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REMARKS ON GARSIA ENTROPY AND MULTIDIMENSIONAL ERDÖS MEASURES

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Abstract. We conjecture that the Garsia entropy coincides with the entropy of the invariant multidimensional Erdös measure. This conjecture is true for all Garsia numbers. We also specify the algebraic unit being non-Pisot number, for which this conjecture is true.

We prove a theorem, which generalizes the Garsia theorem on the absolute continuity of the infinite Bernoulli convolution for the Garsia numbers. The proof uses relations between the multidimensional Erdös problem and the one-dimensional Erdös problem.

We discuss a connection between the entropy of the invariant Erdös measure and the conditional Ledrappier–Young entropies. We also formulate three conjectures and obtain some consequences from them. In particular, we conjecture that the Hausdorff dimension of the Erdös measure is equal to the Ledrappier–Young dimension of conditional measure for the multidimensional invariant Erdös measure along the unstable foliation corresponding to the top Lyapunov exponent of multiplicity 1. For 2-numbers, we obtain formulae for the Hausdorff dimension of Erdös measures on the unstable plane.

Keywords: Garsia entropy, Hausdorff dimension of the measure, Erdös measure, Hochman formula, Lyapunov exponent.

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1. Introduction

In this paper we consider infinite Bernoulli convolutions associated with a number $\beta \in (1,2)$ and a parameter $p_0 \in (0,1)$. We define a random variable as

$$\zeta(\beta, p_0) = \sum_{k=1}^{\infty} \beta^{-k} \varepsilon_k,$$

where $\{\varepsilon_k\}_{k\in\mathbb{N}}$ are independent identically distributed (i.i.d.) random variables:

$$\mathrm{P}(\varepsilon_k = 0) = p_0, \qquad \mathrm{P}(\varepsilon_k = 1) = p_1 = 1 - p_0, \qquad 0 < p_0 < 1.$$

We observe that the series in the previous formula converges for each realization of a sequence of independent variables $\{\varepsilon_k\}_{k\in\mathbb{N}}$.

We define the following measure: $\mu(\Delta) = \mathrm{P}(\zeta \in \Delta)$. The measure μ is called an infinite Bernoulli convolution; we shall also call it Erdös measure.

According to the Jessen–Wintner theorem (1935), the distribution of the random variable ζ is either absolutely continuous or purely singular, that is, the mixed case is impossible. It is naturally to ask, for which values of β the distribution of ζ is singular and for which values of β it is absolutely continuous. This problem was formulated by Erdös 83 years ago [1] and it seems to be very simple at the first glance. But in fact, this issue is very difficult. In 1995, B. Solomyak proved that for almost all β, $1 < \beta < 2$ and $p_0 = \frac{1}{2}$, the distribution of ζ is

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absolutely continuous. Erdös proved in 1939 that if β is a Pisot number, then the distribution of ζ is singular. For example, if β is the golden ratio, then the distribution of ζ is singular.

Another proof of the singularity of the Erdös measure for the case of the golden ratio was given in [2]. The proof used the notion of an invariant Erdös measure for the transformation T_1 of the interval $[0,1]$: $x \to T_1 x = \{\beta x\}$, where $\{x\}$ denotes the fractional part of the number x.

Definition 1.1. *An invariant Erdös measure μ_1 is a T_1-invariant probability measure on $[0,1]$ such that the restriction of the Erdös measure on $[0,1]$ is absolutely continuous with respect to μ_1 [2].*

For each Pisot number β there is a unique T_1-invariant Erdös measure (see [2], [3]).

Conjecture 1.1. *There is a unique T_1-invariant Erdös measure for each number β.*

We denote $\rho = \beta^{-1}$ and define the contractions $S_0 : x \to \rho x,\ \ S_1 : x \to \rho x + \rho,\ x \in \mathbb{R}^+$. Let K_1 be (a closed) support of μ. It is well known that K_1 is a unique solution of the Hutchinson equation (see [4])

$$K_1 = S_0 K_1 \cup S_1 K_1.$$

It is easy to see that

$$K_1 = \left[0; \frac{1}{\beta - 1}\right].$$

The set K_1 is the attractor of the Iterated Function System (IFS) (S_0, S_1), that is the set K_1 is the attractor for Markov chain on real line with the transition probabilities p_0 and p_1 for the following transitions:

$$x \to S_0 x, \qquad x \to S_1 x.$$

The Erdös measure μ is a unique stationary distribution for this Markov chain.

Remark 1.1. *There is a connection with a first order autoregresion model $AR(1)$. Let $AR(1)$ model be*

$$X_{n+1} = \beta^{-1} X_n + \beta^{-1} \varepsilon_{n+1}.$$

We consider a stationary Markov process X_n satisfying this relation. Then the Erdös measure μ is the one-dimensional distribution of this process. That is, the Erdös measure is the stationary distribution for a first-order autoregressive model with the Bernoulli noise.

The measure μ satisfy the following equation, which is a self-similarity equation:

$$\mu = p_0 \mu \circ S_0^{-1} + p_1 \mu \circ S_1^{-1},$$

where $0 < p_0 < 1$, $p_0 + p_1 = 1$. Another form of the self-similarity equation reads as

$$\mu(\Delta) = p_0 \mu(\beta \Delta) + p_1 \mu(\beta \Delta - 1).$$

We observe that the self-similarity equation is simply the equation for the invariant measure of our Markov chain. The solution of this equation is the Erdös measure with parameters β, p_0. We note that here we deal with the simplest random dynamical system. We recall that Erdös proved that the Erdös measure is singular if β is a Pisot algebraic integer; no other cases of singularity are known.

Let us recall the following definitions.

- An algebraic integer β is the root of an irreducible (minimal) polynomial $p(x)$ with integer coefficients and the highest coefficient 1.
- If $\beta > 1$ and the absolute values of all other roots are less than 1, then β is called Pisot number.

We also recall that Hausdorff dimension of any probability measure μ is defined by the formula

$$\dim_H(\mu) = \inf(\dim_H(\Delta) : \mu(\Delta) = 1).$$

Determining the dimension of self-similar measures is a fundamental problem in fractal geometry.

Remark 1.2. *It is well known that the Erdös measure is exact dimensional, that is,*

$$\dim_H(\mu) = \lim_{\varepsilon\to+0} \frac{\log(\mu(\Delta_x))}{\log\varepsilon},$$

where Δ_x denotes the ε-neighborhood of x: $\Delta_x = (x-\varepsilon, x+\varepsilon)$, $\varepsilon > 0$ and the limit is independent of x for almost all x with respect to the measure μ.

Hausdorff dimension of μ and Garsia entropy The Garsia entropy is defined by the formula [5]:

$$h_G(\beta, p_0) = \lim_{n\to\infty} \frac{H(\zeta_n)}{n},$$

where

$$\zeta_n = \sum_{k=1}^{n} \beta^{-k}\varepsilon_k$$

and $H(\zeta_n)$ is the Shennon entropy of the random variable ζ_n.

For algebraic integer β, the Garsia entropy is the absolute value of top Lyapunov exponent of some sequence of independent random matrices, see [6]. For the Pisot numbers, the Garsia entropy is equal to the entropy of some hidden Markov chain, that is, this is the absolute value of the top Lyapunov exponent with respect to the Markov measure, see [2], [3].

The paper is organized as follows. In Section 2 we prove a theorem, which generalizes the Garsia theorem on the absolute continuity of the infinite Bernoulli convolution for the Garsia numbers. In Sections 3 and 4 we discuss the entropy of the invariant Erdös measure and the conditional Ledrappier–Young entropies. We also formulate some conjectures and obtain some consequences from them. In Section 4, we obtain formulas for the Hausdorff dimension of Erdös measures on an unstable plane for 2-numbers.

2. Garsia theorem and related issues

Garsia [5] proved that the Erdös measure is absolutely continuous as β is an algebraic integer number, the absolute value of all the roots of the minimal polynomial are greater than 1, the absolute value of the free coefficient of the minimal polynomial is equal to 2 and $p_0 = 0.5$. We call such numbers Garsia numbers. Here we give an alternative proof of the Garsia theorem.

For algebraic integers β, the Hausdorff dimension of the Erdös measure can be calculated by the Hochman formula [7]:

$$\delta_1 = \dim_H(\mu) = \min\left\{1, \frac{h_G(\beta, p_0)}{\log\beta}\right\}.$$

For Garsia numbers the Garsia entropy reads as

$$h_G(\beta, p_0) = -p_0\log p_0 - p_1\log p_1.$$

Indeed, if

$$\sum_{k=1}^{n} x_k\beta^{-k} = \sum_{k=1}^{n} y_k\beta^{-k},$$

where $x_k, y_k \in \{0, 1\}$, $k = 1, \ldots, n$, then the polynomial

$$f(x) = \sum_{k=1}^{n} (x_k - y_k) x^{n-k}$$

has the root β. Hence, $f(x)$ is divisible by minimal polynomial $p(x)$ of β. But this is impossible because $p(0) = \pm 2$ and $x_k - y_k = 0, \pm 1$.

From the equation $h_G(\beta, p_0^{cr}) = \log \beta$ we find the critical value $p_0^{cr} < 0.5$. For Garsia numbers, the Erdös measure is singular if $p_0 \in (0, p_0^{cr}) \cup (1 - p_0^{cr}, 1)$. Otherwise, the Hausdorff dimension of the Erdös measure is 1. This follows from the Hochman formula.

We shall prove the theorem, which generalizes the Garsia theorem on the absolute continuity of the infinite Bernoulli convolution for the Garsia numbers. The proof uses the connection between the multidimensional Erdös problem and the one-dimensional Erdös problem.

Multidimensional Erdös problem. Multidimensional Erdös problem is the problem on the distribution of a random variable

$$\eta = \sum_{k=1}^{\infty} A^{-k} \xi_k, \qquad \xi_k \in \mathbb{Z}^d, \qquad d \geqslant 2,$$

where A is an expanding matrix, the absolute values of all its eigenvalues are greater than 1 and ξ_k are i.i.d. random variables.

For the minimal polynomial of number β

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{d-1} x^{d-1} + x^d,$$

we consider the residue field of polynomial ring $\mathbb{R}[x]$ modulo $p(x)$. This field is a linear d-dimensional space and $(1, x, \ldots, x^{d-1})$ is the basis in this space.

The matrix of the linear multiplication operator by x with respect to this basis has the form

$$A = \begin{pmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & \ldots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -a_{d-1} \end{pmatrix}.$$

By definition this is the companion matrix for the polynomial $p(x)$.

For Garsia number $|\det A| = 2$. Let ξ_k be $\xi_k = \varepsilon_k e$, where e is a non-zero solution of the equation

$$\left[A^{-1} r\right] = 0, \qquad r \in \mathbb{Z}^d.$$

Proposition 2.1. *The distribution of the random variable η is absolutely continuous at $p_0 = 0.5$ and singular at $p_0 \neq 0.5$.*

This proposition follows from [8].

Let us explain the connection between the multidimensional and one-dimensional Erdös problems for the algebraic integer β. Let $\beta_1 = \beta, \beta_2, \ldots, \beta_d$ be the roots of the polynomial $p(x)$. We define η, l_i, $i \in \{1, \ldots, d\}$ provided $\xi_k = \varepsilon_k e$:

$$\eta = \sum_{k=1}^{\infty} \varepsilon_k A^{-k} e,$$

$$l_i \in \mathbb{R}^d : \ l_i A = \beta_i l_i, \ i = 1, \ldots, d.$$

Proposition 2.2. *The left eigenvector (row) l_i of the companion matrix A, corresponding to the eigenvalue β_i, has the form*

$$l_i = (1, \beta_i, \beta_i^2, \ldots, \beta_i^{d-1}).$$

The Garsia absolute continuity theorem is a special case of the following theorem. In particular, it provides an alternative proof of the Garsia theorem.

Theorem 2.1. *Distribution of the random variable* $\zeta(\beta_i, 0.5)$ *is absolutely continuous.*

Proof. We have

$$l_i \eta = l_i e \zeta(\beta_i, p_0).$$

The numbers $l_i e$ are non-zero for all $i = 1, \dots, d$. Indeed, if for some i the identity $l_i e = 0$ holds, then according to Proposition 2.2 there exists a polynomial of x of degree $d-1$, which has the root β_i. But this is impossible, since $p(x)$ is the minimal polynomial of degree d.

Let us recall that the distribution of the random variable η is absolutely continuous at $p_0 = 0.5$. Hence, it follows that the distribution of random variable $\zeta(\beta_i, 0.5) = \frac{l_i \eta}{l_i e}$ is absolutely continuous. The proof is complete. □

3. Invariant Erdős measure on torus $\mathbb{T}^d$

Let A be a hyperbolic matrix, $A \in GL(d, \mathbb{Z})$. Let L_u be the unstable subspace in $\mathbb{R}^d$ for the matrix A corresponding to the points λ, $|\lambda| > 1$ of the spectrum of the matrix A. Similarly, L_s is the stable subspace in $\mathbb{R}^d$ for the matrix A corresponding to the points λ, $|\lambda| < 1$ of the spectrum of the matrix A. We have

$$\mathbb{R}^d = L_u \oplus L_s, \qquad AL_u = L_u, \qquad AL_s = L_s.$$

We consider $VAR(1)$-model

$$X_{n+1} = A^{-1} X_n + \varepsilon_{n+1} e,$$

where $\{\varepsilon_k,\ k \in \mathbb{Z}\}$ are i.i.d.,

$$\mathrm{P}(\varepsilon_k = 0) = p_0, \qquad \mathrm{P}(\varepsilon_k = 1) = p_1, \qquad p_0 + p_1 = 1, \qquad p_0, p_1 > 0,$$

and $e \neq 0$, $e \in \mathbb{Z}^d$, that is, $e = (1, 0, \dots, 0)^T$.

Let π_s be the projector on L_s, $\pi_s L_u = 0$. Similarly, let π_u be the projector on L_u, $\pi_u L_s = 0$. Note that $A\pi_s = \pi_s A$ and $A\pi_u = \pi_u A$. In addition, we denote $e_s = \pi_s e$, $e_u = \pi_u e$, $e_s + e_u = e$.

We also consider the equations

$$\pi_u X_{n+1} = A^{-1} \pi_u X_n + \varepsilon_{n+1} e_u$$
$$\pi_s X_{n+1} = A^{-1} \pi_s X_n + \varepsilon_{n+1} e_s$$

and its particular solutions

$$X_n^u = \sum_{k=0}^{\infty} A^{-k} e_u \varepsilon_{n-k},$$
$$X_n^s = -\sum_{k=1}^{\infty} A^{k} e_s \varepsilon_{n+k}.$$

We consider a particular solution to the VAR-model:

$$X_n = X_n^s + X_n^u, \qquad X_n^s = \pi_s X_n, \qquad X_n^u = \pi_u X_n. \tag{3.1}$$

In particular,

$$X_0 = \ldots - A^2 e_s \varepsilon_2 - A e_s \varepsilon_1 + A^0 e_u \varepsilon_0 + A^{-1} e_u \varepsilon_{-1} + \ldots .$$

Proposition 3.1. *Process* $\{X_n,\ n \in Z\}$ *is a stationary process.*

The proof follows immediately from formulae (3.1).

Let $\mathbb{T}^d$ be $\mathbb{R}^d/\mathbb{Z}^d$. We define the algebraic endomorphism T_d of $\mathbb{T}^d$:

$$T_d : x \to Ax \mod \mathbb{Z}^d.$$

We have

$$AX_{n+1} = X_n + \varepsilon_{n+1} Ae.$$

We define $\widetilde{X}_n = X_n \bmod \mathbb{Z}^d$ which satisfies the equation

$$\widetilde{X}_n = A\widetilde{X}_{n+1} \mod \mathbb{Z}^d.$$

Process $\widetilde{X}_n$ is $\mathbb{T}^d$-valued stationary process. Let μ_A denote distribution of the random variable $\widetilde{X}_0 = X_0 \bmod \mathbb{Z}^d$. The measure μ_A is T_d-invariant measure.

Remark 3.1. *Let* $\det A = \pm 1$. *If in the previous formulae we replace A by A^{-1}, then on the torus we get a new stationary process $\widetilde{Y}_n = -\widetilde{X}_n$. Hence, this implies that*

$$\mu_{A^{-1}}(\Delta) = \mu_A(-\Delta),$$

where $-\Delta = \{-x \mid x \in \Delta \subset \mathbb{T}^d\}$.

Proposition 3.2. *The entropy of algebraic automorphism $T_d : x \to Ax$ of $\mathbb{T}^d$ with respect to the measure μ_A is equal to the entropy of algebraic automorphism T_d with respect to the measure $\mu_{A^{-1}}$ and*

$$h(T_d, \mu_A) = h(T_d^{-1}, \mu_{A^{-1}}) = h(T_d, \mu_{A^{-1}}).$$

Proof. The map $x \to -x$ establishes an isomorphism of the transformation T_d with the invariant measure μ_A and the transformation T_d with the invariant measure $\mu_{A^{-1}}$. Hence, the coincidence of entropies follows. The proof is complete. □

Definition 3.1. *We call the distribution of the random variable $\widetilde{X}_0$ multidimensional invariant Erdös measure for the number β.*

Conjecture 3.1. *The Garsia entropy $h_G(\beta, p_0)$ is equal to the entropy of the transformation T_d with respect to the invariant Erdös measure μ_A, where A is the companion matrix of the number β.*

Recall the definition of the invariant Erdös measure μ_1 for the transformation $T_1 : x \to \{\beta x\}$, $x \in [0, 1]$. An invariant Erdös measure μ_1 is a T_1-invariant probability measure on $[0, 1]$ such that the restriction of the Erdös measure on $[0, 1]$ is absolutely continuous with respect to μ_1.

Conjecture 3.2. *The endomorphism T_1 with an invariant Erdös measure μ_1 is isomorphic to the factor-endomorphism of the transformation T_d with an invariant Erdös measure μ_A such that its entropy is equal to the Ledrappier–Young entropy h_1 along the unstable foliation corresponding the top Lyapunov exponent logβ. Moreover, the Hausdorff dimension of the measure μ_1 ($\dim_H(\mu_1) = \delta_1 = \dim_H(\mu)$) is equal to the Ledrappier–Young dimension γ_1 of measure μ_A along the unstable foliation corresponding to the top Lyapunov exponent* $\log \beta$ *of multiplicity 1.*

Remark 3.2. *By the Ledrappier–Young formula* [9],

$$h(T_d, \mu_A) = \gamma_1 \lambda_1 + \gamma_2 \lambda_2 + \cdots + \gamma_r \lambda_r,$$

where r is the number of positive Lyapunov exponents for T_d. In our case, the Lyapunov exponents are the logarithms of the absolute values of the roots of the minimal polynomial for the number β.

Hence, if $p_0 = 0.5$ *then* β *is a Pisot number if and only if* $h_G(\beta, 0.5) < \log \beta$. *The identity* $h_G(\beta, 0.5) = \log \beta$ *is impossible for the case of the hyperbolic automorphism* T_d. *If* β *is not a Pisot number, then* $h_G(\beta, 0.5) > \log \beta$. *Then according to the Hochman formula*

$$\gamma_1 = \delta_1 = \min \left\{ 1, \frac{h_G(\beta, 0.5)}{\log \beta} \right\}$$

we obtain $\delta_1 = 1$.

For Pisot number and for each p_0 by the Hochman formula we obtain $\delta_1 < 1$. Hence, $h_G(\beta, p_0) = h(T_1, \mu_1)$. According Conjecture 3, $h(T_1, \mu_1) = h(T_d, \mu_A)$, i.e. $h_G(\beta, p_0) = h(T_d, \mu_A)$. So, for Pisot number, Conjecture 3.1 follows from Conjecture 3.2.

Definition 3.2. *An algebraic integer* β *is called* k*-number if* $\dim L_u = k$.

Let $\det A = \pm 1$ and A be a hyperbolic matrix, $A \in GL(d, \mathbb{Z})$. We define the dual multidimensional Erdös problem by replacing A with A^{-1}. Interesting cases are $k = d-2, d-1$. Let $k = d-2$ and l be the left α-eigenvector: $lA = \alpha l$, $|\alpha| < 1$, $\alpha \neq \overline{\alpha}$, $l = (1, \alpha, \ldots, \alpha^{d-1})$. Then for multidimensional Erdös problem with the matrix A^{-1} and corresponding vector η we have

$$l\,\eta = l\,e\,\zeta(\alpha^{-1}, p_0) = \zeta(\alpha^{-1}, p_0).$$

The number α^{-1} is an example of a complex Pisot number.

Let $\widetilde{\beta} = \alpha^{-1}$. Denote the companion matrix of $\widetilde{\beta}$ by B. The matrices B and A^{-1} are conjugate in the group $GL(d, \mathbb{Z})$. Let $B = CA^{-1}C^{-1}$, $C \in GL(d, \mathbb{Z})$.

The map $T_C : x \to Cx \bmod \mathbb{Z}^d$ gives semiconjugacy for the automorphisms of multiplication by matrices B and A^{-1} on the torus $\mathbb{T}^d$.

We consider VAR-model for A^{-1}:

$$X_{n+1} = AX_n + \varepsilon_{n+1} e.$$

Let $Y_n = CX_n$, then

$$Y_{n+1} = BY_n + \varepsilon_{n+1} Ce.$$

The distribution of the random variable $\widetilde{Y}_0 = Y_0 \bmod \mathbb{Z}^d$ is denoted by ν. We observe that $\widetilde{Y}_0 = T_C \widetilde{X}_0$.

Proposition 3.3. *The automorphism* T_d^{-1} *with invariant measure* $\mu_{A^{-1}}$ *has the same entropy as the automorphism of multiplication by the matrix* B *with invariant measure* ν.

Proof. The proposition follows from the semi-conjugation of the automorphisms of multiplication by matrices B and A^{-1} on the torus $\mathbb{T}^d$ and from the fact that

$$\#\{T_C^{-1} x\} = |\det C|.$$

□

Let X_n be a stationary $VAR(1)$-process: $X_{n+1} = A^{-1} X_n + \varepsilon_{n+1} e$, which we denote by $X_n(A)$.

Definition 3.3. *The distribution* μ_A^u *of random variable* $\pi_u X_0(A)$ *is called the Erdös measure on* L_u.

Let β be a cubic ($\deg p(x) = 3$) complex 2-number, $\operatorname{Im} \beta \neq 0$, i.e. $\beta_1 = \beta$, $\beta_2 = \overline{\beta}$, β is a complex Pisot number, $|\beta| > 1$ and $0.5 < \beta_3 < 1$, where β_3 is the second Galois conjugate. Let A be the companion matrix for the real Pisot number β_3^{-1}. Then the number β corresponds to the matrix A^{-1} and the measure $\mu_{A^{-1}}$. For a real Pisot number β_3^{-1} we have

$$h(T_3, \mu_A) = \dim_H(\mu_1) \log(\beta_3^{-1}).$$

Similarly for the β:

$$h(T_3^{-1}, \mu_{A^{-1}}) = \dim_H(\mu_{A^{-1}}^u) \log |\beta|.$$

Proposition 3.4. *For a 2-number β satisfying the conditions described above, the Hausdorff dimension of the Erdös measure on L_u is equal to the doubled dimension of the Erdös measure for the number β_3^{-1}.*

Proof. The proposition follows from the identity $h(T_3, \mu_A) = h(T_3^{-1}, \mu_{A^{-1}})$. □

4. Examples with 2-numbers

Example 4.1. *Consider the 2-number β with the minimal polynomial:*

$$p(x) = x^3 + x^2 + x - 1,$$

$$\alpha \approx -0.420 + 0.606\imath, \qquad |\alpha| < 1, \qquad |\alpha|^{-1} < 2, \qquad \beta = \alpha^{-1},$$

β satisfies the assumptions of Proposition 3.4. In addition, β_3^{-1} is the tribonacci number. In particular, the Hausdorff dimension of the Erdös measure for the number β_3^{-1} can be calculated with a high precision for each p_0, see [10].

By Proposition 3.4, the largest dimension $\dim_H(\mu^u_{A^{-1}})$ for $p_0 = 0.5$ is approximately equal to 1.96081862.

Example 4.2. *We consider positive 2-number β with the minimal polynomial:*

$$p(x) = x^3 - x^2 - 2x + 1.$$

The roots of this polynomial are:

$$\beta = \beta_1 \approx 1.8019, \qquad \beta_2 \approx -1.2470, \qquad \beta_3 \approx 0.4450.$$

We have $\beta_3^{-1} \approx 2.2470$. This is the Pisot number.

Let A be the companion matrix of the polynomial $p(x)$. The companion matrix for β_3^{-1} is similar to the matrix A^{-1} in the group $GL(3, \mathbb{Z})$. Since $2 < \beta_3^{-1} < 3$, in the usual Erdös problem three digits 0, 1, 2 are used and the corresponding IFS is an overlapping system. In our case, one of the nonzero digits is missing, and the corresponding IFS has no overlaps. This yields

$$h(T_3^{-1}, \mu_{A^{-1}}) = -p_0 \log p_0 - p_1 \log p_1$$

since β_3^{-1} is Pisot number and the Erdös measure with two digits 0, 1 is invariant under the endomorphism $T_1 : x \to \{\beta_3^{-1} x\}$.

In this example

$$h(T_3, \mu_A) = \gamma_1 \log \beta_1 + \gamma_2 \log |\beta_2|.$$

Depending on p_0, if $h(T_3, \mu_A) < \log \beta_1$, then

$$\gamma_1 \log \beta_1 = h_G(\beta_1, p_0) = h(T_3, \mu_A).$$

Hence, $\gamma_2 = 0$. If $h(T_3, \mu_A) \geqslant \log \beta_1$, then $\gamma_1 = 1$. This implies

$$\gamma_2 = \frac{H(p_0) - \log \beta_1}{\log |\beta_2|}, \qquad \dim(\mu^u_A) = 1 + \gamma_2.$$

Hence, the largest dimension $\dim_H(\mu^u_A)$ (for $p_0 = 0.5$) is approximately equal to 1.473.

Conclusion

We associated the endomorphism T_1 with the one-dimensional unstable foliation corresponding to the top Lyapunov exponent. We also define the transformation T_r, $r = \dim L_u$ with the invariant Erdös measure μ_r corresponding to the Erdös measure μ^u_A. It can be shown that the measure μ_r is a sofic measure and Conjecture 3.2 can be generalized to this case. We also conjecture that for $p_0 = 0.5$ Erdös measure is singular if and only if β is Pisot number.

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