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ON ESTIMATES FOR ORDERS OF BEST M-TERM APPROXIMATIONS OF MULTIVARIATE FUNCTIONS IN ANISOTROPIC LORENTZ–KARAMATA SPACES

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Abstract. In the paper we consider a well-known class of weakly varying functions and by these functions we define an anisotropic Lorentz-Karamata space of 2π-periodic functions of many variables. Particular cases of these spaces are anisotropic Lorentz-Zygmund and Lorentz spaces. In the anisotropic Lorentz-Karamata space we define an analogue of Nikolskii-Besov space. The main aim of the paper is to find sharp orders of best M-term trigonometric approximation of functions from Nikolskii-Besov space by the norm of another anisotropic Lorentz-Karamata space. In the paper we establish order sharp two-sided estimates of best M-term trigonometric approximations for the functions from the Nikolskii-Besov space in the anisotropic Lorentz-Karamata space in various metrics. In order to prove an upper bound for M-term approximations, we employ an idea of the greedy algorithms proposed by V.N. Temlyakov and we modify it for the anisotropic Lorentz-Karamata space.

Keywords: Lorentz-Karamata space, Nikolskii-Besov space, M–term approximation.

Mathematics Subject Classification: 41A10, 41A25, 42A05

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ denote the set of natural, integer and real numbers respectively and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, let $\mathbb{R}^m$ be an m-dimensional Euclideatn space of points $\overline{x} = (x_1, \ldots, x_m)$ with real coordinates and

$$I^m = \{\bar{x} \in \mathbb{R}^m;\ 0 \leqslant x_j < 1;\ j = 1, \ldots, m\} = [0, 1)^m$$

be an m-dimensional cube.

We recall the definition of a non-decreasing permutation of a function.

Definition 1.1. *Let f be a Lebesgue measurable function of one variable on* $[0, 1)$. *A distribution function for $|f|$ is defined as the Lebesgue measure, see, for instance,* [1, Ch. 2, Sec. 2]*:*

$$\mu_f(y) := \mu\{x \in [0, 1) :\ |f(x)| > y\}, \qquad 0 \leqslant y < \infty.$$

Two non-negative measurable functions f and g are called equimeasurable if their distribution functions are equal, see, for instance, [1, Ch. 2, Sec. 2].

Definition 1.2. *A non-decreasing permutation of a function f of one variable is non-decreasing on* $[0, 1)$ *function $f^*(t)$ equimeasurable with the function* $|f(x)|$*, see, for instance,* [1, Ch. 2, Sec. 2].

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A non-decreasing permutation f^* of a function f of one variable on $[0,1)$ is defined by the formula,

$$f^*(t) := \inf\{y > 0 : \ \mu_f(y) \leqslant t\}, \qquad t \in [0,1),$$

see, for instance, [1, Ch. 2, Sec. 2].

Now we recall the definition of a non-decreasing permutation of a function of m variables.

Definition 1.3 (see [2]–[4]). *Let $f(x_1,\dots,x_m)$ be a Lebesgue measurable function of m variables on $I^m = [0,1)^m$. A non-decreasing permutation of a function $|f(x_1,\dots,x_m)|$ in the the first variable is the function $f^{*_1}(t_1,x_2,\dots,x_m)$ equimeasurable on I^m, non-decreasing in t_1 and such that the functions $|f(x_1,\dots,x_m)|$ and $f^{*_1}(t_1,x_2,\dots,x_m)$ are equimeasurable as functions of one variable for almost all fixed $x_2,\dots,x_m$.*

In the same way, considering a non-decreasing permutation of the function $f^{*_1}(t_1,x_2,\dots,x_m)$ in the variable x_2, for fixed $t_1,x_3,\dots,x_m$ we define a function $f^{*_1*_2}(t_1,t_2,x_3,\dots,x_m)$ equimeasurable with the function $|f(x_1,\dots,x_m)|$. Continuing this process, we define a non-decreasing permutation $f^{*_1*_2\cdots*_m}(t_1,t_2,\dots,t_m)$ equimeasurable with the function $|f(x_1,\dots,x_m)|$.

Definition 1.4 (see, for instance, [5]). *A function $\varphi(t)$ is called almost increasing on $[1,\infty)$ if there exists a fixed constant C such that $\varphi(t_1) \leqslant C\varphi(t_2)$ for $1 \leqslant t_1 < t_2 < \infty$.*

A function $\varphi(t)$ is called almost decreasing on $[1,\infty)$ if there exists a fixed constant C such that $\varphi(t_1) \geqslant C\varphi(t_2)$ for $1 \leqslant t_1 < t_2 < \infty$.

Definition 1.5. *A positive Lebesgue measurble function $b(t)$ is called slowly varying on $[1,+\infty)$ in the Karamata sense if for each $\varepsilon > 0$ the function $t^\varepsilon b(t)$ almost increases on $[1,\infty)$ and the function $t^{-\varepsilon} b(t)$ almost decreases on $[1,\infty)$, see* [6, Ch. 3, Subsec. 3.4.3], [7, Ch. 1, Sec. 1.1].

The set of such functions is denoted by $SV[1,\infty)$. For a given slowly varying function v on $[1,\infty)$ we let $V(t) = v(1/t)$ for $t \in (0,1]$.

Given numbers $p,\tau \in (1,\infty)$ and a function $v \in SV[1,\infty)$, a Lorentz-Karamata space $L_{p,V,\tau}(\mathbb{T})$ is a set of all Lebesgue measurable 2π-periodic functions f obeying

$$\|f\|_{p,V,\tau} := \left\{ \int_0^1 \big(f^*(t)\big)^\tau V^\tau(t) t^{\frac{\tau}{p}-1} dt \right\}^{\frac{1}{\tau}} < +\infty,$$

where $f^*(t)$ is a non-decreasing permutation of the function $|f(2\pi x)|$, $x \in [0,1)$, $\mathbb{T} = [0,2\pi)$, see, for instance, [6, Ch. 3, Subsec. 3.4.3].

It is known that $L_{p,V,\tau}(\mathbb{T})$ is a symmetric space, see, for instance, [6, Thm. 3.4.41].

We mention that as $V(t) = 1$, the Lorentz-Karamata space $L_{p,V,\tau}(\mathbb{T}^m)$ coincides with a known Lorentz space denoted by the symbol $L_{p,\tau}(\mathbb{T})$, $1 < p$, $\tau < \infty$ (see [8, Ch. 5, Sec. 3]) with the norm

$$\|f\|_{p,\tau} = \left(\frac{\tau}{p} \int\limits_0^1 \big(f^*(t)\big)^\tau t^{\frac{\tau}{p}-1} dt \right)^{1/\tau} < \infty,$$

for $1 < p < \infty$, $1 \leqslant \tau < +\infty$.

Given functions $v_j \in SV[1,\infty)$, $j = 1,\dots,m$, we let $V_j(t) = v_j(1/t)$, for $t \in (0,1]$, $j = 1,\dots,m$ and $\overline{V}(t) = (V_1(t),\dots,V_m(t))$, $\overline{p} = (p_1,\dots p_m)$, $\overline{\tau} = (\tau_1,\dots \tau_m)$, where $p_j,\tau_j \in (1,\infty)$. Anisotropic Lorentz-Karamata space $L^*_{\overline{p},\overline{V},\overline{\tau}}(\mathbb{T}^m)$, consists of Lebesgue measurable functions of m variable f having period 2π in each variable, for which the quantity [2]

$$\|f\|^*_{\overline{p},\overline{V},\overline{\tau}} := \| \dots \|f^{*_1,\dots,*_m}\|_{p_1,V_1,\tau_1} \dots \|_{p_m,V_m,\tau_m}$$

$$=\left[\int_0^1\left[\dots\left[\int_0^1 (f^{*_1,\dots,*_m}(t_1,\dots,t_m))^{\tau_1}\left(\prod_{j=1}^m V_j(t_j)t_j^{\frac{1}{p_j}-\frac{1}{\tau_j}}\right)^{\tau_1}dt_1\right]^{\frac{\tau_2}{\tau_1}}\dots\right]^{\frac{\tau_m}{\tau_{m-1}}}dt_m\right]^{\frac{1}{\tau_m}}$$

is finite, where $f^{*_1,\dots,*_m}(t_1,\dots,t_m)$ is a non-increasing permutation of the function $|f(2\pi\overline{x})|$ in each variable $x_j\in[0,1)$ for fixed other variables.

By $\mathring{L}^*_{\overline{p},\overline{V},\overline{\tau}}(\mathbb{T}^m)$ we denote the set of all functions $f\in L^*_{\overline{p},\overline{V},\overline{\tau}}(\mathbb{T}^m)$ such that

$$\int_0^{2\pi} f(\overline{x})\,dx_j=0,\qquad j=1,\dots,m.$$

For $\overline{p}=(p_1,\dots,p_m)$ by $l_{\overline{p}}$ we denote the space of sequences $\{a_{\overline{n}}\}_{\overline{n}\in\mathbb{Z}^m_+}$ of real numbers with the norm

$$\left\|\{a_{\overline{n}}\}_{\overline{n}\in\mathbb{Z}^m_+}\right\|_{l_{\overline{p}}(\mathbb{Z}^m_+)}=\left\{\sum_{n_m=0}^{\infty}\left[\dots\left[\sum_{n_1=0}^{\infty}|a_{\overline{n}}|^{p_1}\right]^{\frac{p_2}{p_1}}\dots\right]^{\frac{p_m}{p_{m-1}}}\right\}^{\frac{1}{p_m}}<+\infty,$$

for $1\leqslant p_j<+\infty$, $j=1,2,\dots,m$ and

$$\left\|\{a_{\overline{n}}\}\right\|_{l_\infty(\mathbb{Z}^m_+)}=\sup_{\overline{n}\in\mathbb{Z}^m_+}|a_{\overline{n}}|$$

for $p_j=\infty$, $j=1,\dots,m$.

We introduce the notations: $a_{\overline{n}}(f)$ are the Fourier coefficients of a function $f\in L_1(\mathbb{T}^m)$ over a multiple trigonometric system $\{e^{i\langle\overline{n},\overline{x}\rangle}\}$,

$$\delta_{\overline{s}}(f,\overline{x})=\sum_{\overline{n}\in\rho(\overline{s})}a_{\overline{n}}(f)e^{i\langle\overline{n},\overline{x}\rangle},\qquad\text{where}\qquad\langle\overline{y},\overline{x}\rangle=\sum_{j=1}^m y_jx_j,$$

$$\rho(\overline{s})=\left\{\overline{k}=(k_1,\dots,k_m)\in\mathbb{Z}^m:\ [2^{s_j-1}]\leqslant|k_j|<2^{s_j},\ j=1,\dots,m\right\},$$

$[y]$ is an integer part of a real number y and $s_j\in\mathbb{Z}_+$.

We consider a functional Nikolskii-Besov class

$$S^{\overline{r}}_{\overline{p},\overline{V},\overline{\tau}}B=\left\{f\in\mathring{L}^*_{\overline{p},\overline{V},\overline{\tau}}(\mathbb{T}^m):\ \|f\|^*_{\overline{p},\overline{V},\overline{\tau}}+\left\|\left\{\prod_{j=1}^m 2^{s_jr_j}\|\delta_{\overline{s}}(f)\|^*_{\overline{p},\overline{V},\overline{\tau}}\right\}_{\overline{s}\in\mathbb{Z}^m_+}\right\|_{l_{\overline{\theta}}}\leqslant 1\right\},$$

where $\overline{\theta}=(\theta_1,\dots,\theta_m)$, $\overline{r}=(r_1,\dots,r_m)$, $1\leqslant\theta_j\leqslant+\infty$, $0<r_j<+\infty$, $j=1,\dots,m$.

In the case $V_j(t)=1$ and $\tau_j=p_j=p$, $j=1,\dots,m$ the class $S^{\overline{r}}_{\overline{p},\overline{V},\overline{\tau}}B$ coincides with the known Nikolskii-Besov space in the Lebesgue space $L_p(\mathbb{T}^m)$, $1\leqslant p<\infty$, see [9]–[11].

Given a vector $\overline{\gamma}=(\gamma_1,\dots,\gamma_m)$, $\gamma_j>0$, $j=1,\dots,m$, we let

$$Q^{\overline{\gamma}}_n=\cup_{\langle\bar{s},\bar{\gamma}\rangle<n}\rho(\overline{s}),\qquad T(Q^{\overline{\gamma}}_n)=\{t(\overline{x})=\sum_{\overline{k}\in Q^{\overline{\gamma}}_n}b_{\overline{k}}e^{i\langle\overline{k},\overline{x}\rangle}\},$$

and let $E^{(\overline{\gamma})}_n(f)_{\overline{p},\overline{V},\overline{\tau}}$ be the best approximation of the function $f\in L^*_{\overline{p},\overline{V},\overline{\tau}}(\mathbb{T}^m)$ by the polynomials in the set $T(Q^{\overline{\gamma}}_n)$.

We denote

$$E^{(\overline{\gamma})}_n(S^{\overline{r}}_{\overline{p},\overline{V}^{(1)},\overline{\tau}^{(1)},\overline{\theta}}B)_{\overline{q},\overline{V}^{(2)},\overline{\tau}^{(2)}}:=\sup_{f\in S^{\overline{r}}_{\overline{p},\overline{V}^{(1)},\overline{\tau}^{(1)},\overline{\theta}}B}E^{(\overline{\gamma})}_n(f)_{\overline{q},\overline{V}^{(2)},\overline{\tau}^{(2)}},$$

where $\overline{q}=(q_1,\dots,q_m)$, $\overline{V}^{(i)}(t)=(V^{(i)}_1(t),\dots,V^{(i)}_m(t))$, $t\in(0,1]$, $\overline{\tau}^{(i)}=(\tau^{(i)}_1,\dots,\tau^{(i)}_m)$ and $1<q_j,\ \tau^{(i)}_j<\infty$, $i=1,2$, $j=1,\dots,m$.

Let X be a normed space of 2π-periodic functions of many variables. For a function $f \in X$, the best M-term trigonometric approximation is the quantity [12]

$$e_M(f)_X = \inf_{\overline{k}^j, b_j} \Big\| f - \sum_{j=1}^{M} b_j e^{i\langle \overline{k}^j, \overline{x}\rangle} \Big\|_X,$$

where $\{\overline{k}^j\}_{j=1}^M$ is the system of vectors $\overline{k}^j = (k_1^j, \ldots, k_m^j)$ with integer coordinates, b_j are real or complex numbers. If F is some functional class in the space X, we let

$$e_M(F)_X = \sup_{f \in F} e_M(f)_X.$$

Estimates for the order of the quantity $e_M(F)_Y$ in the space $Y = L_q(\mathbb{T}^m)$ for the Sobolev class $F = W_p^r$, Nikolskii-Besov class $F = S_{p,\theta}^r B$, Lizorkin-Triebel class were studied by R.S. Ismagilov [12], E.S. Belinsky [13]–[15], Yu. Makovoz [16], V.E. Maiorov [17], R.A. DeVore [18], V.N. Temlyakov [19]–[21], A.S. Romanyuk [22], M. Hansen and W. Sickel [23], S.A. Stasyuk [24], D.B. Bazarkhanov [25]–[26] and by other authors. A more detailed survey on this field can be found in [27].

The estimates of best M-term approximations of the functions from the Nikolskii-Besov class in the Lorentz and Lebesgue spaces with anisotropic norms were studied in [28]–[34].

The methods for proving upper and lower bounds for the best M-term approximations for the classes of periodic functions of many variables of mixed smoothness were developed by V.N. Temlyakov [19]–[21] and they were used by other authors, see, for instance, [22], [24]–[26], [28]–[32] as well as a bibliography in [27]. In the present paper for the proof of our main result, Theorem 3.1, we employ an idea of the method by V.N. Temlyakov with a modification for the anisotropic Lorentz-Karamata space.

The paper consists of two sections. In the first section we give some auxiliary statements needed for proving main results. In the second section we formulate and prove main results of the paper. We note that in the case

$$V_j^{(1)}(t) = V_j^{(2)}(t) = 1, \qquad t \in (0, 1]$$

and $p_j = \tau_j^{(1)} = p$, $q_j = \tau_j^{(2)} = q$, $\theta_j = \theta$ for $j = 1, \ldots, m$ and $r_1 = \ldots = r_\nu < r_{\nu+1} \leqslant \ldots \leqslant r_m$ Theorem 3.1 coincide with earlier results by V.N. Temlyakov [19, Thm. 2.2] and A.S. Romanyuk [22, Thm. 3.1] for the Lebesgue spaces and in the general case, our theorem generalizes these results for anisotropic Lorentz-Karamata spaces.

By $C(p, q, y, \ldots)$ we denote positive quantities depending on the indicated parameters in the brackets, sometimes without indicating the parameters, but independent of M-term of the best M-term approximation. The writing $A_n \asymp B_n$ means that there exist positive numbers C_1, C_2 such that $C_1 A_n \leqslant B_n \leqslant C_2 A_n$ for $n \in \mathbb{N}$. To shorten the writing, instead of inequalities $B_n \geqslant C_1 A_n$ or $B_n \leqslant C_2 A_n$ we shall often write $B \gg A$ or $B \ll A$, respectively.

2. Auxiliary statements

In this section we define one class of functions and provide several auxiliary statements.

By $SVL[1, \infty)$ we denote the set of all positive Lebesgue measurable on $[1, \infty)$ functions $v(t)$, for which the function $(\log 2t)^{-\varepsilon} v(t)$ almost decrease and the function $(\log 2t)^{\varepsilon} v(t)$ almost increases on $[1, \infty)$ for each number $\varepsilon > 0$.

It is clear that $SVL[1, \infty) \subset SV[1, \infty)$.

Example. *The function* $v(t) = (1 + \log(1 + \log t))^\alpha \in SVL[1, \infty)$, $\alpha \in \mathbb{R}$. *Hereinafter* $\log t$ *stands for the logarithm of a number* $t > 0$ *with the base* 2.

Theorem 2.1 (see [35]). *Let*

$$\overline{p}=(p_1,\dots,p_m),\qquad \overline{q}=(q_1,\dots,q_m),\qquad \overline{\tau}^{(1)}=(\tau_1^{(1)},\dots,\tau_m^{(1)}),$$
$$\overline{\tau}^{(2)}=(\tau_1^{(2)},\dots,\tau_m^{(2)}),\qquad \overline{r}=(r_1,\dots,r_m),\qquad \overline{\theta}=(\theta_1,\dots,\theta_m)$$

and

$$1<\tau_j^{(1)},\qquad \tau_j^{(2)}<+\infty,\qquad 1<p_j<q_j<+\infty,$$
$$v_j^{(i)}\in SV[1,\infty),\qquad V_j^{(i)}(t)=v_j^{(i)}\left(\frac{1}{t}\right),\qquad t\in(0,1],\quad j=1,\dots,m,$$
$$\overline{V}^{(i)}(t)=(V_1^{(i)}(t),\dots,V_m^{(i)}(t)),\quad i=1,2.$$

If $f\in L^*_{\overline{p},\overline{V}^{(1)},\overline{\tau}^{(1)}}(\mathbb{T}^m)$ *and*

$$\left\{\prod_{j=1}^m 2^{s_j(\frac{1}{p_j}-\frac{1}{q_j})}\frac{V_j^{(2)}(2^{-s_j})}{V_j^{(1)}(2^{-s_j})}\|\delta_{\overline{s}}(f)\|^*_{\overline{p},\overline{V}^{(1)},\overline{\tau}^{(1)}}\right\}_{\overline{s}\in\mathbb{Z}_+^m}\in l_{\overline{\tau}^{(2)}},$$

then $f\in L^*_{\overline{q},\overline{V}^{(2)},\overline{\tau}^{(2)}}(\mathbb{T}^m)$ *and an inequality holds:*

$$\|f\|^*_{\overline{q},\overline{V}^{(2)},\overline{\tau}^{(2)}}\leqslant C\left\|\left\{\prod_{j=1}^m 2^{s_j(\frac{1}{p_j}-\frac{1}{q_j})}\frac{V_j^{(2)}(2^{-s_j})}{V_j^{(1)}(2^{-s_j})}\|\delta_{\overline{s}}(f)\|^*_{\overline{p},\overline{V}^{(1)},\overline{\tau}^{(1)}}\right\}_{\overline{s}\in\mathbb{Z}_+^m}\right\|_{l_{\overline{\tau}^{(2)}}}.$$

Theorem 2.2 ([37]). *Let*

$$\overline{p}=(p_1,\dots,p_m),\quad \overline{q}=(q_1,\dots,q_m),\quad \overline{\tau}^{(1)}=(\tau_1^{(1)},\dots,\tau_m^{(1)}),\quad \overline{\tau}^{(2)}=(\tau_1^{(2)},\dots,\tau_m^{(2)}),$$
$$\overline{\gamma}=(\gamma_1,\dots,\gamma_m),\quad \overline{\gamma}'=(\gamma_1',\dots,\gamma_m'),\quad \overline{r}=(r_1,\dots,r_m),\quad \bar{\theta}=(\theta_1,\dots,\theta_m)$$

and

$$0<\theta_j\leqslant\infty,\qquad 1<\tau_j^{(1)},\tau_j^{(2)}<+\infty,\qquad 1<p_j<q_j<+\infty,\qquad r_j>\frac{1}{p_j}-\frac{1}{q_j},$$
$$\gamma_j=\frac{r_j+\frac{1}{q_j}-\frac{1}{p_j}}{r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}},\qquad 1\leqslant\gamma_j'\leqslant\gamma_j,\qquad j=1,\dots,m,$$
$$r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}=\min\left\{r_j+\frac{1}{q_j}-\frac{1}{p_j}:j=1,\dots,m\right\},$$
$$A=\{j:\frac{\gamma_j}{\gamma_j'}=1,\ j=1,\dots,m\},\qquad j_1=\min\{j\in A\},$$
$$v_j^{(i)}\in SV[1,\infty),\qquad V_j^{(i)}(t)=v_j^{(i)}\left(\frac{1}{t}\right),\qquad t\in(0,1],\qquad j=1,\dots,m,$$
$$\bar{V}^{(i)}(t)=(V_1^{(i)}(t),\dots,V_m^{(i)}(t)),\qquad i=1,2.$$

1. If $1\leqslant\tau_j^{(2)}<\theta_j\leqslant+\infty$ *and* $\frac{v_j^{(2)}}{v_j^{(1)}}\in SVL[1,\infty)$, $j=1,\dots,m$, *then*

$$E_n^{(\overline{\gamma}')}\left(S^{\overline{r}}_{\overline{p},\overline{V}^{(1)},\overline{\tau}^{(1)},\overline{\theta}}B\right)_{\overline{q},\overline{V}^{(2)},\overline{\tau}^{(2)}}\asymp 2^{-n\left(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}\right)}\prod_{j\in A}\frac{V_j^{(2)}(2^{-n})}{V_j^{(1)}(2^{-n})}n^{\sum_{j\in A\setminus\{j_1\}}(\frac{1}{\tau_j^{(2)}}-\frac{1}{\theta_j})},$$

for $n\in\mathbb{N}$ *such that* $n>n_0$.

2. If $1 \leqslant \theta_j \leqslant \tau_j^{(2)} < +\infty$, $j = 1, \dots, m$, *then the relation holds*

$$E_n^{(\overline{\gamma}')} \left(S^{\overline{r}}_{\overline{p}, \overline{V}^{(1)}, \overline{\tau}^{(1)}, \overline{\theta}} B \right)_{\overline{q}, \overline{V}^{(2)}, \overline{\tau}^{(2)}} \asymp 2^{-n\left(r_{j_0} + \frac{1}{q_{j_0}} - \frac{1}{p_{j_0}}\right)} \prod_{j \in A} \frac{V_j^{(2)}(2^{-n})}{V_j^{(1)}(2^{-n})}$$

in the cases $\frac{v_j^{(2)}}{v_j^{(1)}} \in SVL[1, \infty)$, $j = 1, \dots, m$ *and* $A \setminus \{j_1\} = \emptyset$ *or* $\frac{v_j^{(2)}(t)}{v_j^{(1)}(t)}$ *almost increases and* $\frac{v_j^{(2)}(t)}{v_j^{(1)}(t)} t^{-\varepsilon}$ *almost decreases on* $[1, \infty)$, $\varepsilon > 0$, $j = 1, \dots, m$ *and* $A \setminus \{j_1\} \neq \emptyset$ *for* $n \in \mathbb{N}$, *such that* $n > n_0$, *where* n_0 *is some positive number greater than* 1.

The next statement will be employed in proving lower bound of M-term approximation of the functions from the Nikolskii-Besov class in Theorem 3.1.

Theorem 2.3. *Let*

$$v_j \in SV[1, \infty), \qquad V_j(t) = v_j\left(\frac{1}{t}\right), \qquad t \in (0, 1],$$
$$1 < \tau_j, \beta_j < +\infty, \qquad 1 < p_j < \lambda_j < \infty, \qquad j = 1, \dots, m.$$

If $f \in L^*_{\overline{p}, \overline{V}, \overline{\tau}}(\mathbb{T}^m)$, *then an inequality holds:*

$$\|f\|^*_{\overline{p}, \overline{V}, \overline{\tau}} \gg \left\{ \sum_{s_m=1}^{\infty} \left[\dots \left[\sum_{s_1=1}^{\infty} \Big(\prod_{j=1}^{m} 2^{s_j(\frac{1}{\lambda_j} - \frac{1}{p_j})} V_j(2^{-s_j}) \Big)^{\tau_1} \left(\|\delta_{\overline{s}}(f)\|^*_{\overline{\lambda}, \overline{\beta}} \right)^{\tau_1} \right]^{\frac{\tau_2}{\tau_1}} \dots \right]^{\frac{\tau_m}{\tau_{m-1}}} \right\}^{\frac{1}{\tau_m}}.$$

This theorem can be probed in the same way as Theorem 4 in [33] and Theorem 4 in [37]. We consider sets

$$Y^m(\overline{\gamma}, n) = \{\overline{s} \in \mathbb{Z}^m_+ : \langle \overline{s}, \overline{\gamma} \rangle \geqslant n\}, \qquad \kappa^m(n, \overline{\gamma}) = \{\overline{s} \in \mathbb{Z}^m_+ : \langle \overline{s}, \overline{\gamma} \rangle = n\}, \quad n \in \mathbb{N}.$$

Lemma 2.1. *Let we be given functions* $V_j(t) = v_j(1/t)$, $t \in (0, 1]$, $j = 1, \dots, m$ *and*

$$\overline{\gamma}' = (\gamma_1', \dots, \gamma_m'), \qquad \overline{\gamma} = (\gamma_1, \dots, \gamma_m), \qquad \overline{\theta} = (\theta_1, \dots, \theta_m),$$
$$0 < \gamma_j' \leqslant \gamma_j, \qquad 1 \leqslant \theta_j \leqslant \infty, \qquad j = 1, \dots, m,$$

and

$$\alpha \in (0, \infty), \qquad \delta = \min\left\{ \frac{\gamma_j}{\gamma_j'} : j = 1, \dots, m \right\},$$
$$A = \left\{ j = 1, \dots, m : \frac{\gamma_j}{\gamma_j'} = \delta \right\}, \qquad j_1 = \min\{j : j \in A\}.$$

Then the inequality

$$\left\| \left\{ 2^{-\alpha\langle \overline{s}, \overline{\gamma} \rangle} \prod_{j=1}^{m} V_j(2^{-s_j}) \right\}_{\overline{s} \in Y^m(n, \overline{\gamma}')} \right\|_{l_{\overline{\theta}}} \leqslant C 2^{-n\alpha\delta} \prod_{j \in A} V_j(2^{-n/\gamma_j'}) n^{\sum\limits_{j \in A \setminus \{j_1\}} \frac{1}{\theta_j}}$$

in the following cases:

if $1 \leqslant \theta_j < \infty$ *and* $v_j \in SV[1, \infty)$,, $j = 1, \dots, m$;

or if $\theta_j = \infty$, $j = 1, \dots, m$ *and* $v_j \in SV[1, \infty)$, $j = 1, \dots, m$ *and* $A \setminus \{j_1\} = \emptyset$;

or if $\theta_j = \infty$, $j = 1, \dots, m$ *and the functions* $v_j(t)$ *almost increases and* $t^{-\varepsilon} v_j(t)$ *almost decrease on* $[1, \infty)$ *for* $\varepsilon > 0$, $j = 1, \dots, m$ *and* $A \setminus \{j_1\} \neq \emptyset$.

Proof. In the case $1 \leqslant \theta_j < \infty$ and $v_j \in SV[1,\infty)$, $j = 1,\dots,m$ this lemma was in fact proved in [37].

We consider the case $\theta_j = \infty$, $j = 1,\dots,m$. We estimate the quantity

$$I_n := \sup_{\bar{s}\in Y^m(n,\bar{\gamma}')} 2^{-\alpha\langle\bar{s},\bar{\gamma}\rangle}\prod_{j=1}^{m} V_j(2^{-s_j}). \tag{2.1}$$

Let $m = 2$. The set $Y^2(n,\bar{\gamma}')$ can be represented as follows:

$$Y^2(n,\bar{\gamma}') = \left\{\bar{s} = (s_1,s_2)\in\mathbb{Z}_+^2:\ 0\leqslant s_2 < \frac{n}{\gamma_2'}, s_1 \geqslant \frac{n - s_2\gamma_2'}{\gamma_1'}\right\} \cup \left\{\bar{s} = (s_1,s_2)\in\mathbb{Z}_+^2:\ s_2\geqslant \frac{n}{\gamma_2'}, s_1\geqslant 0\right\}.$$

By the assumptions of the lemma, $v_2 \in SV[1,\infty)$. Hence, the function $v_2(t)t^{-\varepsilon}$ almost decreases on $[1,\infty)$ for $\varepsilon > 0$. This is why

$$\begin{aligned}2^{-\alpha\langle\bar{s},\bar{\gamma}\rangle}\prod_{j=1}^{2} V_j(2^{-s_j}) &= 2^{-\alpha\langle\bar{s},\bar{\gamma}\rangle}\prod_{j=1}^{2} v_j\left(2^{s_j}\right)\\ &\leqslant C2^{-n\frac{\gamma_1}{\gamma_1'}\alpha}2^{-s_2\gamma_2'\left(\frac{\gamma_2}{\gamma_2'}-\frac{\gamma_1}{\gamma_1'}\right)\alpha}v_2(2^{s_2})v_1\left(2^{\frac{n-s_2\gamma_2'}{\gamma_1'}}\right)\end{aligned} \tag{2.2}$$

for $s_1 \geqslant \frac{n-s_2\gamma_2'}{\gamma_1'}$.

Let $\frac{\gamma_2}{\gamma_2'} - \frac{\gamma_1}{\gamma_1'} > 0$. We choose a number $\eta \in \left(0, \left(\frac{\gamma_2}{\gamma_2'} - \frac{\gamma_1}{\gamma_1'}\right)\alpha\right)$. Since the function $v_1(t)t^{\eta}$ almost decreases on $[1,\infty)$ for $\eta > 0$ and $v_2(t)t^{-\varepsilon}$ almost increases on $[1,\infty)$ for $\varepsilon > 0$ and $0 < n - s_2\gamma_2' \leqslant n$ for $0 \leqslant s_2 < n/\gamma_2'$, then by inequality (2.2) we obtain

$$\begin{aligned}2^{-\alpha\langle\bar{s},\bar{\gamma}\rangle}\prod_{j=1}^{2} V_j(2^{-s_j}) &\leqslant C2^{-n\frac{\gamma_1}{\gamma_1'}\alpha}2^{-n\eta}2^{(n-s_2\gamma_2')\eta}v_1(2^{\frac{n-s_2\gamma_2'}{\gamma_1'}})2^{-s_2\gamma_2'((\frac{\gamma_2}{\gamma_2'}-\frac{\gamma_1}{\gamma_1'})\alpha-\eta)}v_2(2^{s_2})\\ &\leqslant C2^{-n\frac{\gamma_1}{\gamma_1'}\alpha}2^{-n\eta}2^{n\eta}v_1(2^{\frac{n}{\gamma_1'}})2^{-s_2\gamma_2'((\frac{\gamma_2}{\gamma_2'}-\frac{\gamma_1}{\gamma_1'})\alpha-\eta)}v_2(2^{s_2})\\ &\leqslant C2^{-n\frac{\gamma_1}{\gamma_1'}\alpha}v_1(2^{\frac{n}{\gamma_1'}}) = C2^{-n\frac{\gamma_1}{\gamma_1'}\alpha}V_1(2^{-\frac{n}{\gamma_1'}})\end{aligned} \tag{2.3}$$

for $0 \leqslant s_2 < \frac{n}{\gamma_2'}$ and $s_1 \geqslant \frac{n-s_2\gamma_2'}{\gamma_1'}$.

Let $\frac{\gamma_2}{\gamma_2'} - \frac{\gamma_1}{\gamma_1'} < 0$. Choosing then a number $\eta \in (0, -(\frac{\gamma_2}{\gamma_2'} - \frac{\gamma_1}{\gamma_1'})\alpha)$, similar to (2.3) we prove that

$$2^{-\alpha\langle\bar{s},\bar{\gamma}\rangle}\prod_{j=1}^{2} V_j(2^{-s_j}) \leqslant C2^{-n\frac{\gamma_2}{\gamma_2'}\alpha}V_2(2^{-\frac{n}{\gamma_2'}}) \tag{2.4}$$

for $0 \leqslant s_2 < \frac{n}{\gamma_2'}$ and $s_1 \geqslant \frac{n-s_2\gamma_2'}{\gamma_1'}$.

Let $\frac{\gamma_2}{\gamma_2'} - \frac{\gamma_1}{\gamma_1'} = 0$. Then by (2.2) we have

$$2^{-\alpha\langle\bar{s},\bar{\gamma}\rangle}\prod_{j=1}^{2} V_j(2^{-s_j}) \leqslant C2^{-n\frac{\gamma_1}{\gamma_1'}\alpha}v_2(2^{s_2})v_1(2^{\frac{n-s_2\gamma_2'}{\gamma_1'}}) \tag{2.5}$$

for $0 \leqslant s_2 < \frac{n}{\gamma_2'}$ and $s_1 \geqslant \frac{n-s_2\gamma_2'}{\gamma_1'}$. Since the functions v_j, $j = 1, 2$, almost increases, then

$$v_2(2^{s_2}) \leqslant Cv_2(2^{\frac{n}{\gamma_2'}}), \qquad 0 \leqslant s_2 < \frac{n}{\gamma_2'}$$

and

$$v_1(2^{\frac{n-s_2\gamma_2'}{\gamma_1'}})\leqslant Cv_1(2^{\frac{n}{\gamma_1'}}),\qquad 0<\frac{n-s_2\gamma_2'}{\gamma_1'}\leqslant\frac{n}{\gamma_1'}.$$

This is why it follows from (2.5) that

$$2^{-\alpha\langle\overline{s},\overline{\gamma}\rangle}\prod_{j=1}^{2}V_j(2^{-s_j})\leqslant C2^{-n\frac{\gamma_1}{\gamma_1'}\alpha}v_1(2^{\frac{n}{\gamma_1'}})v_2(2^{\frac{n}{\gamma_2'}})=C2^{-n\frac{\gamma_1}{\gamma_1'}\alpha}V_1(2^{-\frac{n}{\gamma_1'}})V_2(2^{-\frac{n}{\gamma_2'}}) \tag{2.6}$$

for $0\leqslant s_2<\frac{n}{\gamma_2'}$ and $s_1\geqslant\frac{n-s_2\gamma_2'}{\gamma_1'}$ in the case $\frac{\gamma_2}{\gamma_2'}-\frac{\gamma_1}{\gamma_1'}=0$.

Let $s_1\geqslant 0$ and $s_2\geqslant\frac{n}{\gamma_2'}$. Then by the fact that the function $v_j(t)t^{-\varepsilon}$ almost decreases on $[1,\infty)$, for $\varepsilon>0$ we get:

$$\begin{aligned}2^{-\alpha\langle\overline{s},\overline{\gamma}\rangle}\prod_{j=1}^{2}V_j(2^{-s_j})&=2^{-s_1\gamma_1\alpha}v_1(2^{s_1})2^{-s_2\gamma_2\alpha}v_2(2^{s_2})\\&\leqslant C2^{-n\frac{\gamma_1}{\gamma_1'}\alpha}v_1(1)v_2(2^{\frac{n}{\gamma_2'}})=C2^{-n\frac{\gamma_2}{\gamma_2'}\alpha}V_1(1)V_2(2^{-\frac{n}{\gamma_2'}}).\end{aligned} \tag{2.7}$$

Now it follows from inequalities (2.1), (2.3) and (2.7) that

$$I_n\leqslant C\Big(2^{-n\frac{\gamma_1}{\gamma_1'}\alpha}V_1(2^{-\frac{n}{\gamma_1'}})+2^{-n\frac{\gamma_2}{\gamma_2'}\alpha}V_2(2^{-\frac{n}{\gamma_2'}})\Big)\leqslant C2^{-n\frac{\gamma_1}{\gamma_1'}\alpha}V_1(2^{-\frac{n}{\gamma_1'}}), \tag{2.8}$$

if $\frac{\gamma_2}{\gamma_2'}-\frac{\gamma_1}{\gamma_1'}>0$.

If $\frac{\gamma_2}{\gamma_2'}-\frac{\gamma_1}{\gamma_1'}<0$, then it follows from inequalities (2.1), (2.4) and (2.7) that

$$I_n\leqslant C2^{-n\frac{\gamma_2}{\gamma_2'}\alpha}V_2(2^{-\frac{n}{\gamma_2'}}). \tag{2.9}$$

If $\frac{\gamma_2}{\gamma_2'}-\frac{\gamma_1}{\gamma_1'}=0$, then by inequalities (2.1), (2.6) and (2.7) we find:

$$I_n\leqslant C\Big\{2^{-n\frac{\gamma_1}{\gamma_1'}\alpha}V_1(2^{-\frac{n}{\gamma_1'}})V_2(2^{-\frac{n}{\gamma_2'}})+2^{-n\frac{\gamma_2}{\gamma_2'}\alpha}V_2(2^{-\frac{n}{\gamma_2'}})\Big\}\leqslant C2^{-n\frac{\gamma_1}{\gamma_1'}\alpha}V_2(2^{-\frac{n}{\gamma_2'}})V_1(2^{-\frac{n}{\gamma_1'}}) \tag{2.10}$$

for almost increasing on $[1,\infty)$ functions v_j, $j=1,2$.

This statement of the lemma was proved for $m=2$ in the case $\theta_j=\infty$, $j=1,2$.

Applying then the induction and inequalities (2.8)–(2.10), the statement can be also proved for $m>2$. The proof is complete. □

Lemma 2.2. [30, Lm. 4]. *Let $\overline{\tau}=(\tau_1,\dots,\tau_m)$, $1\leqslant\tau_j<+\infty$, and $j=1,\dots,m$. Then a relation holds:*

$$\Big\|\big\{\chi_{\varkappa(n)}(\overline{s})\big\}_{\overline{s}\in\varkappa(n)}\Big\|_{l_{\overline{\tau}}}\asymp n^{\sum\limits_{j=2}^{m}\frac{1}{\tau_j}},\quad n\in\mathbb{N}.$$

Hereafter $\chi_{\varkappa(n)}(\overline{s})$ is the characteristic function of a set:

$$\varkappa(n)=\{\overline{s}=(s_1,\dots,s_m)\in\mathbb{Z}_+^m:\ \langle\overline{s},\overline{\gamma}\rangle=n\}.$$

3. Main result

In this section we prove the main result of the paper.

Theorem 3.1. *Let*

$$\begin{aligned}&\overline{p}=(p_1,\dots,p_m),\quad \overline{q}=(q_1,\dots,q_m),\quad \overline{\tau}^{(1)}=(\tau_1^{(1)},\dots,\tau_m^{(1)}),\quad \overline{\tau}^{(2)}=(\tau_1^{(2)},\dots,\tau_m^{(2)}),\\&\overline{\gamma}=(\gamma_1,\dots,\gamma_m),\quad \overline{\gamma}'=(\gamma_1',\dots,\gamma_m'),\quad \overline{r}=(r_1,\dots,r_m),\quad \overline{\theta}=(\theta_1,\dots,\theta_m)\end{aligned}$$

and

$$0<\theta_j\leqslant\infty,\qquad 1<\tau_j^{(1)},\tau_j^{(2)}<+\infty,\qquad 1<p_j<q_j\leqslant 2,\qquad r_j>\frac{1}{p_j}-\frac{1}{q_j},$$

$$\gamma_j = \frac{r_j + \frac{1}{q_j} - \frac{1}{p_j}}{r_{j_0} + \frac{1}{q_{j_0}} - \frac{1}{p_{j_0}}}, \qquad j = 1, \dots, m,$$
$$r_{j_0} + \frac{1}{q_{j_0}} - \frac{1}{p_{j_0}} = \min\left\{ r_j + \frac{1}{q_j} - \frac{1}{p_j} : j = 1, \dots, m \right\},$$
$$A = \left\{ j : \frac{\gamma_j}{\gamma_j'} = 1,\ j = 1, \dots, m \right\}, \qquad j_1 = \min\{j \in A\},$$
$$v_j^{(i)} \in SV[1,\infty), \qquad V_j^{(i)}(t) = v_j^{(i)}\left(\frac{1}{t}\right), \qquad t \in (0,1], \qquad j = 1, \dots, m,$$
$$\bar V^{(i)}(t) = (V_1^{(i)}(t), \dots, V_m^{(i)}(t)), \qquad i = 1,2.$$

1. If $1 < \tau_j^{(2)} < \theta_j \leqslant +\infty$ and $v_j^{(2)}/v_j^{(1)} \in SVL[1,\infty)$, $j \in A$, then

$$e_M\big(S^{\overline{r}}_{\overline{p},\overline{V}^{(1)},\overline{\tau}^{(1)},\overline{\theta}}B\big)_{\overline{q},\overline{V}^{(2)},\overline{\tau}^{(2)}} \asymp M^{-(r_{j_0} + \frac{1}{q_{j_0}} - \frac{1}{p_{j_0}})} \prod_{j\in A} \frac{V_j^{(2)}(M^{-1})}{V_j^{(1)}(M^{-1})} \cdot (\log M)^{(|A|-1)(r_{j_0} + \frac{1}{q_{j_0}} - \frac{1}{p_{j_0}})} (\log M)^{\sum\limits_{j\in A\setminus\{j_1\}} (\frac{1}{\tau_j^{(2)}} - \frac{1}{\theta_j})}.$$

2. If $1 \leqslant \theta_j = \theta \leqslant \tau_j^{(2)} = \tau^{(2)} < +\infty$, $j = 1, \dots, m$, then

$$e_M\big(S^{\overline{r}}_{\overline{p},\overline{V}^{(1)},\overline{\tau}^{(1)},\overline{\theta}}B\big)_{\overline{q},\overline{V}^{(2)},\overline{\tau}^{(2)}} \asymp M^{-\left(r_{j_0} + \frac{1}{q_{j_0}} - \frac{1}{p_{j_0}}\right)} \prod_{j\in A} \frac{V_j^{(2)}(M^{-1})}{V_j^{(1)}(M^{-1})} \cdot (\log M)^{(|A|-1)(r_{j_0} + \frac{1}{q_{j_0}} - \frac{1}{p_{j_0}} + \frac{1}{\tau^{(2)}} - \frac{1}{\theta})_+},$$

in the cases $\frac{v_j^{(2)}}{v_j^{(1)}} \in SVL[1,\infty)$, $j \in A$ and $A \setminus \{j_1\} = \emptyset$ or $\frac{v_j^{(2)}(t)}{v_j^{(1)}(t)} t^{-\varepsilon}$ almost decreases for $\varepsilon > 0$ and $\frac{v_j^{(2)}(t)}{v_j^{(1)}(t)}$ almost increases on $[1,\infty)$ for $j \in A$ and $A \setminus \{j_1\} \neq \emptyset$ for $M \in \mathbb{N}$ such that $M > M_0$, where M_0 is some positive number exceeding 1.

Proof. Let $f \in S^{\overline{r}}_{\overline{p},\overline{V}^{(1)},\overline{\tau}^{(1)},\overline{\theta}}B$. We let

$$r_{j_0} + \frac{1}{q_{j_0}} - \frac{1}{p_{j_0}} = \min\left\{ r_j + \frac{1}{q_j} - \frac{1}{p_j} : j = 1, \dots, m \right\} \quad \text{and} \quad \gamma_j = \frac{r_j + \frac{1}{q_j} - \frac{1}{p_j}}{r_{j_0} + \frac{1}{q_{j_0}} - \frac{1}{p_{j_0}}}, \quad j = 1, \dots, m.$$

We choose numbers $\gamma_j' > 0$ so that $\gamma_j' \leqslant \gamma_j$, $j = 1, \dots, m$ and we introduce the notations

$$A = \left\{ j : \frac{\gamma_j}{\gamma_j'} = 1,\ j = 1, \dots, m \right\}, \quad j_1 = \min\{j \in A\}.$$

We note that $\gamma_j' = \gamma_j = 1$ for $j \in A$. For a number $M \in \mathbb{N}$ there exists a natural number n such that $M \asymp 2^n n^{|A|-1}$, where $|A|$ is the number of the elements in the set A.

Then by the definition of the quantity $e_M(f)_{\overline{q},\overline{V}^{(2)},\overline{\tau}^{(2)}}$ and by Theorem 2.2 we have

$$e_M(f)_{\overline{q},\overline{V}^{(2)},\overline{\tau}^{(2)}} \leqslant E_n^{(\overline{\gamma}')}(f)_{\overline{q},\overline{V}^{(2)},\overline{\tau}^{(2)}} \ll 2^{-n\left(r_{j_0} + \frac{1}{q_{j_0}} - \frac{1}{p_{j_0}}\right)} \prod_{j\in A} \frac{V_j^{(2)}(2^{-n})}{V_j^{(1)}(2^{-n})} n^{\sum\limits_{j\in A\setminus\{j_1\}} (\frac{1}{\tau_j^{(2)}} - \frac{1}{\theta_j})}, \tag{3.1}$$

in the case $\tau_j^{(2)} \leqslant \theta_j \leqslant \infty$, $j=1,\ldots,m$. Since $\frac{v_j^{(2)}}{v_j^{(1)}} \in SVL[1,\infty)$, $j=1,\ldots,m$ and $\frac{\log t}{t} \to 0$ as $t \to +\infty$, then

$$\begin{aligned} \frac{V_j^{(2)}(2^{-n})}{V_j^{(1)}(2^{-n})} &= \frac{v_j^{(2)}(2^n)}{v_j^{(1)}(2^n)} = \frac{v_j^{(2)}(2^n)}{v_j^{(1)}(2^n)} \left(\log 2^{n+1}\right)^{\varepsilon} \left(\log 2^{n+1}\right)^{-\varepsilon} \\ &\ll \frac{v_j^{(2)}(2^n n^{|A|-1})}{v_j^{(1)}(2^n n^{|A|-1})} \left(\log(2^{n+1} n^{|A|-1})\right)^{\varepsilon} \left(\log 2^{n+1}\right)^{-\varepsilon} \\ &= \frac{v_j^{(2)}(2^n n^{|A|-1})}{v_j^{(1)}(2^n n^{|A|-1})} \left(n+1+\log n^{|A|-1}\right)^{\varepsilon} (n+1)^{-\varepsilon} \ll \frac{v_j^{(2)}(2^n n^{|A|-1})}{v_j^{(1)}(2^n n^{|A|-1})} \\ &\ll \frac{V_j^{(2)}((2^n n^{|A|-1})^{-1})}{V_j^{(1)}((2^n n^{|A|-1})^{-1})} \ll \frac{V_j^{(2)}(M^{-1})}{V_j^{(1)}(M^{-1})}. \end{aligned} \tag{3.2}$$

By the relation $M \asymp 2^n n^{|A|-1}$ we have $n \asymp \log M$ and

$$\begin{aligned} 2^{-n\left(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}\right)} n^{\sum\limits_{j\in A\setminus\{j_1\}}\left(\frac{1}{\tau_j^{(2)}}-\frac{1}{\theta_j}\right)} =& 2^{-n\left(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}\right)} n^{-(|A|-1)\left(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}\right)} \\ &\cdot n^{(|A|-1)\left(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}\right)} n^{\sum\limits_{j\in A\setminus\{j_1\}}\left(\frac{1}{\tau_j^{(2)}}-\frac{1}{\theta_j}\right)} \\ \asymp& M^{-\left(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}\right)} \\ &\cdot (\log M)^{(|A|-1)\left(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}\right)+\sum\limits_{j\in A\setminus\{j_1\}}\left(\frac{1}{\tau_j^{(2)}}-\frac{1}{\theta_j}\right)} \end{aligned} \tag{3.3}$$

for $n > n_0$, where n_0 is some positive number exceeding 1.

Then it follows from inequalities (3.1)–(3.2) and relation (3.3) that

$$e_M(f)_{\overline{q},\overline{V}^{(2)},\overline{\tau}^{(2)}} \ll M^{-\left(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}\right)} \prod_{j\in A} \frac{V_j^{(2)}(M^{-1})}{V_j^{(1)}(M^{-1})} (\log M)^{(|A|-1)\left(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}\right)+\sum\limits_{j\in A\setminus\{j_1\}}\left(\frac{1}{\tau_j^{(2)}}-\frac{1}{\theta_j}\right)}$$

for $M > M_0 > 1$ in the case $\tau_j^{(2)} < \theta_j \leqslant \infty$, $j=1,\ldots,m$. This proves the upper bound in the first item.

We consider the second case $1 \leqslant \theta_j = \theta < \tau_j^{(2)} = \tau^{(2)} < \infty$, $j=1,\ldots,m$, and instead of $S^{\overline{r}}_{\overline{p},\overline{V}^{(1)},\overline{\tau}^{(1)},\overline{\theta}}B$ and $\|f\|^*_{\overline{q},\overline{V}^{(2)},\overline{\tau}^{(2)}}$ we shall write $S^{\overline{r}}_{\overline{p},\overline{V}^{(1)},\overline{\tau}^{(1)},\theta}B$ and $\|f\|^*_{\overline{q},\overline{V}^{(2)},\tau^{(2)}}$. For a function $f \in S^{\overline{r}}_{\overline{p},\overline{V}^{(1)},\overline{\tau}^{(1)},\theta}B$ we construct the first approximating polynomial $P(\Omega_M,\overline{x})$, which gives a required estimate for the approximation.

Let a natural number n satisfies the relations

$$M \asymp 2^n n^{|A|-1}, \qquad n_0 = [n + (|A|-1)\log n].$$

We construct the polynomial $P(\Omega_M,\bar{x})$ as

$$P(\Omega_M,\overline{x}) = \tilde{R}(\overline{x}) + \tilde{Q}(\overline{x}),$$

where $\tilde{R}(\overline{x}) = \sum\limits_{\langle \overline{s},\overline{\gamma}'\rangle < n} \delta_{\overline{s}}(f,\overline{x})$, while $\tilde{Q}(\overline{x})$ will be constructed in what follows. For a natural number l we let

$$\tilde{S}_l = \left(\sum_{l \leqslant \langle \overline{s},\overline{\gamma}'\rangle < l+1} 2^{\langle \overline{s},\overline{r}\rangle\theta} \left(\|\delta_{\overline{s}}(f)\|^*_{\overline{p},\overline{V}^{(1)},\overline{\tau}^{(1)}}\right)^{\theta} \right)^{\frac{1}{\theta}}$$

and we denote

$$\tilde{m}_l = \left[2^n n^{|A|-1} 2^{-l} \tilde{S}_l^{\theta}\right] + 1,$$

where $[a]$ is an integer part of a number a. It is easy to confirm that the number of harmonics K, which form the polynomial $P(\Omega_M, \overline{x})$ does not exceed M by the order. By the properties of the norm,

$$\|f - P(\Omega_M)\|^*_{\overline{q},\bar{V}^{(2)},\tau^{(2)}} \leqslant \Big\| f - \sum_{\langle \overline{s}, \overline{\gamma}'\rangle < n_0} \delta_{\overline{s}}(f) \Big\|^*_{\overline{q},\bar{V}^{(2)},\tau^{(2)}} + \|R^* - \tilde{Q}\|_{\overline{q},\bar{V}^{(2)},\tau^{(2)}} = J_1 + J_2. \tag{3.4}$$

According to Theorem 2.1, Jensen inequality [9, Lm. 3.3.3], Lemma 2.1 as $\theta_j = \infty$, $j = 1, \ldots, m$ and taking into consideration that $0 < \gamma_j' \leqslant \gamma_j$ for $j = 1, \ldots, m$, we have

$$\begin{aligned}
J_1 \leqslant & C \left\{ \sum_{\langle \overline{s}, \overline{\gamma}'\rangle \geqslant n_0} \prod_{j=1}^{m} 2^{s_j\left(\frac{1}{p_j} - \frac{1}{q_j}\right)\tau^{(2)}} \left(\frac{V_j^{(2)}(2^{-s_j})}{V_j^{(1)}(2^{-s_j})}\right)^{\tau^{(2)}} \left(\|\delta_{\overline{s}}(f)\|^*_{\overline{p},\bar{V}^{(1)},\bar{\tau}^{(1)}}\right)^{\tau^{(2)}} \right\}^{\frac{1}{\tau^{(2)}}} \\
\ll & \left\{ \sum_{\langle \overline{s}, \overline{\gamma}'\rangle \geqslant n_0} \prod_{j=1}^{m} 2^{s_j r_j \tau^{(2)}} \left(\|\delta_{\overline{s}}(f)\|^*_{\overline{p},\bar{V}^{(1)},\bar{\tau}^{(1)}}\right)^{\tau^{(2)}} \right\}^{\frac{1}{\tau^{(2)}}} \sup_{\langle \overline{s}, \overline{\gamma}'\rangle \geqslant n_0} \prod_{j=1}^{m} 2^{-s_j(r_j + \frac{1}{q_j} - \frac{1}{p_j})} \frac{V_j^{(2)}(2^{-s_j})}{V_j^{(1)}(2^{-s_j})} \\
\ll & \left\{ \sum_{\overline{s} \in \mathbb{Z}_+^m} \prod_{j=1}^{m} 2^{s_j r_j \theta} \left(\|\delta_{\overline{s}}(f)\|^*_{\overline{p},\bar{V}^{(1)},\bar{\tau}^{(1)}}\right)^{\theta} \right\}^{\frac{1}{\theta}} \sup_{\langle \overline{s}, \overline{\gamma}'\rangle \geqslant n_0} 2^{-(r_{j_0} + \frac{1}{q_{j_0}} - \frac{1}{p_{j_0}})\langle \overline{s}, \overline{\gamma}\rangle} \prod_{j=1}^{m} \frac{V_j^{(2)}(2^{-s_j})}{V_j^{(1)}(2^{-s_j})} \\
\leqslant & C 2^{-n_0(r_{j_0} + \frac{1}{q_{j_0}} - \frac{1}{p_{j_0}})} \prod_{j \in A} \frac{V_j^{(2)}(2^{-n_0})}{V_j^{(1)}(2^{-n_0})} \leqslant C 2^{-n(r_{j_0} + \frac{1}{q_{j_0}} - \frac{1}{p_{j_0}})} \prod_{j \in A} \frac{V_j^{(2)}(2^{-n})}{V_j^{(1)}(2^{-n})}
\end{aligned}$$

for each function $f \in S^{\overline{r}}_{\overline{p},\bar{V}^{(1)},\bar{\tau}^{(1)},\theta}B$ in the case $1 \leqslant \theta \leqslant \tau^{(2)} < \infty$.

Now taking into consideration the definition of the number n_0 and the fact that the functions $\frac{v_j^{(2)}}{v_j^{(1)}} t^{\varepsilon}$ almost increase on $[1, \infty)$, $j = 1, \ldots, m$, we get:

$$J_1 \ll (2^n n^{|A|-1})^{-(r_{j_0} + \frac{1}{q_{j_0}} - \frac{1}{p_{j_0}})} \prod_{j \in A} \frac{V_j^{(2)}((2^n n^{|A|-1})^{-1})}{V_j^{(1)}((2^n n^{|A|-1})^{-1})} \tag{3.5}$$

for each function $f \in S^{\overline{r}}_{\overline{p},\bar{V}^{(1)},\bar{\tau}^{(1)},\theta}B$ in the case $1 \leqslant \theta \leqslant \tau^{(2)} < \infty$.

In order to estimate J_2, we shall employ arguing from [19]. To each natural number l, $n \leqslant l < n_0$ we associate the sum

$$\sum_{l \leqslant \langle \overline{s}, \overline{\gamma}'\rangle < l+1} \delta_{\overline{s}}(f, \overline{x}). \tag{3.6}$$

Let $\alpha_i(f, l)$, $i = 1, 2, \ldots, \tilde{m}_l$, stand for the numbers

$$\prod_{j=1}^{m} 2^{s_j(\frac{1}{p_j} - \frac{1}{q_j})} \frac{V_j^{(2)}(2^{-s_j})}{V_j^{(1)}(2^{-s_j})} \|\delta_{\overline{s}}(f)\|^*_{\overline{p},\overline{V}^{(1)},\bar{\tau}^{(1)}}$$

taken in the decreasing order, for which the blocks $\delta_{\overline{s}}(f, \overline{x})$ are involved in sum (3.6). The sum of such obtained "blocks" $\delta_{\overline{s}}(f, \overline{x})$ over all $l \in [n, n_0)$ is denoted by $\tilde{Q}(\overline{x})$. Let D_f stand for the set of the vectors $\bar{s}$ obeying the condition $n \leqslant \langle \overline{s}, \overline{\gamma}'\rangle < n_0$, for which the "blocks" $\delta_{\overline{s}}(f, \overline{x})$ are not in $\tilde{Q}(\overline{x})$.

Since the functions $\frac{v_j^{(2)}}{v_j^{(1)}}$ for $j=1,\ldots,m$ satisfy the assumptions of Lemma 2.1, then taking into consideration that $\gamma_j'=1$ for $j\in A$, we have

$$2^{\langle\bar{s},\bar{\gamma}\rangle(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})}\prod_{j=1}^{m}\frac{V_j^{(1)}(2^{-s_j})}{V_j^{(2)}(2^{-s_j})}\gg 2^{l(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})}\prod_{j\in A}\frac{V_j^{(1)}(2^{-l})}{V_j^{(2)}(2^{-l})}$$

for $\bar{s}$ obeying the inequality $\langle\bar{s},\bar{\gamma}'\rangle\geqslant l$. Employing this inequality and taking into consideration that

$$0<\gamma_j'\leqslant\gamma_j=\frac{r_j+\frac{1}{q_j}-\frac{1}{p_j}}{r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}}$$

for $j=1,\ldots,m$ we have

$$\begin{aligned}\tilde{S}_l&=\left(\sum_{l\leqslant\langle\bar{s},\bar{\gamma}'\rangle<l+1}2^{\langle\bar{s},\bar{r}\rangle\theta}\left(\|\delta_{\bar{s}}(f)\|^*_{\bar{p},\bar{V}^{(1)},\bar{\tau}^{(1)}}\right)^{\theta}\right)^{\frac{1}{\theta}}\\
&=\left(\sum_{l\leqslant\langle\bar{s},\bar{\gamma}'\rangle<l+1}2^{\langle\bar{s},\bar{\gamma}\rangle(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})\theta}\left(\prod_{j=1}^{m}\frac{V_j^{(1)}(2^{-s_j})}{V_j^{(2)}(2^{-s_j})}\right)^{\theta}\right.\\
&\quad\cdot\left.\left(\prod_{j=1}^{m}2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)}\frac{V_j^{(2)}(2^{-s_j})}{V_j^{(1)}(2^{-s_j})}\|\delta_{\bar{s}}(f)\|^*_{\bar{p},\overline{V}^{(1)},\bar{\tau}^{(1)}}\right)^{\theta}\right)^{\frac{1}{\theta}}\\
&\geqslant C2^{l(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})}\prod_{j\in A}\frac{V_j^{(1)}(2^{-l})}{V_j^{(2)}(2^{-l})}\\
&\quad\cdot\left(\sum_{l\leqslant\langle\bar{s},\bar{\gamma}'\rangle<l+1}\left(\prod_{j=1}^{m}2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)}\frac{V_j^{(2)}(2^{-s_j})}{V_j^{(1)}(2^{-s_j})}\|\delta_{\bar{s}}(f)\|^*_{\bar{p},\bar{V}^{(1)},\bar{\tau}^{(1)}}\right)^{\theta}\right)^{\frac{1}{\theta}}\\
&=C2^{l(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})}\prod_{j\in A}\frac{V_j^{(1)}(2^{-l})}{V_j^{(2)}(2^{-l})}\left[\sum_{k=1}^{\tilde{m}_l}\alpha_k^{\theta}(f,l)\right]^{\frac{1}{\theta}}\gg 2^{l(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})}\prod_{j\in A}\frac{V_j^{(1)}(2^{-l})}{V_j^{(2)}(2^{-l})}j^{\frac{1}{\theta}}\alpha_i(f,l)\end{aligned}$$

for $i=1,2,\ldots,\tilde{m}_l$.

Thus,

$$\alpha_i(f,l)\leqslant i^{-\frac{1}{\theta}}2^{-l(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})}\prod_{j\in A}\frac{V_j^{(2)}(2^{-l})}{V_j^{(1)}(2^{-l})}\tilde{S}_l$$

for $i=1,2,\ldots,\tilde{m}_l$. Employing then Theorem 2.1, the above inequality, the definition of the numbers $\tilde{S}_l$, $\tilde{m}_l$, we obtain

$$\begin{aligned}J_2&=\Big\|\sum_{\bar{s}\in D_f}\delta_{\bar{s}}(f)\Big\|^*_{\bar{q},\bar{V}^{(2)},\tau^{(2)}}\\
&\ll\left\{\sum_{\bar{s}\in D_f}\left(\prod_{j=1}^{m}2^{s_j(\frac{1}{p_j}-\frac{1}{q_j})}\frac{V_j^{(2)}(2^{-s_j})}{V_j^{(1)}(2^{-s_j})}\|\delta_{\bar{s}}(f)\|^*_{\bar{p},\bar{V}^{(1)},\bar{\tau}^{(1)}}\right)^{\tau^{(2)}}\right\}^{\frac{1}{\tau^{(2)}}}\\
&=C\left\{\sum_{l=n}^{n_0}\sum_{j\geqslant\tilde{m}_l}\alpha_j^{\tau^{(2)}}(f,l)\right\}^{\frac{1}{\tau^{(2)}}}\end{aligned}$$

$$\ll \left\{\sum_{l=n}^{n_0} 2^{-l(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})\tau^{(2)}} \Big(\prod_{j\in A}\frac{V_j^{(2)}(2^{-l})}{V_j^{(1)}(2^{-l})}\Big)^{\tau^{(2)}} \tilde{S}_l^{\tau^{(2)}} \sum_{j\geqslant \tilde{m}_l} j^{-\frac{\tau^{(2)}}{\theta}}\right\}^{\frac{1}{\tau^{(2)}}}$$

$$\ll \left(2^n n^{|A|-1}\right)^{\frac{1}{\tau^{(2)}}-\frac{1}{\theta}} \left\{\sum_{l=n}^{n_0} 2^{-l\tau^{(2)}\left(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}-\frac{1}{\theta}+\frac{1}{\tau^{(2)}}\right)} \Big(\prod_{j\in A}\frac{V_j^{(2)}(2^{-l})}{V_j^{(1)}(2^{-l})}\Big)^{\tau^{(2)}} \tilde{S}_l^{\theta}\right\}^{\frac{1}{\tau^{(2)}}}.$$

Thus,

$$J_2 \ll \left(2^n n^{|A|-1}\right)^{\frac{1}{\tau^{(2)}}-\frac{1}{\theta}} \left\{\sum_{l=n}^{n_0} 2^{-l\tau^{(2)}(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}-\frac{1}{\theta}+\frac{1}{\tau^{(2)}})} \Big(\prod_{j\in A}\frac{V_j^{(2)}(2^{-l})}{V_j^{(1)}(2^{-l})}\Big)^{\tau^{(2)}} \tilde{S}_l^{\theta}\right\}^{\frac{1}{\tau^{(2)}}} \tag{3.7}$$

in the case $1 \leqslant \theta \leqslant \tau^{(2)} < \infty$.

Then we consider two cases

a) $r_{j_0} \geqslant \frac{1}{p_{j_0}} - \frac{1}{q_{j_0}} + \frac{1}{\theta} - \frac{1}{\tau^{(2)}}$;

b) $\frac{1}{p_{j_0}} - \frac{1}{q_{j_0}} < r_{j_0} < \frac{1}{p_{j_0}} - \frac{1}{q_{j_0}} + \frac{1}{\theta} - \frac{1}{\tau^{(2)}}$.

We introduce the notation

$$J_3 = \left\{\sum_{l=n}^{n_0} 2^{-l\tau^{(2)}(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}-\frac{1}{\theta}+\frac{1}{\tau^{(2)}})} \Big(\prod_{j\in A}\frac{V_j^{(2)}(2^{-l})}{V_j^{(1)}(2^{-l})}\Big)^{\tau^{(2)}} \tilde{S}_l^{\theta}\right\}^{\frac{1}{\tau^{(2)}}}.$$

In Case a), by the definition of $\tilde{S}_l$ and in view of the fact that the functions $\frac{v_j^{(2)}(t)}{v_j^{(1)}(t)}t^{-\varepsilon}$ almost decrease on $[1,\infty)$ for $\varepsilon > 0$, $j = 1,\dots,m$, we have

$$J_3 \leqslant 2^{-n\left(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}-\frac{1}{\theta}+\frac{1}{\tau^{(2)}}\right)} \prod_{j\in A}\frac{V_j^{(2)}(2^{-n})}{V_j^{(1)}(2^{-n})} \|f\|^{\frac{\theta}{\tau^{(2)}}}_{S^{\bar r}_{\bar p,\bar V^{(1)},\bar\tau^{(1)},\theta}B}$$

for each function $f \in S^{\bar r}_{\bar p,\bar V^{(1)},\bar\tau^{(1)},\theta}B$. Therefore, by inequality (3.7) we obtain

$$J_2 \ll 2^{-n(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})} \prod_{j\in A}\frac{V_j^{(2)}(2^{-n})}{V_j^{(1)}(2^{-n})} n^{(|A|-1)(\frac{1}{\tau^{(2)}}-\frac{1}{\theta})} \tag{3.8}$$

for each function $f \in S^{\bar r}_{\bar p,\bar V^{(1)},\bar\tau^{(1)},\theta}B$ in the case

$$r_{j_0} \geqslant \frac{1}{p_{j_0}} - \frac{1}{q_{j_0}} + \frac{1}{\theta} - \frac{1}{\tau^{(2)}}, \qquad \theta < \tau^{(2)}.$$

We proceed to Case b). By the definition of $\tilde{S}_l$ and the choice of numbers n_0 as well as by the fact that $r_{j_0} + \frac{1}{q_{j_0}} - \frac{1}{p_{j_0}} - \frac{1}{\theta} + \frac{1}{\tau^{(2)}} < 0$ and the functions $\frac{v_j^{(2)}(t)}{v_j^{(1)}(t)}t^{\varepsilon}$ almost increase on $[1,\infty)$ for $\varepsilon > 0$, $j = 1,\dots,m$, we have

$$\begin{aligned} J_3 &\leqslant 2^{-n_0\left(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}-\frac{1}{\theta}+\frac{1}{\tau^{(2)}}\right)} \prod_{j\in A}\frac{V_j^{(2)}(2^{-n_0})}{V_j^{(1)}(2^{-n_0})} \left\{\sum_{l=n}^{n_0}\tilde{S}_l^{\theta}\right\}^{\frac{1}{\tau^{(2)}}} \\ &\ll 2^{-n_0\left(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}-\frac{1}{\theta}+\frac{1}{\tau^{(2)}}\right)} \prod_{j\in A}\frac{V_j^{(2)}(2^{-n_0})}{V_j^{(1)}(2^{-n_0})} \|f\|^{\frac{\theta}{\tau^{(2)}}}_{S^{\bar r}_{\bar p,\bar V^{(1)},\bar\tau^{(1)},\theta}B} \end{aligned} \tag{3.9}$$

for each function $f \in S^{\bar r}_{\bar p,\bar V^{(1)},\bar\tau^{(1)},\theta}B$.

Since $n_0 = [n + (|A| - 1)\log n]$, then $2^{n_0} \leqslant 2^n n^{|A|-1} < 2^{n_0+1}$. This is why, by estimate (3.9) we obtain

$$J_3 \ll (2^n n^{|A|-1})^{-(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}-\frac{1}{\theta}+\frac{1}{\tau^{(2)}})} \prod_{j\in A} \frac{V_j^{(2)}((2^n n^{|A|-1})^{-1})}{V_j^{(1)}((2^n n^{|A|-1})^{-1})}$$

for each function $f \in S^{\bar{r}}_{\bar{p},\bar{V}^{(1)},\bar{\tau}^{(1)},\theta}B$ in the case

$$\frac{1}{p_{j_0}} - \frac{1}{q_{j_0}} < r_{j_0} < \frac{1}{p_{j_0}} - \frac{1}{q_{j_0}} + \frac{1}{\theta} - \frac{1}{\tau^{(2)}}, \qquad \theta < \tau^{(2)}.$$

Therefore, taking into consideration that $M \asymp 2^n n^{|A|-1}$, by (3.7) we obtain

$$\begin{aligned} J_2 &\leqslant C(2^n n^{|A|-1})^{-(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})} \prod_{j\in A} \frac{V_j^{(2)}((2^n n^{|A|-1})^{-1})}{V_j^{(1)}((2^n n^{|A|-1})^{-1})} \\ &\asymp M^{-(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})} \prod_{j\in A} \frac{V_j^{(2)}(M^{-1})}{V_j^{(1)}(M^{-1})} \end{aligned} \tag{3.10}$$

for each function $f \in S^{\bar{r}}_{\bar{p},\bar{V}^{(1)},\bar{\tau}^{(1)},\theta}B$, in Case b), $\theta \leqslant \tau^{(2)}$.

Thus, in view of estimates (3.7), (3.8) and (3.10) we conclude that

$$J_2 \leqslant CM^{-(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})} (\log^{|A|-1} M)^{(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}+\frac{1}{\theta}-\frac{1}{\tau^{(2)}})_+} \prod_{j\in A} \frac{V_j^{(2)}(M^{-1})}{V_j^{(1)}(M^{-1})} \tag{3.11}$$

for each function $f \in S^{\bar{r}}_{\bar{p},\bar{V}^{(1)},\bar{\tau}^{(1)},\theta}B$, in the case $1 \leqslant \theta \leqslant \tau^{(2)} < \infty$ and $\frac{1}{p_{j_0}} - \frac{1}{q_{j_0}} < r_{j_0}$.

Now in view estimates (3.5) and (3.11) by inequality (3.4) we obtain

$$\|f - P(\Omega_M)\|^*_{\bar{q},\bar{V}^{(2)},\tau^{(2)}} \ll M^{-(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})} (\log^{|A|-1} M)^{(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}+\frac{1}{\theta}-\frac{1}{\tau^{(2)}})_+} \prod_{j\in A} \frac{V_j^{(2)}(M^{-1})}{V_j^{(1)}(M^{-1})}$$

for each function $f \in S^{\bar{r}}_{\bar{p},\bar{V}^{(1)},\bar{\tau}^{(1)},\theta}B$, $1 \leqslant \theta \leqslant \tau^{(2)} < \infty$, $\frac{1}{p_{j_0}} - \frac{1}{q_{j_0}} < r_{j_0}$. We recall that $y_+ = \max\{0, y\}$. This completes the proof of upper bounds.

We proceed to proving lower bounds. For a number $M \in \mathbb{N}$ we choose a natural number n such that $M \asymp 2^n n^{m-1}$ and $2^n n^{m-1} \geqslant 4M$.

We consider a function

$$f_0(\bar{x}) = n^{-\sum\limits_{j=2}^{m}\frac{1}{\theta_j}} \sum_{\langle \bar{s},\bar{\gamma}\rangle = n} \prod_{j=1}^{m} \frac{2^{-s_j(r_j+1-\frac{1}{p_j})}}{V_j^{(1)}(2^{-s_j})} \sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle \bar{k},\bar{x}\rangle}.$$

Employing a relation [35]

$$\Big\| \sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle \bar{k},\bar{x}\rangle} \Big\|^*_{\bar{p},\bar{V}^{(1)},\bar{\tau}} \asymp \prod_{j=1}^{m} 2^{s_j(1-\frac{1}{p_j})} V_j^{(1)}(2^{-s_j}),$$

for $1 < p_j, \tau_j < +\infty,\ j = 1, \ldots, m$ and Lemma 2.2, we obtain

$$\begin{aligned} &\Big\| \Big\{ 2^{\langle \bar{s},\bar{r}\rangle} \|\delta_{\bar{s}}(f_0)\|^*_{\bar{p},\bar{V}^{(1)},\bar{\tau}^{(1)}} \Big\}_{\langle \bar{s},\bar{r}\rangle = n} \Big\|_{l_{\bar{\theta}}} \\ &= n^{-\sum\limits_{j=2}^{m}\frac{1}{\theta_j}} \Big\| \Big\{ 2^{\langle \bar{s},\bar{r}\rangle} \prod_{j=1}^{m} \frac{2^{-s_j(r_j+1-\frac{1}{p_j})}}{V_j^{(1)}(2^{-s_j})} \Big\| \sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle \bar{k},\bar{x}\rangle} \Big\|^*_{\bar{p},\bar{V}^{(1)},\bar{\tau}^{(1)}} \Big\}_{\langle \bar{s},\bar{r}\rangle = n} \Big\|_{l_{\bar{\theta}}} \end{aligned}$$

$$\ll n^{-\sum\limits_{j=2}^{m}\frac{1}{\theta_j}} \|\{1\}_{\langle \bar{s},\bar{r}\rangle=n}\|_{l_{\bar{\theta}}} \leqslant C_0.$$

Thus, $f_0 \in S^{\bar{r}}_{\bar{p},\bar{V}^{(1)},\bar{\tau}^{(1)},\bar{\theta}}B$.

Let Ω_M be a set of M m-dimensional vectors $\{\overline{k}^{(1)},\dots,\overline{k}^{(M)}\}$ with integer coordinates. For each vector $\bar{s}$ obeying $\langle \bar{s},\bar{\gamma}\rangle = n$ we consider the sets $\Omega_M \cap \rho(\bar{s})$. Then by the choice of the number n the set S of the vectors $\bar{s}$ such that $\langle \bar{s},\bar{\gamma}\rangle = n$ and $|\Omega_M \cap \rho(\bar{s})| \leqslant \frac{1}{2}|\rho(\bar{s})|$ contains at least half of all $\bar{s}$ such that $\langle \bar{s},\bar{\gamma}\rangle = n$ and therefore, $|S| \asymp n^{m-1}$.

Let $T(\bar{x})$ be an arbitrary trigonometric polynomial with indices of the harmonics from Ω_M. Then by Theorem 2.3 as $\beta_j = \lambda_j = 2$, $j = 1,\dots,m$ we have

$$\begin{aligned}
\|f_0 - T\|^*_{\bar{q},\bar{V}^{(2)},\bar{\tau}^{(2)}} &\gg \left\| \left\{ \prod_{j=1}^{m} 2^{s_j(\frac{1}{2}-\frac{1}{q_j})} V_j^{(2)}(2^{-s_j})\, \|\delta_{\bar{s}}(f_0 - T)\|_2 \right\}_{\langle \bar{s},\bar{\gamma}\rangle=n} \right\|_{l_{\bar{\tau}^{(2)}}} \\
&\gg \left\| \left\{ \prod_{j=1}^{m} 2^{s_j(\frac{1}{2}-\frac{1}{q_j})} V_j^{(2)}(2^{-s_j})\, \|\delta_{\bar{s}}(f_0 - T)\|_2 \right\}_{\bar{s}\in S} \right\|_{l_{\bar{\tau}^{(2)}}} \\
&\gg n^{-\sum\limits_{j=2}^{m}\frac{1}{\theta_j}} \left\| \left\{ \prod_{j=1}^{m} 2^{s_j(\frac{1}{2}-\frac{1}{q_j})} V_j^{(2)}(2^{-s_j}) \prod_{j=1}^{m} \frac{2^{-s_j(r_j+1-\frac{1}{p_j})}}{V_j^{(1)}(2^{-s_j})} \prod_{j=1}^{m} 2^{\frac{s_j}{2}} \right\}_{\bar{s}\in S} \right\|_{l_{\bar{\tau}^{(2)}}} \qquad (3.12) \\
&\gg n^{-\sum\limits_{j=2}^{m}\frac{1}{\theta_j}} \left\| \left\{ 2^{-(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})\langle \bar{s},\bar{\gamma}\rangle} \prod_{j=1}^{m} \frac{V_j^{(2)}(2^{-s_j})}{V_j^{(1)}(2^{-s_j})} \right\}_{\bar{s}\in S} \right\|_{l_{\bar{\tau}^{(2)}}} \\
&= Cn^{-\sum\limits_{j=2}^{m}\frac{1}{\theta_j}} 2^{-n(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})} \left\| \left\{ \prod_{j=1}^{m} \frac{V_j^{(2)}(2^{-s_j})}{V_j^{(1)}(2^{-s_j})} \right\}_{\bar{s}\in S} \right\|_{l_{\bar{\tau}^{(2)}}}.
\end{aligned}$$

It is easy to make sure that if $v_j \in SVL[1,\infty)$, $j = 1,\dots,m$, then $1/v_j \in SVL[1,\infty)$, $j = 1,\dots,m$. This is why in accordance with Lemma 1 in [37] we have

$$\left\| \left\{ \prod_{j=1}^{m} \frac{1}{V_j(2^{-s_j})} \right\}_{\langle \bar{s},\bar{\gamma}\rangle=n} \right\|_{l_{\bar{\tau}'}} \ll \prod_{j=1}^{m} \frac{1}{V_j(2^{-n})} (n+1)^{\sum\limits_{j=2}^{m}\frac{1}{\tau'_j}}, \qquad (3.13)$$

where $\bar{\tau}' = (\tau'_1,\dots,\tau'_m)$, $\frac{1}{\tau_j} + \frac{1}{\tau'_j} = 1$, $1 < \tau_j < \infty$, $j = 1,\dots,m$.

Applying the Hölder inequality with $\frac{1}{\tau_j^{(2)}} + \frac{1}{\tau_j^{(2)\prime}} = 1$, $j = 1,\dots,m$ and letting $V_j(t) = \frac{V_j^{(2)}(t)}{V_j^{(1)}(t)}$, $t \in (0,1]$, $j = 1,\dots,m$, in inequality (3.13), we get

$$\begin{aligned}
|S| = \sum_{\bar{s}\in S} 1 &\leqslant \left\| \left\{ \prod_{j=1}^{m} \frac{V_j^{(2)}(2^{-s_j})}{V_j^{(1)}(2^{-s_j})} \right\}_{\bar{s}\in S} \right\|_{l_{\bar{\tau}^{(2)}}} \left\| \left\{ \prod_{j=1}^{m} \frac{V_j^{(1)}(2^{-s_j})}{V_j^{(2)}(2^{-s_j})} \right\}_{\bar{s}\in S} \right\|_{l_{\bar{\tau}^{(2)\prime}}} \\
&\leqslant \left\| \left\{ \prod_{j=1}^{m} \frac{V_j^{(2)}(2^{-s_j})}{V_j^{(1)}(2^{-s_j})} \right\}_{\bar{s}\in S} \right\|_{l_{\bar{\tau}^{(2)}}} \left\| \left\{ \prod_{j=1}^{m} \frac{V_j^{(1)}(2^{-s_j})}{V_j^{(2)}(2^{-s_j})} \right\}_{\langle \bar{s},\gamma\rangle=n} \right\|_{l_{\bar{\tau}^{(2)\prime}}}
\end{aligned}$$

$$\ll n^{\sum\limits_{j=2}^{m}\frac{1}{\tau_j^{(2)'}}}\left\|\left\{\prod_{j=1}^{m}\frac{V_j^{(2)}(2^{-s_j})}{V_j^{(1)}(2^{-s_j})}\right\}_{\bar{s}\in S}\right\|_{l_{\bar{\tau}^{(2)}}}.$$

Since $|S|\asymp n^{m-1}$, we then obtain

$$n^{m-1}\leqslant C(n+1)^{\sum\limits_{j=2}^{m}\frac{1}{\tau_j^{(2)'}}}\prod_{j=1}^{m}\frac{V_j^{(1)}(2^{-n})}{V_j^{(2)}(2^{-n})}\left\|\left\{\prod_{j=1}^{m}\frac{V_j^{(2)}(2^{-s_j})}{V_j^{(1)}(2^{-s_j})}\right\}_{\bar{s}\in S}\right\|_{l_{\bar{\tau}^{(2)}}}.$$

Therefore,

$$(n+1)^{\sum\limits_{j=2}^{m}\frac{1}{\tau_j^{(2)}}}\prod_{j=1}^{m}\frac{V_j^{(2)}(2^{-n})}{V_j^{(1)}(2^{-n})}\ll\left\|\left\{\prod_{j=1}^{m}\frac{V_j^{(2)}(2^{-s_j})}{V_j^{(1)}(2^{-s_j})}\right\}_{\bar{s}\in S}\right\|_{l_{\bar{\tau}^{(2)}}}.$$

This is why by inequality (3.12) we obtain

$$\|f_0-T\|^*_{\bar{q},\bar{V}^{(2)},\bar{\tau}^{(2)}}\gg 2^{-n(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})}\prod_{j=1}^{m}\frac{V_j^{(2)}(2^{-n})}{V_j^{(1)}(2^{-n})}(n+1)^{\sum\limits_{j=2}^{m}(\frac{1}{\tau_j^{(2)}}-\frac{1}{\theta_j})}$$

for each polynomial $T(\bar{x})$ with the indices of harmonics in Ω_M. Therefore,

$$\begin{aligned}e_M\left(S^{\bar{r}}_{\bar{p},\bar{V}^{(1)},\bar{\tau}^{(1)},\bar{\theta}}B\right)_{\bar{q},\bar{V}^{(2)},\bar{\tau}^{(2)}}&\gg 2^{-(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})}\prod_{j=1}^{m}\frac{V_j^{(2)}(2^{-n})}{V_j^{(1)}(2^{-n})}(n+1)^{\sum\limits_{j=2}^{m}(\frac{1}{\tau_j^{(2)}}-\frac{1}{\theta_j})}\\ &\gg M^{-\left(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}\right)}\prod_{j=1}^{m}\frac{V_j^{(2)}(M^{-1})}{V_j^{(1)}(M^{-1})}(\log M)^{(m-1)\left(r_{j_0}-\frac{1}{p_{j_0}}+\frac{1}{q_{j_0}}\right)+\sum\limits_{j=2}^{m}\left(\frac{1}{\tau_j^{(2)}}-\frac{1}{\theta_j}\right)}.\end{aligned}\tag{3.14}$$

Let $\tau_j^{(2)}<\theta_j$, $j=1,\ldots,m$. Since

$$|A|\leqslant m,\qquad r_{j_0}-\frac{1}{p_{j_0}}+\frac{1}{q_{j_0}}>0$$

and $\frac{v_j^{(2)}}{v_j^{(1)}}\in SVL[1,\infty)$, $j=1,\ldots,m$, then

$$\begin{aligned}&\prod_{j\notin A}\frac{V_j^{(2)}(M^{-1})}{V_j^{(1)}(M^{-1})}(\log M)^{(m-|A|)\left(r_{j_0}-\frac{1}{p_{j_0}}+\frac{1}{q_{j_0}}\right)+\sum\limits_{j\notin A\setminus\{j_1\}}\left(\frac{1}{\tau_j^{(2)}}-\frac{1}{\theta_j}\right)}\\ &\gg\prod_{j\notin A}\frac{V_j^{(2)}(2^{-1})}{V_j^{(1)}(2^{-1})}(\log 2)^{(m-|A|)\left(r_{j_0}-\frac{1}{p_{j_0}}+\frac{1}{q_{j_0}}\right)+\sum\limits_{j\notin A\setminus\{j_1\}}\left(\frac{1}{\tau_j^{(2)}}-\frac{1}{\theta_j}\right)},\end{aligned}\tag{3.15}$$

for natural numbers $M\geqslant 2$. This is why by inequality (3.14) we obtain

$$\begin{aligned}e_M\left(S^{\bar{r}}_{\bar{p},\bar{V}^{(1)},\bar{\tau}^{(1)},\bar{\theta}}B\right)_{\bar{q},\bar{V}^{(2)},\bar{\tau}^{(2)}}\gg&M^{-(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})}\\ &\cdot\prod_{j\in A}\frac{V_j^{(2)}(M^{-1})}{V_j^{(1)}(M^{-1})}(\log M)^{(|A|-1)(r_{j_0}-\frac{1}{p_{j_0}}+\frac{1}{q_{j_0}})+\sum\limits_{j\in A\setminus\{j_1\}}(\frac{1}{\tau_j^{(2)}}-\frac{1}{\theta_j})}\end{aligned}$$

for natural numbers $M\geqslant 2$ in the case $\tau_j^{(2)}<\theta_j$, $j=1,\ldots,m$.

Let

$$\theta<\tau^{(2)}<\infty,\qquad r_{j_0}\geqslant\frac{1}{p_{j_0}}-\frac{1}{q_{j_0}}+\frac{1}{\theta}-\frac{1}{\tau^{(2)}}.$$

Taking then into consideration that $\frac{v_j^{(2)}}{v_j^{(1)}} \in SVL[1,\infty)$, $j=1,\dots,m$, and $|A| \leqslant m$, we have (see (3.13)):

$$\begin{aligned}\prod_{j=1}^{m} \frac{V_j^{(2)}(M^{-1})}{V_j^{(1)}(M^{-1})} (\log M)^{(m-1)\left(r_{j_0}-\frac{1}{p_{j_0}}+\frac{1}{q_{j_0}}\right)+\sum_{j=2}^{m}\left(\frac{1}{\tau_j^{(2)}}-\frac{1}{\theta_j}\right)} \\ \gg \prod_{j\in A} \frac{V_j^{(2)}(M^{-1})}{V_j^{(1)}(M^{-1})} (\log M)^{(|A|-1)\left(r_{j_0}-\frac{1}{p_{j_0}}+\frac{1}{q_{j_0}}+\frac{1}{\tau^{(2)}}-\frac{1}{\theta}\right)}\end{aligned} \tag{3.16}$$

for natural numbers $M \geqslant 2$. Then it follows from inequalities (3.14) and (3.16) that

$$e_M\left(S^{\overline{r}}_{\overline{p},\overline{V}^{(1)},\overline{\tau}^{(1)},\theta}B\right)_{\overline{q},\overline{V}^{(2)},\tau^{(2)}} \gg M^{-\left(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}\right)} \prod_{j\in A} \frac{V_j^{(2)}(M^{-1})}{V_j^{(1)}(M^{-1})} (\log M)^{(|A|-1)\left(r_{j_0}-\frac{1}{p_{j_0}}+\frac{1}{q_{j_0}}+\frac{1}{\tau^{(2)}}-\frac{1}{\theta}\right)}$$

for natural numbers $M \geqslant 2$ in the case

$$\theta < \tau^{(2)} < \infty \quad \text{and} \quad r_{j_0} \geqslant \frac{1}{p_{j_0}} - \frac{1}{q_{j_0}} + \frac{1}{\theta} - \frac{1}{\tau^{(2)}}.$$

Let

$$\theta < \tau^{(2)} < \infty \quad \text{and} \quad \frac{1}{p_{j_0}} - \frac{1}{q_{j_0}} < r_{j_0} < \frac{1}{p_{j_0}} - \frac{1}{q_{j_0}} + \frac{1}{\theta} - \frac{1}{\tau^{(2)}}.$$

We consider a function

$$f_1(\overline{x}) = \prod_{j=1}^{m} \frac{2^{-s_j^0(r_j+1-\frac{1}{p_j})}}{V_j^{(1)}(2^{-s_j^0})} \sum_{\overline{k}\in\rho(\overline{s}^0)} e^{i\langle \overline{k},\overline{x}\rangle},$$

where $\overline{s}^0 = (s_1^0,\dots,s_m^0)$, $s_j^0 = s_j$ if $j \in A$ and $s_j^0 = 0$ if $j \notin A$. It is easy to confirm that $f_1 \in S^{\overline{r}}_{\overline{p},\bar{V}^{(1)},\overline{\tau}^{(1)},\theta}B$.

Let Ω_M be a set of M m-dimensional vectors $\{\overline{k}^{(1)},\dots,\overline{k}^{(M)}\}$ with integer coordinates. There exists $\overline{s}^0 \in \mathbb{Z}_+^m$ such that $|\rho(\overline{s}^0)| \asymp M$ and $|\rho(\overline{s}^0)| \geqslant 2M$. Then $|\rho(\overline{s}^0)\cap\Omega| \leqslant \frac{|\rho(\overline{s}^0)|}{2}$.

Let $T(\overline{x})$ stands for an arbitrary trigonometric polynomial with the indices of the harmonics in Ω_M. Then by Theorem 2.3 as $\beta_j = \lambda_j = 2$, $j=1,\dots,m$, we have

$$\begin{aligned}\|f_0 - T\|^*_{\overline{q},\bar{V}^{(2)},\overline{\tau}^{(2)}} &\gg \prod_{j=1}^{m} 2^{s_j^0(\frac{1}{2}-\frac{1}{q_j})} V_j^{(2)}(2^{-s_j^0})\, \|\delta_{\overline{s}^0}(f_0 - T)\|_2 \\ &= C\prod_{j=1}^{m} 2^{s_j^0(\frac{1}{2}-\frac{1}{q_j})} V_j^{(2)}(2^{-s_j^0}) \prod_{j=1}^{m} \frac{2^{-s_j^0(r_j+1-\frac{1}{p_j})}}{V_j^{(1)}(2^{-s_j^0})} (|\rho(\overline{s}^0)| - M)^{1/2} \\ &\gg \prod_{j=1}^{m} 2^{-s_j^0(r_j+\frac{1}{q_j}-\frac{1}{p_j})} \frac{V_j^{(2)}(2^{-s_j^0})}{V_j^{(1)}(2^{-s_j^0})} \gg M^{-(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})} \prod_{j\in A} \frac{V_j^{(2)}(M^{-1})}{V_j^{(1)}(M^{-1})}.\end{aligned}$$

The proof is complete. □

Corollary 3.1. *Let the numbers* $q_j, p_j, \tau_j^{(1)}, \tau_j^{(2)}, r_j \in \mathbb{R}$, $j=1,\dots,m$, *satisfy the assumptions of Theorem 3.1 and the functions*

$$v_j^{(1)}(t) = (1+\log t)^{a_j}(1+\log(1+\log t))^{b_j},$$
$$v_j^{(1)}(t) = (1+\log t)^{a_j}(1+\log(1+\log t))^{c_j}, \qquad a_j, b_j, c_j \in \mathbb{R}, \qquad c_j > b_j, \qquad j=1,\dots,m.$$

1. *If* $1 < \tau_j^{(2)} < \theta_j \leqslant \infty$, $j = 1, \ldots, m$, *then*

$$e_M\big(S^{\overline{r}}_{\overline{p},\bar{V}^{(1)},\overline{\tau}^{(1)},\overline{\theta}}B\big)_{\overline{q},\bar{V}^{(2)},\overline{\tau}^{(2)}} \asymp M^{-(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})}\prod_{j\in A}\Big(1+\log(1+\log M)\Big)^{c_j-b_j}$$
$$\cdot (\log M)^{(|A|-1)(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}})}(\log M)^{\sum\limits_{j\in A\setminus\{j_1\}}(\frac{1}{\tau_j^{(2)}}-\frac{1}{\theta_j})}.$$

2. *If* $1 \leqslant \theta \leqslant \tau^{(2)} < +\infty$, *then*

$$e_M\big(S^{\overline{r}}_{\overline{p},\bar{V}^{(1)},\overline{\tau}^{(1)},\theta}B\big)_{\overline{q},\bar{V}^{(2)},\overline{\tau}^{(2)}} \asymp M^{-\left(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}\right)}\prod_{j\in A}\Big(1+\log(1+\log M)\Big)^{c_j-b_j}$$
$$\cdot (\log M)^{(|A|-1)(r_{j_0}+\frac{1}{q_{j_0}}-\frac{1}{p_{j_0}}+\frac{1}{\tau^{(2)}}-\frac{1}{\theta})_+}.$$

Proof. It is known, see, for instance, [6, Ch. 3, Subsec. 3.4.3], that the functions

$$v_j^{(1)}(t) = (1+\log t)^{a_j}(1+\log(1+\log t))^{b_j}, \qquad v_j^{(1)}(t) = (1+\log t)^{a_j}(1+\log(1+\log t))^{c_j},$$

belong to the class $SV[1,\infty)$ for $j = 1, \ldots, m$ and their quotient

$$\frac{v_j^{(2)}(t)}{v_j^{(1)}(t)} = (1+\log(1+\log t))^{c_j-b_j}$$

increases, while $\frac{v_j^{(2)}(t)}{v_j^{(1)}(t)}t^{-\varepsilon}$ almost decreases on $[1,\infty)$, $\varepsilon > 0$, $j = 1, \ldots, m$. This is why the statements of the corollary are true according to Theorem 3.1. The proof is complete. □

Remark 3.1. *In the case* $V_j^{(1)}(t) = V_j^{(2)}(t) = 1$, $t \in (0,1]$, *and* $p_j = \tau_j^{(1)} = p$, $q_j = \tau_j^{(2)} = q$, $\theta_j = \theta$ *for* $j = 1, \ldots, m$ *and* $r_1 = \ldots = r_\nu < r_{\nu+1} \leqslant \ldots \leqslant r_m$ *Theorem 3.1 coincides with earlier results by V.N. Temlyakov* [19, Thm. 2.2] *and A.S. Romanyuk* [22, Thm. 3.1] *for the Lebesgue spaces and in the general case it extends these results to anisotropic Lorentz-Karamata spaces. For* $V_j^{(1)}(t) = V_j^{(2)}(t) = 1$, $t \in (0,1]$, $j = 1, \ldots, m$, *and* $r_1 = \ldots = r_\nu < r_{\nu+1} \leqslant \ldots \leqslant r_m$ *Theorem 3.1 coincides with* [28, Thm. 3] *and* [34, Thm. 5].

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