$$\mathbf{u}_{w\boldsymbol{\eta}_i}=\mathbf{u}_i,\quad\dot{\mathbf{u}}_{\lambda\boldsymbol{\eta}_i}=\dot{\mathbf{u}}_i,\quad\theta_{\lambda\boldsymbol{\eta}_i}=\theta_i,\quad\zeta_{\mu_i}=\zeta_i\quad\text{for}\quad i=1,2.$$

We use similar arguments as in the proof of relations (4.45) and (4.46) to find that

$$\begin{aligned}\|\Pi_\lambda(\boldsymbol{\eta}_1,\mu_1)(t)-\Pi_\lambda(\boldsymbol{\eta}_2,\mu_2)(t)\|_{Q\times L^2(\Omega)}^2\leqslant&C(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V^2\\&+\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}^2+\|\zeta_1(t)-\zeta_2(t)\|_{L^2(\Omega)}^2),\end{aligned}\tag{4.47}$$

for all $t \in [0,T]$. We employ (4.24), (4.16), (4.34) and (4.35) to obtain

$$\|\Pi_\lambda(\boldsymbol{\eta}_1,\mu_1)(t)-\Pi_\lambda(\boldsymbol{\eta}_2,\mu_2)(t)\|^2_{Q\times L^2(\Omega)} \leqslant C\int_0^t \|(\boldsymbol{\eta}_1,\mu_1)(s)-(\boldsymbol{\eta}_2,\mu_2)(s)\|^2_{Q\times L^2(\Omega)}ds \quad \forall t\in[0,T].$$

Reiterating this inequality m times, we get:

$$\|\Pi^m_\lambda(\boldsymbol{\eta}_1,\mu_1)-\Pi^m_\lambda(\boldsymbol{\eta}_2,\mu_2)\|^2_{C([0,T];Q\times L^2(\Omega))} \leqslant \frac{C^mT^m}{m!}\|(\boldsymbol{\eta}_1,\mu_1)-(\boldsymbol{\eta}_2,\mu_2)\|^2_{C([0,T];Q\times L^2(\Omega))}.$$

Thus, for m sufficiently large, Π^m_λ is a contraction on the Banach space $C([0,T];Q)$, and so Π_λ has a unique fixed point. The proof is complete. □

Let $\lambda \in C([0,T];L^2(\Gamma_3))$. At the fifth step, we consider the following variational problem.

Problem $\mathcal{P}^V_\lambda$. *Find a displacement field $\mathbf{u}_\lambda : \Omega\times[0,T]\to\mathbb{R}^d$, a stress field $\boldsymbol{\sigma}_\lambda : \Omega\times[0,T]\to\mathbb{S}^d$, a temperature field $\theta_\lambda : \Omega\times[0,T]\to\mathbb{R}$, and a damage field $\zeta_\lambda : \Omega\times[0,T]\to\mathbb{R}$ such that*

$$\boldsymbol{\sigma}_\lambda(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\lambda(t)) + \mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}_\lambda(t))) - \mathcal{M}\theta_\lambda(t) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}_\lambda(s),\boldsymbol{\varepsilon}(\mathbf{u}_\lambda(s)),\zeta_\lambda(s))\,ds, \tag{4.48}$$

$$\begin{aligned}(\boldsymbol{\sigma}_\lambda(t),\boldsymbol{\varepsilon}(\mathbf{v}-\dot{\mathbf{u}}_\lambda(t)))_Q + P(\lambda(t),\mathbf{u}_\lambda(t),\mathbf{v}) - P(\lambda(t),\mathbf{u}_\lambda(t),\dot{\mathbf{u}}_\lambda(t))&\\ \geqslant (\mathbf{f}(t),\mathbf{v}-\dot{\mathbf{u}}_\lambda(t))_V \quad \forall \mathbf{v}\in V,&\end{aligned} \tag{4.49}$$

$$\begin{aligned}(\dot{\zeta}_\lambda(t),\vartheta-\zeta_\lambda(t))_{L^2(\Omega)} + a(\zeta(t),\vartheta-\zeta_\lambda(t))&\\ \geqslant (\psi(\boldsymbol{\sigma}_\lambda,\boldsymbol{\varepsilon}(\mathbf{u}_\lambda(t)),\theta_\lambda(t),\zeta_w(t)),\vartheta-\zeta_x(t))_{L^2(\Omega)} \quad \zeta_\lambda(t)\in K \quad \forall\vartheta\in K,&\end{aligned} \tag{4.50}$$

$$\dot{\theta}_\lambda(t) + \mathcal{Z}\theta_\lambda(t) = \mathcal{R}\dot{u}_\lambda(t) + S(t) \qquad \text{in} \quad E', \tag{4.51}$$

$$\mathbf{u}_\lambda(0)=\mathbf{u}_0, \quad \theta_\lambda(0)=\theta_0, \quad \zeta_\lambda(0)=\zeta_0. \tag{4.52}$$

Lemma 4.6. *Problem $\mathcal{P}^V_\lambda$ has a unique solution $(\mathbf{u}_\lambda,\boldsymbol{\sigma}_\lambda,\theta_\lambda,\zeta_\lambda)$ satisfying* (4.1)–(4.5).

Proof. Let $(\boldsymbol{\eta}_\lambda,\mu_\lambda)\in C([0,T];Q\times L^2(\Omega))$ the fixed point of Π_λ defined by (4.42)–(4.44) and $\mathbf{u}_\lambda=\mathbf{u}_{\lambda\boldsymbol{\eta}_\lambda}$, $\theta_\lambda=\theta_{\lambda\boldsymbol{\eta}_\lambda}$, $\zeta_\lambda=\zeta_{\mu_\lambda}$ be the solutions to problems $\mathcal{P}^V_{\lambda\boldsymbol{\eta}}$ and $\mathcal{P}^V_\mu$ obtained in Lemmata 4.4 and 4.2 for $(\boldsymbol{\eta},\mu)=(\boldsymbol{\eta}_\lambda,\mu_\lambda)$. Let

$$\boldsymbol{\sigma}_\lambda(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\lambda(t)) + \mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}_\lambda(t))) - \mathcal{M}\theta_\lambda(t) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}_\lambda(s),\boldsymbol{\varepsilon}(\mathbf{u}_\lambda(s)),\zeta_\lambda)\,ds.$$

Equation $\Pi^0_\lambda(\boldsymbol{\eta}_\lambda)=\boldsymbol{\eta}_\lambda$ and $\Pi^1_\lambda(\boldsymbol{\eta}_\lambda,\mu_\lambda)=\mu_\lambda$ combined with (4.43) and (4.44) shows that $(\mathbf{u}_\lambda,\boldsymbol{\sigma}_\lambda,\theta_\lambda,\zeta_\lambda)$ satisfies (4.48)-(4.51). Then (4.52) and the regularities (4.1)-(4.4) follow from Lemmata 4.2, 4.4 and assumptions on $\mathcal{A},\mathcal{B}$, $\mathcal{G}$ and $\mathcal{M}$ which concludes the existence part of the Lemma 4.6.

The uniqueness part of Lemma 4.6 is a consequence of the uniqueness of the fixed point of the operator Π_λ defined by (4.42)–(4.44) and the unique solvability of problems $\mathcal{P}^V_{\lambda\boldsymbol{\eta}_\lambda}$ and $\mathcal{P}^V_{\mu_\lambda}$. The proof is complete. □

Now we consider an operator $\Phi : C([0,T];L^2(\Gamma_3))\to C([0,T];L^2(\Gamma_3))$ defined by

$$\Phi\lambda = k_w\|\mathbf{v}^*\|\int_0^t p_\nu(u_\nu-\lambda)ds \qquad \forall t\in[0,T]. \tag{4.53}$$

Lemma 4.7. *The operator Φ has a unique fixed point $\lambda^*\in C([0,T];L^2(\Gamma_3))$.*

Proof. Let $\lambda_1, \lambda_2 \in C([0,T]; L^2(\Gamma_3))$ and denote by $\mathbf{u}_i$, $i = 1, 2$, the solution of (4.49) in problem $\mathcal{P}_\lambda^V$ for $\lambda = \lambda_i$, that is, $\mathbf{u}_i = \mathbf{u}_{\lambda_i}$ and we also let $\dot{\mathbf{u}}_i = \dot{\mathbf{u}}_{\lambda_i}$. Furthermore, in what follows by C we denote various positive constants which may depend on k_w and $\mathbf{v}^*$. Taking into consideration (4.53), (2.5) and (3.26), we deduce that

$$\begin{aligned} \|\Phi\lambda_1(t) - \Phi\lambda_2(t)\|^2_{L^2(\Gamma_3)} \leqslant & C\Big(\int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 \, ds \\ & + \int_0^t \|\lambda_1(s) - \lambda_2(s)\|^2_{L^2(\Gamma_3)} \, ds\Big). \end{aligned} \tag{4.54}$$

We use similar arguments that those used in the proof of the relation (4.23) to find that

$$\begin{aligned} \int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 ds \leqslant C\Big(& \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 \, ds \\ & + \int_0^t \|\lambda_1(s) - \lambda_2(s)\|^2_{L^2(\Gamma_3)} \, ds\Big). \end{aligned} \tag{4.55}$$

Since $\mathbf{u}_1(0) = \mathbf{u}_2(0) = \mathbf{u}_0$, by using (4.55) we obtain

$$\begin{aligned} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 & \leqslant C \int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V^2 \, ds \\ & \leqslant C \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 \, ds + C \int_0^t \|\lambda_1(s) - \lambda_2(s)\|^2_{L^2(\Gamma_3)} \, ds. \end{aligned} \tag{4.56}$$

Applying Gronwall inequality, we deduce

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 \leqslant C \int_0^t \|\lambda_1(s) - \lambda_2(s)\|^2_{L^2(\Gamma_3)} ds. \tag{4.57}$$

It follows from this inequality that

$$\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V^2 ds \leqslant C \int_0^t \int_0^s \|\lambda_1(r) - \lambda_2(r)\|^2_{L^2(\Gamma_3)} \, dr ds.$$

Since $s \leqslant t$,

$$\begin{aligned} \int_0^t \int_0^s \|\lambda_1(r) - \lambda_2(r)\|^2_{L^2(\Gamma_3)} \, dr ds & \leqslant C \int_0^t \int_0^t \|\lambda_1(r) - \lambda_2(r)\|^2_{L^2(\Gamma_3)} \, dr ds \\ & = C \int_0^t \|\lambda_1(r) - \lambda_2(r)\|^2_{L^2(\Gamma_3)} dr \int_0^t \, ds. \end{aligned}$$

Then

$$\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V^2 ds \leqslant C \int_0^t \|\lambda_1(s) - \lambda_2(s)\|^2_{L^2(\Gamma_3)} \, ds. \tag{4.58}$$

Combining now (4.54) with (4.58), we obtain

$$\|\Phi\lambda_1(t) - \Phi\lambda_2(t)\|^2_{L^2(\Gamma_3)} \leqslant C \int_0^t \|\lambda_1(s) - \lambda_2(s)\|^2_{L^2(\Gamma_3)}\, ds. \tag{4.59}$$

Reiterating this inequality n times, we arrive at

$$\|\Phi^n\lambda_1 - \Phi^n\lambda_2\|^2_{C([0,T];L^2(\Gamma_3))} \leqslant \frac{C^nT^n}{n!}\|\lambda_1 - \lambda_2\|^2_{C([0,T];L^2(\Gamma_3))}.$$

Thus, for n sufficiently large, Φ^n is a contraction on $C([0,T];L^2(\Gamma_3))$, and so Φ has a unique fixed point λ^* in this Banach space. The proof is complete. □

We now have all ingredients needed to provide the proof of Theorem 4.1.

Proof. Let λ^* be the fixed point of the operator $\mathcal{L}$ given by (4.53). With (4.48)–(4.53) it is easy to verify that $(\mathbf{u}_{\lambda^*}, \boldsymbol{\sigma}_{\lambda^*}, \theta_{\lambda^*}, \zeta_{\lambda^*}, \lambda^*)$ is the unique solution to problem $\mathcal{P}^V$ possessing regularities (4.1)–(4.5). □

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