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ON A CLASS OF HYPERBOLIC EQUATIONS WITH THIRD-ORDER INTEGRALS

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Abstract. We consider a Goursat problem on classification nonlinear second order hyperbolic equations integrable by the Darboux method. In the work we study a class of hyperbolic equations with second order y-integral reduced by an differential substitution to equations with first order y-integral. It should be noted that Laine equations are in the considered class of equations. In the work we provide a second order y-integral for the second Laine equation and we find a differential substitution relating this equation with one of the Moutard equations.

We consider a class of nonlinear hyperbolic equations possessing first order y-integrals and third order x-integrals. We obtain three conditions under which the equations in this class possess first order and third order integrals. We find the form of such equations and obtain the formulas for x- and y-integrals. In the paper we also provide differential substitutions relating Laine equations.

Keywords: Laplace invariants, x- and y-integrals, differential substitutions.

Mathematics Subject Classification: 35Q51, 37K60

1. Introduction

For a complete classification of nonlinear hyperbolic equations

$$u_{xy} = f(x, y, u, u_x, u_y)$$

one needs to classify equations in a special class, which were not studied in work [1], namely, the following equations:

$$u_{xy} = \frac{p - \varphi_u}{\varphi_{u_y}} u_x + \frac{q}{\varphi_{u_y}} \sqrt{u_x}. \tag{1.1}$$

Here p, q are the functions of the variables x, y, u, while φ is a function of the variables x, y, u, u_y.

In 1926 Laine constructed two equations [2]–[4]

$$u_{xy} = \left(\frac{u_y}{u-x} + \frac{u_y}{u-y}\right) u_x + \frac{u_y}{u-x}\sqrt{u_x}, \tag{1.2}$$

$$u_{xy} = 2\left[(u+Y)^2 + u_y + (u+Y)\sqrt{(u+Y)^2 + u_y}\right] \cdot \left[\frac{\sqrt{u_x} + u_x}{u-x} - \frac{u_x}{\sqrt{(u+Y)^2 + u_y}}\right], \tag{1.3}$$

where $Y = Y(y)$, which possessed a second order y-integral $\bar{w} = \bar{w}(x, y, u, u_y, u_{yy})$ and a third order x-integral $w = w(x, y, u, u_x, u_{xx}, u_{xxx})$ ($D\bar{w} = 0$, $\bar{D}w = 0$). Here D (respectively, $\bar{D}$) is an operator of total differentiation in x (respectively, in y).

We note that equations (1.2) and (1.3) are in the class of equations (1.1). Indeed, as

$$q = \frac{1}{u-x}, \qquad p = \frac{1}{u-x} + \frac{1}{u-y}, \qquad \varphi = \ln u_y$$

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equation (1.2) coincides with equation (1.1), while as

$$p = q = \frac{1}{u-x}, \qquad \varphi = \ln\left[(u+Y) + \sqrt{u_y + (u+Y)^2}\right]$$

equation (1.1) becomes (1.3).

In work [5] the following statement was proved.

Lemma 1.1. *If equation* (1.1) *possesses a second order y-integral, then the function φ is independent of the variable x.*

Hence, the y-interal can be represented as

$$\bar{W} = \bar{D}r + \beta(x,y,r)$$

and this is why the differential substitution

$$r = \varphi(y,u,u_y) - h(x,y,u), \qquad p = h_u, \tag{1.4}$$

maps solutions of equation (1.1) into solutions of the equation

$$D\bar{D}r + D\beta = 0. \tag{1.5}$$

Let us provide differential substitutions (1.4), equations (1.5) and integrals for Laine equations, see [2]–[4]. The differential substitution

$$r = \ln\frac{u_y}{(u-x)(u-y)} \tag{1.6}$$

relates equation (1.2) with the Moutard equation

$$r_{xy} + \frac{1}{2}(x-y)r_x e^r + \frac{1}{2}e^r = 0. \tag{1.7}$$

The above equation possesses a third order x-integral

$$w = \frac{r_{xxx} - 3r_x\cdot r_{xx} + r_x^3}{r_{xx} - r_x^2}. \tag{1.8}$$

Then equation (1.2) possesses an x-integral of form

$$W = \frac{z_x}{z} + z, \tag{1.9}$$

where

$$z = \frac{u_{xx}}{2(u_x + \sqrt{u_x})} - \frac{u_x + \sqrt{u_x}}{u-x}.$$

Equation (1.2) also possesses a second order y-integral:

$$\bar{W} = \frac{u_{yy}}{u_y} - \frac{u_y}{2}\left(\frac{1}{u-x} + \frac{3}{u-y}\right) + \frac{1}{u-y}.$$

A differential substitution

$$r = \ln\left[\frac{u + Y(y) + \sqrt{u_y + (u+Y(y))^2}}{u-x}\right] \tag{1.10}$$

maps solutions of equation (1.3) into the solutions of the equation

$$r_{xy} - \frac{d}{dx}\left[e^r(x+Y(y))\right] = 0. \tag{1.11}$$

Equation (1.11) possesses a third order x-integral (1.8), while equation (1.3) possesses integral (1.9), that is, it coincides with the x-integral of equation (1.2).

It was also found an y-integral of equation (1.3) in the form

$$\begin{aligned}\bar{W} =& \frac{u_{yy}}{2u_y}\left(1 - \frac{u+Y}{\sqrt{u_y + (u+Y)^2}}\right)\\ &- \frac{u_y + (u+Y)^2 + (u+Y)\sqrt{u_y+(u+Y)^2}}{u-x} + u + \frac{(u+Y)^2 + 2u_y + Y'}{\sqrt{u_y + (u+Y)^2}}.\end{aligned}$$

The aim of the present work is the description of equations (1.5) possessing first order y-integral and a third order x-integral.

2. x-INTEGRALS OF EQUATION (1.5)

Let us study equation (1.5) possessing third order x-integrals. We make the change $r \to u$, $\beta \to -p$. Then equation (1.5) is rewritten in the form

$$D\bar{D}u = Dp, \qquad p = p(x,y,u). \tag{2.1}$$

For the sake of convenience of the presentation we introduce the notations

$$u_1 = u_x, \qquad u_2 = u_{xx}, \qquad \ldots, \qquad \bar{u}_1 = u_y, \qquad \bar{u}_2 = u_{yy}, \qquad \ldots.$$

We note that an y-integral of equation (2.1) is given by the formula

$$\bar{W} = \bar{u}_1 - p.$$

Let $W = W(x,y,u,u_1,u_2,u_3)$ be a x-integral of equation (2.1). In view of the expression

$$\bar{D}W = W_y + W_u \cdot \bar{u}_1 + W_{u_1} \cdot Dp + W_{u_2} \cdot D^2 p + W_{u_3} \cdot D^3 p = 0, \tag{2.2}$$

it is clear that $W_u = 0$. It is known that if there exists an integral of order n, $n \geqslant 2$, we can suppose that it is linear in the higher variable. We let

$$W = A(x,y,u_1,u_2) \cdot u_3 + B(x,y,u_1,u_2).$$

Expression (2.2) is rewritten as

$$A\left(p_u \cdot u_3 + 3p_{uu} \cdot u_1 u_2 + 3u_2 \cdot p_{ux} + u_1^3 \cdot p_{uuu} + 3u_1^2 \cdot p_{uux} + 3u_1 \cdot p_{xxu} + p_{xxx}\right) + \bar{D}B = 0$$

or

$$\bar{D}A + p_u A = 0, \tag{2.3}$$

$$A\left(3p_{uu}u_1u_2 + 3u_2 p_{ux} + u_1^3 p_{uuu} + 3u_1^2 p_{uux} + 3u_1 p_{xxu} + p_{xxx}\right) + \bar{D}B = 0. \tag{2.4}$$

We consider equation (2.3) and the first case when $A = A(x,y)$. Then by expression (2.3) we find that

$$p = -\frac{A_y}{A} \cdot u + E(x,y).$$

By means of the change $u = v + Q(x,y)$, where $-\frac{A_y}{A}Q + E - Q_y = 0$, we obtain the equation

$$D\bar{D}v = D\left(a(x,y) \cdot v\right), \tag{2.5}$$

in which $a(x,y) = -\frac{A_y}{A}$.

Now we proceed to the case when $A = A(x,y,u_1)$, $A_{u_1} \neq 0$. Differentiating expression (2.3) in u_1, we obtain

$$\bar{D}A_{u_1} + 2A_{u_1} \cdot p_u = 0$$

and taking into consideration that $\bar{D}A + p_u A = 0$, we have

$$p_u = -\frac{\bar{D}A}{A} = -\frac{1}{2}\frac{\bar{D}A_{u_1}}{A_{u_1}},$$

that is,

$$\bar{D}\ln\frac{A_{u_1}}{A^2} = 0.$$

Since we consider a third order x-integral, then

$$\frac{A_{u_1}}{A^2} = a(x), \quad a(x) \neq 0.$$

This yields

$$A = \frac{\tilde{a}(x)}{u_1 + b(x,y)}.$$

We can suppose that $\tilde{a}(x) = 1$, and the change $u \to u - \int b(x,y)\,dx$ allows us to represent A as

$$A = \frac{1}{u_1}.$$

By identity (2.3) we find $p_x = 0$, that is, in this case we have

$$A = \frac{1}{u_1}, \qquad D\bar{D}u = Dp(y,u).$$

It remains to consider the case $A = A(x, y, u_1, u_2)$, $A_{u_2} \neq 0$. Differentiating expression (2.3) in the variable u_2, we find that

$$\bar{D}A_{u_2} + 2p_u \cdot A_{u_2} = 0.$$

This implies

$$p_u = -\frac{\bar{D}A}{A} = -\frac{1}{2}\frac{\bar{D}A_{u_2}}{A_{u_2}}.$$

Then

$$A = \frac{1}{u_2 + b(x, y, u_1)}. \tag{2.6}$$

Substituting the found A into (2.3), we obtain

$$p_{uu} \cdot u_1^2 + 2u_1 \cdot p_{ux} + p_{xx} + b_y + b_{u_1} \cdot Dp - p_u \cdot b = 0. \tag{2.7}$$

Differentiating this identity in the variable u_1, we find

$$2p_{uu} \cdot u_1 + 2p_{ux} + \bar{D}b_{u_1} = 0.$$

Then

$$\bar{D}b_{u_1 u_1 u_1} + 2p_u \cdot b_{u_1 u_1 u_1} = 0.$$

If $b_{u_1 u_1 u_1} \neq 0$, then $p_u = -\frac{1}{2}\bar{D}\ln b_{u_1 u_1 u_1}$. And since $p_u = -\bar{D}\ln A$, we get

$$\bar{D}\left(\ln\frac{1}{u_2 + b} - \frac{1}{2}\ln b_{u_1 u_1 u_1}\right) = 0.$$

Hence, there exists a second order integral, which contradicts to the assumption that the order of the x-integral is three. Thus, $b_{u_1 u_1 u_1} = 0$ and

$$b = \frac{\alpha}{2} \cdot u_1^2 + \gamma \cdot u_1 + \delta, \tag{2.8}$$

where α, γ, δ are the functions of the variables x and y. We substitute function (2.8) into equation (2.7) and we obtain the identities

$$p_{uu} + \frac{\alpha_y}{2} + \frac{\alpha}{2} \cdot p_u = 0, \tag{2.9}$$

$$2p_{ux} + \gamma_y + \alpha \cdot p_x = 0, \tag{2.10}$$

$$p_{xx} + \delta_y + \gamma \cdot p_x - \delta \cdot p_u = 0. \tag{2.11}$$

A solution to equation (2.9) is given by the formula

$$p = -\frac{2}{\alpha}Ce^{-\frac{\alpha}{2}u} - \frac{\alpha_y}{\alpha}u + \kappa(y), \tag{2.12}$$

as $\alpha \neq 0$.

If $\alpha = 0$, then $p_{uu} = 0$, $p_u = \mu(x, y)$ and

$$\bar{D}\left(\ln A + \int \mu dy\right) = 0,$$

that is, there exists a second order x-integral. Thus, if $A = A(x, y, u_1, u_2)$, then formulas (2.6), (2.8), (2.9)–(2.12) hold true.

To simplify the function p in (2.12), in equation (2.1) we make the change

$$u = \beta(y) \cdot v + \mu(x, y).$$

After simple transformations we obtain an equation ($v \to u$)

$$D\bar{D}u = D(e^u + d(x, y)),$$

where $p = e^u + d(x, y)$. Then conditions (2.9)–(2.11) become

$$\alpha = -2, \qquad \delta = 0, \qquad \gamma_{xy} = -\gamma \cdot \gamma_y, \qquad d_x = \frac{1}{2}\gamma_y.$$

Thus, we have proved the following statement.

Lemma 2.1. *Let equation* (2.1) *has a third order x-integral*

$$W = A(x,y,u_1,u_2)\cdot u_3 + B(x,y,u_1,u_2).$$

Then of the following conditions hold:

$$A = A(x,y), \quad p = a(x,y)\cdot u, \quad a = -\frac{A_y}{A}, \tag{2.13}$$

$$A = \frac{1}{u_1}, \quad p = p(y,u), \tag{2.14}$$

$$A = \frac{1}{u_2+b}, \quad b = -u_1^2 + \gamma u_1, \quad p = e^u + d(x,y), \tag{2.15}$$

$$\gamma_{xy} = -\gamma\cdot\gamma_y, \quad d_x = \frac{1}{2}\gamma_y.$$

Under conditions (2.13)–(2.15), *identity* (2.3) *is true and vice versa, condition* (2.3) *is reduced to one of* (2.13), (2.14), (2.15).

We then consider equation (2.4) in case (2.13):

$$A\cdot(3u_2\cdot a_x + 3u_1\cdot a_{xx} + a_{xxx}\cdot u) + \bar{D}B = 0. \tag{2.16}$$

Differentiating (2.16) by the variable u_2, we obtain

$$3a_x\cdot A + \bar{D}B_{u_2} + a\cdot B_{u_2} = 0,$$
$$\bar{D}B_{u_2u_2} + 2a\cdot B_{u_2u_2} = 0.$$

We note that $a_x \neq 0$. If $a_x = 0$, then $B = B(x)$ and there exists a first order x-integral $W = A\cdot u_1$. We also have $B_{u_2}\neq 0$, otherwise $a_x = 0$.

If $B_{u_2u_2} = 0$, then

$$B = \alpha(x,y,u_1)\cdot u_2 + \beta(x,y,u_1). \tag{2.17}$$

By substituting (2.17) into expression (2.16) we obtain the relation

$$3A\cdot a_x + \alpha\cdot a + \alpha_y + \alpha_{u_1}(a_x\cdot u + a\cdot u_1) = 0, \tag{2.18}$$

$$A\cdot a_{xxx} + \alpha\cdot a_{xx} + a_x\cdot\beta_{u_1} = 0, \tag{2.19}$$

$$3A\cdot a_{xx}\cdot u_1 + 2\alpha\cdot a_x\cdot u_1 + \beta_y + \beta_{u_1}\cdot a\cdot u_1 = 0. \tag{2.20}$$

Since $a_x\neq 0$, then $\alpha_{u_1} = 0$, that is, $\alpha = \alpha(x,y)$ and expression (2.18) is rewritten as

$$3A\cdot a_x + \alpha\cdot a + \alpha_y = 0. \tag{2.21}$$

By (2.19) we find

$$\beta = -\frac{1}{a_x}\left(A\cdot a_{xxx} + \alpha\cdot a_{xx}\right)\cdot u_1 + \gamma(x,y). \tag{2.22}$$

Then expression (2.20) becomes

$$3A\cdot a_{xx} + 2\alpha\cdot a_x - \frac{\partial}{\partial y}\left[\frac{1}{a_x}\left(Aa_{xxx} + \alpha a_{xx}\right)\right] - a\left[\frac{1}{a_x}\left(Aa_{xxx}+\alpha a_{xx}\right)\right] = 0 \tag{2.23}$$

and $\gamma_y = 0$. Since $W = Au_3 + \alpha u_2 + \beta$, we can suppose that $\gamma\equiv 0$.

By equation (2.23) we find α in the form

$$\alpha = -\frac{\left(6a_{xx} - \left(\frac{a_{xxx}}{a_x}\right)'_y\right)\cdot A}{2a_x - \left(\frac{a_{xx}}{a_x}\right)'_y}, \tag{2.24}$$

the denominator satisfies $2a_x - \left(\frac{a_{xx}}{a_x}\right)'_y \neq 0$ since otherwise there exists a second order x-integral $W = A\left(u_2 - \frac{a_{xx}}{a_x}u_1\right)$.

Thus, it follows from (2.21), (2.22) and (2.24) that in the case $B_{u_2u_2} = 0$ a third order x-integral can be represented as

$$W = e^{-b} \cdot \left(u_3 - \frac{E}{Fa_x}(a_x u_2 - a_{xx} u_1) - \frac{a_{xxx}}{a_x} u_1 \right),$$

where $b_y = a$, $E = 6a_{xx} - \left(\frac{a_{xxx}}{a_x}\right)'_y$, $F = 2a_x - \left(\frac{a_{xx}}{a_x}\right)'_y$ and the condition

$$\frac{E}{F} - 3b_x + \kappa(x) = 0 \tag{2.25}$$

holds true, where $\kappa(x)$ is an arbitrary function.

Now let $B_{u_2u_2} \neq 0$. Then

$$\bar{D} \ln B_{u_2u_2} = -2a = 2\frac{A_y}{A}$$

or

$$B_{u_2u_2} = \gamma(x) \cdot A^2,$$

or

$$B = \frac{\gamma(x)}{2} A^2 u_2^2 + \varepsilon(x, y, u, u_1) u_2 + \mu(x, y, u, u_1),$$

$\gamma \neq 0$. Then

$$W = Au_3 + \frac{\gamma}{2} A^2 u_2^2 + \varepsilon u_2 + \mu$$

and using the change $\gamma \cdot A \to A$, we can rewrite the integral as

$$W = Au_3 + \frac{1}{2} A^2 u_2^2 + \varepsilon u_2 + \mu,$$

where ε, μ are the functions of the variables x, y, u, u_1. Thus,

$$B = \frac{A^2}{2} u_2^2 + \varepsilon u_2 + \mu. \tag{2.26}$$

Now we write condition (2.16) for the above function B. We obtain the relations

$$\varepsilon_u = 0, \qquad \mu_u = 0,$$

$$A^2 a_{xx} + \varepsilon_{u_1} a_x = 0, \tag{2.27}$$

$$3Aa_x + 2A^2 a_x u_1 + \varepsilon_y + \varepsilon_{u_1} a u_1 + \varepsilon a = 0, \tag{2.28}$$

$$Aa_{xxx} + \varepsilon a_{xx} + \mu_{u_1} a_x = 0, \tag{2.29}$$

$$3Aa_{xx} u_1 + 2\varepsilon a_x u_1 + \mu_y + \mu_{u_1} a u_1 = 0. \tag{2.30}$$

We note that $a_x \neq 0$. By (2.27) we find

$$\varepsilon = -A^2 \cdot \frac{a_{xx}}{a_x} \cdot u_1 + \delta(x, y), \tag{2.31}$$

while by (2.29) we get

$$\mu = \left(\frac{a_{xx}}{a_x}\right)^2 \frac{A^2}{2} u_1^2 - \left(A\frac{a_{xxx}}{a_x} + \frac{a_{xx}}{a_x}\delta \right) u_1 + \gamma(x, y). \tag{2.32}$$

In view of (2.31), (2.32) relations (2.28), (2.30) are rewritten as

$$3Aa_x + \delta_y + a\delta = 0, \tag{2.33}$$

$$2A^2 a_x - \left(A^2 \frac{a_{xx}}{a_x} \right)'_y - 2aA^2 \frac{a_{xx}}{a_x} = 0, \tag{2.34}$$

$$3Aa_{xx} + 2a_x\delta - \left(A\frac{a_{xxx}}{a_x} + \frac{a_{xx}}{a_x}\delta \right)'_y - a\left(A\frac{a_{xxx}}{a_x} + \frac{a_{xx}}{a_x}\delta \right) = 0, \tag{2.35}$$

$$-2A^2 a_{xx} + \frac{1}{2}\left[\left(\frac{a_{xx}}{a_x} A \right)^2 \right]'_y + a\left(\frac{a_{xx}}{a_x} A \right)^2 = 0, \tag{2.36}$$

$\gamma_y = 0$. We can suppose that $\gamma(x) \equiv 0$. After simple transformations, relations (2.33)–(2.36) can be represented as

$$3Aa_x + \delta_y + a\delta = 0,$$
$$2a_x - \left(\frac{a_{xx}}{a_x}\right)'_y = 0,$$
$$6a_{xx} - \left(\frac{a_{xxx}}{a_x}\right)'_y = 0.$$

But if

$$2a_x - \left(\frac{a_{xx}}{a_x}\right)'_y = 0,$$

original equation (2.1) possesses a second order x-integral

$$W = A\left(u_2 - \frac{a_{xx}}{a_x}u_1\right), \qquad a = -\frac{A_y}{A}.$$

Since we seek a third order x-integral, such scenario can not be realized.

We proceed to the case (2.14). Equation (2.4) is written as

$$3p_{uu}u_2 + u_1^2 p_{uuu} + B_y + B_{u_1}(p_u u_1) + B_{u_2}(p_u u_2 + p_{uu}u_1^2) = 0. \tag{2.37}$$

By differentiating in the variable u_2, we obtain

$$3p_{uu} + \bar{D}B_{u_2} + p_u \cdot B_{u_2} = 0. \tag{2.38}$$

If $B_{u_2} = 0$, then $p_{uu} = 0$, that is, $p = \alpha(y)u + \beta(y)$. In this case there exists a first order x-integral $W = \gamma(y) \cdot u_1$, where $\gamma' + \gamma \cdot \alpha = 0$.

Now let $B_{u_2} \neq 0$, $B_{u_2u_2} = 0$, that is,

$$B = \alpha(x, y, u_1) \cdot u_2 + \beta(x, y, u_1).$$

Expression (2.37) becomes

$$3p_{uu} + \alpha_y + \alpha_{u_1}p_u u_1 + \alpha p_u = 0, \tag{2.39}$$
$$u_1^2 p_{uuu} + \alpha p_{uu}u_1^2 + \bar{D}\beta = 0. \tag{2.40}$$

Differentiating (2.39) in the variable u_1, we obtain:

$$\bar{D}\alpha_{u_1} + 2p_u \cdot \alpha_{u_1} = 0.$$

If $\alpha_{u_1} = 0$, then $\alpha = \alpha(x, y)$ and

$$3p_{uu} + \alpha_y + \alpha \cdot p_u = 0. \tag{2.41}$$

A solution to equation (2.41) is given by the formula

$$p = -\frac{\alpha_y}{\alpha} \cdot u - 3\frac{\kappa(x, y)}{\alpha} \cdot e^{-\frac{1}{3}\alpha u} + \mu(x, y).$$

Since $p_x = 0$, we have either

$$\kappa = 0, \qquad \frac{\alpha_y}{\alpha} = \delta(y), \qquad \mu = \mu(y),$$
$$p = -\delta(y) \cdot u + \mu(y), \tag{2.42}$$

or

$$\kappa = \kappa(y) \neq 0, \quad \frac{\alpha_y}{\alpha} = \delta(y), \quad \alpha = \alpha(y), \quad \mu = \mu(y),$$
$$p = -\delta(y) \cdot u - 3\frac{\kappa(y)}{\alpha(y)} \cdot e^{-\frac{1}{3}\alpha u} + \mu(y). \tag{2.43}$$

In case (2.39), (2.40), (2.42) there exists a first order x-integral $W = \gamma(y) \cdot u_1$. And in case (2.39), (2.40), (2.43) there exists a second order x-integral $W = \frac{u_2}{u_1} + \frac{\alpha(y)}{3} \cdot u_1$. Thus, both these situations are not realized.

If $\alpha_{u_1} \neq 0$, then

$$\bar{D} \ln \alpha_{u_1} + 2p_u = 0$$

or

$$\bar{D} \ln \alpha_{u_1} + 2\bar{D} \ln u_1 = 0.$$

This implies

$$\alpha = -\frac{\varepsilon(x)}{u_1} + \gamma(x, y). \tag{2.44}$$

In view of the above identity relation (2.44) becomes

$$3p_{uu} + \gamma_y + \gamma p_u = 0. \tag{2.45}$$

Since $p_x = 0$, then $\gamma = \gamma(y)$. Equation (2.45) coincides with (2.41) ($\alpha \to \gamma$). Hence, this case also is not realized.

We finally consider the case $B_{u_2u_2} \neq 0$. Differentiating equation (2.38) in the variable u_2, we find

$$\bar{D}B_{u_2u_2} + 2p_u \cdot B_{u_2u_2} = 0$$

or

$$\bar{D} \ln B_{u_2u_2} + 2\bar{D} \ln u_1 = 0.$$

This yields

$$B = \alpha(x) \cdot \left(\frac{u_2}{u_1}\right)^2 + \beta(x, y, u_1) \cdot u_2 + \gamma(x, y, u_1). \tag{2.46}$$

Substituting (2.46) into (2.37), we obtain

$$(3 + 2\alpha) \cdot p_{uu} + (\beta + u_1\beta_{u_1}) \cdot p_u + \beta_y = 0, \tag{2.47}$$

$$u_1^2 \cdot p_{uuu} + \gamma_y + p_u \cdot u_1 \cdot \gamma_{u_1} + p_{uu} \cdot u_1^2 \cdot \beta = 0. \tag{2.48}$$

Then $\frac{\partial}{\partial u_1}(\beta + u_1\beta_{u_1}) = 0$, otherwise $p_{uu} = 0$ and $B_{u_2} = 0$. We find

$$\beta = \varepsilon(x, y) + \frac{\delta(x, y)}{u_1}$$

and substitute the expression for β into (2.47). This gives $\delta_y = 0$ and

$$(3 + 2\alpha(x)) \cdot p_{uu} + \varepsilon(x, y) \cdot p_u + \varepsilon_y = 0.$$

If $3 + 2\alpha = 0$, then $\varepsilon = 0$ and $\beta = \frac{\delta(x)}{u_1}$. Now we consider (2.48):

$$u_1^2 \cdot p_{uuu} + \gamma_y + \gamma_{u_1} \cdot u_1 \cdot p_u + p_{uu} \cdot u_1 \cdot \delta = 0.$$

For $\delta(x) \neq 0$ we have

$$p_{uu} = c_1p_u + c_2, \quad p_{uuu} = a_1p_u + a_2, \quad c_i = c_i(y), \quad a_i = a_i(y), \quad i = 1, 2.$$

Since $p_u \neq 0$, then $c_1^2 = a_1$, $c_1c_2 = a_2$ and

$$p_{uu} = c_1p_u + c_2, \quad p_{uuu} = c_1^2p_u + c_1c_2. \tag{2.49}$$

Substituting (2.49) into identity (2.48), we obtain the following relations

$$\gamma_{u_1} = -c_1^2u_1 - c_1\delta, \quad \gamma_y = -c_1c_2u_1^2 - c_2\delta u_1.$$

This implies $c_1' = c_2$. Then

$$p_{uu} = c_1p_u + c_1', \quad p_{uuu} = c_1^2p_u + c_1c_1'.$$

In this case equation (2.1) possesses a second order x-integral $W = \frac{u_2}{u_1} - c_1(y) \cdot u_1$ and this case can not be realized.

Let $\delta(x) = 0$, then $\beta = 0$ and relation (2.48) becomes

$$p_{uuu} + \frac{\gamma_y}{u_1^2} + \frac{\gamma_{u_1}}{u_1} \cdot p_u = 0.$$

Then

$$\frac{\gamma_{u_1}}{u_1} = \mu(x,y), \quad \frac{\gamma_y}{u_1^2} = \kappa(x,y), \tag{2.50}$$
$$p_{uuu} + \kappa(x,y) + \mu(x,y)\cdot p_u = 0.$$

Since $p_x = 0$, then $\mu_x = 0$ and $\kappa_x = 0$. It follows from (2.50) that $\mu' = 2\kappa$, $\gamma = \frac{\mu(y)}{2}u_1^2$ and

$$p_{uuu} + \mu(y)\cdot p_u + \frac{1}{2}\mu'(y) = 0.$$

In this case we represent a third order x-integral in the form

$$W = \frac{u_3}{u_1} - \frac{3}{2}\cdot\left(\frac{u_2}{u_1}\right)^2 + \frac{\mu(y)}{2}\cdot u_1^2.$$

Let $3+2\alpha \neq 0$. Then by equation (2.47) we obtain

$$\frac{\beta + u_1\beta_{u_1}}{3+2\alpha(x)} = \mu(y), \qquad \frac{\beta_y}{3+2\alpha(x)} = \kappa(y), \tag{2.51}$$
$$p_{uu} + \mu(y)\cdot p_u + \kappa(y) = 0.$$

By relations (2.51) we find $\mu'(y) = \kappa(y)$. This case is not realized since equation (2.1) possesses a x-integral

$$W = \frac{u_2}{u_1} - \mu(y)\cdot u_1.$$

We finally consider case (2.15). We make the change $B = A\cdot C$, and then by (2.3), $\bar{D}B = A\cdot(\bar{D}C - e^u\cdot C)$ and equation (2.4) becomes

$$3e^u\cdot u_1u_2 + u_1^3\cdot e^u + d_{xxx} + \bar{D}C - e^u\cdot C = 0. \tag{2.52}$$

This yields

$$\bar{D}C_{u_2u_2} + e^u\cdot C_{u_2u_2} = 0. \tag{2.53}$$

If $C_{u_2u_2} = 0$, that is, $C = \alpha(x,y,u_1)\cdot u_2 + \beta(x,y,u_1)$, by relation (2.52) we obtain the identity

$$3u_1 + u_1\cdot\alpha_{u_1} = 0, \tag{2.54}$$
$$u_1^3 + \alpha\cdot u_1^2 + u_1\cdot\beta_{u_1} - \beta = 0, \tag{2.55}$$
$$\alpha_y + \alpha_{u_1}\cdot d_x = 0, \tag{2.56}$$
$$d_{xxx} + \alpha\cdot d_{xx} + \beta_y + \beta_{u_1}\cdot d_x = 0. \tag{2.57}$$

By (2.54), (2.56) we find α in the form

$$\alpha = -3u_1 + 3\cdot\int d_x(x,y)\,dy.$$

By equation (2.55), (2.57) we easily get

$$\beta = u_1^3 - \varepsilon\cdot u_1^2 + \mu(x,y)\cdot u_1,$$

where

$$\mu = -\frac{d_{xxx}}{d_x} + 3\frac{d_{xx}}{d_x}\cdot\int d_x(x,y)\,dy,$$

and also the relation

$$\left(\frac{d_{xx}}{d_x}\right)'_y + 2d_x = 0$$

holds. Then a third order x-integral becomes

$$W = \frac{1}{u_2 - u_1^2 - \frac{d_{xx}}{d_x}u_1}\left(u_3 - 3u_1u_2 + u_1^3 - \frac{d_{xxx}}{d_x}u_1\right) + 3\int d_x(x,y)\,dy$$

and at the same time,

$$d_{xy} + 2d\cdot d_x = \varepsilon(y)\cdot d_x.$$

It remains to treat the case $C_{u_2u_2} \neq 0$. By identity (2.53) we find

$$C_{u_2u_2} = \frac{\varphi(x)}{u_2+b}, \qquad \varphi(x) \neq 0.$$

Then

$$C = \varphi(x)\cdot((u_2+b)\cdot\ln(u_2+b)-u_2)+\alpha(x,y,u_1)u_2+\beta(x,y,u_1).$$

We substitute the latter expression for C into equation (2.54) and we get $\varphi(x) = 0$, which is a contradiction. Thus, this case is not realized. As a result, we have proved the following theorem.

Theorem 2.1. *If equation* (2.1) *possesses a third order x-integral and a first order y-integral $\bar{W} = \bar{u}_1 - p$, then one of the following three cases is realized:*

1) $p = a(x,y)\cdot u, \qquad W = e^{-b}\cdot\left(u_3 - \dfrac{E}{Fa_x}(a_xu_2 - a_{xx}u_1) - \dfrac{a_{xxx}}{a_x}u_1\right),$

where $b_y = a, \quad E = 6a_{xx} - \left(\dfrac{a_{xxx}}{a_x}\right)'_y, \quad F = 2a_x - \left(\dfrac{a_{xx}}{a_x}\right)'_y$ *and condition* (2.25) *holds*;

2) $p_{uuu} + \mu(y)\cdot p_u + \dfrac{1}{2}\mu'(y) = 0, \qquad W = \dfrac{u_3}{u_1} - \dfrac{3}{2}\cdot\left(\dfrac{u_2}{u_1}\right)^2 + \dfrac{\mu(y)}{2}\cdot u_1^2;$

3) $p = e^u + d(x,y), \qquad d_{xy} + 2d\cdot d_x = \varepsilon(y)\cdot d_x,$

$$W = \frac{1}{u_2 - u_1^2 - \frac{d_{xx}}{d_x}u_1}\left(u_3 - 3u_1u_2 + u_1^3 - \frac{d_{xxx}}{d_x}u_1\right) + 3\int d_x(x,y)\,dy,$$

where $\mu(y)$, $\varepsilon(y)$ are arbitrary functions.

3. Differential substitutions of Laine equations (1.2), (1.3)

In this section we consider differential substitutions relating equations (1.2), (1.3). In order to do this, in equation (1.2) we change the variable y by z:

$$u_{xz} = \left(\frac{u_z}{u-x} + \frac{u_z}{u-z}\right)u_x + \frac{u_z}{u-x}\sqrt{u_x}. \tag{3.1}$$

By the differential substitution

$$r = \ln\frac{u_z}{(u-x)(u-z)} \tag{3.2}$$

this equation is reduced to the Moutard equation

$$D\bar{D}r = \frac{1}{2}D\left[e^r(z-x)\right]. \tag{3.3}$$

The second Laine equation

$$v_{xy} = 2\left[(v+Y)^2 + v_y + (v+Y)\sqrt{(v+Y)^2+v_y}\right]\times\left[\frac{\sqrt{v_x}+v_x}{v-x} - \frac{v_x}{\sqrt{(v+Y)^2+v_y}}\right] \tag{3.4}$$

is reduced by the differential substitution

$$s = \ln\left[\frac{v+Y(y)+\sqrt{v_y+(v+Y(y))^2}}{v-x}\right] \tag{3.5}$$

to the equation

$$D\bar{D}s = D\left[e^s(x+Y(y))\right]. \tag{3.6}$$

Let us show that equations (3.6) and (3.3) are mutually related. We let $z = -Y(y)$, then

$$s(x,y) = q(x,z).$$

We rewrite equation (3.6) as

$$q_{xz} = D\left[(z-x)e^{q-\ln Y'(y)}\right].$$

We let $\ln Y'(y) = a(z)$,

$$r = q - a(z) + \ln 2. \tag{3.7}$$

Then we obtain equation (3.3)

$$r_{xz} = \frac{1}{2} D\left[e^r(z-x)\right].$$

We substitute (3.2) into expression (3.7)

$$\ln \frac{u_z}{(u-x)(u-z)} = q - \ln Y' + \ln 2,$$

make the change $z = -Y(y)$ and we get

$$s = \frac{u_y}{2(x-u)(u+Y)}.$$

In view of (3.5) we obtain

$$\frac{u_y}{2(x-u)(u+Y(y))} = \frac{v+Y(y)+\sqrt{v_y+(v+Y(y))^2}}{v-x}. \tag{3.8}$$

We differentiate expression (3.8) in x and replace u_{xz} and v_{xy} by equations (3.1) and (3.4). We obtain the relation

$$\frac{\sqrt{u_x}+1}{u-x} = \frac{\sqrt{v_x}+1}{v-x}. \tag{3.9}$$

Thus, we have obtained that equations (3.1) and (3.4) are related by differential expression (3.8), (3.9).

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