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CONDITIONS FOR ABSENCE OF SOLUTIONS TO SOME HIGHER ORDER ELLIPTIC INEQUALITIES WITH SINGULAR COEFFICIENTS IN $\mathbb{R}^n$

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Abstract. In this paper we study Liouville type theorems for elliptic higher order inequalities with singular coefficients and gradient terms in $\mathbb{R}^n$. Our approach is based on the Pokhozhaev nonlinear capacity method, which is widely used for studying various nonlinear elliptic inequalities. We obtain apriori estimates for solutions of an elliptic inequality using the method of test functions. An optimal choice of the test function leads us to a nonlinear minimax problem, which generates a nonlinear capacity induced by a corresponding nonlinear problem. The existence of the zero limit of the corresponding apriori estimate ensures the absence of a nontrivial solution to the problem. Our result provide a new view on the behavior of solutions of higher order elliptic inequalities with singular coefficients and gradient terms and this approach can be useful in studying nonlinear elliptic inequalities of other types.

Keywords: Liouville type theorems, apriori estimate, nonlinear capacity, singular coefficients, gradient terms.

Mathematics Subject Classification: 35J30, 35J62.

1. Introduction

Over past ten years the absence of solutions to various nonlinear differential inequalities and systems was studied by many mathematicians, in particular, for elliptic inequalities and systems of partial differential equations with singular coefficients and gradient terms, what is equivalent to the absence of stationary states for the corresponding parabolic inequalities and systems, see [1]–[25]. Here we can mention a result by B. Gidas and J. Spruck [13] on absence of positive solutions to the equation

$$-\Delta u = u^q \qquad (x \in \mathbb{R}^n)$$

for $1 < q < \frac{n+2}{n-2}$. Later a comparison method became a main approach for studying this problem, which allowed to obtain the sufficient conditions for the absence of solutions in terms of the critical nonlinearity index for many second order equations like

$$\begin{aligned} -\Delta u &= |x|^{-2} u^q \qquad (x \in \mathbb{R}^n \setminus \{0\}), \\ -\Delta_p u &= u^q - |\nabla u|^s \qquad (x \in \mathbb{R}^n), \end{aligned}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, and similar ones as well as for corresponding inequalities, see, for instance, [2]–[3], [14]–[16] and the references therein. However, in the general case, the comparison methods are not applicable for higher order operators.

E. Mitidieri and S.I. Pokhozhaev [17] developed a new effective approach to these issues called a nonlinear capacity method. It is based on a special choice of a parametric family of

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test functions, which allows one to obtain apriori estimates by applying corresponding algebraic inequalities to the integral formulation of the considered elliptic inequality and to get then the results on the absence of solutions by letting the parameter tend to zero or to infinity. This approach provides simple and exact results. In particular, it was shown in [23] that the inequality

$$-\Delta_p u \geqslant |x|^{-\alpha} u^q \qquad (x \in \mathbb{R}^n)$$

with $p > 1$, $\alpha < p$ and $p - 1 < q \leqslant \frac{(n-\alpha)(p-1)}{n-p}$ has no positive solutions. This method was successfully applied to inequalities with more general operators like a mean curvature operator [1], [9], a wide class of anisotropic quasilinear operators [5], [6], [8], [11], [12], [14]–[19], [21]–[24], as well as for systems of inequalities [7]–[12], [20]. Later, by employing a more sophisticated technique, R. Filippucci, P. Pucci, M. Rigoli [2]–[7] obtained rather essential results on existence and absence of solutions for coercive inequalities with the opposite sign at Δ_p including inequalities with gradient terms of form $|\nabla u|^s$.

In the present paper we prove a Liouville type theorem for some higher order elliptic inequalities with singular coefficients and gradient terms in $\mathbb{R}^n$, which were not studied before.

The rest of the papers consists of Sections 2 and 3. In Section 2 we formulate a result on the absence of non-constant solutions to an elliptic inequality, while in Section 3 we prove this statement. Throughout the paper the letter c denotes for positive constants, which can depend on the parameters of the considered inequalities.

2. Formulation of result

We establish the absence of non-constant entire solutions to inequalities of form

$$\pm\Delta^k u(x) \geqslant (1+|x|)^{-\alpha}\, |\nabla u(x)|^q - (1+|x|)^{-\beta} |\nabla u(x)|^s, \qquad x \in \mathbb{R}^n, \tag{2.1}$$

where $k \in \mathbb{N}$, α, $\beta \in \mathbb{R}$ and

$$s > 0, \qquad q > \max(1, s). \tag{2.2}$$

Definition 2.1. *A weak solution to inequality (2.1) is a function $u \in W^{1,q}_{loc}(\mathbb{R}^n)$ such that*

$$(1+|x|)^{-\alpha}\, |\nabla u|^q \in L^1_{loc}(\mathbb{R}^n), \qquad (1+|x|)^{-\beta} |\nabla u|^s \in L^1_{loc}(\mathbb{R}^n),$$

and

$$\int\limits_{\mathbb{R}^n} (1+|x|)^{-\alpha}\, |\nabla u(x)|^q\, \varphi(x)\, dx \leqslant \pm \int\limits_{\mathbb{R}^n} u(x) \Delta^k \varphi(x)\, dx + \int\limits_{\mathbb{R}^n} (1+|x|)^{-\beta}\, |\nabla u(x)|^s\, \varphi(x)\, dx \tag{2.3}$$

holds for each non-negative test function $\varphi \in C^{2k}_0(\mathbb{R}^n)$.

Theorem 2.1. *Let (2.2) holds. Assume that $\theta_1 \leqslant 0$ and $\theta_2 < 0$, and*

$$\theta_1 = \frac{n(q-1) - (2k-1)q + \alpha}{q-1}, \qquad \theta_2 = \frac{n(q-s) - \beta q + \alpha s}{q-s}. \tag{2.4}$$

Then each weak solution to equation (2.1) is identically constant almost everywhere in $\mathbb{R}^n$.

3. Proof of the result

We use definition (2.3) of a weak solution to inequality (2.1) and integrate by parts in the first term in the right hand side:

$$\pm \int\limits_{\mathbb{R}^n} u(x) \Delta^k \varphi(x)\, dx = -\int\limits_{\mathbb{R}^n} \nabla u(x) \cdot \nabla(\Delta^{k-1} \varphi(x))\, dx.$$

This implies:

$$\int\limits_{\mathbb{R}^n} (1+|x|)^{-\alpha} |\nabla u(x)|^q \varphi(x)\,dx \leqslant \int\limits_{\mathbb{R}^n} |\nabla u(x)| \cdot |\nabla(\Delta^{k-1}\varphi(x))|\,dx + \int\limits_{\mathbb{R}^n} (1+|x|)^{-\beta} |\nabla u(x)|^s \varphi(x)\,dx. \tag{3.1}$$

Now we are going to estimate the terms in the right hand side of (3.1). Applying the Young inequality with the exponents $q>1$ and $q' = \frac{q}{q-1} > 1$ to the first term in the right hand side of (3.1), we obtain the following estimate:

$$\int\limits_{\mathbb{R}^n} |\nabla u(x)| \cdot |\nabla(\Delta^{k-1}\varphi(x))| \leqslant \frac{1}{3}\int\limits_{\mathbb{R}^n} (1+|x|)^{-\alpha} |\nabla u(x)|^q \varphi(x)\,dx + c \int\limits_{\operatorname{supp}\varphi} (1+|x|)^{\alpha\frac{q'}{q}} |\nabla(\Delta^{k-1}\varphi(x))|^{q'} \varphi^{-\frac{q'}{q}}(x)\,dx. \tag{3.2}$$

In the same way was we estimate the second term in the right hand side of (3.1). Applying the Young inequality with the exponents

$$r = \frac{q}{s} > 1, \qquad r' = \frac{q}{q-s} > 1,$$

we obtain:

$$\int\limits_{\mathbb{R}^n} (1+|x|)^{-\beta} |\nabla u(x)|^s \varphi(x)\,dx \leqslant \frac{1}{3}\int\limits_{\mathbb{R}^n} (1+|x|)^{-\alpha} |\nabla u(x)|^q \varphi(x)\,dx + c\int\limits_{\mathbb{R}^n} (1+|x|)^{(-\beta+\frac{\alpha}{r})r'} \varphi(x)\,dx. \tag{3.3}$$

Combining (3.1)–(3.3), we arrive at

$$\int\limits_{\mathbb{R}^n} (1+|x|)^{-\alpha} |\nabla u(x)|^q \varphi(x)\,dx \leqslant c \int\limits_{\operatorname{supp}\varphi} (1+|x|)^{\alpha\frac{q'}{q}} |\nabla(\Delta^{k-1}\varphi(x))|^{q'} \varphi^{-\frac{q'}{q}}(x)\,dx + c\int\limits_{\mathbb{R}^n} (1+|x|)^{(-\beta+\frac{\alpha}{r})r'} \varphi(x)\,dx. \tag{3.4}$$

We choose a test function φ of form

$$\varphi(x) = \varphi_R(x) = \psi^\lambda\left(\frac{|x|^2}{R^2}\right), \tag{3.5}$$

where $\lambda > 2kq'$ and an non-negative function $\psi \in C_0^{2k}\left(\overline{\mathbb{R}_+}\right)$ is such that

$$\psi(s) = \begin{cases} 1, & 0 \leqslant s \leqslant 1, \\ 0, & s \geqslant 4. \end{cases} \tag{3.6}$$

We then make the change of variables

$$x \to \xi, \quad \text{where} \quad x = R\xi. \tag{3.7}$$

Considering the right hand sides of inequalities (3.4) and (3.7) for $R \geqslant 1$, we obtain

$$\begin{aligned}\int\limits_{\mathbb{R}^n} (1+|x|)^{\alpha\frac{q'}{q}} \left|\nabla(\Delta^{k-1}\varphi_R(x))\right|^{q'} \varphi_R^{-\frac{q'}{q}}(x)\,dx \\ \leqslant R^{\theta_1} \int_{1\leqslant|\xi|\leqslant 2} (1+|\xi|)^{\alpha\frac{q'}{q}} \left|\nabla(\Delta^{k-1}\varphi_1(\xi))\right|^{q'} \varphi_1^{-\frac{q'}{q}}(\xi)d\xi,\end{aligned} \tag{3.8}$$

where $\theta_1 = n-(2k-1)q' + \alpha\frac{q'}{q} \leqslant 0$, and

$$\int\limits_{\mathbb{R}^n} (1+|x|)^{(-\beta+\frac{\alpha}{r})r'} \varphi_R(x)\,dx \leqslant R^{\theta_2} \int_{|\xi|\leqslant 2} (1+|\xi|)^{(-\beta+\frac{\alpha}{r})r'} \varphi_1(\xi)\,d\xi, \tag{3.9}$$

where $\theta_2 = n + (-\beta+\frac{\alpha}{r})r' < 0$. Then by the choice of the test function $\varphi_1(\xi) = \psi^\lambda(|\xi|)$ with $\lambda > 2kq'$, we have

$$\int_{1\leqslant|\xi|\leqslant 2} (1+|\xi|)^{\alpha\frac{q'}{q}} \left|\nabla(\Delta^{k-1}\varphi_1(\xi))\right|^{q'} \varphi_1^{-\frac{q'}{q}}(\xi)\,d\xi < \infty$$

and

$$\int_{|\xi|\leqslant 2} (1+|\xi|)^{(-\beta+\frac{\alpha}{r})r'} \varphi_1(\xi)\,d\xi < \infty,$$

since the integral in the right hand side of (3.4) is finite. Then it follows from (3.4) that

$$\int\limits_{\mathbb{R}^n} (1+|x|)^{-\alpha} |\nabla u(x)|^q \varphi_R(x)\,dx \leqslant cR^\theta, \tag{3.10}$$

where $\theta = \max(\theta_1, \theta_2)$. Now we consider the following two cases with different values of θ_1.

Case 1: If $\theta_1 < 0$, we pass to the limit as $R \to \infty$ in (3.10) and we get:

$$\int\limits_{\mathbb{R}^n} (1+|x|)^{-\alpha} |\nabla u(x)|^q \varphi_R(x)\,dx \to 0. \tag{3.11}$$

Thus, u is constant almost everywhere in $\mathbb{R}^n$.

Case 2: $\theta_1 = 0$. In this case it follows from relation (3.8) that

$$\begin{aligned}\int\limits_{\mathbb{R}^n} (1+|x|)^{\alpha\frac{q'}{q}} \left|\nabla(\Delta^{k-1}\varphi_R(x))\right|^{q'} \varphi_R^{-\frac{q'}{q}}(x)\,dx \\ \leqslant \int_{1\leqslant|\xi|\leqslant 2} (1+|\xi|)^{\alpha\frac{q'}{q}} \left|\nabla(\Delta^{k-1}\varphi_1(\xi))\right|^{q'} \varphi_1^{-\frac{q'}{q}}(\xi)\,d\xi := c\end{aligned} \tag{3.12}$$

and since $\theta_2 < 0$,

$$\lim_{R\to\infty} \int\limits_{\mathbb{R}^n} (1+|x|)^{(-\beta+\frac{\alpha}{r})r'} \varphi_R(x)\,dx = \lim_{R\to\infty} cR^{\theta_2} \int_{|\xi|\leqslant 2} (1+|\xi|)^{(-\beta+\frac{\alpha}{r})r'} \varphi_1(\xi)\,d\xi = 0. \tag{3.13}$$

This is why by (3.4) we have

$$\int\limits_{\mathbb{R}^n} (1+|x|)^{-\alpha} |\nabla u(x)|^q \varphi_R(x)\,dx \leqslant c.$$

Passing to limit as $R \to \infty$, we obtain

$$\int\limits_{\mathbb{R}^n} (1+|x|)^{-\alpha} |\nabla u(x)|^q \,dx \leqslant c. \tag{3.14}$$

We return back to inequality (3.1). We observe that

$$\operatorname{supp}\{\nabla(\Delta^{k-1}\varphi_R)\} \subseteq \{x \in \mathbb{R}^n \mid R \leqslant |x| \leqslant 2R\}.$$

Then by the Hölder inequality with the corresponding exponents from relation (3.1) we obtain

$$\begin{aligned}\int\limits_{\mathbb{R}^n} (1+|x|)^{-\alpha}\,|\nabla u(x)|^q\,\varphi_R(x)\,dx \leqslant \left(\int_{R\leqslant|x|\leqslant 2R} (1+|x|)^{-\alpha}\,|\nabla u(x)|^q\,\varphi_R(x)\,dx\right)^{\frac{1}{q}}&\\ \cdot\left(\int_{R\leqslant|x|\leqslant 2R} (1+|x|)^{\alpha\frac{q'}{q}}\left|\nabla(\Delta^{k-1}\varphi_R(x))\right|^{q'}\varphi_R^{-\frac{q'}{q}}(x)\,dx\right)^{\frac{1}{q'}}&\\ +\left(\int_{R\leqslant|x|\leqslant 2R} (1+|x|)^{-\alpha}\,|\nabla u(x)|^q\,\varphi_R(x)\,dx\right)^{\frac{1}{r}}&\\ \cdot\left(\int_{R\leqslant|x|\leqslant 2R} (1+|x|)^{(-\beta+\frac{\alpha}{r})r'}\varphi_R(x)d\xi\right)^{\frac{1}{r'}}&.\end{aligned} \tag{3.15}$$

However, by (3.14) and the absolute convergence of the integral in this relation we have

$$\int_{R\leqslant|x|\leqslant 2R} (1+|x|)^{-\alpha}\,|\nabla u(x)|^q\,dx \to 0$$

as $R\to\infty$. Passing to the limit as $R\to\infty$ in (3.15) and taking into considered (3.12) and (3.13), we again obtain (3.11). Thus, the function u is constant in $\mathbb{R}^n$ in this case as well. The proof is complete.

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