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ON TWO-ORDER FRACTIONAL BOUNDARY VALUE PROBLEM WITH GENERALIZED RIEMANN-LIOUVILLE DERIVATIVE

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Abstract. In this paper we focus our study on the existence, uniqueness and Hyers-Ulam stability for the following problem involving generalized Riemann-Liouville operators:

$$\mathcal{D}_{0^+}^{\rho_1,\Psi}\Big(\mathcal{D}_{0^+}^{\rho_2,\Psi}+\nu\Big)\mathrm{u}(\mathfrak{t})=\mathfrak{f}(\mathfrak{t},\mathrm{u}(\mathfrak{t})).$$

It is well known that the existence of solutions to the fractional boundary value problem is equivalent to the existence of solutions to some integral equation. Then it is sufficient to show that the integral equation has only one fixed point. To prove the uniqueness result, we use Banach fixed point Theorem, while for the existence result, we apply two classical fixed point theorems due to Krasnoselskii and Leray-Scauder. Then we continue by studying the Hyers-Ulam stability of solutions which is a very important aspect and attracted the attention of many authors. We adapt some sufficient conditions to obtain stability results of the Hyers-Ulam type.

Keywords: fractional derivatives, generalized Riemann-Liouville derivative, fixed point theorem, fractional Boundary value problem, Hyers-Ulam stability.

Mathematics Subject Classification: 34A08, 26A33, 34A12, 34B15

1. Introduction

Fractional calculus is a subject that has lately spread rapidly and its applications are used in several fields of applied sciences [22], [29], [36]. It plays essential roles, for example in engineering [1], [23], [24], [32], structures [19], optimal control [6], chaotic systems [17], epidemiological models [12], [30]. The fractional structures of boundary value problems and initial value problems generally give a great diversity of mathematical models for the description of certain physical, chemical and biological processes that can be referred to in recently published papers [2], [7], [10], [11], [21], [25], [26], [31], [33]. Parallely to these real patterns caused by real phenomena, many researchers studied the existence theory of solutions for general constructions of fractional boundary value problems involving boundary conditions implying a multi-point nonlocal integral [3], [8], [9], [13], [14], [15].

S. Rezapour et al. [27] discussed the existence of numerical solutions via DGJIM and ADM methods for the follwing fractional boundary value problem implying the generalized Ψ-Riemann-Liouville operators:

$$\begin{aligned}&\mathcal{D}_{0^+}^{\varrho;\psi}u(\mathfrak{t})=\zeta\big(\mathfrak{t},u(\mathfrak{t}),\mathcal{D}_{0^+}^{\delta_1;\psi}u(\mathfrak{t}),\mathcal{D}_{0^+}^{\delta_2;\psi}u(\mathfrak{t}),\dots,\mathcal{D}_{0^+}^{\delta_n;\psi}u(\mathfrak{t})\big),\\ &u(0)=0,\qquad u(1)=p\mathcal{I}_{0^+}^{\mu;\psi}k_1(\xi,u(\xi))+q\mathcal{I}_{0^+}^{\nu;\psi}k_2(\eta,u(\eta)),\end{aligned}$$

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where

$$0 \leqslant \mathfrak{t} \leqslant 1, \qquad 1 < \varrho < 2, \qquad 0 < \delta_1 < \delta_2 < \cdots < \delta_n < 1, \qquad \varrho > \delta_n + 1,$$

and

$$\zeta : [0,1] \times \mathbb{R}^{n+1} \to \mathbb{R}, \qquad k_j : [0,1] \times \mathbb{R} \to \mathbb{R}, \qquad j = 1, 2,$$

are continuous functions, $\mathcal{D}_{0^+}^{\varrho;\psi}, \mathcal{D}_{0^+}^{\delta_1;\psi}, \dots, \mathcal{D}_{0^+}^{\delta_n;\psi}$ are the ψ-Riemann-Liouville derivatives depending on an increasing function ψ of orders $\varrho, \delta_1, \dots, \delta_n$, respectively, and $\mathcal{I}_{0^+}^{\gamma;\psi}$ is the ψ-Riemann-Liouville integral depending on the special function ψ of order $\gamma \in \{\mu, \nu\}$ with $\mu, \nu, p, q > 0$ and $0 < \xi, \eta \leqslant 1$. Thabet et al. [34] considered and studied the existence of solutions of the following coupled system of the Caputo conformable fractional boundary value problems of the Pantograph differential equations formulated as

$$\begin{aligned} {}^{\mathcal{CC}}\mathfrak{D}_{\mathfrak{t}_0}^{\varrho,\sigma_1^*} v(\mathfrak{t}) &= \tilde{\mathcal{P}}_1(\mathfrak{t}, m(\mathfrak{t}), m(\ell\mathfrak{t})), \qquad z \in [\mathfrak{t}_0, \tilde{K}], \qquad \mathfrak{t}_0 \geqslant 0, \\ {}^{\mathcal{CC}}\mathfrak{D}_{\mathfrak{t}_0}^{\varrho,\sigma_2^*} m(\mathfrak{t}) &= \tilde{\mathcal{P}}_2(\mathfrak{t}, v(\mathfrak{t}), v(\ell\mathfrak{t})), \end{aligned}$$

via three-point-RL-conformable integral conditions

$$\begin{aligned} &v(\mathfrak{t}_0) = 0, && c_1 v(\tilde{K}) + c_2 {}^{\mathcal{RC}}\mathfrak{I}_{\mathfrak{t}_0}^{\varrho,\theta^*} v(\delta) = w_1^*, \\ &m(\mathfrak{t}_0) = 0, && c_1^* m(\tilde{K}) + c_2^* {}^{\mathcal{RC}}\mathfrak{I}_{\mathfrak{t}_0}^{\varrho,\theta^*} m(\nu) = w_2^*, \end{aligned}$$

where

$$\begin{aligned} &\varrho \in (0,1], \qquad \sigma_1^*, \sigma_2^* \in (1,2), \qquad \delta, \nu \in (\mathfrak{t}_0, \tilde{K}), \\ &c_1, c_2, c_1^*, c_2^*, w_1^*, w_2^* \in \mathbb{R}, \qquad \ell \in (0,1), \qquad \tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2 \in C([\mathfrak{t}_0, \tilde{K}] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}). \end{aligned}$$

Based on some ideas and techniques used in the above cited works, we are interested in certain criteria of existence, uniqueness and Ulam-stability of the generalized fractional boundary value problem

$$\begin{aligned} &\mathcal{D}_{0^+}^{\rho_1,\Psi}\Big(\mathcal{D}_{0^+}^{\rho_2,\Psi} + \nu\Big)\mathrm{u}(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}, \mathrm{u}(\mathfrak{t})), \qquad \mathfrak{t} \in \mathcal{O} = [0,1], \\ &\mathrm{u}(0) = 0, \qquad \mathrm{u}(1) = \mathbf{p}\mathcal{I}_{0^+}^{\rho_3,\Psi}\Phi_1\big(\eta, \mathrm{u}(\eta)\big) + \mathbf{q}\mathcal{I}_{0^+}^{\rho_4,\Psi}\Phi_2\big(\sigma, \mathrm{u}(\sigma)\big), \end{aligned} \tag{1.1}$$

where $0 < \rho_1, \rho_2 < 1$, $\rho_1 + \rho_2 > 1$, $\mathfrak{f}, \Phi_j : \mathcal{O} \times \mathbb{R} \to \mathbb{R}$, $(j = 1, 2)$ are three continuous functions, $\mathcal{D}_{0^+}^{\varrho,\Psi}$ is the Ψ-Riemann-Liouville fractional derivative depending on an increasing function Ψ of order $\varrho \in \{\rho_1, \rho_2\}$, $\mathcal{I}_{0^+}^{\delta,\Psi}$ is the Ψ-Riemann-Liouville integral which depends on the function Ψ of order $\delta \in \{\rho_3, \rho_4\}$, with $\mathbf{p}, \mathbf{q}, \rho_3, \rho_4, \nu > 0$ and $0 < \eta, \sigma < 1$.

The structure of the paper is as follows. In Section 2, we present some useful definitions, lemmas and theorems used throughout our work: the Riemann-Liouville fractional derivative of a function with respect to another function, fixed point Theorems due to Banach, Krasnoselskii and Leray-Schauder. Section 3 contains result of the existence and uniqueness with an illustrative example. We finish our paper by Section 4 where we study two problems on existence using Krasnoselskii fixed point theorem and Leray-Schauder nonlinear alternative.

2. Basic notions of fractional calculus

Before starting the proofs of our main results, we should remind the notion of fractional derivatives of a function with respect to another function as well as its essential properties. To this end, in the following and throughout this section, $\varrho > 0$ is a real constant number, $\mathcal{O} = [\mathbf{a}, \mathbf{b}]$ is a finite or infinite interval, $\mathbf{x}$ an integrable function and $\Psi \in C^n(\mathcal{O})$ is an increasing function such that $\Psi'(\mathbf{t}) \neq 0$ for each $\mathbf{t} \in \mathcal{O}$.

The Ψ-Riemann-Liouville fractional integral of order ϱ of the function $\mathbf{x}$ is defined as

$$\mathcal{I}_{\mathbf{a}^+}^{\varrho,\Psi}\mathbf{x}(\mathbf{t}) = \frac{1}{\Gamma(\varrho)}\int_{\mathbf{a}}^{\mathbf{t}} \Psi'(\mathbf{s})\big(\Psi(\mathbf{t}) - \Psi(\mathbf{s})\big)^{\varrho-1}\mathbf{x}(\mathbf{s})\,\mathbf{ds},$$

and the Ψ-Riemann-Liouville fractional derivative of order ϱ of the function $\mathbf{x}$ is defined as

$$\begin{aligned}\mathcal{D}_{\mathbf{a}^+}^{\varrho,\Psi}\mathbf{x}(\mathbf{t}) =&\Big(\frac{1}{\Psi'(\mathbf{t})}\frac{\mathbf{d}}{\mathbf{dt}}\Big)^n\mathcal{I}_{\mathbf{a}^+}^{n-\varrho,\Psi}\mathbf{x}(\mathbf{t})\\ =&\frac{1}{\Gamma(n-\varrho)}\Big(\frac{1}{\Psi'(\mathbf{t})}\frac{\mathbf{d}}{\mathbf{dt}}\Big)^n\int_{\mathbf{a}}^{\mathbf{t}} \Psi'(\mathbf{s})\big(\Psi(\mathbf{t}) - \Psi(\mathbf{s})\big)^{n-\varrho-1}\mathbf{x}(\mathbf{s})\,\mathbf{ds},\end{aligned}$$

where $n = [\varrho] + 1$. In particular, if we choose $\Psi(\mathbf{t}) = \mathbf{t}$, $\Psi(\mathbf{t}) = \ln\mathbf{t}$, $\Psi(\mathbf{t}) = \mathbf{t}^\nu$, we find respectively the well-known fractional operators of Riemann-Liouville, Hadamard and Erdélyi-Kober type.

The semigroup property holds for fractional integrals, in other words, for $\rho, \nu > 0$, we have

$$\mathcal{I}_{\mathbf{a}^+}^{\rho,\Psi}\mathcal{I}_{\mathbf{a}^+}^{\nu,\Psi}\mathbf{x}(\mathbf{t}) = \mathcal{I}_{\mathbf{a}^+}^{\rho+\nu,\Psi}\mathbf{x}(\mathbf{t}).$$

Definition 2.1. [22] *Let $\Psi \in C^n[\mathbf{a},\mathbf{b}]$ such that $\Psi'(\mathbf{t}) \neq 0$ for all $\mathbf{t} \in [\mathbf{a},\mathbf{b}]$. Then we define*

$$AC^{n;\Psi}[\mathbf{a},\mathbf{b}] = \left\{\theta : [\mathbf{a},\mathbf{b}] \to \mathbb{R},\quad \theta^{[n-1]} = \left(\frac{1}{\Psi'(\mathbf{t})}\frac{d}{d\mathbf{t}}\right)^{n-1}\theta,\ \theta^{[n-1]} \in AC[\mathbf{a},\mathbf{b}]\right\}.$$

Lemma 2.1. [22] *Let $\varrho > 0$ and $\nu > 0$. If $u(s) = \big[\Psi(s) - \Psi(\mathbf{a})\big]^{\nu-1}$, then*

$$\Big(\mathcal{D}_{\mathbf{a}^+}^{\varrho;\Psi}u(s)\Big)(\mathbf{t}) = \frac{\Gamma(\nu)}{\Gamma(\nu-\varrho)}\big[\Psi(\mathbf{t}) - \Psi(\mathbf{a})\big]^{\nu-\varrho-1}, \tag{2.1}$$

and

$$\Big(\mathcal{I}_{\mathbf{a}^+}^{\varrho;\Psi}u(s)\Big)(\mathbf{t}) = \frac{\Gamma(\nu)}{\Gamma(\nu+\varrho)}\big[\Psi(\mathbf{t}) - \Psi(\mathbf{a})\big]^{\varrho+\nu-1}. \tag{2.2}$$

As a particular case of (2.1) and (2.2) we have respectively the following expressions

$$\Big(\mathcal{D}_{0^+}^{\varrho;\mathbf{t}^\sigma}s^{\sigma(\nu-1)}\Big)(\mathbf{t}) = \frac{\Gamma(\nu)}{\Gamma(\nu-\varrho)}\mathbf{t}^{\sigma(\nu-\varrho-1)},$$

and

$$\Big(\mathcal{I}_{0^+}^{\varrho;\mathbf{t}^\sigma}s^{\sigma(\nu-1)}\Big)(\mathbf{t}) = \frac{\Gamma(\nu)}{\Gamma(\nu+\varrho)}\mathbf{t}^{\sigma(\varrho+\nu-1)}.$$

Lemma 2.2. [20] *Let $\varrho > n$ with $n \in \mathbb{N}$. Then*

$$\left(\frac{1}{\Psi'(\mathbf{t})}.\frac{d}{d\mathbf{t}}\right)^n\mathcal{I}_{\mathbf{a}^+}^{\varrho;\Psi}\Phi(\mathbf{t}) = \mathcal{I}_{\mathbf{a}^+}^{\varrho-n;\Psi}\Phi(\mathbf{t}).$$

Lemma 2.3. [20] *Let $\varrho > \nu$, $n-1 < \nu < n$, $n \in \mathbb{N}$. Then*

$$\mathcal{D}_{\mathbf{a}^+}^{\nu;\Psi}\mathcal{I}_{\mathbf{a}^+}^{\varrho;\Psi}\Phi(\mathbf{t}) = \mathcal{I}_{\mathbf{a}^+}^{\varrho-\nu;\Psi}\Phi(\mathbf{t}).$$

In particular,

$$\mathcal{D}_{\mathbf{a}^+}^{\varrho;\Psi}\mathcal{I}_{\mathbf{a}^+}^{\varrho;\Psi}\Phi(\mathbf{t}) = \Phi(\mathbf{t}).$$

Lemma 2.4. [20] *Let* $\varrho > 0$, $n = [\varrho] + 1$, $\Phi \in L[\mathbf{a}, \mathbf{b}]$ *and* $\mathcal{I}_{\mathbf{a}^+}^{\varrho;\Psi}\Phi \in AC^{n;\Psi}[\mathbf{a}, \mathbf{b}]$. *Then*

$$\left(\mathcal{I}_{\mathbf{a}^+}^{\varrho;\Psi}\mathcal{D}_{\mathbf{a}^+}^{\varrho;\Psi}\right)\Phi(\mathbf{t}) = \Phi(\mathbf{t}) - \sum_{j=1}^{n} \frac{\mathcal{I}_{\mathbf{a}^+}^{j-\varrho;\Psi}\Phi(\mathbf{a})}{\Gamma(\varrho - j + 1)}\big(\Psi(\mathbf{t}) - \Psi(\mathbf{a})\big)^{\varrho - j}.$$

In the special case $0 < \varrho < 1$ *we have*

$$\left(\mathcal{I}_{\mathbf{a}^+}^{\varrho;\Psi}\mathcal{D}_{\mathbf{a}^+}^{\varrho;\Psi}\right)\Phi(\mathbf{t}) = \Phi(\mathbf{t}) - \frac{\mathcal{I}_{\mathbf{a}^+}^{1-\varrho;\Psi}\Phi(\mathbf{a})}{\Gamma(\varrho)}\big(\Psi(\mathbf{t}) - \Psi(\mathbf{a})\big)^{\varrho - 1}.$$

Lemma 2.5. [20] *Let* $\varrho > 0$ *and* $\mathcal{D}_{\mathbf{a}^+}^{\varrho;\Psi}\Phi \in AC^{n;\Psi}[\mathbf{a}, \mathbf{b}] \cap L^1[\mathbf{a}, \mathbf{b}]$, *then*

$$\mathcal{I}_{\mathbf{a}^+}^{\varrho;\Psi}\mathcal{D}_{\mathbf{a}^+}^{\varrho;\Psi}\Phi(\mathbf{t}) = \Phi(\mathbf{t}) + k_1\big(\Psi(\mathbf{t}) - \Psi(\mathbf{a})\big)^{\varrho-1} + k_2\big(\Psi(\mathbf{t}) - \Psi(\mathbf{a})\big)^{\varrho-2} + \cdots + k_n\big(\Psi(\mathbf{t}) - \Psi(\mathbf{a})\big)^{\varrho-n},$$

where $k_1, \ldots, k_n \in \mathbb{R}$ *and* $n = [\varrho] + 1$.

Theorem 2.1. [16]*(Banach fixed point theorem). Let* $(\mathfrak{E}, d)$ *be a complete metric space and* $T : \mathfrak{E} \to \mathfrak{E}$ *be a contraction mapping. Then* T *has unique fixed point in* $\mathfrak{E}$.

Theorem 2.2. [16] *(Krasnoselskii fixed point theorem). Let* $\mathcal{M}$ *be a nonempty, bounded, closed and convex subset of a Banach space* $\mathfrak{E}$. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two operators such that:*

- $\mathcal{A}x + \mathcal{B}y \in M$ *whenever* $x, y \in \mathcal{M}$.
- $\mathcal{A}$ *is compact and continuous.*
- $\mathcal{B}$ *is a contraction mapping.*

Then there exists $z \in \mathcal{M}$ *such that* $z = \mathcal{A}z + \mathcal{B}z$.

Theorem 2.3. [16] *(Leray-Schauder nonlinear alternative). Let* $\mathfrak{E}$ *be a Banach space,* $\mathcal{C}$ *a closed, convex subset of* $\mathfrak{E}$, $\mathcal{U}$ *an open subset of* $\mathcal{C}$ *and* $0 \in \mathcal{U}$.

Suppose that $T : \overline{\mathcal{U}} \to \mathcal{C}$ *is a continuous, compact map (that is,* $T(\overline{\mathcal{U}})$ *is a relatively compact subset of* $\mathcal{C}$*). Then either:*

- T *has a fixed point in* $\overline{\mathcal{U}}$, *or*
- *There are a* $x \in \partial\mathcal{U}$ *(the boundary of* $\mathcal{U}$ *in* $\mathcal{C}$*) and* $\lambda \in (0, 1)$ *with* $\lambda T(x) = x$.

Lemma 2.6. *Let*

$$0 < \rho_1, \rho_2 < 1, \qquad \rho_1 + \rho_2 > 1, \qquad \mathbf{p}, \mathbf{q}, \rho_3, \rho_4, \nu > 0, \qquad 0 < \eta, \sigma < 1$$

and $\mathfrak{f}, \Phi_j : \mathcal{O} \times \mathbb{R} \to \mathbb{R}$, $j = 1, 2$, *are three continuous functions. Then the fractional boundary value problem*

$$\begin{aligned} &\mathcal{D}_{0^+}^{\rho_1,\Psi}\left(\mathcal{D}_{0^+}^{\rho_2,\Psi} + \nu\right)\mathrm{u}(\mathrm{t}) = \mathfrak{f}(\mathrm{t}, \mathrm{u}(\mathrm{t})), \qquad \mathrm{t} \in \mathcal{O} = [0, 1], \\ &\mathrm{u}(0) = 0, \qquad \mathrm{u}(1) = \mathbf{p}\mathcal{I}_{0^+}^{\rho_3,\Psi}\Phi_1\big(\eta, \mathrm{u}(\eta)\big) + \mathbf{q}\mathcal{I}_{0^+}^{\rho_4,\Psi}\Phi_2\big(\sigma, \mathrm{u}(\sigma)\big) \end{aligned} \tag{2.3}$$

is equivalent to the following integral equation

$$
\begin{aligned}
\mathrm{u}(\mathrm{t}) = &- \frac{\nu}{\Gamma(\rho_2)} \int_0^{\mathrm{t}} \Psi'(\mathbf{s})\big(\Psi(\mathrm{t}) - \Psi(\mathbf{s})\big)^{\rho_2-1} \mathbf{u}(\mathbf{s})\, \mathbf{ds} \\
&+ \frac{1}{\Gamma(\rho_1+\rho_2)} \int_0^{\mathrm{t}} \Psi'(\mathbf{s})\big(\Psi(\mathrm{t}) - \Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1} \mathfrak{f}(\mathfrak{s}, \mathrm{u}(\mathfrak{s}))\, \mathbf{ds} \\
&- \xi_\Psi(\mathrm{t})\Bigg(- \frac{\nu}{\Gamma(\rho_2)} \int_0^1 \Psi'(\mathbf{s})\big(\Psi(1) - \Psi(\mathbf{s})\big)^{\rho_2-1} \mathbf{u}(\mathbf{s})\, \mathbf{ds} \\
&+ \frac{1}{\Gamma(\rho_1+\rho_2)} \int_0^1 \Psi'(\mathbf{s})\big(\Psi(1) - \Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1} \mathfrak{f}(\mathfrak{s}, \mathrm{u}(\mathfrak{s}))\, \mathbf{ds} \\
&- \frac{\mathbf{p}}{\Gamma(\rho_3)} \int_0^{\eta} \Psi'(\mathbf{s})\big(\Psi(\eta) - \Psi(\mathbf{s})\big)^{\rho_3-1} \Phi_1\big(\mathbf{s}, \mathrm{u}(\mathbf{s})\big)\, \mathbf{ds} \\
&- \frac{\mathbf{q}}{\Gamma(\rho_4)} \int_0^{\sigma} \Psi'(\mathbf{s})\big(\Psi(\sigma) - \Psi(\mathbf{s})\big)^{\rho_4-1} \Phi_2\big(\mathbf{s}, \mathrm{u}(\mathbf{s})\big)\, \mathbf{ds}\Bigg),
\end{aligned}
\tag{2.4}
$$

where

$$\xi_\Psi(\mathrm{t}) = -\Big(\frac{\Psi(\mathrm{t}) - \Psi(0)}{\Psi(1) - \Psi(0)}\Big)^{\rho_1+\rho_2-1}.$$

Proof. From (2.3) and by applying the integral operator $\mathcal{I}_{0^+}^{\rho_1,\Psi}$ to both sides we get

$$\mathcal{I}_{0^+}^{\rho_1,\Psi}\mathcal{D}_{0^+}^{\rho_1,\Psi}\Big(\mathcal{D}_{0^+}^{\rho_2,\Psi} + \nu\Big)\mathrm{u}(\mathrm{t}) = \mathcal{I}_{0^+}^{\rho_1,\Psi}\mathfrak{f}(\mathrm{t}, \mathrm{u}(\mathrm{t})). \tag{2.5}$$

Then by using Lemma 2.5 we see that

$$\mathcal{D}_{0^+}^{\rho_2,\Psi}\mathrm{u}(\mathrm{t}) = -\nu\mathrm{u}(\mathrm{t}) + \mathcal{I}_{0^+}^{\rho_1,\Psi}\mathfrak{f}(\mathrm{t}, \mathrm{u}(\mathrm{t})) - c_1\big(\Psi(\mathrm{t}) - \Psi(0)\big)^{\rho_1-1}. \tag{2.6}$$

Applying the Ψ-Riemann-Liouville fractional operator $\mathcal{I}_{0^+}^{\rho_2,\Psi}$ to both sides of (2.6), we find:

$$\mathcal{I}_{0^+}^{\rho_2,\Psi}\mathcal{D}_{0^+}^{\rho_2,\Psi}\mathrm{u}(\mathrm{t}) = -\nu\mathcal{I}_{0^+}^{\rho_2,\Psi}\mathrm{u}(\mathrm{t}) + \mathcal{I}_{0^+}^{\rho_1+\rho_2,\Psi}\mathfrak{f}(\mathrm{t}, \mathrm{u}(\mathrm{t})) - c_1\mathcal{I}_{0^+}^{\rho_2,\Psi}\big(\Psi(\mathrm{t}) - \Psi(0)\big)^{\rho_1-1}.$$

In view of Lemma 2.5, the last equation implies

$$
\begin{aligned}
\mathrm{u}(\mathrm{t}) = &- \nu\mathcal{I}_{0^+}^{\rho_2,\Psi}\mathrm{u}(\mathrm{t}) + \mathcal{I}_{0^+}^{\rho_1+\rho_2,\Psi}\mathfrak{f}(\mathrm{t}, \mathrm{u}(\mathrm{t})) - c_1\frac{\Gamma(\rho_1)}{\Gamma(\rho_1+\rho_2)}\big(\Psi(\mathrm{t}) - \Psi(0)\big)^{\rho_1+\rho_2-1} \\
&- c_2\big(\Psi(\mathrm{t}) - \Psi(0)\big)^{\rho_2-1}.
\end{aligned}
\tag{2.7}
$$

From the first boundary condition, we get $c_2 = 0$ and then (2.7) becomes

$$\mathrm{u}(\mathrm{t}) = -\nu\mathcal{I}_{0^+}^{\rho_2,\Psi}\mathrm{u}(\mathrm{t}) + \mathcal{I}_{0^+}^{\rho_1+\rho_2,\Psi}\mathfrak{f}(\mathrm{t}, \mathrm{u}(\mathrm{t})) - c_1\frac{\Gamma(\rho_1)}{\Gamma(\rho_1+\rho_2)}\big(\Psi(\mathrm{t}) - \Psi(0)\big)^{\rho_1+\rho_2-1}. \tag{2.8}$$

Employing then the second boundary condition, we have on the one hand

$$\begin{aligned}\mathrm{u}(1) =& -\nu\mathcal{I}_{0^+}^{\rho_2,\Psi}\mathrm{u}(\mathfrak{t})\big|_{\mathfrak{t}=1} + \mathcal{I}_{0^+}^{\rho_1+\rho_2,\Psi}\mathfrak{f}(\mathfrak{t},\mathrm{u}(\mathfrak{t}))\big|_{\mathfrak{t}=1} - c_1\frac{\Gamma(\rho_1)}{\Gamma(\rho_1+\rho_2)}\big(\Psi(1)-\Psi(0)\big)^{\rho_1+\rho_2-1} \\ =& -\frac{\nu}{\Gamma(\rho_2)}\int_0^1 \Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_2-1}\mathrm{u}(\mathbf{s})\,\mathbf{ds} \\ &+\frac{1}{\Gamma(\rho_1+\rho_2)}\int_0^1 \Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds} \\ &- c_1\frac{\Gamma(\rho_1)}{\Gamma(\rho_1+\rho_2)}\big(\Psi(1)-\Psi(0)\big)^{\rho_1+\rho_2-1},\end{aligned} \tag{2.9}$$

and on the other hand

$$\begin{aligned}\mathrm{u}(1) =& \mathbf{p}\mathcal{I}_{0^+}^{\rho_3,\Psi}\Phi_1\big(\eta,\mathrm{u}(\eta)\big) + \mathbf{q}\mathcal{I}_{0^+}^{\rho_4,\Psi}\Phi_2\big(\sigma,\mathrm{u}(\sigma)\big) \\ =& \frac{\mathbf{p}}{\Gamma(\rho_3)}\int_0^\eta \Psi'(\mathbf{s})\big(\Psi(\eta)-\Psi(\mathbf{s})\big)^{\rho_3-1}\Phi_1\big(\mathbf{s},\mathrm{u}(\mathbf{s})\big)\,\mathbf{ds} \\ &+\frac{\mathbf{q}}{\Gamma(\rho_4)}\int_0^\sigma \Psi'(\mathbf{s})\big(\Psi(\sigma)-\Psi(\mathbf{s})\big)^{\rho_4-1}\Phi_2\big(\mathbf{s},\mathrm{u}(\mathbf{s})\big)\,\mathbf{ds}.\end{aligned} \tag{2.10}$$

Using then expressions (2.9), (2.10), after some computations we get

$$\begin{aligned}c_1 =& \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\big(\Psi(1)-\Psi(0)\big)^{\rho_1+\rho_2-1}}\Bigg(-\frac{\nu}{\Gamma(\rho_2)}\int_0^1 \Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_2-1}\mathbf{u}(\mathbf{s})\,\mathbf{ds} \\ &+\frac{1}{\Gamma(\rho_1+\rho_2)}\int_0^1 \Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds} \\ &-\frac{\mathbf{p}}{\Gamma(\rho_3)}\int_0^\eta \Psi'(\mathbf{s})\big(\Psi(\eta)-\Psi(\mathbf{s})\big)^{\rho_3-1}\Phi_1\big(\mathbf{s},\mathrm{u}(\mathbf{s})\big)\,\mathbf{ds} \\ &-\frac{\mathbf{q}}{\Gamma(\rho_4)}\int_0^\sigma \Psi'(\mathbf{s})\big(\Psi(\sigma)-\Psi(\mathbf{s})\big)^{\rho_4-1}\Phi_2\big(\mathbf{s},\mathrm{u}(\mathbf{s})\big)\,\mathbf{ds}\Bigg).\end{aligned}$$

Replacing c_1 by its value in (2.8), we obtain immediately integral equation (2.4).

For the reverse case, we just write $\mathrm{u}(\mathfrak{t})$ in the form

$$\begin{aligned}\mathrm{u}(\mathfrak{t}) =& -\nu\mathcal{I}_{0^+}^{\rho_2,\Psi}\mathrm{u}(\mathfrak{t}) + \mathcal{I}_{0^+}^{\rho_1+\rho_2,\Psi}\mathfrak{f}(\mathfrak{t},\mathrm{u}(\mathfrak{t})) - \xi_\Psi(\mathfrak{t})\Big[-\nu\mathcal{I}_{0^+}^{\rho_2,\Psi}\mathrm{u}(1) \\ &+\mathcal{I}_{0^+}^{\rho_1+\rho_2,\Psi}\mathfrak{f}(1,\mathrm{u}(1)) - \mathbf{p}\mathcal{I}_{0^+}^{\rho_3,\Psi}\Phi_1\big(\eta,\mathrm{u}(\eta)\big) - \mathbf{q}\mathcal{I}_{0^+}^{\rho_4,\Psi}\Phi_2\big(\sigma,\mathrm{u}(\sigma)\big)\Big].\end{aligned} \tag{2.11}$$

Applying the Ψ-Riemann-Liouville fractional operator $\mathcal{D}_{0^+}^{\rho_2,\Psi}$ to both sides of (2.11) and using Lemmas 2.1 and 2.3, after some manipulations we get

$$\begin{aligned}\Big(\mathcal{D}_{0^+}^{\rho_2,\Psi}+\nu\Big)\mathrm{u}(\mathfrak{t}) =&\mathcal{I}_{0^+}^{\rho_1,\Psi}\mathfrak{f}(\mathfrak{t},\mathrm{u}(\mathfrak{t}))-\frac{\Gamma(\rho_1+\rho_2)\big(\Psi(\mathfrak{t})-\Psi(0)\big)^{\rho_1-1}}{\big(\Psi(1)-\Psi(0)\big)^{\rho_1+\rho_2-1}}\Big(-\nu\mathcal{I}_{0^+}^{\rho_2,\Psi}\mathrm{u}(1)\\ &+\mathcal{I}_{0^+}^{\rho_1+\rho_2,\Psi}\mathfrak{f}(1,\mathrm{u}(1))-\mathbf{p}\mathcal{I}_{0^+}^{\rho_3,\Psi}\Phi_1\big(\eta,\mathrm{u}(\eta)\big)-\mathbf{q}\mathcal{I}_{0^+}^{\rho_4,\Psi}\Phi_2\big(\sigma,\mathrm{u}(\sigma)\big)\Big).\end{aligned} \tag{2.12}$$

Applying the fractional derivative $\mathcal{D}_{0^+}^{\rho_1,\Psi}$ to both sides of (2.12), due to the property

$$\mathcal{D}_{0^+}^{\rho_1,\Psi}\big(\Psi(\mathfrak{t})-\Psi(0)\big)^{\rho_1-1}=0,$$

we obtain

$$\mathcal{D}_{0^+}^{\rho_1,\Psi}\Big(\mathcal{D}_{0^+}^{\rho_2,\Psi}+\nu\Big)\mathrm{u}(\mathfrak{t})=\mathfrak{f}(\mathfrak{t},\mathrm{u}(\mathfrak{t})).$$

To check boundary conditions, it is easy to confirm by (2.11) that

$$\mathrm{u}(0)=0,\quad \text{and}\quad \mathrm{u}(1)=\mathbf{p}\mathcal{I}_{0^+}^{\rho_3,\Psi}\Phi_1\big(\eta,\mathrm{u}(\eta)\big)+\mathbf{q}\mathcal{I}_{0^+}^{\rho_4,\Psi}\Phi_2\big(\sigma,\mathrm{u}(\sigma)\big).$$

Therefore, $\mathrm{u}(\mathfrak{t})$ is a solution of the problem (1.1) and this completes the proof. □

In what follows, we introduce some new notations based on Lemma 2.6. In addition, we consider the Banach space $\mathfrak{C}=\mathcal{C}\big([0,1],\mathbb{R}\big)$ equipped with the norm $\|\mathrm{u}\|=\max_{\mathfrak{t}\in[0,1]}|\mathrm{u}(\mathfrak{t})|$ and we define an operator $\mathcal{N}:\mathfrak{C}\to\mathfrak{C}$

$$\begin{aligned}(\mathcal{N}\mathrm{u})(\mathfrak{t})=&-\frac{\nu}{\Gamma(\rho_2)}\int_0^{\mathfrak{t}}\Psi'(\mathrm{s})\big(\Psi(\mathfrak{t})-\Psi(\mathrm{s})\big)^{\rho_2-1}\mathbf{u}(\mathrm{s})\,\mathbf{ds}\\ &+\frac{1}{\Gamma(\rho_1+\rho_2)}\int_0^{\mathfrak{t}}\Psi'(\mathrm{s})\big(\Psi(\mathfrak{t})-\Psi(\mathrm{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds}\\ &-\xi_\Psi(\mathfrak{t})\Big(-\frac{\nu}{\Gamma(\rho_2)}\int_0^{1}\Psi'(\mathrm{s})\big(\Psi(1)-\Psi(\mathrm{s})\big)^{\rho_2-1}\mathbf{u}(\mathrm{s})\,\mathbf{ds}\\ &+\frac{1}{\Gamma(\rho_1+\rho_2)}\int_0^{1}\Psi'(\mathrm{s})\big(\Psi(1)-\Psi(\mathrm{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds}\\ &-\frac{\mathbf{p}}{\Gamma(\rho_3)}\int_0^{\eta}\Psi'(\mathrm{s})\big(\Psi(\eta)-\Psi(\mathrm{s})\big)^{\rho_3-1}\Phi_1\big(\mathrm{s},\mathrm{u}(\mathrm{s})\big)\,\mathbf{ds}\\ &-\frac{\mathbf{q}}{\Gamma(\rho_4)}\int_0^{\sigma}\Psi'(\mathrm{s})\big(\Psi(\sigma)-\Psi(\mathrm{s})\big)^{\rho_4-1}\Phi_2\big(\mathrm{s},\mathrm{u}(\mathrm{s})\big)\,\mathbf{ds}\Big).\end{aligned} \tag{2.13}$$

The fixed point equation

$$\mathcal{N}\mathrm{u}=\mathrm{u},\quad \mathrm{u}\in\mathfrak{C},$$

is equivalent to integral equation (2.4), and the continuity of the functions $\mathfrak{f},\Phi_1,\Phi_2$ ensures that for the operator $\mathcal{N}$.

We are in position to formulate an existence and uniqueness theorem.

3. Unique solvability

Theorem 3.1. *Suppose that the following assertions hold*
$(\mathcal{H}_1)$: *There exists a real constant* $\mathcal{M}_{\mathfrak{f}} > 0$ *such that*

$$|\mathfrak{f}(\mathfrak{t},\mathrm{u}) - \mathfrak{f}(\mathfrak{t},\mathrm{v})| \leqslant \mathcal{M}_{\mathfrak{f}}|\mathrm{u}-\mathrm{v}|, \quad \mathfrak{t}\in[0,1], \quad \mathrm{u},\mathrm{v}\in\mathbb{R}.$$

$(\mathcal{H}_2)$: *There exist two* Ψ*-Riemann-Liouville integrable functions* $\omega_1,\omega_2 : [0,1]\to\mathbb{R}^+$ *such that*

$$|\Phi_i(\mathfrak{t},\mathrm{u}) - \Phi_i(\mathfrak{t},\mathrm{v})| \leqslant \omega_i(\mathfrak{t})|\mathrm{u}-\mathrm{v}|, \quad i\in\{1,2\}, \quad \mathfrak{t}\in[0,1], \quad \mathrm{u},\mathrm{v}\in\mathbb{R}.$$

$(\mathcal{H}_3)$: *A real constant* γ

$$\gamma = \frac{2\nu\big(\Psi(1)-\Psi(0)\big)^{\rho_2}}{\Gamma(\rho_2+1)} + \frac{2\mathcal{M}_{\mathfrak{f}}\big(\Psi(1)-\Psi(0)\big)^{\rho_1+\rho_2}}{\Gamma(\rho_1+\rho_2+1)} + \mathbf{p}\mathcal{I}_{0^+}^{\rho_3,\Psi}\omega_1(\eta) + \mathbf{q}\mathcal{I}_{0^+}^{\rho_4,\Psi}\omega_2(\sigma)$$

satisfies $0<\gamma<1$.
Then fractional boundary value problem (1.1) *possesses a unique solution.*

Proof. In view of the above, we know that the solvability of fractional boundary value problem (1.1) is equivalent to the solvability of integral equation (2.4). Hence, it is sufficient to show that integral equation (2.4) has only one fixed point.

First, since Ψ is an increasing function and $\rho_1+\rho_2>1$, we have $0\leqslant \xi_\Psi(\mathfrak{t})\leqslant 1$. Therefore, for each $\mathfrak{t}\in[0,1]$ we can write

$$\begin{aligned}
\big|(\mathcal{N}\mathrm{u})(\mathfrak{t}) - (\mathcal{N}\mathrm{v})(\mathfrak{t})\big| \leqslant & \frac{\nu}{\Gamma(\rho_2)}\int_0^{\mathfrak{t}} \Psi'(\mathbf{s})\big(\Psi(\mathfrak{t})-\Psi(\mathbf{s})\big)^{\rho_2-1}|\mathbf{u}(\mathbf{s})-\mathbf{v}(\mathbf{s})|\mathbf{ds} \\
& + \frac{1}{\Gamma(\rho_1+\rho_2)}\int_0^{\mathfrak{t}} \Psi'(\mathbf{s})\big(\Psi(\mathfrak{t})-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}|\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))-\mathfrak{f}(\mathfrak{s},\mathrm{v}(\mathfrak{s}))|\mathbf{ds} \\
& + \frac{\nu}{\Gamma(\rho_2)}\int_0^{1} \Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_2-1}|\mathbf{u}(\mathbf{s})-\mathbf{v}(\mathbf{s})|\mathbf{ds} \\
& + \frac{1}{\Gamma(\rho_1+\rho_2)}\int_0^{1} \Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}|\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))-\mathfrak{f}(\mathfrak{s},\mathrm{v}(\mathfrak{s}))|\mathbf{ds} \\
& + \frac{\mathbf{p}}{\Gamma(\rho_3)}\int_0^{\eta} \Psi'(\mathbf{s})\big(\Psi(\eta)-\Psi(\mathbf{s})\big)^{\rho_3-1}|\Phi_1\big(\mathbf{s},\mathrm{u}(\mathbf{s})\big)-\Phi_1\big(\mathbf{s},\mathrm{v}(\mathbf{s})\big)|\mathbf{ds} \\
& + \frac{\mathbf{q}}{\Gamma(\rho_4)}\int_0^{\sigma} \Psi'(\mathbf{s})\big(\Psi(\sigma)-\Psi(\mathbf{s})\big)^{\rho_4-1}|\Phi_2\big(\mathbf{s},\mathrm{u}(\mathbf{s})\big)-\Phi_2\big(\mathbf{s},\mathrm{v}(\mathbf{s})\big)|\mathbf{ds} \\
\leqslant & \frac{\nu\|\mathrm{u}-\mathrm{v}\|\big(\Psi(\mathfrak{t})-\Psi(\mathbf{0})\big)^{\rho_2}}{\Gamma(\rho_2+1)} + \frac{\mathcal{M}_{\mathfrak{f}}\|\mathrm{u}-\mathrm{v}\|\big(\Psi(\mathfrak{t})-\Psi(\mathbf{0})\big)^{\rho_1+\rho_2}}{\Gamma(\rho_1+\rho_2+1)} \\
& + \frac{\nu\|\mathrm{u}-\mathrm{v}\|\big(\Psi(1)-\Psi(\mathbf{0})\big)^{\rho_2}}{\Gamma(\rho_2+1)} + \frac{\mathcal{M}_{\mathfrak{f}}\|\mathrm{u}-\mathrm{v}\|\big(\Psi(1)-\Psi(\mathbf{0})\big)^{\rho_1+\rho_2}}{\Gamma(\rho_1+\rho_2+1)} \\
& + \mathbf{p}\|\mathrm{u}-\mathrm{v}\|\mathcal{I}_{0^+}^{\rho_3,\Psi}\omega_1(\eta) + \mathbf{q}\|\mathrm{u}-\mathrm{v}\|\mathcal{I}_{0^+}^{\rho_4,\Psi}\omega_2(\sigma) \\
\leqslant & \Bigg(\frac{2\nu\big(\Psi(1)-\Psi(\mathbf{0})\big)^{\rho_2}}{\Gamma(\rho_2+1)} + \frac{2\mathcal{M}_{\mathfrak{f}}\big(\Psi(1)-\Psi(\mathbf{0})\big)^{\rho_1+\rho_2}}{\Gamma(\rho_1+\rho_2+1)} + \mathbf{p}\mathcal{I}_{0^+}^{\rho_3,\Psi}\omega_1(\eta)
\end{aligned}$$

$$+ \mathbf{q}\mathcal{I}_{0^+}^{\rho_4,\Psi}\omega_2(\sigma)\Big)\|\mathrm{u}-\mathrm{v}\|.$$

Thus,

$$\big\|\mathcal{N}\mathrm{u}-\mathcal{N}\mathrm{v}\big\| \leqslant \gamma\|\mathrm{u}-\mathrm{v}\|, \quad 0<\gamma<1, \tag{3.1}$$

and $\mathcal{N}$ is a contraction. Therefore, by applying the Banach principle, we conclude that $\mathcal{N}$ has only one fixed point and this implies the existence of a unique solution for the FBVP (1.1). The proof is complete. □

Example 3.1. *Consider a fractional boundary value problem of the following form*

$$\begin{aligned} &\mathcal{D}_{0^+}^{\frac{1}{2},\mathfrak{t}^2}\Big(\mathcal{D}_{0^+}^{\frac{2}{3},\mathfrak{t}^2}+\frac{1}{4}\Big)\mathrm{u}(\mathfrak{t}) = \mathfrak{t}+\frac{1}{16}\sin(2u(\mathfrak{t})), \qquad \mathfrak{t}\in[0,1],\\ &u(0)=0,\\ &u(1)=\int_0^{\frac{1}{2}} 2s\Big(\frac{1}{4}-s^2\Big)^3\big(1+10u(s)e^s\big)\,ds+\int_0^{\frac{1}{4}} 2s\Big(\frac{1}{16}-s^2\Big)^4\big(1+\frac{1}{2}e^s\sin\big(u(s)\big)\big)\,ds. \end{aligned} \tag{3.2}$$

In this case we have

$$\Psi(\mathfrak{t})=\mathfrak{t}^2, \qquad \rho_1=\frac{1}{2}, \qquad \rho_2=\frac{2}{3}, \qquad \rho_3=4, \qquad \rho_4=5,$$

$$\nu=\frac{1}{4}, \qquad \mathbf{p}=\Gamma(4), \qquad \mathbf{q}=\Gamma(5), \qquad \eta=\frac{1}{2}, \qquad \sigma=\frac{1}{4},$$

and

$$\begin{aligned} &\mathfrak{f}(\mathfrak{t},\mathrm{u}(\mathfrak{t}))=\mathfrak{t}+\frac{1}{16}\sin(2u(\mathfrak{t})),\\ &\Phi_1(\mathfrak{t},\mathrm{u}(\mathfrak{t}))=1+10u(\mathfrak{t})e^{\mathfrak{t}},\\ &\Phi_2(\mathfrak{t},\mathrm{u}(\mathfrak{t}))=1+\frac{1}{2}e^{\mathfrak{t}}\sin\big(u(\mathfrak{t})\big). \end{aligned}$$

Then

$$\mathcal{M}_{\mathfrak{f}}=\frac{1}{8}, \qquad \omega_1(\mathfrak{t})=10e^{\mathfrak{t}}, \qquad \omega_2(\mathfrak{t})=\frac{1}{2}e^{\mathfrak{t}}.$$

Therefore, by simple computations we get

$$\begin{aligned} \gamma =&\frac{2\nu\big(\Psi(1)-\Psi(0)\big)^{\rho_2}}{\Gamma(\rho_2+1)}+\frac{2\mathcal{M}\big(\Psi(1)-\Psi(0)\big)^{\rho_1+\rho_2}}{\Gamma(\rho_1+\rho_2+1)}+\mathbf{p}\mathcal{I}_{0^+}^{\rho_3,\Psi}\omega_1(\eta)+\mathbf{q}\mathcal{I}_{0^+}^{\rho_4,\Psi}\omega_2(\sigma)\\ \approx&0.9031<1. \end{aligned}$$

Hence, by Theorem 3.1 *we conclude that the considered fractional boundary value problem is uniquely solvable.*

4. Existence results

In order to apply the Krasnoselskii fixed point theorem, it is useful to decompose the operator $\mathcal{N}$as $\mathcal{N}=\mathcal{N}_1+\mathcal{N}_2$, where,

$$(\mathcal{N}_1\mathrm{u})(\mathfrak{t}) =\frac{1}{\Gamma(\rho_1+\rho_2)}\int_0^{\mathfrak{t}} \Psi'(\mathbf{s})\big(\Psi(\mathfrak{t})-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds}$$

$$- \frac{\xi_\Psi(\mathrm{t})}{\Gamma(\rho_1+\rho_2)} \int_0^1 \Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1} \mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds},$$

and

$$\begin{aligned}(\mathcal{N}_2\mathrm{u})(\mathrm{t}) = &- \frac{\nu}{\Gamma(\rho_2)} \int_0^{\mathrm{t}} \Psi'(\mathbf{s})\big(\Psi(\mathrm{t})-\Psi(\mathbf{s})\big)^{\rho_2-1}\mathbf{u}(\mathbf{s})\,\mathbf{ds}\\ &+ \xi_\Psi(\mathrm{t})\Bigg(\frac{\nu}{\Gamma(\rho_2)} \int_0^1 \Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_2-1}\mathbf{u}(\mathbf{s})\,\mathbf{ds}\\ &+ \frac{\mathbf{p}}{\Gamma(\rho_3)} \int_0^{\eta} \Psi'(\mathbf{s})\big(\Psi(\eta)-\Psi(\mathbf{s})\big)^{\rho_3-1}\Phi_1\big(\mathbf{s},\mathrm{u}(\mathbf{s})\big)\,\mathbf{ds}\\ &+ \frac{\mathbf{q}}{\Gamma(\rho_4)} \int_0^{\sigma} \Psi'(\mathbf{s})\big(\Psi(\sigma)-\Psi(\mathbf{s})\big)^{\rho_4-1}\Phi_2\big(\mathbf{s},\mathrm{u}(\mathbf{s})\big)\,\mathbf{ds}\Bigg).\end{aligned}$$

To simplify the futher calculations, we use the following parameters:

$$\Omega_1 = \frac{2\big(\Psi(1)-\Psi(0)\big)^{\rho_1+\rho_2}}{\Gamma(\rho_1+\rho_2+1)}, \qquad \Omega_2 = \frac{2\nu\big(\Psi(1)-\Psi(0)\big)^{\rho_2}}{\Gamma(\rho_2+1)}$$

$$\Omega_3 = \frac{\mathbf{p}\big(\Psi(\eta)-\Psi(0)\big)^{\rho_3}}{\Gamma(\rho_3+1)}, \qquad \Omega_4 = \frac{\mathbf{q}\big(\Psi(\sigma)-\Psi(0)\big)^{\rho_4}}{\Gamma(\rho_4+1)}.$$

The main result in this section is the following theorem.

Theorem 4.1. *Assume that Conditions* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$ *hold. In addition, let there exist three functions* $\Upsilon, \varphi_j \in \mathcal{C}\big([0,1],\mathbb{R}_+\big)$, $j \in \{1,2\}$ *such that*
$(\mathcal{H}_4)$: $|\mathfrak{f}(\mathrm{t},\mathrm{u})| \leqslant \Upsilon(\mathrm{t})$ *for all* $\mathrm{t} \in \mathcal{O}$, $\mathrm{u} \in \mathbb{R}$*;*
$(\mathcal{H}_5)$: $|\Phi_i(\mathrm{t},\mathrm{u})| \leqslant \varphi_i(\mathrm{t})$, $i \in \{1,2\}$, *for all* $\mathrm{t} \in \mathcal{O}$, $\mathrm{u} \in \mathbb{R}$.
Then boundary value problem (1.1) *is solvable if*

$$\Omega_2 + \Omega_3\|\omega_1\| + \Omega_4\|\omega_2\| < 1. \tag{4.1}$$

Proof. We begin with considering the following nonempty closed convex ball

$$\mathcal{B}_\varrho = \{\mathrm{u} \in \mathfrak{C} : \quad \|\mathrm{u}\| \leqslant \varrho\},$$

where ϱ satisfies the inequality

$$\varrho \geqslant \frac{\Omega_1\|\Upsilon\| + \Omega_3\|\varphi_1\| + \Omega_4\|\varphi_2\|}{1-\Omega_2},$$

with $\|\Upsilon\| = \sup_{\mathrm{t}\in\mathcal{O}} \Upsilon(\mathrm{t})$ and $\|\varphi_j\| = \sup_{\mathrm{t}\in\mathcal{O}} \varphi_j(\mathrm{t})$.
First step: We are going to show that $\mathcal{N}_1\mathrm{u} + \mathcal{N}_2\mathrm{v} \in \mathcal{B}_\varrho$ for every $\mathrm{u},\mathrm{v} \in \mathcal{B}_\varrho$. Let $\mathrm{u} \in \mathcal{B}_\varrho$, then we can write

$$\begin{aligned}|(\mathcal{N}_1\mathrm{u})(\mathrm{t})| =& \frac{1}{\Gamma(\rho_1+\rho_2)}\Bigg|\Bigg(\int_0^{\mathrm{t}} \Psi'(\mathrm{s})\big(\Psi(\mathrm{t})-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds}\\ &- \xi_\Psi(\mathrm{t}) \int_0^1 \Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds}\Bigg)\Bigg|\end{aligned}$$

$$\leqslant \frac{1}{\Gamma(\rho_1+\rho_2)}\Big(\int_0^{t} \Psi'(s)\big(\Psi(t)-\Psi(s)\big)^{\rho_1+\rho_2-1}|\mathfrak{f}(\mathfrak{s},u(\mathfrak{s}))|\,ds$$
$$+\xi_\Psi(t)\int_0^1 \Psi'(s)\big(\Psi(1)-\Psi(s)\big)^{\rho_1+\rho_2-1}|\mathfrak{f}(\mathfrak{s},u(\mathfrak{s}))|ds\Big)$$
$$\leqslant \frac{\|\Upsilon\|}{\Gamma(\rho_1+\rho_2+1)}\Big(\big(\Psi(t)-\Psi(0)\big)^{\rho_1+\rho_2}+\xi_\Psi(t)\big(\Psi(1)-\Psi(0)\big)^{\rho_1+\rho_2}\Big).$$

Since Ψ is an increasing function, we have

$$\begin{aligned}\|\mathcal{N}_1 u\| \leqslant& \frac{\|\Upsilon\|}{\Gamma(\rho_1+\rho_2+1)}\sup_{t\in\mathcal{O}}\Big(\big(\Psi(t)-\Psi(0)\big)^{\rho_1+\rho_2}+\xi_\Psi(t)\big(\Psi(1)-\Psi(0)\big)^{\rho_1+\rho_2}\Big)\\ \leqslant& \frac{2\|\Upsilon\|}{\Gamma(\rho_1+\rho_2+1)}\big(\Psi(1)-\Psi(0)\big)^{\rho_1+\rho_2}=\Omega_1\|\Upsilon\|.\end{aligned}\tag{4.2}$$

In the same way, for $v\in\mathcal{B}_\varrho$ we get

$$\begin{aligned}|(\mathcal{N}_2 v)(t)| =&\Big|-\frac{\nu}{\Gamma(\rho_2)}\int_0^{t}\Psi'(s)\big(\Psi(t)-\Psi(s)\big)^{\rho_2-1}v(s)\,ds\\
&+\xi_\Psi(t)\Big(\frac{\nu}{\Gamma(\rho_2)}\int_0^1\Psi'(s)\big(\Psi(1)-\Psi(s)\big)^{\rho_2-1}v(s)\,ds\\
&+\frac{p}{\Gamma(\rho_3)}\int_0^{\eta}\Psi'(s)\big(\Psi(\eta)-\Psi(s)\big)^{\rho_3-1}\Phi_1\big(s,v(s)\big)\,ds\\
&+\frac{q}{\Gamma(\rho_4)}\int_0^{\sigma}\Psi'(s)\big(\Psi(\sigma)-\Psi(s)\big)^{\rho_4-1}\Phi_2\big(s,v(s)\big)\,ds\Big)\Big|\\
\leqslant&\frac{\nu}{\Gamma(\rho_2)}\Big(\int_0^{t}\Psi'(s)\big(\Psi(t)-\Psi(s)\big)^{\rho_2-1}|v(s)|ds\\
&+\xi_\Psi(t)\int_0^1\Psi'(s)\big(\Psi(1)-\Psi(s)\big)^{\rho_2-1}|v(s)|ds\Big)\\
&+\xi_\Psi(t)\Big(\frac{p}{\Gamma(\rho_3)}\int_0^{\eta}\Psi'(s)\big(\Psi(\eta)-\Psi(s)\big)^{\rho_3-1}|\Phi_1\big(s,v(s)\big)|\,ds\\
&+\frac{q}{\Gamma(\rho_4)}\int_0^{\sigma}\Psi'(s)\big(\Psi(\sigma)-\Psi(s)\big)^{\rho_4-1}|\Phi_2\big(s,v(s)\big)|ds\Big).\end{aligned}$$

Then

$$\begin{aligned}\|\mathcal{N}_2 v\| \leqslant&\frac{2\nu\|v\|}{\Gamma(\rho_2+1)}\big(\Psi(1)-\Psi(0)\big)^{\rho_2}+\frac{p\|\varphi_1\|}{\Gamma(\rho_3+1)}\big(\Psi(\eta)-\Psi(0)\big)^{\rho_3}\\
&+\frac{q\|\varphi_2\|}{\Gamma(\rho_4+1)}\big(\Psi(\sigma)-\Psi(0)\big)^{\rho_4}\\
\leqslant&\Omega_2\varrho+\Omega_3\|\varphi_1\|+\Omega_4\|\varphi_2\|.\end{aligned}\tag{4.3}$$

By combining (4.2) and (4.3) we get

$$\begin{aligned}\|\mathcal{N}_1 u+\mathcal{N}_2 v\| \leqslant&\|\mathcal{N}_1 u\|+\|\mathcal{N}_2 v\| \\ &\leqslant \Omega_2\varrho+\Omega_1\|\Upsilon\|+\Omega_3\|\varphi_1\|+\Omega_4\|\varphi_2\| \leqslant \varrho,\end{aligned}$$

which shows that $\mathcal{N}_1 u+\mathcal{N}_2 v \in \mathcal{B}_\varrho$ for every $u, v \in \mathcal{B}_\varrho$.

Second step: We are going to prove that $\mathcal{N}_2$ is a contraction on $\mathcal{B}_\varrho$. For every $u, v \in \mathcal{B}_\varrho$ we have

$$\begin{aligned}|(\mathcal{N}_2 u)(t)-(\mathcal{N}_2 v)(t)| =&\Bigg|-\frac{\nu}{\Gamma(\rho_2)}\int_0^t \Psi'(s)\big(\Psi(t)-\Psi(s)\big)^{\rho_2-1}\big(u(s)-v(s)\big)\,ds \\ &+\frac{\nu\xi_\Psi(t)}{\Gamma(\rho_2)}\int_0^1 \Psi'(s)\big(\Psi(1)-\Psi(s)\big)^{\rho_2-1}\big(u(s)-v(s)\big)\,ds \\ &+\frac{p\xi_\Psi(t)}{\Gamma(\rho_3)}\int_0^\eta \Psi'(s)\big(\Psi(\eta)-\Psi(s)\big)^{\rho_3-1}\big(\Phi_1(s,u(s))-\Phi_1(s,v(s))\big)\,ds \\ &+\frac{q\xi_\Psi(t)}{\Gamma(\rho_4)}\int_0^\sigma \Psi'(s)\big(\Psi(\sigma)-\Psi(s)\big)^{\rho_4-1}\big(\Phi_2(s,u(s))-\Phi_2(s,v(s))\big)\,ds\Bigg| \\ \leqslant&\frac{\nu}{\Gamma(\rho_2)}\int_0^t \Psi'(s)\big(\Psi(t)-\Psi(s)\big)^{\rho_2-1}\big|u(s)-v(s)\big|\,ds \\ &+\frac{\nu\xi_\Psi(t)}{\Gamma(\rho_2)}\int_0^1 \Psi'(s)\big(\Psi(1)-\Psi(s)\big)^{\rho_2-1}\big|u(s)-v(s)\big|\,ds \\ &+\frac{p\xi_\Psi(t)}{\Gamma(\rho_3)}\int_0^\eta \Psi'(s)\big(\Psi(\eta)-\Psi(s)\big)^{\rho_3-1}\big|\Phi_1(s,u(s))-\Phi_1(s,v(s))\big|\,ds \\ &+\frac{q\xi_\Psi(t)}{\Gamma(\rho_4)}\int_0^\sigma \Psi'(s)\big(\Psi(\sigma)-\Psi(s)\big)^{\rho_4-1}\big|\Phi_2(s,u(s))-\Phi_2(s,v(s))\big|\,ds, \\ \leqslant&\frac{\nu\|u-v\|}{\Gamma(\rho_2+1)}\big(\Psi(t)-\Psi(0)\big)^{\rho_2}+\frac{\nu\xi_\Psi(t)\|u-v\|}{\Gamma(\rho_2+1)}\big(\Psi(1)-\Psi(0)\big)^{\rho_2} \\ &+\frac{p\xi_\Psi(t)\|\omega_1\|\|u-v\|}{\Gamma(\rho_3+1)}\big(\Psi(\eta)-\Psi(0)\big)^{\rho_3} \\ &+\frac{q\xi_\Psi(t)\|\omega_2\|\|u-v\|}{\Gamma(\rho_4+1)}\big(\Psi(\sigma)-\Psi(0)\big)^{\rho_4}.\end{aligned}$$

Then

$$\begin{aligned}\|\mathcal{N}_2 u-\mathcal{N}_2 v\| \leqslant&\Bigg(\frac{2\nu}{\Gamma(\rho_2+1)}\big(\Psi(1)-\Psi(0)\big)^{\rho_2}+\frac{p\|\omega_1\|}{\Gamma(\rho_3+1)}\big(\Psi(\eta)-\Psi(0)\big)^{\rho_3} \\ &+\frac{q\|\omega_2\|}{\Gamma(\rho_4+1)}\big(\Psi(\sigma)-\Psi(0)\big)^{\rho_4}\Bigg)\|u-v\| \\ =&\big(\Omega_2+\Omega_3\|\omega_1\|+\Omega_4\|\omega_2\|\big)\|u-v\|.\end{aligned}$$

Hence, it follows from condition (4.1) that $\mathcal{N}_2$ is a contraction.

Third step: We are going to show that $\mathcal{N}_1$ is compact and continuous.

i) It follows from the definition of the operator $\mathcal{N}_1$ that the continuity of $\mathfrak{f}$ implies that of $\mathcal{N}_1$.

ii) The uniform boundedness of the operator $\mathcal{N}_1$ on B_ϱ is due to expression (4.2), where we have shown that for any $\mathrm{u} \in B_\varrho$,

$$\|\mathcal{N}_1\mathrm{u}\| \leqslant \Omega_1\|\Upsilon\|.$$

iii) In view of $(\mathcal{H}_4)$, for all $\mathrm{u} \in B_\varrho$ and for each $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathcal{O}$ such that $\mathfrak{t}_1 < \mathfrak{t}_2$ we have:

$$\begin{aligned}
|(\mathcal{N}_1\mathrm{u})(\mathfrak{t}_2) - (\mathcal{N}_1\mathrm{u})(\mathfrak{t}_1)| =& \frac{1}{\Gamma(\rho_1+\rho_2)}\Bigg| \int_0^{\mathfrak{t}_2} \Psi'(\mathbf{s})\big(\Psi(\mathfrak{t}_2) - \Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds} \\
&- \int_0^{\mathfrak{t}_1} \Psi'(\mathbf{s})\big(\Psi(\mathfrak{t}_1) - \Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds} \\
&- \xi_\Psi(\mathfrak{t}_2)\int_0^1 \Psi'(\mathbf{s})\big(\Psi(1) - \Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds} \\
&+ \xi_\Psi(\mathfrak{t}_1)\int_0^1 \Psi'(\mathbf{s})\big(\Psi(1) - \Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds}\Bigg| \\
=& \frac{1}{\Gamma(\rho_1+\rho_2)}\Bigg| \int_0^{\mathfrak{t}_1} \Psi'(\mathbf{s})\big(\Psi(\mathfrak{t}_2) - \Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds} \\
&+ \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \Psi'(\mathbf{s})\big(\Psi(\mathfrak{t}_2) - \Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds} \\
&- \int_0^{\mathfrak{t}_1} \Psi'(\mathbf{s})\big(\Psi(\mathfrak{t}_1) - \Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds} \\
&- \Big(\xi_\Psi(\mathfrak{t}_2) - \xi_\Psi(\mathfrak{t}_1)\Big)\int_0^1 \Psi'(\mathbf{s})\big(\Psi(1) - \Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds}\Bigg| \\
=& \frac{1}{\Gamma(\rho_1+\rho_2)}\Bigg| \int_0^{\mathfrak{t}_1} \Psi'(\mathbf{s})\Big(\big(\Psi(\mathfrak{t}_2) - \Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1} \\
&\qquad - \big(\Psi(\mathfrak{t}_1) - \Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\Big)\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds} \\
&+ \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \Psi'(\mathbf{s})\big(\Psi(\mathfrak{t}_2) - \Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds} \\
&- \Big(\xi_\Psi(\mathfrak{t}_2) - \xi_\Psi(\mathfrak{t}_1)\Big)\int_0^1 \Psi'(\mathbf{s})\big(\Psi(1) - \Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds}\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{\Gamma(\rho_1+\rho_2)}\Bigg(\int_0^{\mathfrak{t}_1} \Psi'(\mathbf{s})\Big(\big(\Psi(\mathfrak{t}_2)-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1} \\
&\qquad\qquad - \big(\Psi(\mathfrak{t}_1)-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\Big)|\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))|\,\mathbf{ds} \\
&\quad + \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \Psi'(\mathbf{s})\big(\Psi(\mathfrak{t}_2)-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}|\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))|\,\mathbf{ds} \\
&\quad + \Big(\xi_\Psi(\mathfrak{t}_2)-\xi_\Psi(\mathfrak{t}_1)\Big)\int_0^1 \Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}|\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))|\mathbf{ds}\Bigg) \\
&\leqslant \frac{\|\Upsilon\|}{\Gamma(\rho_1+\rho_2+1)}\Big(\big(\Psi(\mathfrak{t}_2)-\Psi(0)\big)^{\rho_1+\rho_2} - \big(\Psi(\mathfrak{t}_1)-\Psi(0)\big)^{\rho_1+\rho_2} \\
&\quad + \Big(\xi_\Psi(\mathfrak{t}_2)-\xi_\Psi(\mathfrak{t}_1)\Big)\big(\Psi(1)-\Psi(0)\big)^{\rho_1+\rho_2}\Big).
\end{aligned}
$$

We observe that the right hand side in the above inequality is independent on u and tends to zero as $\mathfrak{t}_2 \to \mathfrak{t}_1$. Hence, $\mathcal{N}_1$ is equicontinuous. By Arzelà-Ascoli theorem this yields the compactness of the operator $\mathcal{N}_1$. As a consequence of Krasnoselskii fixed point theorem, problem (1.1) has at least one solution. The proof is complete. □

Theorem 4.2. *Assume that there exist three functions $\chi, \chi_1, \chi_2 \in \mathcal{C}\big([0,1], \mathbb{R}_+\big)$ and three non-decreasing functions $\Xi, \Xi_1, \Xi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that*
$(\mathcal{H}_6)$: $|\mathfrak{f}(\mathfrak{t},\mathrm{u})| \leqslant \chi(\mathfrak{t})\Xi\big(\|\mathrm{u}\|\big)$, *for all* $\mathfrak{t} \in \mathcal{O}$, $\mathrm{u} \in \mathbb{R}$.
$(\mathcal{H}_7)$: $|\Phi_i(\mathfrak{t},\mathrm{u})| \leqslant \chi_i(\mathfrak{t})\Xi_i\big(\|\mathrm{u}\|\big)$, $i \in \{1,2\}$, *for all* $\mathfrak{t} \in \mathcal{O}$, $\mathrm{u} \in \mathbb{R}$.
$(\mathcal{H}_8)$: *There exists a real constant $\overline{\omega} > 0$ satisfying*

$$
\frac{\overline{\omega}(1-\Omega_2)}{\|\chi\|\Xi(\overline{\omega}|)\Omega_1 + \|\chi_1\|\Xi_1(\overline{\omega})\Omega_3 + \|\chi_2\|\Xi_2(\overline{\omega})\Omega_4} > 1.
$$

Then fractional boundary value problem (1.1) *possesses a solution.*

Proof. **First step:** We are going to show that $\mathcal{N}$ maps bounded sets into bounded sets in $\mathfrak{C}$. For a positive number $\mathbf{r}$ we define the bounded sets $\mathcal{B}_{\mathbf{r}}$ of $\mathfrak{C}$ as follows:

$$
\mathcal{B}_{\mathbf{r}} = \{\mathrm{u} \in \mathfrak{C} : \quad \|\mathrm{u}\| \leqslant \mathbf{r}\}.
$$

For $\mathrm{u} \in \mathcal{B}_{\mathbf{r}}$, we have

$$
\begin{aligned}
|(\mathcal{N}\mathrm{u})(\mathfrak{t})| = \Bigg| &- \frac{\nu}{\Gamma(\rho_2)}\int_0^{\mathfrak{t}} \Psi'(\mathbf{s})\big(\Psi(\mathfrak{t})-\Psi(\mathbf{s})\big)^{\rho_2-1}\mathbf{u}(\mathbf{s})\,\mathbf{ds} \\
&+ \frac{1}{\Gamma(\rho_1+\rho_2)}\int_0^{\mathfrak{t}} \Psi'(\mathbf{s})\big(\Psi(\mathfrak{t})-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds} \\
&- \xi_\Psi(\mathfrak{t})\Bigg(- \frac{\nu}{\Gamma(\rho_2)}\int_0^1 \Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_2-1}\mathbf{u}(\mathbf{s})\,\mathbf{ds} \\
&+ \frac{1}{\Gamma(\rho_1+\rho_2)}\int_0^1 \Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\mathbf{p}}{\Gamma(\rho_3)} \int_0^\eta \Psi'(\mathbf{s})\big(\Psi(\eta) - \Psi(\mathbf{s})\big)^{\rho_3-1} \Phi_1\big(\mathbf{s}, \mathrm{u}(\mathbf{s})\big)\, \mathbf{ds} \\
&- \frac{\mathbf{q}}{\Gamma(\rho_4)} \int_0^\sigma \Psi'(\mathbf{s})\big(\Psi(\sigma) - \Psi(\mathbf{s})\big)^{\rho_4-1} \Phi_2\big(\mathbf{s}, \mathrm{u}(\mathbf{s})\big)\, \mathbf{ds}\Big)\Bigg| \\
\leqslant & \frac{\nu}{\Gamma(\rho_2)} \int_0^{\mathfrak{t}} \Psi'(\mathbf{s})\big(\Psi(\mathfrak{t}) - \Psi(\mathbf{s})\big)^{\rho_2-1} |\mathbf{u}(\mathbf{s})|\, \mathbf{ds} \\
&+ \frac{1}{\Gamma(\rho_1+\rho_2)} \int_0^{\mathfrak{t}} \Psi'(\mathbf{s})\big(\Psi(\mathfrak{t}) - \Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1} |\mathfrak{f}(\mathfrak{s}, \mathrm{u}(\mathfrak{s}))|\, \mathbf{ds} \\
&+ \xi_\Psi(\mathfrak{t})\Bigg(- \frac{\nu}{\Gamma(\rho_2)} \int_0^1 \Psi'(\mathbf{s})\big(\Psi(1) - \Psi(\mathbf{s})\big)^{\rho_2-1} |\mathbf{u}(\mathbf{s})|\, \mathbf{ds} \\
&+ \frac{1}{\Gamma(\rho_1+\rho_2)} \int_0^1 \Psi'(\mathbf{s})\big(\Psi(1) - \Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1} |\mathfrak{f}(\mathfrak{s}, \mathrm{u}(\mathfrak{s}))|\, \mathbf{ds} \\
&+ \frac{\mathbf{p}}{\Gamma(\rho_3)} \int_0^\eta \Psi'(\mathbf{s})\big(\Psi(\eta) - \Psi(\mathbf{s})\big)^{\rho_3-1} |\Phi_1\big(\mathbf{s}, \mathrm{u}(\mathbf{s})\big)|\, \mathbf{ds} \\
&+ \frac{\mathbf{q}}{\Gamma(\rho_4)} \int_0^\sigma \Psi'(\mathbf{s})\big(\Psi(\sigma) - \Psi(\mathbf{s})\big)^{\rho_4-1} |\Phi_2\big(\mathbf{s}, \mathrm{u}(\mathbf{s})\big)|\, \mathbf{ds}\Bigg) \\
\leqslant & \frac{\nu\|\mathrm{u}\|}{\Gamma(\rho_2+1)}\big(\Psi(\mathfrak{t}) - \Psi(0)\big)^{\rho_2} + \frac{\|\chi\|\varXi\big(\|\mathrm{u}\|\big)}{\Gamma(\rho_1+\rho_2+1)}\big(\Psi(\mathfrak{t}) - \Psi(0)\big)^{\rho_1+\rho_2} \\
&+ \xi_\Psi(\mathfrak{t})\bigg(\frac{\nu\|\mathrm{u}\|}{\Gamma(\rho_2+1)}\big(\Psi(1) - \Psi(0)\big)^{\rho_2} + \frac{\|\chi\|\varXi\big(\|\mathrm{u}\|\big)}{\Gamma(\rho_1+\rho_2+1)}\big(\Psi(1) - \Psi(0)\big)^{\rho_1+\rho_2} \\
&+ \frac{\mathbf{p}\|\chi_1\|\varXi_1\big(\|\mathrm{u}\|\big)}{\Gamma(\rho_3+1)}\big(\Psi(\eta) - \Psi(0)\big)^{\rho_3} + \frac{\mathbf{q}\|\chi_2\|\varXi_2\big(\|\mathrm{u}\|\big)}{\Gamma(\rho_4+1)}\big(\Psi(\sigma) - \Psi(0)\big)^{\rho_4}\bigg).
\end{aligned}
$$

Then

$$
\begin{aligned}
\|\mathcal{N}\mathrm{u}\| \leqslant & \frac{2\nu\|\mathrm{u}\|}{\Gamma(\rho_2+1)}\big(\Psi(1) - \Psi(0)\big)^{\rho_2} + \frac{2\|\chi\|\varXi\big(\|\mathrm{u}\|\big)}{\Gamma(\rho_1+\rho_2+1)}\big(\Psi(1) - \Psi(0)\big)^{\rho_1+\rho_2} \\
&+ \frac{\mathbf{p}\|\chi_1\|\varXi_1\big(\|\mathrm{u}\|\big)}{\Gamma(\rho_3+1)}\big(\Psi(\eta) - \Psi(0)\big)^{\rho_3} + \frac{\mathbf{q}\|\chi_2\|\varXi_2\big(\|\mathrm{u}\|\big)}{\Gamma(\rho_4+1)}\big(\Psi(\sigma) - \Psi(0)\big)^{\rho_4} \\
\leqslant & \mathbf{r}\Omega_2 + \|\chi\|\varXi(\mathbf{r})\Omega_1 + \|\chi_1\|\varXi_1(\mathbf{r})\Omega_3 + \|\chi_2\|\varXi_2(\mathbf{r})\Omega_4.
\end{aligned}
\tag{4.4}
$$

Second step: We are going to show that $\mathcal{N}$ maps bounded sets into equicontinuous sets of $\mathfrak{C}$. Based on Assumptions $(\mathcal{H}_6) - (\mathcal{H}_7)$, for all $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathcal{O}$ with $\mathfrak{t}_1 < \mathfrak{t}_2$ and each $\mathrm{u} \in \mathcal{B}_{\mathbf{r}}$ we can write

$$
|(\mathcal{N}\mathrm{u})(\mathfrak{t}_2) - (\mathcal{N}\mathrm{u})(\mathfrak{t}_1)| = \Bigg| - \frac{\nu}{\Gamma(\rho_2)}\Bigg(\int_0^{\mathfrak{t}_2} \Psi'(\mathbf{s})\big(\Psi(\mathfrak{t}_2) - \Psi(\mathbf{s})\big)^{\rho_2-1} \mathbf{u}(\mathbf{s})\, \mathbf{ds}
$$

$$
\begin{aligned}
&- \int_0^{t_1} \Psi'(s)\big(\Psi(t_1) - \Psi(s)\big)^{\rho_2-1}\mathbf{u}(s)\,\mathbf{ds}\Bigg) \\
&+ \frac{1}{\Gamma(\rho_1+\rho_2)}\Bigg(\int_0^{t_2} \Psi'(s)\big(\Psi(t_2) - \Psi(s)\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds} \\
&- \int_0^{t_1} \Psi'(s)\big(\Psi(t_1) - \Psi(s)\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds}\Bigg) \\
&- \big(\xi_\Psi(t_2) - \xi_\Psi(t_1)\big)\frac{\nu}{\Gamma(\rho_2)}\int_0^1 \Psi'(s)\big(\Psi(1) - \Psi(s)\big)^{\rho_2-1}\mathbf{u}(s)\,\mathbf{ds}\Bigg| \\
=&\Bigg| - \frac{\nu}{\Gamma(\rho_2)}\Bigg(\int_0^{t_1} \Psi'(s)\Big(\big(\Psi(t_2) - \Psi(s)\big)^{\rho_2-1} - \big(\Psi(t_1) - \Psi(s)\big)^{\rho_2-1}\Big)\mathbf{u}(s)\,\mathbf{ds} \\
&+ \int_{t_1}^{t_2} \Psi'(s)\big(\Psi(t_2) - \Psi(s)\big)^{\rho_2-1}\mathbf{u}(s)\,\mathbf{ds}\Bigg) \\
&+ \frac{1}{\Gamma(\rho_1+\rho_2)}\Bigg(\int_0^{t_1} \Psi'(s)\Big(\big(\Psi(t_2) - \Psi(s)\big)^{\rho_1+\rho_2-1} \\
&\qquad - \big(\Psi(t_1) - \Psi(s)\big)^{\rho_1+\rho_2-1}\Big)\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds} \\
&+ \int_{t_1}^{t_2} \Psi'(s)\big(\Psi(t_2) - \Psi(s)\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\mathrm{u}(\mathfrak{s}))\,\mathbf{ds}\Bigg) \\
&- \big(\xi_\Psi(t_2) - \xi_\Psi(t_1)\big)\frac{\nu}{\Gamma(\rho_2)}\int_0^1 \Psi'(s)\big(\Psi(1) - \Psi(s)\big)^{\rho_2-1}\mathbf{u}(s)\,\mathbf{ds}\Bigg| \\
\leqslant& \frac{\nu}{\Gamma(\rho_2+1)}\Big(\big(\Psi(t_2) - \Psi(0)\big)^{\rho_2} - \big(\Psi(t_1) - \Psi(0)\big)^{\rho_2}\Big) \\
&+ \frac{\|\chi\|\varXi(\mathbf{r})}{\Gamma(\rho_1+\rho_2+1)}\Big(\big(\Psi(t_2) - \Psi(0)\big)^{\rho_1+\rho_2} - \big(\Psi(t_1) - \Psi(0)\big)^{\rho_1+\rho_2}\Big) \\
&+ \big(\xi_\Psi(t_2) - \xi_\Psi(t_1)\big)\frac{\nu\mathbf{r}}{\Gamma(\rho_2)}\big(\Psi(1) - \Psi(0)\big)^{\rho_2}.
\end{aligned}
$$

We observe that as $t_1 \to t_2$, the right-hand side in the above relations is independent of u and goes to zero uniformly. Therefore, the operator $\mathcal{N} : \mathfrak{C} \to \mathfrak{C}$ is equicontinuous and thus the operator $\mathcal{N}$ is completely continuous.

We are going to confirm that the set of all solutions to the equation $\lambda\mathcal{N}\mathrm{u} = \mathrm{u}$ is bounded for $\lambda \in (0,1)$.

By computations similar to ones used in the first step we get

$$\|\mathrm{u}\| \leqslant \|\mathrm{u}\|\Omega_2 + \|\chi\|\varXi(\|\mathrm{u}\|)\Omega_1 + \|\chi_1\|\varXi_1(\|\mathrm{u}\|)\Omega_3 + \|\chi_2\|\varXi_2(\|\mathrm{u}\|)\Omega_4,$$

which leads to

$$\frac{\|\mathrm{u}\|(1-\Omega_2)}{\|\chi\|\varXi(\|\mathrm{u}\|)\Omega_1 + \|\chi_1\|\varXi_1(\|\mathrm{u}\|)\Omega_3 + \|\chi_2\|\varXi_2(\|\mathrm{u}\|)\Omega_4} \leqslant 1.$$

According to Assumption $(\mathcal{H}_8)$, there exists a real constant $\overline{\omega} > 0$ such that $\|\mathrm{u}\| \neq \overline{\omega}$ and

$$\frac{\overline{\omega}(1-\Omega_2)}{\|\chi\|\Xi(\overline{\omega}|)\Omega_1 + \|\chi_1\|\Xi_1(\overline{\omega})\Omega_3 + \|\chi_2\|\Xi_2(\overline{\omega})\Omega_4} > 1.$$

We introduce the set

$$\mathcal{U} = \{\mathrm{u} \in \mathfrak{C} : \quad \|\mathrm{u}\| < M\},$$

and note that $\mathcal{N} : \overline{\mathcal{U}} \to \mathcal{C}$ is continuous and completely continuous. Then the choice of u implies that there is no $\mathrm{u} \in \partial\mathcal{U}$ such that $\lambda\mathcal{N}(u) = u$ for some $\lambda \in (0,1)$. Then by Leray-Schauder nonlinear alternative we conclude that $\mathcal{N}$ has a fixed point $u \in \overline{\mathcal{U}}$ which corresponds to a solution of fractional boundary value problem (1.1). The proof is complete. □

5. Hyers-Ulam stability outcomes

Fractional differential equations play a very important role in mathematical analysis and especially in the modeling of physical phenomenons and these have been widely studied from different sides. Among these, the stability analysis in the Hyers-Ulam sense is a very important aspect which attracted the attention of many authors [18], [35], [28]. Based on the definition of Hyers-Ulam stability, then this notion was modified into more general types [4], [5]. In this section, we will adapt some sufficient conditions to obtain stability results of the Hyers-Ulam type for our main problem.

Definition 5.1. [18], [28] *Let us consider a Banach space $\mathbb{S}$ and an operator $\mathbf{K} : \mathbb{S} \to \mathbb{S}$. The operator equation*

$$\mathbf{K}\mathrm{u} = \mathrm{u}, \tag{5.1}$$

is said to be Hyers-Ulam stable if the inequality

$$\big|\mathrm{u}(\mathfrak{t}) - \mathbf{K}\mathrm{u}(\mathfrak{t})\big| \leqslant \epsilon,$$

which holds for all $\mathfrak{t} \in \mathcal{O} = [0,1]$, implies that there exists a constant $\pi_{\mathbf{K}} > 0$ such that for each $\mathrm{u} \in \mathcal{C}\big(\mathcal{O}, \mathbb{R}\big)$ satisfying (5.1) one can find a unique solution $\widehat{\mathrm{u}} \in \mathcal{C}\big(\mathcal{O}, \mathbb{R}\big)$ of operator equation (5.1) provided that for each $\mathfrak{t} \in \mathcal{O}$ we have

$$\big|\mathrm{u}(\mathfrak{t}) - \widehat{\mathrm{u}}(\mathfrak{t})\big| \leqslant \pi_{\mathbf{K}}\epsilon.$$

Definition 5.2. [18], [28] *Consider the operator $\mathcal{N} : \mathfrak{C} \to \mathfrak{C}$. We say that the operator equation*

$$\mathrm{u}(\mathfrak{t}) = \mathcal{N}\mathrm{u}(\mathfrak{t}), \tag{5.2}$$

is Hyers-Ulam stable if for the inequality

$$\big|\mathrm{u}(\mathfrak{t}) - \mathcal{N}\mathrm{u}(\mathfrak{t})\big| \leqslant \epsilon, \quad \mathfrak{t} \in \mathcal{O}, \tag{5.3}$$

we can find a constant $\pi_{\mathcal{N}}$ such that for each u satisfying (5.2) there exists a unique solution $\widehat{\mathrm{u}}$ of the operator equation (5.2) provided that

$$\big|\mathrm{u}(\mathfrak{t}) - \widehat{\mathrm{u}}(\mathfrak{t})\big| \leqslant \pi_{\mathcal{N}}\epsilon$$

for each $\mathfrak{t} \in \mathcal{O}$

Theorem 5.1. *Assume that $\vartheta \in \mathcal{C}\big(\mathcal{O}, \mathbb{R}\big)$ is a solution of the inequality (5.3) satisfying the conditions*
(i) $|\vartheta(\mathfrak{t})| \leqslant \epsilon$ for all $\mathfrak{t} \in \mathcal{O}$;
(ii) $\mathcal{D}^{\rho_1,\Psi}_{0^+}\big(\mathcal{D}^{\rho_2,\Psi}_{0^+} + \nu\big)\mathrm{u}(\mathfrak{t}) - \mathfrak{f}(\mathfrak{t}, \mathrm{u}(\mathfrak{t})) + \vartheta(\mathfrak{t}) = 0$ for all $\mathfrak{t} \in \mathcal{O}$.
Then

$$\big|\vartheta(\mathfrak{t}) - \mathcal{N}\vartheta(\mathfrak{t})\big| \leqslant \ell\epsilon, \quad \mathfrak{t} \in \mathcal{O},$$

where

$$\ell = \frac{2}{\Gamma(\rho_1+\rho_2+1)}\big(\Psi(1)-\Psi(0)\big)^{\rho_1+\rho_2},$$

and $\mathcal{N}$ is the operator given by (2.13).

Proof. According to Condition (ii), for each $\mathfrak{t}\in\mathcal{O}$ we have

$$\begin{aligned}&\mathcal{D}_{0^+}^{\rho_1,\Psi}\Big(\mathcal{D}_{0^+}^{\rho_2,\Psi}+\nu\Big)\vartheta(\mathfrak{t})-\mathfrak{f}(\mathfrak{t},\vartheta(\mathfrak{t}))+\phi(\mathfrak{t})=0,\\ &\vartheta(0)=0,\qquad \vartheta(1)=\mathbf{p}\mathcal{I}_{0^+}^{\rho_3,\Psi}\Phi_1\big(\eta,\vartheta(\eta)\big)+\mathbf{q}\mathcal{I}_{0^+}^{\rho_4,\Psi}\Phi_2\big(\sigma,\vartheta(\sigma)\big).\end{aligned}\tag{5.4}$$

In view of Lemma 2.6, the solution of fractional boundary value problem (5.4) can be expressed as

$$\begin{aligned}\vartheta(\mathfrak{t})=&-\frac{\nu}{\Gamma(\rho_2)}\int_0^{\mathfrak{t}}\Psi'(\mathbf{s})\big(\Psi(\mathfrak{t})-\Psi(\mathbf{s})\big)^{\rho_2-1}\vartheta(\mathbf{s})\,\mathbf{ds}\\ &+\frac{1}{\Gamma(\rho_1+\rho_2)}\int_0^{\mathfrak{t}}\Psi'(\mathbf{s})\big(\Psi(\mathfrak{t})-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\vartheta(\mathfrak{s}))\,\mathbf{ds}\\ &+\frac{1}{\Gamma(\rho_1+\rho_2)}\int_0^{\mathfrak{t}}\Psi'(\mathbf{s})\big(\Psi(\mathfrak{t})-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\phi(\mathfrak{s})\,\mathbf{ds}\\ &-\xi_\Psi(\mathfrak{t})\Big(-\frac{\nu}{\Gamma(\rho_2)}\int_0^{1}\Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_2-1}\vartheta(\mathbf{s})\,\mathbf{ds}\\ &+\frac{1}{\Gamma(\rho_1+\rho_2)}\int_0^{1}\Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\vartheta(\mathfrak{s}))\,\mathbf{ds}\\ &+\frac{1}{\Gamma(\rho_1+\rho_2)}\int_0^{1}\Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\phi(\mathfrak{s})\,\mathbf{ds}\\ &-\frac{\mathbf{p}}{\Gamma(\rho_3)}\int_0^{\eta}\Psi'(\mathbf{s})\big(\Psi(\eta)-\Psi(\mathbf{s})\big)^{\rho_3-1}\Phi_1\big(\mathbf{s},\vartheta(\mathbf{s})\big)\,\mathbf{ds}\\ &-\frac{\mathbf{q}}{\Gamma(\rho_4)}\int_0^{\sigma}\Psi'(\mathbf{s})\big(\Psi(\sigma)-\Psi(\mathbf{s})\big)^{\rho_4-1}\Phi_2\big(\mathbf{s},\vartheta(\mathbf{s})\big)\,\mathbf{ds}\Big).\end{aligned}$$

Since $\mathfrak{t}\in\mathcal{O}$, the above relations imply

$$\begin{aligned}\Bigg|\vartheta(\mathfrak{t})-\Big(&-\frac{\nu}{\Gamma(\rho_2)}\int_0^{\mathfrak{t}}\Psi'(\mathbf{s})\big(\Psi(\mathfrak{t})-\Psi(\mathbf{s})\big)^{\rho_2-1}\vartheta(\mathbf{s})\,\mathbf{ds}\\ &+\frac{1}{\Gamma(\rho_1+\rho_2)}\int_0^{\mathfrak{t}}\Psi'(\mathbf{s})\big(\Psi(\mathfrak{t})-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\vartheta(\mathfrak{s}))\,\mathbf{ds}\\ &-\xi_\Psi(\mathfrak{t})\Big(-\frac{\nu}{\Gamma(\rho_2)}\int_0^{1}\Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_2-1}\vartheta(\mathbf{s})\,\mathbf{ds}\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\rho_1+\rho_2)}\int_0^1 \Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\vartheta(\mathfrak{s}))\,\mathbf{ds}\\
&-\frac{\mathbf{p}}{\Gamma(\rho_3)}\int_0^\eta \Psi'(\mathbf{s})\big(\Psi(\eta)-\Psi(\mathbf{s})\big)^{\rho_3-1}\Phi_1\big(\mathbf{s},\vartheta(\mathbf{s})\big)\,\mathbf{ds}\\
&-\frac{\mathbf{q}}{\Gamma(\rho_4)}\int_0^\sigma \Psi'(\mathbf{s})\big(\Psi(\sigma)-\Psi(\mathbf{s})\big)^{\rho_4-1}\Phi_2\big(\mathbf{s},\vartheta(\mathbf{s})\big)\,\mathbf{ds}\Big)\Big)\Big|\\
=&\Big|\frac{1}{\Gamma(\rho_1+\rho_2)}\int_0^{\mathfrak{t}} \Psi'(\mathbf{s})\big(\Psi(\mathfrak{t})-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\phi(\mathfrak{s})\,\mathbf{ds}\\
&-\frac{\xi_\Psi(\mathfrak{t})}{\Gamma(\rho_1+\rho_2)}\int_0^1 \Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\phi(\mathfrak{s})\,\mathbf{ds}\Big|\\
\leqslant&\frac{1}{\Gamma(\rho_1+\rho_2)}\int_0^{\mathfrak{t}} \Psi'(\mathbf{s})\big(\Psi(\mathfrak{t})-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\big|\phi(\mathfrak{s})\big|\,\mathbf{ds}\\
&-\frac{\xi_\Psi(\mathfrak{t})}{\Gamma(\rho_1+\rho_2)}\int_0^1 \Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\big|\phi(\mathfrak{s})\big|\,\mathbf{ds}\\
\leqslant&\frac{2}{\Gamma(\rho_1+\rho_2+1)}\big(\Psi(1)-\Psi(0)\big)^{\rho_1+\rho_2}\epsilon,
\end{aligned}
$$

which can also be written

$$
\big|\vartheta(\mathfrak{t})-\mathcal{N}\vartheta(\mathfrak{t})\big|\leqslant \ell.\epsilon,\quad \mathfrak{t}\in\mathcal{O}.
$$

This completes the proof. □

Theorem 5.2. *Let Assumptions* $(\mathcal{H}_1)-(\mathcal{H}_3)$ *hold. Then the solution of fractional boundary value problem* (1.1) *is Hyers-Ulam stable.*

Proof. Let $\vartheta\in\mathcal{C}\big(\mathcal{O},\mathbb{R}\big)$ be an arbitrary solution of the following inequality

$$
\Big|\mathcal{D}_{0^+}^{\rho_1,\Psi}\Big(\mathcal{D}_{0^+}^{\rho_2,\Psi}+\nu\Big)\mathrm{u}(\mathfrak{t})-\mathfrak{f}(\mathfrak{t},\mathrm{u}(\mathfrak{t}))\Big|\leqslant\epsilon,\qquad \mathfrak{t}\in\mathcal{O},
$$

and let $\widehat{\vartheta}\in\mathcal{C}\big(\mathcal{O},\mathbb{R}\big)$ be the unique solution of the problem

$$
\begin{aligned}
&\mathcal{D}_{0^+}^{\rho_1,\Psi}\Big(\mathcal{D}_{0^+}^{\rho_2,\Psi}+\nu\Big)\widehat{\vartheta}(\mathfrak{t})=\mathfrak{f}(\mathfrak{t},\widehat{\vartheta}(\mathfrak{t})),\qquad \mathfrak{t}\in\mathcal{O},\\
&\widehat{\vartheta}(0)=0,\qquad \widehat{\vartheta}(1)=\mathbf{p}\mathcal{I}_{0^+}^{\rho_3,\Psi}\Phi_1\big(\eta,\widehat{\vartheta}(\eta)\big)+\mathbf{q}\mathcal{I}_{0^+}^{\rho_4,\Psi}\Phi_2\big(\sigma,\widehat{\vartheta}(\sigma)\big).
\end{aligned}\tag{5.5}
$$

Thanks to Lemma 2.6, the solution of fractional boundary value problem (1.1) can be expressed as

$$\begin{aligned}\widehat{\vartheta}(\mathfrak{t}) = & -\frac{\nu}{\Gamma(\rho_2)}\int_0^{\mathfrak{t}} \Psi'(\mathbf{s})\big(\Psi(\mathfrak{t})-\Psi(\mathbf{s})\big)^{\rho_2-1}\widehat{\vartheta}(\mathbf{s})\,\mathbf{ds} \\ & + \frac{1}{\Gamma(\rho_1+\rho_2)}\int_0^{\mathfrak{t}} \Psi'(\mathbf{s})\big(\Psi(\mathfrak{t})-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\widehat{\vartheta}(\mathfrak{s}))\,\mathbf{ds} \\ & - \xi_\Psi(\mathfrak{t})\Bigg(-\frac{\nu}{\Gamma(\rho_2)}\int_0^{1} \Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_2-1}\widehat{\vartheta}(\mathbf{s})\,\mathbf{ds} \\ & + \frac{1}{\Gamma(\rho_1+\rho_2)}\int_0^{1} \Psi'(\mathbf{s})\big(\Psi(1)-\Psi(\mathbf{s})\big)^{\rho_1+\rho_2-1}\mathfrak{f}(\mathfrak{s},\widehat{\vartheta}(\mathfrak{s}))\,\mathbf{ds} \\ & - \frac{\mathbf{p}}{\Gamma(\rho_3)}\int_0^{\eta} \Psi'(\mathbf{s})\big(\Psi(\eta)-\Psi(\mathbf{s})\big)^{\rho_3-1}\Phi_1\big(\mathbf{s},\widehat{\vartheta}(\mathbf{s})\big)\,\mathbf{ds} \\ & - \frac{\mathbf{q}}{\Gamma(\rho_4)}\int_0^{\sigma} \Psi'(\mathbf{s})\big(\Psi(\sigma)-\Psi(\mathbf{s})\big)^{\rho_4-1}\Phi_2\big(\mathbf{s},\widehat{\vartheta}(\mathbf{s})\big)\,\mathbf{ds}\Bigg).\end{aligned} \tag{5.6}$$

Based on (5.6), we can write

$$\begin{aligned}\big|\vartheta(\mathbf{t})-\widehat{\vartheta}(\mathbf{t})\big| &= \big|\vartheta(\mathbf{t})-\mathcal{N}\widehat{\vartheta}(\mathbf{t})\big| \\ &= \big|\vartheta(\mathbf{t})-\mathcal{N}\vartheta(\mathbf{t})+\mathcal{N}\vartheta(\mathbf{t})-\mathcal{N}\widehat{\vartheta}(\mathbf{t})\big| \\ &\leqslant \big|\vartheta(\mathbf{t})-\mathcal{N}\vartheta(\mathbf{t})\big| + \big|\mathcal{N}\vartheta(\mathbf{t})-\mathcal{N}\widehat{\vartheta}(\mathbf{t})\big|.\end{aligned}$$

Then by (3.1) and Theorem 5.1 we arrive at

$$\big\|\vartheta-\widehat{\vartheta}\big\| \leqslant \ell\epsilon + \gamma\big\|\vartheta-\widehat{\vartheta}\big\|,$$

which gives immediately

$$\big\|\vartheta-\widehat{\vartheta}\big\| \leqslant \frac{\ell}{1-\gamma}\epsilon.$$

Therefore, the solution of fractional boundary value problem (1.1) is Hyers-Ulam stablen. The proof is complete. □

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